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SOME REMARKS ON ORDERED FIELDS

by Masayoshi NAGATA

Let K be a field. It is well known that one can give an order in K so that K becomes an ordered field if, and only if, K is formally real, namely, -1 cannot be the sum of the squares of any finite number of elements in K. It is also well known that a formally real field is of characteristic zero, hence it contains the rational number field Q.

The purpose of the present note is to give some remarks on the structure of an order of an ordered field and they can be stated in two theorems :

THEOREM 1. - *Let K be an ordered field. Set*

$$V = \{x \in K \ ; \ -n < x < n \ \text{ for some natural number } n\}$$

and

$$P = \{x \in K \ ; \ -(1/n) < x < 1/n \ \text{ for every natural number } n\} .$$

Then (V , P) is a valuation ring (i. e., V is a valuation ring with maximal ideal P), and the residue class field V/P is naturally isomorphic to a subfield of the real number field R.

Conversely,

THEOREM 2. - *Let (V , P) be a valuation ring of a field K. Assume that there is an injection φ of the field V/P into the real number field R. Then one can make K an ordered field so that (V , P) coincides with such a pair defined in Theorem 1 with respect to the given order.*

Thus orders for a field K corresponds to valuation rings of K having residue class fields contained in R. But the correspondence is not one-one, as we shall discuss at the end of the article.

Proof of Theorem 1. - Assume that $x \in K$ and $x \notin V$. Then either x or $-x$ is greater than any natural number n, which implies that $-(1/n) < x^{-1} < 1/n$ and therefore $x^{-1} \in P$. Thus we see that (V , P) is a valuation ring of K. For each x in V, we set

$$S_x = \{r \in Q \ ; \ x < r\} .$$

Then $\inf S_x$ exists in the real number field R. Let ψ be the mapping of V in R such that $\psi x = \inf S_x$. Then one sees easily the ψ is a homomorphism whose kernel coincides with P.

Q. E. D.

Before proving Theorem 2, we need

LEMMA 3. - *Let* (W , P') *be a maximally complete valuation ring* [1] *of a field* L *such that*

(i) W/P' *is algebraically closed and*

(ii) *the value group of the valuation is divisible. Then the field* L *is algebraically closed.*

Proof follows immediately from the definition of maximal completeness and we omit the detail.

Q. E. D.

COROLLARY 4. - *Let* (V , P) *be a maximally complete valuation ring of a field* K *such that*

(i) V/P *is real closed, and*

(ii) *the value group of the valuation is divisible. Then* K *is real closed* [2].

By the way, we note that the following is immediate from the definition of maximal completeness.

LEMMA 5. - *Let* (W , P') *be a maximally complete valuation ring of a field* L *such that*

(i) W/P' *is algebraically closed, and*

(ii) *the value group of the valuation is divisible. Then the field* L *is algebraically closed.*

Proof of Theorem 2. - In order to prove the theorem, we may replace K with an extension field. Therefore, first of all, we may assume that V is maximally complete. Then V is henselian and contains Q, and therefore every maximal subfield K^* of V forms a complete set of representatives for V/P. Therefore, extending residue class field, we may assume that $V/P = R$. On the other hand, if the value group G is not divisible, for instance if there is a g in G for which h such that $ph = g$ does not exist (p being a prime number), then we may add g/h to G by adjoining, p-th root of an element whose value is g. Repeating such a process, we may assume that G is divisible. Then corollary 4 implies that K is

(1) It is well known that a field K is real closed if, and only if, (i) K itself is not algebraically closed, and (ii) the algebraic closure of K is of finite degree over K, or if, and only if, $K(\sqrt{-1})$ is algebraically closed besides the condition (i) above.

(2) For the notion of maximal completeness (due to KAPLANSKY), see for instance ; SCHILLING (O. F. G.). - The theory of valuations. - New York, American mathematical Society, 1950 (Mathematical Surveys, 4).

real closed. K^* may be identified with R . Furthermore, setting

$$S = \{x^2 \ ; \ 0 \neq x \in K\} ,$$

we know that K is the disjoint union of S , $-S$, $\{0\}$, and K has a unique structure as an ordered field. Under the order, $a > b$ if, and only if, $a-b \in S$.

(1) Assume that $a \in P \cap S$ and that $a > 1/n$ for a natural number n . Then $a - (1/n) = b^2$ with $b \in K$. This implies that $-(1/n) \equiv b^2$ modulo P , which is impossible because $V/P = R$. Thus $a \in P \cap S$ implies that $a < 1/n$ for every natural number n . Therefore $a' \in P$ implies that $-(1/n) < a' < 1/n$ for every natural number n .

(2) Assume that $b \in S$, $b \notin V$. Then $b^{-1} \in P \cap S$, and therefore $b > n$ for every natural number n by virtue of (1) above.

(3) Assume now that $c \in S \cap V$. Then there is a $c^* \in K^*$, such that $c-c^* \in P$. By (1) above, we see that $-(1/n) < c - c^* < 1/n$ for every natural number n , hence $c^* - (1/n) < c < c^* + (1/n)$.

In view of these (1) ~ (3), we see easily that the valuation ring defined by the order of K coincides with V .

Q. E. D.

Remarks on the correspondance. - Many different orders of field K may give the same valuation ring (V , P) . Roughly speaking, there are two kind of reasons for this.

One is by the injection φ of V/P in R . Namely let ψ be a homomorphism of V into R whose kernel is P . Then, if ψ can be changed, then we surely have a different order in K .

Thus, from now on, we fix ψ also. Then, another cause comes from the structure of the value group G of the valuation defined by V . Namely, let G' be

$$\{2g \ ; \ g \in G\} .$$

For each $g \in G$, let c_g be an element of K whose value is g ; here, if $g \in G'$, we choose c_g to be a square element. Let U be the unit group of V . If $g \in G$, then c_g must be a positive element, hence $c_g u$ $(u \in U)$ is positive if and only if, ψu is positive. Thus

PROPOSITION 6. - *If* $G = G'$ *(i. e., if* G *is 2-divisible), then the pair* (V , ψ) *defines an order of* K *uniquely.*

If $G \neq G'$, then we surely have arbitrarity in adjoining square roots on our way to extend the value group G to a 2-divisible group, and the arbitrarity allows us to alter positivity of certain elements.

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