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Absolutely- p -summing operators in \mathcal{L}_r -spaces II

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ABSOLUTELY-p-SUMMING OPERATORS IN \mathcal{L}_r -SPACES II

by A. PIETSCH

§ 6. THE v_p -NORM (cf. [10], [23], [24]).

In the following let us assume that at least one of the Banach spaces E and F has finite dimension. Then every operator $T \in \mathcal{L}(E, F)$ can be represented in the form

$$Tx = \sum_i \langle x, a_i \rangle y_i \quad \text{for all } x \in E$$

with $a_1, \dots, a_n \in E'$ and $y_1, \dots, y_n \in F$. Now the v_p -norm is defined by

$$v_p(T) := \inf \left[\left\{ \sum_i \|a_i\|^{p'} \right\}^{1/p} \sup_{\|b\| \leq 1} \left\{ \sum_i |\langle y_i, b \rangle|^{p'} \right\}^{1/p'} \right],$$

$1 < p < \infty$, where the infimum is taken over all possible representations. In the case $p = 1$ and $p = \infty$ we put

$$v_1(T) := \inf \left[\sum_i \|a_i\| \|y_i\| \right]$$

and

$$v_\infty(T) := \inf \left[\sup_i \|a_i\| \sup_{\|b\| \leq 1} \sum |\langle y_i, b \rangle| \right].$$

It follows from the well-known relations

$$\pi_p(T) = \sup \{ |\text{trace}(ST)| : S \in \mathcal{L}(F, E), v_p(S) \leq 1 \} \quad \text{for all } T \in \mathcal{L}(E, F)$$

and

$$v_{p'}(S) = \sup \{ |\text{trace}(ST)| : T \in \mathcal{L}(E, F), \pi_p(T) \leq 1 \} \quad \text{for all } S \in \mathcal{L}(F, E)$$

that the inequalities

$$\pi_p(T) \leq c \pi_q(T) \quad \text{for all } T \in \mathcal{L}(E, F)$$

and

$$v_{q'}(S) \leq c v_{p'}(S) \quad \text{for all } S \in \mathcal{L}(F, E)$$

are equivalent.

We have

$$\pi_p(T) \leq v_p(T) \quad \text{for all } T \in \mathcal{L}(E, F),$$

and in the case $p = 2$,

$$\pi_2(T) = v_2(T) \quad \text{for all } T \in \mathcal{L}(E, F) .$$

If at least one of the Banach spaces E and F has the extension property then also the equation

$$\pi_p(T) = v_p(T) \quad \text{for all } T \in \mathcal{L}(E, F)$$

is valid. On the other side A. Pełczyński [21] has shown that there exists no constant $c > 0$ such that for every bounded linear operator T between arbitrary finite dimensional Banach spaces the inequality

$$v_p(T) \leq c \pi_p(T)$$

holds.

Problem : If $1 \leq r, s \leq \infty$ and $1 < p < \infty$, does there exist a constant $c_{rsp} > 0$ such that

$$v_p(T) \leq c_{rsp} \pi_p(T) \quad \text{for all } T \in \mathcal{L}(l_r^n, l_s^n) ?$$

Now we prove further results by duality.

Theorem 1* : Let $T \in \mathcal{L}(E, l_s^n)$. If $2 < s < p \leq \infty$ then

$$v_p(T) \leq c_{s'p'} c_{s'1}^{-1} v_\infty(T) .$$

Proof : If $2 < s < \infty$ and $1 \leq p' < s'$ then by theorem 1 we have

$$\pi_1(S) \leq c_{s'p'} c_{s'1}^{-1} \pi_{p'}(S) \quad \text{for all } S \in \mathcal{L}(l_s^n, E) .$$

Consequently, there holds the dual inequality

$$v_p(T) \leq c_{s'p'} c_{s'1}^{-1} v_\infty(T) \quad \text{for all } T \in \mathcal{L}(E, l_s^n) .$$

Theorem 2* : Let $T \in \mathcal{L}(E, l_s^n)$. If $s = 1$, resp. $1 < s \leq 2$, then

$$v_2(T) \leq c_G v_\infty(T), \text{ resp. } v_2(T) \leq c_{2s}, c_{21}^{-1} v_\infty(T) .$$

Theorem 3* : Let $T \in \mathcal{L}(l_r^n, F)$. If $1 \leq r \leq 2$ and $1 < p \leq 2$ then

$$v_p(T) \leq c_{2p}, c_{21}^{-1} v_2(T) .$$

Theorem 4* (CONJECTURE) : Let $T \in \mathcal{L}(l_r^n, F)$. If $2 < r < \infty$ and $1 < p < q < r'$ then, with a constant $c_{r,pq} > 0$,

$$v_p(T) \leq c_{r,pq} v_q(T) .$$

Finally, we formulate some special cases of theorem 1* and 2*.

Proposition 4 (S. Kwapien [7]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $2 < s < p < \infty$ then

$$v_p(T) \leq c_{s,p}, c_{s,1}^{-1} \|T\| .$$

Proposition 5 (J. Lindenstrauss and A. Pełczyński [8]) : Let $T \in \mathcal{L}(l_\infty^n, l_s^n)$. If $s = 1$, resp. $1 < s \leq 2$, then

$$v_2(T) \leq c_G \|T\|, \text{ resp. } v_2(T) \leq c_{2s}, c_{21}^{-1} \|T\| .$$

Proof : The results follow from the fact that

$$v_\infty(T) = \|T\| \text{ for all } T \in \mathcal{L}(l_\infty^n, F) .$$

Remark : It is easy to prove the following stronger form of lemma 4. Let $T \in \mathcal{L}(E, l_s^n)$. Then

$$v_s(T) \leq \pi_s(T) .$$

One can obtain further results by using this inequality.

§ 7. IDENTITY OPERATORS IN l_r^n -SPACES.

Let I_n be the identity operator from l_r^n into l_s^n . We define the limit order $\lambda_I(r, s, \pi_p)$ to be the infimum of all real numbers λ for which there exists a constant $c_{rs,p} > 0$ such that the inequality

$$\pi_p(I_n : l_r^n \rightarrow l_s^n) \leq c_{rs,p} n^\lambda$$

for all $n=1,2,\dots$ holds. The limit order $\lambda_I(r, s, v_p)$ is defined in the same way.

Historical remark : The π_p - and v_p -norm of the identity operator from l_r^n into itself was determined or estimated by D.J.H. Garling and Y. Gordon (cf. [16], [17], [18]). In the cases v_∞ and π_1 the first result was proved by B. Grünbaum [19] and D. Rutovitz [22]. A. Tong [26] has given necessary and sufficient conditions for a diagonal operator from l_r into l_s to be nuclear (cf. also L. Schwartz [25]).

Lemma 5 : If $\alpha + \beta \leq 1$,

$$\lambda_I(r, s, \pi_p) \leq \alpha \quad \text{and} \quad \lambda_I(s, r, v_p) \leq \beta$$

then

$$\lambda_I(r, s, \pi_p) = \alpha \quad \text{and} \quad \lambda_I(s, r, v_p) = \beta \quad .$$

Proof : Since

$$n = \text{trace}(I_n) \leq \pi_p(I_n : l_r^n \rightarrow l_s^n) v_p(I_n : l_s^n \rightarrow l_r^n)$$

we have

$$1 \leq \lambda_I(r, s, \pi_p) + \lambda_I(s, r, v_p) \leq \alpha + \beta = 1 \quad .$$

Consequently, identity holds.

Lemma 6 :

$$\lambda_I(r, s, \|\cdot\|) \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 0 & \text{if } r \leq s \end{cases} .$$

Proof : The result follows from the well-known inequality

$$\|I_n : l_r^n \rightarrow l_s^n\| \leq \begin{cases} n^{1/s - 1/r} & \text{if } r \geq s \\ 1 & \text{if } r \leq s \end{cases} .$$

Lemma 7 :

$$\lambda_I(1, \infty, v_1) \leq 0 \quad .$$

Proof : If $e = (\varepsilon_i)$ ranges over the set of all n -dimensional vectors with $\varepsilon_i = \pm 1$ then the identity operator I_n has the representation

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n .$$

Consequently,

$$v_1(I_n : l_1^n \rightarrow l_\infty^n) \leq 1 \quad .$$

Lemma 8 : If $1 < p \leq \infty$ then

$$\lambda_I(1, 2, v_p) \leq 0 \quad .$$

Proof : We represent the identity operator I_n in the form

$$I_n x = 2^{-n} \sum_e \langle x, e \rangle e \quad \text{for all } x \in l_1^n .$$

Then

$$\left\{ \sum_e \|e\|_\infty^p \right\}^{1/p} = 2^{n/p} .$$

On the other hand, it follows from Littlewood's inequality (cf. [20]) that

$$\sup_{\|b\|_2 \leq 1} \left\{ \sum_e |\langle e, b \rangle|^{p'} \right\}^{1/p'} \leq 2^{n/p'} c_p \quad .$$

Therefore,

$$v_p(I_n : l_1^n \rightarrow l_2^n) \leq c_p \quad .$$

Lemma 9 :

$$\lambda_I(1, 2, \pi_1) \leq 0 .$$

Proof : From Littlewood's inequality we have

$$\|x\|_2 \leq c_L 2^{-n} \sum_e |\langle x, e \rangle| .$$

Consequently, if $x_1, \dots, x_m \in l_1^n$

$$\sum_i \|x_i\| \leq c_L \sup_{\|a\|_\infty \leq 1} \sum |\langle x_i, a \rangle| ,$$

and therefore,

$$\pi_1(I_n : l_1^n \rightarrow l_2^n) \leq c_L .$$

Remark : Lemma 9 follows also from proposition 2^G

Lemma 10 :

$$\lambda_I(\infty, p, \nu_p) \leq 1/p .$$

Proof : If e_1, \dots, e_n are the usual unit vectors we can represent the identity operator I_n in the form

$$I_n x = \sum_i \langle x, e_i \rangle e_i \quad \text{for all } x \in l_\infty^n .$$

Since

$$\left\{ \sum_i \|e_i\|_1^p \right\}^{1/p} = n^{1/p} \quad \text{and} \quad \sup_{\|a\|_p \leq 1} \left\{ \sum_i |\langle e_i, a \rangle|^{p'} \right\}^{1/p'} = 1$$

we obtain

$$\nu_p(I_n : l_\infty^n \rightarrow l_p^n) \leq n^{1/p} .$$

Lemma 11 : If $1 \leq s < 2$ then

$$\lambda_I(s', s, \pi_1) \leq 1/s$$

Proof : In the case $s = 1$ the result follows from lemma 10. Now we assume $1 < s < 2$. Then there exists ε with $0 < \varepsilon < s - 1$. By lemma 3 and 10 we obtain

$$\begin{aligned} \pi_1(I_n: l_{s'}^n \rightarrow l_s^n) &\leq c_{s, s-\varepsilon} c_{s1}^{-1} \pi_{s-\varepsilon}(I_n: l_{s'}^n \rightarrow l_s^n) \\ &\leq c_{s, s-\varepsilon} c_{s1}^{-1} \|I_n: l_{s'}^n \rightarrow l_\infty^n\| \|\pi_{s-\varepsilon}(I_n: l_\infty^n \rightarrow l_{s-\varepsilon}^n)\| \|I_n: l_{s-\varepsilon}^n \rightarrow l_s^n\| \\ &\leq c_{s, s-\varepsilon} c_{s1}^{-1} n^{1/(s-\varepsilon)} . \end{aligned}$$

Consequently,

$$\lambda_I(s', s, \pi_1) \leq 1/(s-\varepsilon) .$$

The result follows since ε can be made as small as we please.

Remark : It should be possible to determine the exact asymptotic behaviour of $\pi_p(I_n: l_{s'}^n \rightarrow l_s^n)$ as n tends to infinity by using the relation

$$\pi_p(I_n: l_{s'}^n \rightarrow l_s^n) = c_{sp}^{-1} \left\{ \int_{\mathbb{R}^n} \|x\|_s^p d\mu_s^n(x) \right\}^{1/p}, \quad 1 \leq p < s .$$

The limit orders $\lambda_I(r, s, \|\cdot\|)$ and $\lambda_I(s, r, v_1)$

By lemma 6 we have

$$(1) \quad \lambda_I(r, s, \|\cdot\|) \leq \begin{cases} 1/s - 1/r & \text{if } r \geq s \\ 0 & \text{if } r \leq s . \end{cases}$$

On the other hand it follows from lemma 6, 7 and 10 that

$$\lambda_I(s, r, v_1) \leq \lambda_I(s, 1, \|\cdot\|) + \lambda_I(1, \infty, v_1) + \lambda_I(\infty, r, \|\cdot\|) \leq 1/s' + 1/r$$

and

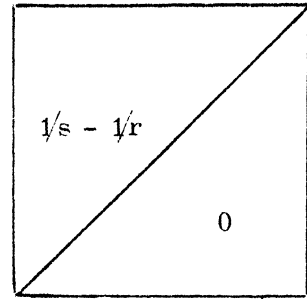
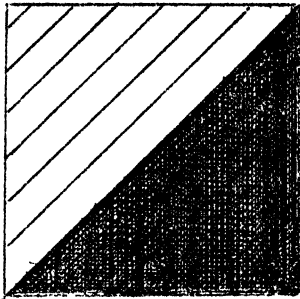
$$\lambda_I(s, r, v_1) \leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, 1, v_1) + \lambda_I(1, r, \|\cdot\|) \leq 1 .$$

In each case, choosing the best result we obtain

$$(1^*) \quad \lambda_I(s, r, v_1) \leq \begin{cases} 1/s' + 1/r & \text{if } r \geq s \\ 1 & \text{if } r \leq s \end{cases}$$

Finally, lemma 5 implies that identity holds in (1) and (1*). In what follows we illustrate our results with pairs of diagrams in the unit square with coordinates $1/r$ and $1/s$. In the left hand diagram we plot the level curves of $\lambda_I(r, s, \pi_p)$. In the right hand diagrams we indicate the algebraic expression for $\lambda_I(r, s, \pi_p)$.

$$\underline{\lambda_I(r, s, \parallel \parallel)}$$



The limit orders $\lambda_I(r, s, \pi_2)$ and $\lambda_I(s, r, \nu_2)$

By lemmas 6, 9 and 10 we have

$$\begin{aligned} \lambda_I(r, s, \pi_2) &\leq \lambda_I(r, \infty, \parallel \parallel) + \lambda_I(\infty, 2, \pi_2) + \lambda_I(2, s, \parallel \parallel) \\ &\leq 0 + 1/2 + \begin{cases} 1/s - 1/2 & \text{if } 1 \leq s \leq 2 \\ 0 & \text{if } 2 \leq s \leq \infty \end{cases} \end{aligned}$$

and

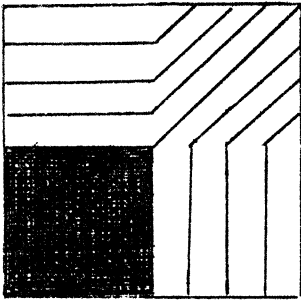
$$\begin{aligned} \lambda_I(r, s, \pi_2) &\leq \lambda_I(r, 1, \parallel \parallel) + \lambda_I(1, 2, \pi_2) + \lambda_I(2, s, \parallel \parallel) \\ &\leq 1/r + 0 + \begin{cases} 1/s - 1/2 & \text{if } 1 \leq s \leq 2 \\ 0 & \text{if } 2 \leq s \leq \infty . \end{cases} \end{aligned}$$

Consequently,

$$(2) \quad \lambda_I(r, s, \pi_2) \leq \begin{cases} 1/r' + 1/s - 1/2 & \text{if } 1 \leq r \leq 2, 1 \leq s \leq 2, \\ 1/s & \text{if } 2 \leq r \leq \infty, 1 \leq s \leq 2, \\ 1/r' & \text{if } 1 \leq r \leq 2, 2 \leq s \leq \infty, \\ 1/2 & \text{if } 2 \leq r \leq \infty, 2 \leq s \leq \infty. \end{cases}$$

Since $\lambda_I(s, r, \nu_2) = \lambda_I(s, r, \pi_2)$ it follows from lemma 5 that identity holds in (2).

$$\underline{\lambda_I(r, s, \pi_2)}$$



$\frac{1}{s}$	$\frac{1}{r'} + \frac{1}{s} - \frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{r'}$

The limit orders $\lambda_I(r, s, \pi_p)$ and $\lambda_I(s, r, \nu_{p'})$ with $1 \leq p < 2$

Since by theorem 2 and 2* for $1 \leq r \leq 2$ we have

$$\lambda_I(r, s, \pi_p) = \lambda_I(r, s, \pi_2) \quad \text{and} \quad \lambda_I(s, r, \nu_{p'}) = \lambda_I(s, r, \nu_2)$$

in the following we need only consider the case $2 < r \leq \infty$.

By lemma 6 and 11 we obtain

$$\lambda_I(r, s, \pi_p) \leq \lambda_I(r, s', \|\cdot\|) + \lambda_I(s', s, \pi_p) \leq 1/s \quad \text{if } r \leq s' \quad \text{and} \quad 1 \leq s \leq 2,$$

and

$$\lambda_I(r, s, \pi_p) \leq \lambda_I(r, r', \pi_p) + \lambda_I(r', s, \|\cdot\|) \leq 1/r' \quad \text{if } r' \leq s \quad \text{and} \quad 1 \leq r' \leq 2$$

On the other hand it follows from lemma 6 and 10 that

$$\begin{aligned} \lambda_I(r, s, \pi_p) &\leq \lambda_I(r, \infty, \|\cdot\|) + \lambda_I(\infty, p, \pi_p) + \lambda_I(p, s, \|\cdot\|) \\ &\leq 0 + 1/p + \begin{cases} 1/s - 1/p & \text{if } p \geq s, \\ 0 & \text{if } p \leq s. \end{cases} \end{aligned}$$

In each case, choosing the best result we obtain

$$(3) \quad \lambda_I(r, s, \pi_p) \leq \begin{cases} 1/s & \text{if } p' \leq r \leq \infty, 1 \leq s \leq p, \\ 1/p & \text{if } p' \leq r \leq \infty, p \leq s \leq \infty, \\ 1/s & \text{if } 2 \leq r \leq p', 1 \leq s \leq r', \\ 1/r' & \text{if } 2 \leq r \leq p', r' \leq s \leq \infty. \end{cases}$$

By lemma 6 and 11

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, p', v_{p'}) + \lambda_I(p', r, \|\cdot\|) \\ &\leq 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if } p' \geq r, \\ 0 & \text{if } p' \leq r. \end{cases} \end{aligned}$$

Moreover,

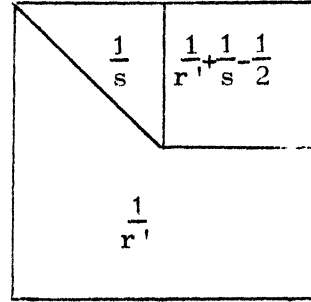
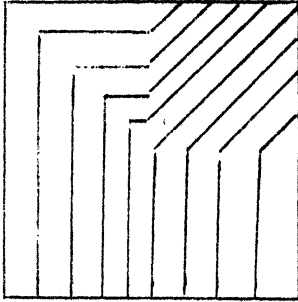
$$\lambda_I(s, r, v_{p'}) \leq \lambda_I(s, r, v_2) = 1/s' \quad \text{if } 1 \leq s \leq 2.$$

Consequently,

$$(3^*) \quad \lambda_I(s, r, v_{p'}) \leq \begin{cases} 1/s' & \text{if } p' \leq r \leq \infty, 1 \leq s \leq p, \\ 1/p' & \text{if } p' \leq r \leq \infty, p \leq s \leq \infty, \\ 1/s' & \text{if } 2 \leq r \leq p', 1 \leq s \leq r', \\ 1/r & \text{if } 2 \leq r \leq p', r' \leq s \leq \infty. \end{cases}$$

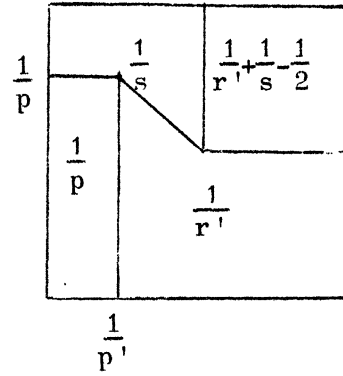
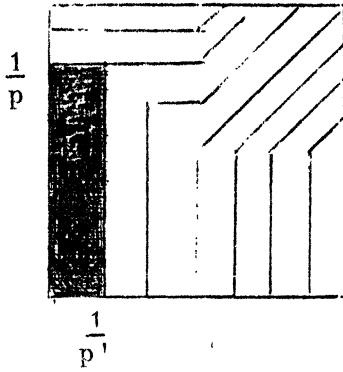
Finally, lemma 5 implies that identity holds in (3) and (3*).

$$\lambda_I(r, s, \pi_1)$$



$$\lambda_I(r, s, \pi_1)$$

$$1 < p < 2$$



The limit orders $\lambda_I(r, s, \pi_p)$ and $\lambda_I(r, s, \nu_{p'})$ with $2 < p < \infty$

Since by theorem 3 and 3* for $1 \leq s \leq 2$ we have

$$\lambda_I(r, s, \pi_p) = \lambda_I(r, s, \pi_2) \quad \text{and} \quad \lambda_I(s, r, \nu_{p'}) = \lambda_I(s, r, \nu_2)$$

in the following we need only consider the case $2 < s \leq \infty$.

Since

$$\lambda_I(r, s, \pi_p) \leq \lambda_I(r, s, \nu_{p'})$$

by (3*) we obtain

$$(4) \quad \lambda_I(r, s, \pi_p) \leq \begin{cases} 1/r' & \text{if } 1 \leq r \leq p', p \leq s \leq \infty, \\ 1/p & \text{if } p' \leq r \leq \infty, p \leq s \leq \infty, \\ 1/r' & \text{if } 1 \leq r \leq s', 2 \leq s \leq p, \\ 1/s & \text{if } s' \leq r \leq \infty, 2 \leq s \leq p. \end{cases}$$

It follows from lemma 6 and 10 that

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, \infty, \|\cdot\|) + \lambda_I(\infty, p', v_p) + \lambda_I(p', r, \|\cdot\|) \\ &\leq 0 + 1/p' + \begin{cases} 1/r - 1/p' & \text{if } p' \geq r, \\ 0 & \text{if } p' \leq r. \end{cases} \end{aligned}$$

On the other hand lemma 6 and 8 imply that

$$\begin{aligned} \lambda_I(s, r, v_{p'}) &\leq \lambda_I(s, 1, \|\cdot\|) + \lambda_I(1, 2, v_{p'}) + \lambda_I(2, r, \|\cdot\|) \\ &\leq 1/s' + 0 + 0 \quad \text{if } 2 \leq r. \end{aligned}$$

In each case, choosing the best result we obtain

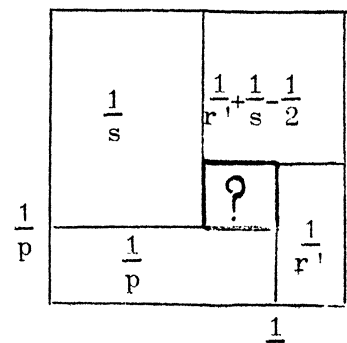
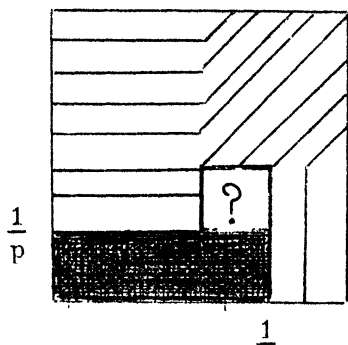
$$(4^*) \quad \lambda_I(s, r, v_{p'}) \leq \begin{cases} 1/r & \text{if } 1 \leq r \leq p', p \leq s \leq \infty, \\ 1/p' & \text{if } p' \leq r \leq \infty, p \leq s \leq \infty, \\ 1/r & \text{if } 1 \leq r \leq p', 2 \leq s \leq p, \\ 1/s' & \text{if } 2 \leq r \leq \infty, 2 \leq s \leq p \end{cases}$$

Because the square

$$Q_{I,p} := \{(1/r, 1/s) : p' < r < 2, 2 < s < p\}$$

does not appear in (4*), we have the open problem whether identity holds for all r and s in (4).

$$\frac{\lambda_I(r, s, \pi_p)}{2 < p < \infty}$$



§ 8. LITTELWOOD OPERATORS IN l_r^n -SPACES.

In the following n ranges over the set of all natural numbers $n = 2^k$ with $k = 1, 2, \dots$. The symmetric Littlewood operators $A_n = (\alpha_{ik}^{(n)})$ are defined inductively by (cf. [20])

$$A_2 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \dots, A_{2n} := \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}, \dots$$

Then

$$A_n^2 = n I_n \quad \text{and} \quad \alpha_{ik}^{(n)} = \pm 1.$$

The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(s, r, \nu_p)$ are introduced in the same way as in the case of identity operators.

Lemma 12 : If $\alpha + \beta \leq 2$,

$$\lambda_A(r, s, \pi_p) \leq \alpha \quad \text{and} \quad \lambda_A(s, r, \nu_p) \leq \beta$$

then

$$\lambda_A(r, s, \pi_p) = \alpha \quad \text{and} \quad \lambda_A(s, r, \nu_p) = \beta.$$

Proof : Since

$$n^2 = \text{trace}(n I_n) \leq \pi_p(A_n : l_r^n \rightarrow l_s^n) \nu_p(A_n : l_s^n \rightarrow l_r^n)$$

we have

$$2 \leq \lambda_A(r, s, \pi_p) + \lambda_A(s, r, \nu_p) \leq \alpha + \beta \leq 2.$$

Consequently, identity holds.

Lemma 13 : If $2 \leq s \leq \infty$ then

$$\lambda_A(r, s, \|\cdot\|) \leq 1/s.$$

Proof : Since the operator $n^{-1/2} A_n$ is unitary we have

$$\|A_n : l_2^n \rightarrow l_2^n\| \leq n^{1/2}.$$

On the other hand, because $|\alpha_{ik}^{(n)}| = 1$, it follows that

$$\|A_n : l_1^n \rightarrow l_\infty^n\| \leq 1 .$$

Finally, if $2 \leq s \leq \infty$, the M. Riesz' connexity theorem implies

$$\|A_n : l_s^n \rightarrow l_s^n\| \leq n^{1/s} .$$

Lemma 14 :

$$\lambda_A(1, 2, v_1) \leq 1/2 .$$

Proof : The result follows from

$$\begin{aligned} v_1(A_n : l_1^n \rightarrow l_\infty^n) &= \pi_1(A_n : l_1^n \rightarrow l_\infty^n) \\ &= \pi_1(I_n : l_1^n \rightarrow l_2^n) \|A_n : l_2^n \rightarrow l_\infty^n\| \|I_n : l_2^n \rightarrow l_\infty^n\| \\ &\leq c_L n^{1/2} . \end{aligned}$$

The limit orders $\lambda_A(r, s, \|\cdot\|)$ and $\lambda_A(s, r, v_1)$

By lemma 6 and 13 we have

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_I(r, 2, \|\cdot\|) + \lambda_A(2, 2, \|\cdot\|) + \lambda_I(2, s, \|\cdot\|) \\ &\leq \begin{cases} (1/2 - 1/r) + 1/2 + (1/s - 1/2) & \text{if } r \geq 2, 2 \geq s, \\ 0 + 1/2 + (1/s - 1/2) & \text{if } r \leq 2, 2 \geq s, \\ (1/2 - 1/r) + 1/2 + 0 & \text{if } r \geq 2, 2 \leq s . \end{cases} \end{aligned}$$

On the other hand we obtain

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_I(r, s', \|\cdot\|) + \lambda_A(s', s, \|\cdot\|) \\ &\leq 0 + 1/s \quad \text{if } r \leq s' \quad \text{and } 2 \leq s, \end{aligned}$$

and

$$\begin{aligned} \lambda_A(r, s, \|\cdot\|) &\leq \lambda_A(r, r', \|\cdot\|) + \lambda_I(r', s, \|\cdot\|) \\ &\leq 1/r' + 0 \quad \text{if } r \leq 2 \quad \text{and } r' \leq s . \end{aligned}$$

Summarizing the results we have

$$(5) \quad \lambda_A(r, s, \|\cdot\|) \leq \begin{cases} 1/r' + 1/s - 1/2 & \text{if } 2 \leq r \leq \infty, 1 \leq s \leq 2, \\ 1/s & \text{if } 1 \leq r \leq 2, 1 \leq s \leq r', \\ 1/r' & \text{if } s' \leq r \leq \infty, 2 \leq s \leq \infty. \end{cases}$$

With the known values of $\lambda_I(s, r, v_1)$ we obtain

$$\lambda_A(s, r, v_1) \leq \lambda_I(s, 1, v_1) + \lambda_A(1, r, \|\cdot\|) \leq 1 + 1/r,$$

and

$$\lambda_A(s, r, v_1) \leq \lambda_A(s, \infty, \|\cdot\|) + \lambda_I(\infty, r, v_1) \leq 1/s' + 1.$$

On the other hand it follows from lemma 14 that

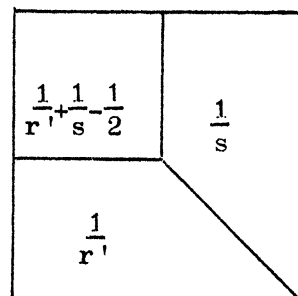
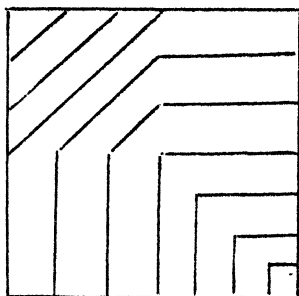
$$\begin{aligned} \lambda_A(s, r, v_1) &\leq \lambda_I(s, 1, \|\cdot\|) + \lambda_A(1, \infty, v_1) + \lambda_I(\infty, r, \|\cdot\|) \\ &\leq 1/s' + 1/2 + 1/r. \end{aligned}$$

In each case, choosing the best result we obtain,

$$(5^*) \quad \lambda_A(s, r, v_1) \leq \begin{cases} 1/r + 1/s' + 1/2 & \text{if } 2 \leq r \leq \infty, 1 \leq s \leq 2, \\ 1/s' + 1 & \text{if } 1 \leq r \leq 2, 1 \leq s \leq r', \\ 1/r + 1 & \text{if } s' \leq r \leq \infty, 2 \leq s \leq \infty. \end{cases}$$

Finally, lemma 12 implies that identity holds in (5) and (5*).

$$\underline{\lambda_A(r, s, \|\cdot\|)}$$



The limit orders $\lambda_A(r, s, \pi_2)$ and $\lambda_A(s, r, \nu_2)$

Since

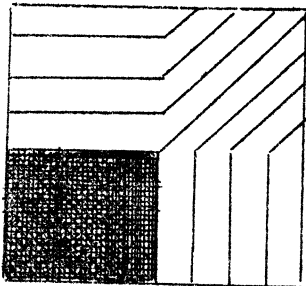
$$\lambda_A(r, s, \pi_2) \leq \lambda_I(r, 2, \pi_2) + \lambda_A(2, s, \|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r, 2, \pi_1)$ and $\lambda_A(2, s, \|\cdot\|)$,

$$(6) \quad \lambda_A(r, s, \pi_2) \leq \begin{cases} 1/r' + 1/s & \text{if } 1 \leq r \leq 2, \quad 1 \leq s \leq 2, \\ 1/2 + 1/s & \text{if } 2 \leq r \leq \infty, \quad 1 \leq s \leq 2, \\ 1/r' + 1/2 & \text{if } 1 \leq r \leq 2, \quad 2 \leq s \leq \infty \\ 1/2 + 1/2 & \text{if } 2 \leq r \leq \infty, \quad 2 \leq s \leq \infty. \end{cases}$$

Finally, it follows from lemma 12 and $\lambda_A(s, r, \nu_2) = \lambda_A(s, r, \pi_2)$ that identity holds in (6)

$$\underline{\lambda_A(r, s, \pi_2)}$$



$\frac{1}{2} + \frac{1}{s}$	$\frac{1}{r'} + \frac{1}{s}$
1	$\frac{1}{r'} + \frac{1}{2}$

The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(s, r, \nu_p)$ with $1 \leq p < 2$

Since by theorem 2 and 2* for $1 \leq r \leq 2$, we have

$$\lambda_A(r, s, \pi_p) = \lambda_A(r, s, \pi_2) \quad \text{and} \quad \lambda_A(s, r, \nu_p) = \lambda_A(s, r, \nu_2)$$

in the following we need only consider the case $2 < r \leq \infty$.

Since

$$\lambda_A(r, s, \pi_p) \leq \lambda_I(r, p, \pi_p) + \lambda_A(p, s, \|\cdot\|)$$

we obtain, using the known values of $\lambda_I(r, p, \pi_p)$ and $\lambda_A(p, s, \|\cdot\|)$,

$$\lambda_A(r, s, \pi_p) \leq 1/p + \begin{cases} 1/p' & \text{if } s \geq p', \\ 1/s & \text{if } s \leq p'. \end{cases}$$

On the other hand it follows from

$$\lambda_A(r, s, \pi_p) \leq \lambda_I(r, r', \pi_p) + \lambda_A(r', s, \|\cdot\|)$$

that

$$\lambda_A(r, s, \pi_p) \leq 1/r' + \begin{cases} 1/r & \text{if } s \geq r, \\ 1/s & \text{if } s \leq r. \end{cases}$$

In each case, choosing the best result we obtain

$$(7) \quad \lambda_A(r, s, \pi_p) \leq \begin{cases} 1 & \text{if } p' \leq r \leq \infty, \quad p' \leq s \leq \infty, \\ 1/p + 1/s & \text{if } p' \leq r \leq \infty, \quad 1 \leq s \leq p', \\ 1 & \text{if } 2 \leq r \leq p', \quad r \leq s \leq \infty, \\ 1/r' + 1/s & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r. \end{cases}$$

Moreover,

$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_A(s, \infty, \|\cdot\|) + \lambda_I(\infty, r, v_{p'}) \\ &\leq 1/s + \begin{cases} 1/r & \text{if } r \leq p', \\ 1/p' & \text{if } r \geq p', \end{cases} \end{aligned}$$

and

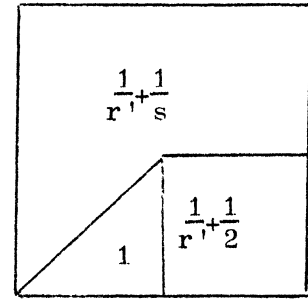
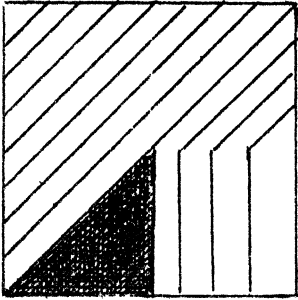
$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_A(s, 2, v_{p'}) + \lambda_I(2, r, \|\cdot\|) \\ &\leq \lambda_A(s, 2, v_2) \leq 1 \quad \text{if } 2 \leq s \leq \infty. \end{aligned}$$

Consequently,

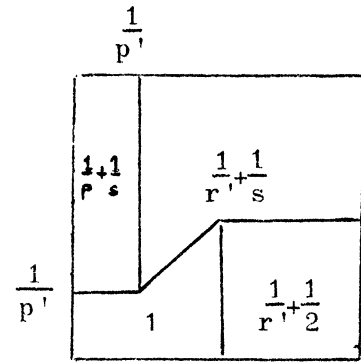
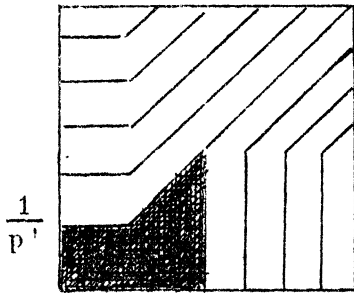
$$(7^*) \quad \lambda_A(s, r, v_{p'}) \leq \begin{cases} 1 & \text{if } p' \leq r \leq \infty, \quad p' \leq s \leq \infty, \\ 1/p' + 1/s' & \text{if } p' \leq r \leq \infty, \quad 1 \leq s \leq p', \\ 1 & \text{if } 2 \leq r \leq p', \quad r \leq s \leq \infty, \\ 1/r + 1/s' & \text{if } 2 \leq r \leq p', \quad 1 \leq s \leq r. \end{cases}$$

Finally, lemma 12 implies that identity holds in (7) and (7*).

$$\lambda_A(r, s, \pi_1)$$



$$\lambda_A(r, s, \pi_p) \quad 1 < p < 2$$



The limit orders $\lambda_A(r, s, \pi_p)$ and $\lambda_A(s, r, \nu_{p'})$ with $2 < p < \infty$

Since by theorem 3 and 3* for $1 \leq s \leq 2$ we have

$$\lambda_A(r, s, \pi_p) = \lambda_A(r, s, \pi_2) \quad \text{and} \quad \lambda_A(s, r, \nu_{p'}) = \lambda_A(s, r, \nu_2)$$

in the following we need only consider the case $2 < s \leq \infty$.

From (7*) and

$$\lambda_A(r, s, \pi_p) \leq \lambda_A(r, s, \nu_p)$$

we obtain

$$(8) \quad \lambda_A(r, s, \pi_p) \leq \begin{cases} 1/r' + 1/s & \text{if } 1 \leq r \leq s, & 2 \leq s \leq p, \\ 1 & \text{if } s \leq r \leq \infty, & 2 \leq s \leq p, \\ 1/r' + 1/p & \text{if } 1 \leq r \leq p, & p \leq s \leq \infty, \\ 1 & \text{if } p \leq r \leq \infty, & p \leq s \leq \infty. \end{cases}$$

On the other hand, we have

$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_I(s, p', v_{p'}) + \lambda_A(p', r, \|\cdot\|) \\ &\leq 1/p' + \begin{cases} 1/p & \text{if } p \leq r, \\ 1/r & \text{if } p \geq r, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lambda_A(s, r, v_{p'}) &\leq \lambda_I(s, 2, \|\cdot\|) + \lambda_A(2, 2, v_{p'}) + \lambda_I(2, r, \|\cdot\|) \\ &\leq (1/2 - 1/s) + 1 + (1/r - 1/2) \quad \text{if } 1 \leq r \leq 2 \text{ and } 2 \leq s \leq \infty. \end{aligned}$$

In each case, choosing the best result we obtain

$$(8^*) \quad \lambda_A(s, r, v_{p'}) \leq \begin{cases} 1/r + 1/s' & \text{if } 1 \leq r \leq 2, 2 \leq s \leq p, \\ 1 & \text{if } p \leq r \leq \infty, 2 \leq s \leq p, \\ 1/r + 1/p' & \text{if } 1 \leq r \leq p, p \leq s \leq \infty, \\ 1 & \text{if } p \leq r \leq \infty, p \leq s \leq \infty. \end{cases}$$

Because the square

$$Q_{A,p} := \{(1/r', 1/s) : 2 < r < p, 2 < s < p\}$$

does not appear in (8*), we have the open problem whether identity holds for all r and s in (8).

$$\frac{\lambda_A(r, s, \pi_p)}{2 < p < \infty}$$

