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LOCALIZATION AND WAVE FRONTS

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Let $P = P(\zeta)$ be a homogeneous polynomial hyperbolic with respect to $\vartheta \in \dot{R}^n = R^n \setminus \{0\}$, i.e. $P(\xi+it\vartheta) \neq 0$ when $\xi \in R^n$ and $t \in \dot{R}$. Then

$$(1) \qquad E(x) = (2\pi)^{-n} \int_{R^n - i\vartheta} e^{i<x,\zeta>} P(\zeta)^{-1} d\zeta$$

defines the unique fundamental solution $E = E(P)$ for $P(D)$ which has support in $H(\vartheta) = \{x;\ <x,\vartheta> \geq 0\}$. Of course, (1) should be understood in the distribution sense, i.e.

$$<E,\phi> = \int_{R^n - i\vartheta} \tilde{\phi}(\zeta)P(\zeta)^{-1}d\zeta, \quad \text{when} \quad \phi \in C_0^\infty(R^n) \quad \text{and}$$

$$\tilde{\phi}(\zeta) = (2\pi)^{-n} \int e^{i<x,\zeta>}\phi(x)dx \ .$$

The problem of describing the singularities of $E(P)$ has a long history and recently Atiyah, Bott and Gårding [2,3] have studied this question in detail. For the location of the singularities of $E(P)$ only the local behaviour of $P(\zeta)$, in neighbourhoods of real points $\xi \in \dot{R}^n$, is relevant. It is therefore possible to extend many results to a rather general class of operators $P(D)$ which I shall now describe.

As usual, $\mathcal{O}_n$ denotes the ring of germs of functions holomorphic at the origin in C^n. If $h \in \mathcal{O}_n$, we write $\underline{h}$ for the first non-zero term h_k in the expansion $h = \sum_0^\infty h_j$, where h_j is homogeneous of degree j.

Definition. $h \in \mathcal{O}_n$ is called locally hyperbolic with respect to $\vartheta \in R^n$ if

$$(2) \qquad \underline{h}(\vartheta) \neq 0 \quad \text{and}$$

$$(3) \qquad h(\xi+it\vartheta) \neq 0 \quad \text{when} \quad \xi \in R^n,\ t \in \dot{R} \ \text{are small.}$$

The class of h:s locally hyperbolic with respect to ϑ will be denoted by $Hyp_{loc}(\vartheta)$.

Let now $\dot{R}^n \ni \xi \to \vartheta(\xi) \in \dot{R}^n$ be a vectorfield homogeneous of degree zero. A homogeneous polynomial $P(\zeta)$ is then called locally hyperbolic with respect to $\vartheta(\xi)$ if, for every $\xi \in \dot{R}^n$, the polynomial $\zeta \to P(\xi+\zeta)$ belongs to $\mathrm{Hyp}_{loc}(\vartheta(\xi))$. (This is a slight variation of the definition given in [1]).

For a polynomial which is locally hyperbolic with respect to $\vartheta(\xi)$ it is possible to define a fundamental solution by a formula similar to (1). When ξ is close to ξ_0 one integrates over the chain $\zeta = \xi - i\varepsilon\vartheta(\xi_0)$ and the construction is then completed by means of a suitable partition of unity. This construction, which in particular is valid for an arbitrary operator $P(D)$ of real principal type, gives fundamental solutions with optimal regularity properties.

However, rather than describing these results, I want to indicate how the concept of local hyperbolicity can be conveniently used to examine the singularities of the fundamental solution (1) of a hyperbolic operator.

First we note that if $h \in \mathrm{Hyp}_{loc}(\vartheta)$ then $\underline{h} \in \mathrm{Hyp}(\vartheta)$, i.e. $\underline{h}$ is hyperbolic with respect to ϑ. In fact, if $\underline{h} = h_k$ then $\lambda^k h(\lambda^{-1}(\xi+it\vartheta))$ tends to $\underline{h}(\xi+it\vartheta)$ as $\lambda \to +\infty$. Since $\underline{h}(\vartheta) \neq 0$ it thus follows from (3) that $\underline{h} \in \mathrm{Hyp}(\vartheta)$. If $h(\zeta) = P(\xi+\zeta)$ we shall use the notation $\underline{h}(\zeta) = P_\xi(\zeta)$.

Let now P be a homogeneous polynomial hyperbolic with respect to ϑ. Then it follows from basic properties of hyperbolic polynomials that $\zeta \to P(\xi+\zeta)$ belongs to $\mathrm{Hyp}_{loc}(\vartheta)$ for every $\xi \in \dot{R}^n$. (A stronger result is contained in Lemma 1 below). Thus $P_\xi \in \mathrm{Hyp}(\vartheta)$ and we can define $E(P_\xi)$ by the formula (1). If we denote by $WF(u)$ the wave front set of the distribution u, we have the following theorem which is just a special case of a more general result (see Hörmander [5, p.339])

__Theorem 1__. If $\xi \in \dot{R}^n$, then

$$\operatorname{supp} E(P_\xi) \times \{\xi\} \subset WF(E(P)).$$

__Proof__: Put $F_\lambda(x) = e^{-i\lambda\langle x,\xi\rangle} E(P)(x)$, i.e.

$$\langle F_\lambda, \phi\rangle = \int_{R^n - i\vartheta} \tilde{\phi}(\zeta)P(\lambda\xi+\zeta)^{-1}d\zeta =$$

$$= \lambda^{k-m} \int_{R^n - i\vartheta} \tilde{\phi}(\zeta)(P_\xi(\zeta)+\lambda^{-1}R_{\xi,\lambda}(\zeta))^{-1}d\zeta,$$

where m,k are the degrees of P and P_ξ respectively and $R_{\xi,\lambda}(\zeta)$ is a polynomial in λ^{-1} and ζ. Then $\lambda^{m-k}F_\lambda \to E(P_\xi)$ in $\mathcal{D}'(R^n)$, when $\lambda \to \infty$, and the theorem is proved.

In the other direction the following result, with a less precise formulation, is proved in [2].

Theorem 2. $WF_A(E(P)) \subset \bigcup_{\xi \in \dot{R}^n} K_\xi \times \{\xi\}$,

where K_ξ is the convex hull of supp $E(P_\xi)$.

Here WF_A denotes the analytic wave front set defined as follows (see [6]). First one observes that there are bounded sequences ϕ_N in $C_0^\infty(R^n)$ such that $\phi_N = 1$ on a fixed neighbourhood of x_0, independent of N, and

(4) $$\sup_{|\alpha|\leq N} |D^\alpha \phi_N| \leq C(CN)^{|\alpha|}, \quad \text{when} \quad |\alpha| \leq N.$$

If $u \in \mathcal{D}'(R^n)$, $WF_A(u)$ is then defined as the complement, in $R^n \times \dot{R}^n$, of the points (x_0,ξ_0) such that for some sequence of this type there is a conic neighbourhood Δ of ξ_0 in $\dot{R}^n$ with

(5) $$|\widehat{\phi_N u}(\xi)| \leq C(CN)^N(1+|\xi|)^{-N}, \text{ when } \xi \in \Delta.$$

It is easily proved that the projection $(x,\xi) \to x$ maps $WF_A(u)$ onto the complement of the largest open set where u is analytic.

For any $h \in Hyp_{loc}(\vartheta)$ we know that $\underline{h} \in Hyp(\vartheta)$ and thus the component of $R^n \setminus \{\xi;\ \underline{h}(\xi) = 0\}$ containing ϑ is an open convex cone which we denote by $\Gamma(\underline{h},\vartheta)$. In particular, if $h(\zeta) = P(\xi+\zeta)$ it is well-known that K_ξ = c.h. supp $E(P_\xi) = \{x;\ \langle x,\Gamma(P_\xi,\vartheta)\rangle \geq 0\}$. The main step in the proof of Theorem 2 is now the following

Lemma 1. Suppose that $h \in Hyp_{loc}(\vartheta)$ and put $T_\xi h(\zeta) = h(\xi+\zeta)$. If M is a compact subset of $\Gamma(\underline{h},\vartheta)$ then, for small $\xi \in R^n$,

(6) $$\underline{T_\xi h} \not\equiv 0 \quad \text{on} \quad M$$

(7) $$T_\xi h(\zeta+it\eta) \not\equiv 0 \quad \text{when} \quad \zeta \in R^n,\ t \in \dot{R} \text{ are small and } \eta \in M$$

(8) $$M \subset \Gamma(\underline{T_\xi h},\vartheta).$$

Note that (6) and (7) implies that $T_\xi h \in Hyp_{loc}(\eta)$, when $\eta \in M$, so (8) makes sense.

The condition (8) can be expressed by saying that the mapping $\xi \to \Gamma(T_{\underline{\xi}}h,\vartheta)$ is inner continuous for small $\xi \in \dot{R}^n$. In the same way a mapping $\xi \to M_\xi$, M_ξ compact, is called outer continuous if any open set containing M_{ξ_0} also contains M_ξ when ξ is close to ξ_0. (7) can then be sharpened as follows.

Lemma 2. Suppose that $h \in \mathrm{Hyp}_{\mathrm{loc}}(\vartheta)$ and that $S \subset \dot{R}^n$ is compact. Let

$$S \ni \xi \to M_\xi \subset \Gamma(T_{\underline{\xi}}\underline{h},\vartheta)$$

be an outer continuous mapping whose values are compact sets. Then

(9) $\qquad \rho > 0,\ \zeta \in R^n,\ \eta \in M_\xi,\ t \in \dot{R} \implies h(\rho(\xi+\zeta+it\eta)) \neq 0,$

for all $\xi \in S$ when ρ,ζ,t are small enough.

These two lemmas are proved in [4] using only elementary facts about Puiseux series and continuity of zeros of holomorphic functions. We shall now see how Theorem 2 follows from Lemma 1. Lemma 2 will then be used to improve Theorem 2.

Proof of Theorem 2. Suppose that $(x_0,\xi_0) \notin \bigcup_{\xi \in \dot{R}^n} K_\xi \times \{\xi\}$ and put $h(\zeta) = P(\xi_0+\zeta)$. Then $h \in \mathrm{Hyp}_{\mathrm{loc}}(\vartheta)$ and it follows from Lemma 1 and the definition of K_ξ that there is a neighbourhood U of x_0, an open conic neighbourhood Δ_1 of ξ_0 and a vector $\eta \in \Gamma(\underline{h},-\vartheta)$ such that

(10) $\qquad \langle x,\eta\rangle > 0 \qquad$ when $x \in U$

(11) $\qquad P(\xi+i(t|\xi|\eta-\vartheta)) \neq 0$ when $\xi \in \Delta_1$, $|\xi| \geq C$ and $0 \leq t$ small

Let Δ be a conic neighbourhood of ξ_0 with $\bar{\Delta} \setminus \{0\} \subset \Delta_1$ and let $\psi \in C^\infty(\dot{R}^n)$ be positively homogeneous of degree one and such that $\mathrm{supp}\,\psi \subset \Delta_1$, $\psi(\xi) = |\xi|$ on Δ. Put

$$v_t(\xi) = -\vartheta + t\psi(\xi)\eta.$$

Then it follows from (11) that, for some $r > 0$,

(12) $\qquad P(\xi+iv_t(\xi)) \neq 0 \quad$ if $|\xi| \geq C$ and $0 \leq t \leq r$.

To prove that $U \times \Delta \cap WF_A(E(P)) = \emptyset$, we take a bounded sequence ϕ_N in $C_0^\infty(U)$ which satisfies (4). With $E = E(P)$, we then have to prove that

(13) $$|\widehat{\phi_N E}(\theta)| \leq C(CN)^N(1+|\theta|)^{-N}, \quad \theta \quad \Delta.$$

Denote by V_t the chain $\zeta = \xi + iv_t(\xi)$, $|\xi| \geq C$. Since $\operatorname{supp} \phi_N \subset U$, it follows from (4) and (10) that

(14) $$|\tilde{\phi}_N(\zeta-\theta)| \leq C(CN)^N(1+|\zeta-\theta|)^{-N}, \quad \zeta \in V_t .$$

In view of (12) we can apply Stokes' formula and get

$$\widehat{\phi_N E}(\theta) = \int_{R^n - i\vartheta} \tilde{\phi}_N(\zeta-\theta)P(\zeta)^{-1}d\zeta =$$

$$= \int_\gamma \tilde{\phi}_N(\zeta-\theta)P(\zeta)^{-1}d\zeta + \int_{V_r} \tilde{\phi}_N(\zeta-\theta)P(\zeta)^{-1}d\zeta ,$$

where γ is a compact chain.
The first term on the right hand side obviously satisfies estimates of the type (13) and, to estimate the second term, we note that

$$|\zeta-\theta| \geq \varepsilon(|\zeta|+|\theta|), \quad \text{when} \quad \theta \in \Delta \quad \text{and} \quad \zeta \in V_r .$$

Therefore (13) follows from (14).

Denote by $WF_A(u)|_\xi$ the fiber of $WF_A(u)$ over ξ. Then it follows from Theorem 1 and 2 that

(15) $$\operatorname{supp} E(P_\xi) \subset WF_A(E(P))|_\xi \subset \text{c.h. supp } E(P_\xi)$$

In general either of these inclusions may be proper, but there is one important case where the left inclusion reduces to equality. To be able to describe the complete result, we shall first define sharpness of a distribution across a hypersurface.

Let $H(s)$ be the Heaviside function, i.e. the characteristic function of the half-line $s \geq 0$ and assume that Σ is an analytic hypersurface in R^n given by $\sigma(x) = \sigma(x_0)$, where $\operatorname{grad} \sigma(x_0) \neq 0$. We shall then say that the distribution u is <u>normally A-sharp across Σ at x_0</u> if there is a function $g(x)$, holomorphic in a neighbourhood of x_0, such that $(x_0, \operatorname{grad} \sigma(x_0)) \notin WF_A(u-(H\circ\sigma)g)$. Finally we recall the notation $K_{\xi_0} = \text{c.h. supp} E(P_{\xi_0})$

and put $W_{\xi_0} = \bigcup^{(c.h.)} \{\operatorname{supp} E((P_{\xi_0})_\xi);\ 0 \neq \xi \in L(P_{\xi_0})^\perp\}$, where $L(Q)^\perp$ denotes the orthogonal complement of the lineality $L(Q) = \{\eta \in R^n;\ Q(\eta+\zeta) \equiv Q(\zeta)\}$.

Theorem 3. Suppose that $\dim L(P_{\xi_0}) = 1$ and that $x_0 \in K_{\xi_0} \setminus W_{\xi_0}$. Then $E(P)$ is normally A-sharp across K_{ξ_0} at x_0.

If, in addition, x_0 belongs to a lacuna, in K_{ξ_0}, for all powers of P_{ξ_0} and if $\deg P_{\xi_0} < n - 1$ then $(x_0,\xi_0) \notin WF_A(E(P))$.

Remark. For the definition of lacuna we refer to [3] where Theorem 3 is proved in a less precise form using what is called the local Petrovsky condition.

Proof: We can, without restriction, assume that $\xi_0 = (1,0,\dots,0)$ and $\vartheta = (1,1,0,\dots,0)$. Then

$$P(\zeta_1,\zeta') = \zeta_1^{m-k}(P_{\xi_0}(\zeta')-\zeta_1^{-1}R(\zeta)),$$

where m,k are the degrees of P and P_ξ respectively, $\zeta' = (\zeta_2,\dots,\zeta_n)$ and

$$R(\zeta) = R_1(\zeta') + \zeta_1^{-1}R_2(\zeta') + \dots + \zeta_1^{k+1-m}R_{m-k}(\zeta').$$

When $P(\zeta) \not\equiv 0$ and $P_{\xi_0}(\zeta') \not\equiv 0$ we can expand $P(\zeta)^{-1}$ in a finite geometric series

$$P(\zeta)^{-1} = \left(\sum_{j=0}^{N-1} R(\zeta)^j \zeta_1^{k-m-j} P_{\xi_0}(\zeta')^{-j-1}\right) + \left(R(\zeta)^N \zeta_1^{-N} P_{\xi_0}(\zeta')^{-N} P(\zeta)^{-1}\right) =$$

$$= A_N(\zeta) + B_N(\zeta).$$

Next we note that

$$\int_{R^n - i\vartheta} \tilde{\phi}(\zeta) A_N(\zeta)\, d\zeta = \langle a_N, \phi\rangle, \quad \text{where}$$

(16) $$a_N(x) = H(x_1)\cdot \sum_{j=0}^{(m-k)(N-1)} \frac{x_1^{m-k+j-1}}{(m-k+j-1)!} Q_j(D')E(P_{\xi_0}^{j+1})(x')$$

and $Q_j(D')$ is homogeneous of degree $j(k+1)$ and satisfies

(17) $$|Q_j(\zeta')| \le C^{j+1}|\zeta'|^{j(k+1)}$$

If we put $a_N(x) = H(x_1)g_N(x)$, it follows that g_N converges uniformly to a holomorphic function g on some complex neighbourhood of x_0. In fact, since $x_0 = (0,x_0')$ where $x_0' \notin W_{\xi_0}$, we can, as in the proof of Theorem 2, choose a C^∞ vector field $\xi' \to v'(\xi')$ homogeneous of degree one such that, when x' is close to x_0',

(18) $$\langle x',v'(\xi')\rangle \geq \varepsilon|\xi'|$$

(19) $$|P_{\xi_0}(\xi'+iv'(\xi'))| \geq \varepsilon|\xi'|^k$$

(20) $$E(P_{\xi_0}^{j+1})(x') = (2\pi)^{-n} \int_{V'} e^{i\langle x',\zeta'\rangle} P_{\xi_0}(\zeta')^{-j-1} d\zeta'$$

and V' is given by $\zeta' = \zeta' + iv'(\xi')$ when $|\xi'| \geq C$.
From (17) - (20) we get that, if $z' = x' + iy'$ is close to x_0', then

$$|Q_j(D')E(P_{\xi_0}^{j+1})(z')| \leq \int_0^{+\infty} e^{-\varepsilon s} C^{j+1} s^{j-k+n-1} ds \leq C_1^{j+1}\, j!$$

This proves that $g = \lim_{N\to\infty} g_N$ is holomorphic in a neighbourhood of x.

We shall now prove that $(x_0,\xi_0) \notin WF_A(E(P)-H(x_1)g)$. Let therefore U be a small neighbourhood of x_0 and ϕ_N a bounded sequence in $C_0^\infty(U)$ satisfying (4). We have to prove an estimate of type (13) with E replaced by $E(P) - H(x_1)g$ and Δ some conic neighbourhood of $\theta' = 0$. Since

$$E(P) - H(x_1)g = (E(P)-a_N) + H(x_1)(g_N-g) \quad \text{and}$$

$$|D^\alpha\phi_N(x)H(x_1)(g_N(x)-g(x))| \leq C(CN)^{|\alpha|} \quad ,|\alpha| \leq N,$$

when U is small enough, we only have to estimate $\widehat{\phi_N(E(P)-a_N)}(\theta)$.
For this we shall use Lemma 2. Put

$$h(z,\zeta') = z^{m-k}P(z^{-1},\zeta') = P_{\xi_0}(\zeta') + zR_1(\zeta') + \ldots + z^{m-k}R_{m-k}(\zeta').$$

Then $h \in Hyp_{loc}((0,\vartheta'))$, where $(0,\vartheta') = (0,1,0,\ldots,0)$. In fact, $P_{\xi_0}(\vartheta') = P_{\xi_0}(\vartheta)$ and, when $z = 0$, $h(0,\zeta'+it\vartheta') =$
$= P_{\xi_0}(\zeta'+it\vartheta') \neq 0$, $\zeta \in R^n$, $t \in \dot{R}$. Finally $h(z,\zeta') = z^{-k}P(\xi_0+z(0,\zeta'))$,

when $z \neq 0$, and since $(0,\vartheta') \in \Gamma(P_{\xi_0},\vartheta)$ we have, according to (7), $P(\xi_0+\zeta+it(0,\vartheta')) \neq 0$ for $\zeta \in R^n$, $t \in \dot{R}$ small. In particular

$$h((z,\zeta') + it(0,\vartheta')) = z^{-k}P(\xi_0+z(0,\zeta') + izt(0,\vartheta')) \neq 0 ,$$

when $\zeta' \in R^{n-1}$, $z, t \in \dot{R}$ are small.

By assumption $x_0' \notin W_{\xi_0} = \bigcup_{0 \neq \xi'} \text{c.h. supp } E((P_{\xi_0})_{\xi'})$ so it follows from Lemma 2 that we can choose a C^∞ vector field $\xi' \to w'(\xi')$, $|\xi'| = 1$, such that, when x' is close to x_0',

(21) $\qquad \langle x',w'(\xi')\rangle \geq \varepsilon > 0$

(22) $\qquad P(\xi_1,\xi'+i(tw'(\xi')-s\vartheta')) \neq 0$, if $\xi_1 \in R$ is large, $s, t \geq 0$ small and $s + t > 0$.

(23) $\qquad P_{\xi_0}(\xi'+i(tw'(\xi')-s\vartheta')) \neq 0$, when $s,t \geq 0$ small and $s + t > 0$.

In fact, (22) is just another way of writing (9) for our choice of h and M_ξ, and that $w'(\xi')$ can be taken to satisfy (23) follows from Lemma 1.

Let now $\psi \in C^\infty(\dot{R}^n)$ be homogeneous of degree zero and such tha supp $\psi \subset \Delta_2$, $\psi = 1$ on Δ_1 where Δ_1 and Δ_2 are small conic neighbourhoods of $\xi' = 0$. If $\chi \in C^\infty(R^{n-1})$ vanishes when $|\xi'| \leq 1$ and is equal to 1 when $|\xi'| \geq 2$, we put

$$w(\xi) = \chi(\xi')\psi(\xi)(0,|\xi'|w'(\xi'/|\xi'|)) - (1-\chi(\xi')\psi(\xi))\dot{\vartheta} .$$

If Δ_2 and the neighbourhood U around x_0 are small enough, V is the chain given by $\zeta = \xi + iw(\xi)$ and if V_{Δ_1} denotes the same chain over Δ_1, then it follows from (21) - (23) and Stokes' formula that

$$\widehat{\phi_N(E(P)-a_N)}(\theta) = \int_V \tilde{\phi}_N(\zeta-\theta)(P(\zeta)^{-1}-A_N(\zeta))d\zeta =$$

(24) $$\int_{V\setminus V_{\Delta_1}} \tilde{\phi}_N(\zeta-\theta)P(\zeta)^{-1}d\zeta - \int_{V\setminus V_{\Delta_1}} \tilde{\phi}_N(\zeta-\theta)A_N(\zeta)d\zeta + $$
$$+ \int_{V_{\Delta_1}} \tilde{\phi}_N(\zeta-\theta)B_N(\zeta)d\zeta .$$

If Δ is a conic neighbourhood of $\xi' = 0$ with $\bar{\Delta} \setminus \{0\} \subset \Delta_1$, then the estimate for the first term in (24) follows directly. To estimate the second term we apply Stokes' formula once more to push $V \setminus V_{\Delta_1}$, keeping the boundary fixed, to a chain $\tilde{V} \setminus V_{\Delta_1}$ where $|\zeta_1| \geq \rho|\zeta|$ and $|P_{\xi_0}(\zeta')| \geq \rho|\zeta'|^k$, for some $\rho > 0$. This is possible since $x_0' \notin W_{\xi_0}$. Then $|A_N(\zeta)| \leq C^N|\zeta|^{-m}$ on $\tilde{V} \setminus V_{\Delta_1}$ and the estimate for the second term follows when $\theta \in \Delta$.

Finally it follows from (22) and (23) that

$$|B_N(\zeta)| \leq C^N|\zeta'|^N|\zeta_1|^{-N} \qquad \text{on } V_{\Delta_1}$$

and since, in view of (21),

$$|\tilde{\phi}_N(\zeta-\theta)| \leq C(CN)(1+|\zeta-\theta|)^{-N} \qquad \text{on } V_{\Delta_1}$$

this gives the estimate for the third term in (24). In fact, if $|\zeta-\theta| \leq \delta(|\zeta|+|\theta|)$, then $|\theta| \leq C|\zeta_1|$. On the other hand we always have $|\zeta-\theta| \geq \varepsilon|\zeta'| - C$ on V_{Δ_1} and, since the estimate for $\phi_N(\zeta-\theta)B_N(\zeta)$ is trivial when $|\zeta-\theta| \geq \delta(|\zeta|+|\theta|)$, this completes the proof of the first part of Theorem 3.

The second part now follows directly from the fact (see |3|), that, if x_0 belongs to a lacuna for all powers of $P(D)$, then $E(P^j)$ is a polynomial of degree $mj-n$, in a neighbourhood of x_0. Here m is the degree of the homogeneous operator $P(D) = P(D_1,\ldots,D_n)$. Thus $E(P_{\xi_0}^{j+1})(x')$ is a polynomial of degree $(j+1)k - n + 1$ close to x_0 and since $Q_j(D)$ has degree $j(k+1)$ and $k < n - 1$ the proof is finished.

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