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REGULARITY OF THE $\bar{\partial}$ -NEUMANN PROBLEM

by

J.J. KOHN

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a smooth boundary $b\Omega$. If z_1, \dots, z_n are coordinates in \mathbb{C}^n we set $x_j = \text{Re}(z_j), y_j = \text{Im}(z_j)$ and, when u is a differentiable function, we define :

$$(1) \quad \begin{aligned} u_{z_j} &= \frac{1}{2} (u_{x_j} - \sqrt{-1} u_{y_j}) \\ u_{\bar{z}_j} &= \frac{1}{2} (u_{x_j} + \sqrt{+1} u_{y_j}) . \end{aligned}$$

Given $\alpha_1, \dots, \alpha_n$ on Ω we will study the system

$$(2) \quad u_{\bar{z}_j} = \alpha_j \text{ for } j = 1, \dots, n.$$

We will assume that the α_j satisfy the compatibility conditions

$$(3) \quad \alpha_j \bar{z}_k = \alpha_k \bar{z}_j .$$

We shall also assume that $b\Omega$ is pseudo-convex, that is if r is a real C^∞ function defined in a neighborhood of $b\Omega$ such that $dr \neq 0$, $r < 0$ in Ω and $r > 0$ outside of $\bar{\Omega}$, if $P \in b\Omega$ and if $(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ satisfy

$$(4) \quad \sum r_{z_i}(P) \zeta_i = 0$$

then we have

$$(5) \quad \sum r_{z_i \bar{z}_j}(P) \zeta_i \bar{\zeta}_j \geq 0.$$

Definition : If $u \in L_2(\Omega)$ we define the singular support of u , denoted by $s.s.(u)$, to be the subset of $\bar{\Omega}$ defined as follows. The point $P \notin s.s.(u)$ if and only if there exists a neighborhood U of P such that the restriction of u to $U \cap \bar{\Omega}$ is in $C^\infty(U \cap \bar{\Omega})$.

This lecture will be concerned with the following two problems.

Problem I. : Given $\alpha_j \in L_2(\Omega)$, $j = 1, \dots, n$ satisfying (3) does

there exist a $u \in L_2(\Omega)$ such that

$$(5) \quad \text{s.s.}(u) \subset \bigcup_j \text{s.s.}(\alpha_j) \quad ?$$

Problem II. : Let $\mathcal{H}(\Omega)$ denote the subspace of $L_2(\Omega)$ consisting of holomorphic functions. Let u be the unique solution of (2) such that $u \perp \mathcal{H}(\Omega)$. Does u satisfy (5) ?

Observe that the system (2) is elliptic thus $\Omega \cap \text{s.s.}(u)$ is contained in the singular support of the α_j . Hence problem I is only interesting for points in $b\Omega$. It is easy to construct a holomorphic function h which is singular at $b\Omega$, then if u satisfies (2) so does $u + h$, so that (5) cannot hold for all solutions of (2). In case the right side of (5) is empty we have the following result.

Theorem : If Ω is pseudo-convex and if $\alpha_j \in C^\infty(\bar{\Omega})$, $j = 1, \dots, n$ satisfying (3), then there exists $u \in C^\infty(\bar{\Omega})$ which satisfies (2).

It is at present unknown whether, under the hypotheses of the above theorem, the solution of (2) which is orthogonal to $\mathcal{H}(\Omega)$ is also in $C^\infty(\bar{\Omega})$. One of the reasons for the solution orthogonal to $\mathcal{H}(\Omega)$ is because the Bergman projection $B : L_2(\Omega) \rightarrow \mathcal{H}(\Omega)$ can be expressed in terms of it. Namely, if $f \in L_2(\Omega)$, if the $f_{z_j} \in L_2(\Omega)$ and if $u \perp \mathcal{H}(\Omega)$ satisfies $u_{z_j} = f_{z_j}$ for $j = 1, \dots, n$ then clearly Bf , the orthogonal projection of f in $\mathcal{H}(\Omega)$, is given by :

$$(6) \quad Bf = f - u.$$

This (5) implies $\text{s.s.}B(f) \subset \text{s.s.}(f)$. By a result of Bell and Ligocka regularity properties of B imply boundary regularity of biholomorphic maps.

It is known that the solution of Problem I depends on the geometry of Ω . The following results of Catlin show this .

Theorem (Catlin) : If Ω is pseudo-convex and if V is a complex-analytic curve contained in $b\Omega$ then there exists α_j satisfying (3) such that (5) is not satisfied for any solution of (2).

Theorem (Catlin) : There exists a pseudo-convex domain $\Omega \subset \mathbb{C}^3$ such that $b\Omega$ does not contain any complex curves and such that there exist α_j as above, so that (5) does not hold.

The $\bar{\partial}$ -Neumann problem enables one to express the solution of II in terms of a boundary value problem. In particular we have the notion of subellipticity of this problem defined as follows.

Definition : The $\bar{\partial}$ -Neumann problem on Ω is subelliptic at $x_0 \in b\Omega$ if there exists a neighborhood U of x_0 and constants $\varepsilon > 0, c > 0$ such that for all n -tuples $(\varphi_1, \dots, \varphi_n)$ with $\varphi_j \in C^\infty(U \cap \bar{\Omega})$ and

$$(7) \quad \sum r_{z_j} \varphi_j = 0 \text{ on } b\Omega$$

we have

$$(8) \quad \sum \|\varphi_j\|_\varepsilon^2 \leq C \left(\sum_1^n \|\varphi_{jz_k} - \varphi_{kz_j}\|^2 + \|\sum_1^n \varphi_j z_j\|^2 \right),$$

where $\|\cdot\|_\varepsilon$ denotes the ε -Sobolev-norm.

The quadratic form defined by the left side of (8) is connected with the solution u of II as follows. If $(\alpha_1, \dots, \alpha_n)$ satisfies (3) and $(\varphi_1, \dots, \varphi_n)$ with $\varphi_j \in C^\infty(\bar{\Omega})$ satisfies (7) and

$$(9) \quad \sum_{j,k} (\varphi_{j\bar{z}_k} - \varphi_{k\bar{z}_j}, \psi_{j\bar{z}_k} - \psi_{k\bar{z}_j}) + (\sum_j \varphi_{jz_j}, \sum_k \psi_{k\bar{z}_k}) = \sum (\alpha_j, \psi_j),$$

for all (ψ_1, \dots, ψ_n) with $\psi_j \in C^\infty(\bar{\Omega})$ and $\sum r_{z_j} \psi_j = 0$ on $b\Omega$. Then

$$(10) \quad u = - \sum_j \varphi_{jz_j}$$

is the unique solution of (2) which is orthogonal to $\mathcal{K}(\Omega)$.

Proposition : If the $\bar{\partial}$ -Neumann is subelliptic at $x_0 \in b\Omega$, if $(\alpha_1, \dots, \alpha_n)$ satisfies (3) and if $\alpha_j \in H_{loc}^s(x_0)$ then $u \in H_{loc}^{s+1}(x_0)$, where $u \perp \mathcal{K}(\Omega)$ and u satisfies (2). In particular we have

$$(11) \quad U \cap \text{s.s.}(u) \subset U \cap \bigcup_j \text{s.s.}(\alpha_j).$$

Hence if the $\bar{\partial}$ -Neumann problem is subelliptic at all points in $b\Omega$ then II is settled affirmatively. From Catlin's result it then follows that subellipticity cannot hold at $x_0 \in b\Omega$ if there is a complex-analytic curve through x_0 contained in $b\Omega$. Catlin also obtained a quantitative measure of the dependence of subellipticity on how close curves through x_0 can be to $b\Omega$.

Theorem (Catlin) : Suppose Ω is pseudo-convex, $x_0 \in b\Omega$ and (8) holds for some $\varepsilon > 0$. Suppose further that there is a complex-analytic curve V with $x_0 \in V$, which has the property that there exists a neighborhood U of x_0 and constants $C > 0$, $\eta > 0$ such that

$$(12) \quad |r(z)| \leq C|z - x_0|^\eta$$

for all $z \in U \cap V$. Then $\varepsilon \leq \frac{1}{\eta}$.

Definition : Suppose $x_0 \in b\Omega$, let $\mathcal{V}(x_0) = \mathcal{V}$ denote the set of germs of complex analytic curves through x_0 . The order of x_0 denoted by $\mathcal{O}(x_0)$ is defined by

$$(13) \quad \mathcal{O}(x_0) = \sup_{V \in \mathcal{V}} \mathcal{O}(x_0, V),$$

where $\mathcal{O}(x_0, V)$, the order of contact of V to $b\Omega$ at x_0 , is defined by

$$(14) \quad \mathcal{O}(x_0, V) = \sup\{\eta \mid \exists \text{ a neighborhood } U \text{ of } x_0 \text{ and } C > 0 \text{ so that (12) holds for all } z \in U \cap V\}.$$

Denoting by \mathcal{W} the set of germs of non-singular complex-analytic curves at x_0 we define $\text{reg } \mathcal{O}(x_0)$ the regular order of x_0 by

$$(15) \quad \text{reg } \mathcal{O}(x_0, V) = \sup_{V \in \mathcal{W}} \mathcal{O}(x_0, V).$$

It is clear that for $n = 1$ we have $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0) = 1$ and that if $n > 1$ then $\mathcal{O}(x_0) \geq \text{reg } \mathcal{O}(x_0) \geq 2$. Furthermore, it is easy to show that for $n = 2$ and Ω pseudo-convex we have $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0)$. However for $\Omega \subset \mathbb{C}^3$, we have the following example (studied by Bloom and Graham), let Ω be given by

$$(16) \quad r(z) = \text{Re}(z_3) + |z_1^2 - z_2^3|^2 + \exp(-1/|z|^2)$$

then for $x_0 \neq 0$ we have $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0) = 2$ and when $x_0 = 0$, we have $\text{reg } \mathcal{O}(0) = 6$ and $\mathcal{O}(0) = \infty$.

D'Angelo has shown that for pseudo-convex domains if $\text{reg } \mathcal{O}(x_0) \leq 4$ then $\mathcal{O}(x_0) = \text{reg } \mathcal{O}(x_0)$.

Returning to subellipticity for pseudo-convex domains the convex of Catlin's theorem holds if $n \leq 2$ and if $\mathcal{O}(x_0) = 2$, that is in those cases when $\mathcal{O}(x_0) < \infty$, then subellipticity holds at x_0 for all $\varepsilon \leq \frac{1}{\mathcal{O}(x_0)}$. A recent example of d'Angelo shows that this is not true in general. Let $\Omega \subset \mathbb{C}^n$ for $n \geq 3$ defined by :

$$(17) \quad r(z) = \text{Re}(z_n) + \sum_{j=1}^{n-2} |z_j^m - z_n z_{j+1}|^2 + |z_{n-1}^m|^2,$$

for this domain $\mathcal{O}(x_0) = 2m$ when $x_0 = (0, \dots, 0)$, $\mathcal{O}(x_0) = 2$ when $z_{n-1}(x_0) \neq 0$ and $\mathcal{O}(x_0) = (2m)^n$ when $x_0 = (0, \dots, 0, i\delta)$ whenever $\delta \in \mathbb{R} - \{0\}$. Catlin shows that subellipticity at $x_0 = (0, \dots, 0)$ holds for $\varepsilon \leq (2m)^{-n}$ but does not hold for $\varepsilon > (2m)^{-n}$. Using the phenomena involved in d'Angelo's example Catlin has constructed for each ε_0 , with $0 < \varepsilon_0 \leq \frac{1}{4}$, a domain such that subellipticity holds with $\varepsilon = \varepsilon_0$ but does not hold for $\varepsilon > \varepsilon_0$. Furthermore he constructs a domain such that subellipticity holds for all $\varepsilon < \varepsilon_0$ but does not hold for $\varepsilon = \varepsilon_0$.

The major problem now is whether $\mathcal{O}(x_0) < \infty$ implies subellipticity on pseudo-convex domain. Here we will briefly describe the sufficient conditions that are known.

Definition : If $x_0 \in \text{b } \Omega$ we define a succession of ideals $I_1(x_0) \subset I_2(x_0) \subset \dots \subset I_k(x_0) \subset C^\infty(x_0)$, where $C^\infty(x_0)$ denotes the ring of germs of C^∞ functions at x_0 . We define by

$$(18) \quad \lambda = \det \begin{pmatrix} 0 & r_{z_1} & \dots & r_{z_n} \\ r_{\bar{z}_1} & r_{z_1 \bar{z}_1} & \dots & r_{z_n \bar{z}_1} \\ \vdots & \vdots & & \vdots \\ r_{\bar{z}_n} & r_{z_1 \bar{z}_n} & \dots & r_{z_n \bar{z}_n} \end{pmatrix}$$

Let $I_1(x_0) = \sqrt{\mathbb{R}(\lambda)}$, where (λ) denotes the ideal generated by λ and $\sqrt{\mathbb{R}}$ is the real radical defined as follows. If $J \subset C^\infty(x_0)$ is an ideal then $\sqrt{\mathbb{R} J} = \{f \in C^\infty(x_0) \mid \exists m \text{ and } g \in J \text{ with } |f|^m \leq |g|\}$. To define $I_k(x_0)$ consider for any $f^{(1)}, \dots, f^{(n+1)} \in I_{k-1}(x_0)$ the $(n+1) \times 2(n+1)$ matrix

$$(19) \quad \begin{pmatrix} 0 & r_{z_1} & \dots & r_{z_n} \\ r_{\bar{z}_1} & r_{z_1 \bar{z}_1} & \dots & r_{z_n \bar{z}_1} \\ \vdots & \vdots & & \\ r_{\bar{z}_n} & r_{z_1 \bar{z}_n} & \dots & r_{z_n \bar{z}_n} \\ 0 & f_{z_1}^{(1)} & \dots & f_{z_n}^{(1)} \\ 0 & f_{z_1}^{(n+1)} & \dots & f_{z_n}^{(n+1)} \end{pmatrix}$$

Let $D^k(f^{(1)}, \dots, f^{(n+1)})$ denote the set of all determinants of $(n+1) \times (n+1)$ minors of the above matrix. Then we define $I_k(x_0)$ by

$$(20) \quad I_k(x_0) = \sqrt{\mathbb{R}(I_{k-1}(x_0), \cup D^k(f^{(1)}, \dots, f^{(n+1)}))},$$

where the union is taken over all $(n+1)$ -tuples in $I_{k-1}(x_0)$.

Theorem : If Ω is pseudo-convex and if $1 \in I_k(x_0)$ the subellipticity holds at x_0 .

In case r is real-analytic in a neighborhood of x_0 it can be shown (using a result of Diederich and Fornaess) that for pseudo-convex domain the condition $1 \in I_k(x_0)$ for some k is equivalent to the non-existence of complex-analytic curves through x_0 which lie in $b\Omega$.

Recent result of d'Angelo give a new analysis of the notion of order of x_0 and in many cases give sharp bounds for the $\mathcal{O}(x)$ in terms of $\mathcal{O}(x_0)$ when x is near x_0 .

References : The recent results of Catlin and d'Angelo are not yet published. For the other references and a more detailed survey see : "Several complex variables from the point of view of linear partial differential equations" by J.J. Kohn in Proc. Conf. on PDE and Diff. Geom. 1980, Beijing. Acad. Sinica.

