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PSEUDO-DIFFERENTIAL OPERATORS WITH POSITIVE SYMBOLS

by

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§ 1. INTRODUCTION

Let A be a pseudo-differential operator of order 2 with symbol $a(x,\xi)$ which will always be assumed to be real. We shall consider the general problem of determining the spectrum of A, which in the present context can more precisely be divided into three parts :

(a) Find conditions on $a(x,\xi)$ which would imply the positivity of A, i.e., estimates of the type

$$\mathrm{Re} < Au,u > \geq - C\|u\|^2_{(2-\sigma)/2} \qquad (1)$$

for some $\sigma > 0$;

(b) Determine the size of the first eigenvalue of A when it is positive and large ;

(c) When the eigenvalues are discrete, estimate N(K) = {number of eigenvalues < K of A counted with their multiplicities}, or better still, estimate the size of the N-th eigenvalue of A.

Answers to these questions would be useful in many situations, among which the study of subelliptic and energy estimates. For example, let $T^*(\mathbb{R}^n) = \cup Q_\nu$ be a classical partition of phase space into dyadic cubes, and let (x_ν, ξ_ν) denote the center of Q_ν ; then the subelliptic estimate

$$\mathrm{Re} < Au,u > \geq c\|u\|^2_{(\varepsilon)} \qquad \varepsilon > 0 \quad (2)$$

simply asserts that the first eigenvalue of A microlocalized to Q_ν is at least of size $|\xi_\nu|^{2\varepsilon}$. More generally any estimate of the form

$$\sum_{j=1}^{N} \|X_j u\|^2 \geq c \sum_{k=1}^{M} \|Y_k u\|^2$$

is equivalent to the positivity of $A = \sum_{j=1}^{N} X_j^* X_j - \sum_{k=1}^{M} Y_k^* Y_k$ and is thus reduced to a question of lower bounds for pseudo-differential operators.

The classical sharp Gårding inequality corresponds to (1) with $\sigma = 0$ and $a(x,\xi) \geq 0$. The significant improvement $\sigma = 1$ under the same condition was obtained by Hörmander [5] and extended to systems by Lax-Nirenberg [8] in 1966. More recently Hörmander [7] established positivity with $\sigma = 6/5$ under the condition $a + \frac{1}{2}\,\mathrm{Trace}^+ a'' \geq 0$, where $\mathrm{Trace}^+ a''$ is a positive quantity associated to the Hessian of a which had been introduced earlier by Melin [9]. (In fact, Hörmander's result is stronger but we have stated a weaker version for the sake of simplicity). Thus bounds for A can be better than bounds for $a(x,\xi)$, as is best examplified by the Hermite operator with the well known properties

$$a(x,\xi) = \sum_{j=1}^{n} (\lambda_j^2 \xi_j^2 + \mu_j^2 x_j^2) - \sum_{j=1}^{n} \lambda_j\mu_j \ , \ \lambda_j, \ \mu_j \geq 0$$

$$\mathrm{Trace}^+ a'' = 2 \sum_{j=1}^{n} \lambda_j \mu_j$$

$$\mathrm{Re} < Au,u > \geq 0 \qquad (3)$$

upon which the Hörmander-Melin results are built.

Concerning eigenvalue distributions, when $a(x,\xi)$ is elliptic we have

$$N(K) = \left(\frac{1}{2\pi}\right)^n \iint_{\{a(x,\xi) < K\}} dx d\xi + O(K^{(n-1)/2}) \qquad (4)$$

result which is due to Hörmander [6]. Examples of Avakumovic show that the bounds for the error are sharp. The main term on the right hand side then justifies the familiar principle of quantum mechanics that a reasonable set of unit size in phase space corresponds to an eigenstate. A reformulation of this principle will play an important role in this article and we shall return to it later.

When $a(x,\xi)$ is not elliptic, asymptotic expansions for N(K) have been obtained by several authors under various conditions, in particular when $a(x,\xi)$ belongs to a class of subelliptic operators with smoothness and non-degeneracy conditions on the characteristic variety (c.f. Menikoff - Sjöstrand [10], Sjöstrand [14], Mohamed [12]), and when $A = -\sum_{j=1}^{N} X_j^2$ with $\{X_j\}$ a system a vector fields whose Lie brackets span and satisfy a constant rank condition (c.f. Métivier [11]).

§ 2. THE CASE OF POSITIVE SYMBOLS :

The case $a(x,\xi)\geq 0$ is now reasonably well understood (see [1][2][3][4])and answers to Questions (1),(2), and (3) are provided by the following theorems :

Theorem 1 : The estimate (1) holds with $\sigma = 2$ when $a(x,\xi) \geq 0$.

To motivate the next two theorems we first give a heuristic discussion of the uncertainty principle which under the form of bounds for the Hermite operator already played a role in the Hörmander-Melin result. In mathematical terms, with suitable normalizations, it states that only cubes in phase spaces of volume ≥ 1 may carry the essential support of some function. This suggests that as long as the set $S(a,K) =\{(x,\xi);\ a(x,\xi) < K\}$ does not contain any cube of volume ≥ 1 no part of the spectrum of A will lie under K. According to the theorem of Egorov, conjugation with Fourier integral operators preserves bounds for A, and hence not only should cubes in phase space be considered, but also their images by canonical transformations. Thus define a canonically twisted cube to be a set of the form $\Phi(Q_o)$ where Q_o is the unit cube $\{(x,\xi)\ ;\ |x| \leq 1,\ |\xi| \leq 1\}$ and Φ is a canonical transformation ; then we expect $N(K)$ to be roughly the same as the number of canonically twisted cubes which can be disjointly imbedded in the set $S(a,K)$.

In establishing this fact it is perhaps natural to try to adapt the modern microlocal analysis of wave packets to the classical variational method of Courant-Weyl in the following monner. First find a decomposition of $T^*(\mathbb{R}^n) = \bigcup_{\nu\in J_1\cup J_2} Q_\nu$ into disjoint canonically twisted cubes Q_ν with Q_ν, $\nu \in J_1$ essentially fitting in $S(a,K)$ and Q_ν, $\nu \in J_2$ essentially fitting in $T^*(\mathbb{R}^n) \setminus S(a,K)$; then for functions φ_ν essentially supported in Q_ν, suitably nommalized we might hope that

$$L^2(\mathbb{R}^n) \sim \bigoplus_{\nu \in J_1 \cup J_2} \{\varphi_\nu\}$$

$$\mathrm{Re} < A\varphi_\nu, \varphi_\nu > \sim a(x_\nu, \xi_\nu)$$

Setting $H_1 = \bigoplus_{\nu \in J_1} \{\varphi_\nu\}$ and $H_2 = \bigoplus_{\nu \in J_2} \{\varphi_\nu\}$ would thus yield subspaces H_1 and H_2 such that

$$\mathrm{Re} < Au, u > \leq K\|u\|^2 \text{ for all } u \in H_1$$
$$\mathrm{Re} < Au, u > \geq K\|u\|^2 \text{ for all } u \in H_2$$

For the Hermite operator the set $S(a,K)$ is especially simple, the Q_ν's can be chosen to be straight cubes whose sizes vary within good bounds, the φ_ν's can be obtained from translations and dilations of a fixed function intimately related to the operator, namely the gaussian $\varphi(x) = e^{-\frac{1}{2}|x|^2}$, and the program can be carried through. In fact there is practically no loss of volume in fitting Q_ν in $S(a,K)$ and $N(K) \sim \mathrm{Vol}\, S(a,K)$ as expected. For more general symbols $S(a,K)$ could however be quite complicated and the problem of imbedding disjoint canonically twisted cubes in a given set is not well understood. There are further serious difficultes ; for example given a canonically twisted cube Q_ν it is not easy to obtain φ_ν essentially supported in Q_ν without a much more general theory of Fourier integral operators than is presently available.

On the other hand a direct study of the spectral function $e(x,y;\lambda)$ and its cosine transform $K(x,y;t)$

$$e(x,y;\lambda) = \text{Kernel of } E(\lambda), \text{ with } A^{1/2} = \int \lambda\, dE(\lambda)$$

$$K(x,y;t) = \int e^{i\lambda t}\, de(x,y;\lambda) = \text{Kernel of } e^{it\, A^{1/2}}$$

through the wave equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - A^{1/2}\right) K(x,y;t) = 0$$

also meets major difficulties since the symbol $a(x,\xi)$ could be quite degenerate.

At the basis of the proofs of theorems 1,2, and 3 is instead a method of reduction of the number of variables which has the advantage of providing at the same time an algorithm to determine the spectrum of A. In view of applications to subelliptic estimates, we shall state the answer to Question (b), theorem 2, in a localized form. It will also show that not all canonical transformations need be considered, but only those with certain bounds.

<u>Theorem 2</u> : Let $a(x,\xi) \geq 0$ be a C^∞ function satisfying the bounds

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} M^{2-|\beta|}$$

$$a(x,\xi) \geq M^2 \text{ for } |x| + |\xi|/M \geq 1o.$$

Fix $\varepsilon > 0$ and define

$$\lambda = \min_\Phi \left(\max_{(x,\xi)\in\Phi(Q^o)} a(x,\xi) \right)$$

where Φ runs over all canonical transformations with the property that

$$|D_x^\alpha D_\xi^\beta \Psi| \leq C'_{\alpha\beta} M^{-(|\alpha|+|\beta|)\delta}$$

if $\Phi(x,\xi) = (y,\eta),(y_o\eta_o) = \Phi(o)$, Ψ denotes the mapping $\Psi(x,\xi) = (y-y_o,(\eta-\eta_o)/M)$ and δ is a small positive number depending on ε and the dimension.

If $\lambda \geq M^\varepsilon$ then

$$\mathrm{Re} < Au,u > \geq c_\varepsilon \lambda \|u\|^2.$$

Subellipticity with sharp bounds for vector fields whose Lie brackets span follows easily.

<u>Theorem 3</u> : Let Ω be a compact manifold without boundary with a given measure. Then there is on algorithm associating to each symbol $a(x,\xi)$ and number K a set $Q(a,K)$ of disjoint canonically twisted cubes in $S(a,K)$ such that for $K \geq K_o$

1) $N(K) \geq c$ {number of elements in $Q(a,cK)$}

2) $N(K) \leq C$ {number of elements in $Q(a,CK)$}

Here K_o, c, C are constants and N(K) is the number of eigenvalues of the quadratic form Re $< a(x,D)u,u >$.

To conclude this section we mention briefly some questions of symplectic geometry arising from the above theorems. This is a largely unexplored area ; for example, it is not even known whether the set $S = \{(x,\xi)\in \mathbb{R}^2 \times \mathbb{R}^2 \ ; \ |x_1|,|\xi_1| < \varepsilon \}$ contains a canonically twisted cube. Recent results of A. Weinstein indicate that the answer is probably negative. Next theorem 2 provides upper and lower bounds for the N-th eigenvalue which would be interesting to compare. In general we expect them to be of the same size. Finally when $a(x,\xi)$ is elliptic it is readily seen that our estimate for N(K) coincides with the one given by the more precise formula in (4), but relating the present results to those for the various classes of subelliptic symbols mentioned earlier will require more careful arguments.

§ 3. THE CASE OF DIFFERENTIAL OPERATORS :

Despite the fact that there are many unsettled questions in connection with disjoint imbeddings of canonically twisted cubes, the algorithm appearing in Theorem 3 does permit to extract concrete information in specific situations.

We shall consider here the case when A is a differential operator of order 2 with symbol $a(x,\xi) \geq 0$ on a compact manifold without boundary Ω. Fix a metric on Ω and denote by μ the corresponding measure.

Define a vector $X \in T_x(\Omega)$ to be admissible if .

$$< X,\xi >^2 \leq a(x,\xi) \text{ for all } \xi \in T^*(\Omega),. \quad (5)$$

an admissible curve of length ρ to be a curve $\gamma : [0,\rho] \to \Omega$ with $\gamma'(t)$ admissible, and set

$B(x,\rho) = \{y \in \Omega,$ y can be joined to x by a curve of length $\rho\}$.

When $A = - \sum_{j=1}^{k} X_j^2$, X_j real vector fields, then (5) is just equivalent to saying that X is in the subspace spanned by the X_j's, and the balls $B(x,\rho)$ coincide with the ones introduced by Nagel, Stein, and Wainger [13] in a different context.

Subellipticity and eigenvalue distribution for A are then characterized by the following theorem :

<u>Theorem 4</u> : Under the above hypotheses

(1) The subelliptic estimate

$$\mathrm{Re} < Au,u > c\|u\|^2_{(\varepsilon)} - C\|u\|^2_{(o)} \qquad (6)$$

holds if and only if

$$B(x,\rho) \geq B_E(x, c\rho^{1/\varepsilon}) \qquad (7)$$

for small ρ. Here $B_E(x,\rho)$ denotes the usual geodesic ball centered at x of radius ρ with respect to the given metric.

(2) When (7) holds, the eigenvalues are discrete and

$$N(K) \sim \int_\Omega \frac{d\mu}{\mu(B(x,K^{-1/2}))} \; .$$

(Here equivalence $\sim$ between two quantities $A \sim B$ is taken in the sense that $c_1 A \leq B \leq c_2 A$ for some fixed constants c_1 and c_2).

Finally, it should be stated that we have remained rather sketchy throughout for the sake of simplicity. The reader interested in more precise statements and details on the above is referred to the articles [1] [2] [3] [4].

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