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The Sobolev inequality on the Heisenberg group and the Yamabe problem on CR manifolds (d’après travaux avec J. M. Lee)


<http://www.numdam.org/item?id=SEDP_1983-1984____A18_0>
THE SOBOLEV INEQUALITY ON THE HEISENBERG GROUP AND THE
YAMABE PROBLEM ON CR MANIFOLDS

(d'après travaux avec J.M. Lee)

par D. JERISON

Exposé n°XVIII 20 Mars 1984
We begin by recalling the classical Sobolev inequality and the Yamabe problem on Riemannian manifolds.

The Sobolev inequality in $\mathbb{R}^m$, $m \geq 3$, is

$$\left( \int_{\mathbb{R}^m} |\varphi(x)|^{q} \, dx \right)^{2/q} \leq C \int_{\mathbb{R}^m} |\nabla \varphi(x)|^{2} \, dx \quad \text{for all } \varphi \in C^0_0(\mathbb{R}^m)$$

The change of variable by dilation $\psi(x) = \psi(5x)$, $\delta > 0$, shows that the only exponent $q$ for which the inequality can hold is $1/q = 1/2 - 1/m$. The best constant $C$ in the Sobolev inequality can be calculated by considering the variational problem

$$\inf \left\{ \int_{\mathbb{R}^m} |\nabla \psi|^2 / \left( \int_{\mathbb{R}^m} |\psi|^q \right)^{2/q} : \nabla \psi \in L^2(\mathbb{R}^m), \psi \in L^q(\mathbb{R}^m) \right\}$$

Extremal functions exist and satisfy the Euler-Lagrange equation

$$\Delta \psi = \mu \psi^{q-1} \quad \text{on } \mathbb{R}^m, \quad (1)$$

Indeed, the extremal $\mu$ constant functions are translations and dilations of the function $C(1+|x|^2)^{(m-2)/2}$ (See [1,11]).

However, the problem has more symmetries than dilation and translation. For instance, the Kelvin transform $\psi(x) \mapsto |x|^{2-m} \psi(x/|x|^2)$ preserves both numerator and denominator in the infimum above. This is a special case of a more general phenomenon of conformal invariance. Stereographic projection is a conformal transformation of the sphere $S^m$ to $\mathbb{R}^m$. The corresponding extremal problem on the sphere is rotation invariant and the Kelvin transform in $\mathbb{R}^m$ corresponds to reflection across the equator of $S^m$.

Let $(M,g)$ be a Riemannian manifold of dimension $m \geq 3$, with scalar curvature $K$ and Laplace-Beltrami quator $\Delta$. If $\tilde{K}$ is the scalar curvature associated with the metric $\tilde{g} = \varphi^{q-2} g$ ($\varphi \in C^\infty(M), \varphi > 0$) then $\tilde{K}$ is given by

$$a_m \Delta \varphi + K \varphi = \tilde{K} \varphi^{q-1}, a_m = 4(m-1)/(m-2).$$
A key feature of this formula is that the exponent \( q = \frac{2m}{m-2} \) is the same as the exponent \( q \) in the Sobolev inequality.

The (Riemannian) Yamabe problem is to find a metric of constant scalar curvature among all conformal multiples of a given metric. Thus the problem is equivalent to solving

\[
a_m \Lambda \psi + K \psi = \mu \psi^{q-1}
\]

for some \( \psi \in C^\infty(M) \), \( \psi > 0 \), \( \mu \) constant. In the case of \( \mathbb{R}^m \), \( K = 0 \), so that this is the same as equation (1). (On the other hand, \( \mathbb{R}^m \) and \( S^m \) are conformally equivalent, so solving equation (1) is the same as solving (2) in the case of \( S^m \)).

Equation (2) is the Euler-Lagrange equation for the variational problem

\[
\mu(M, g) = \inf \left\{ \int_M \left( a \, d\psi, d\psi + K \psi^2 \right) dV_g : \int_M \psi^q \, dV_g = 1 \right\}
\]

Here \( \left< , \right> \) denotes the (dual) metric induced by \( g \) on 1-forms and \( dV_g \) the volume element for \( g \).

The best result to date is due to T. Aubin [1].

**Theorem** Let \((M, g)\) be a compact, Riemannian manifold of dimension \( \geq 3 \)

(a) \( \mu(M, g) \) depends only on the conformal class of \( g \)

(b) \( \mu(M, g) \leq \mu(S^m, \text{standard metric}) = \mu(\mathbb{R}^m, \text{standard metric}) = \mu_o \)

(c) If \( \mu(M, g) < \mu_o \), then the infimum in (3) is attained by a positive, \( C^\infty \) solution to (2). Thus the Yamabe problem is solved.

(d) If \( m \geq 6 \), then \( \mu(M, g) < \mu_o \) unless \( M \) is everywhere locally conformally equivalent to \( \mathbb{R}^m \).

When \( m < 6 \), much less is known about when \( \mu(M, g) < \mu_o \).

In this article I would like to discuss an analogous theorem in the setting of CR manifolds. I would also like to point out why it
would be useful to solve explicitly the problem on the Heisenberg
group analogous to (1) and to announce a partial result in that
direction. I will be describing joint work with John M. Lee; details
will appear elsewhere.

There is a far-reaching analogy between conformal and CR
geometry. To illustrate this as vividly as possible, we summarize in
a table the most important parallels. This will serve as an outline
of the discussion that follows.

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Conformal change:

\[
\tilde{\gamma} = \varphi^{q-2} \gamma
\]

Conformal invariant \(\mu(M,g)\)

Yamabe equation:

\[
a^m \Delta \varphi + K \varphi = \mu^{q-1}
\]

The Heisenberg group \(\mathbb{H}^n\) is the Lie group whose underlying
manifold is \(\mathbb{C}^n \times \mathbb{R}\) with coordinates \((z,t) = (z_1, \ldots, z_n, t)\) and whose group
law is given by

\[
(z, t) (z', t') = (z + t', t + t' + 2 \text{Im} \sum_{j=1}^{n} \overline{z_j} \overline{z_j'})
\]
We will also denote elements of $\mathbb{H}^n$ by $x$ and $y$. For $x = (z, t)$ the dilations $\delta x = (\delta z, \delta t)$ preserve the group law: $\delta(xy) = (\delta x)(\delta y)$. The vector fields $Z_j = i\partial \bar{z}_j + \partial \bar{z}_j / \partial t$ are left-invariant and homogeneous of degree $-1$ with respect to dilations. The Haar measure $d\mu$ for the group is just Euclidean measure.

Consider the Sobolev-type inequality

$$(\ast) \quad \left( \int_{\mathbb{H}^n} |u(x)|^p d\mu(x) \right)^{2/p} \leq C \int_{\mathbb{H}^n} \sum_{j=1}^n |Z_j u(x)|^2 d\mu(x)$$

for real-valued functions $u$. The change of variable by dilation (noting that $d\mu(\delta x) = 5^{2n+2} d\mu(x)$) shows that the only possible exponent $p$ is given by $1/p = 1/2 - 1/(2n+2)$. Denote

$$\mathcal{L} = -\frac{1}{2} \sum_{j=1}^n Z_j \bar{Z}_j + \bar{Z}_j Z_j$$

The Euler-Lagrange equation associated to the extremal problem for $(\ast)$ is

$$(\ast\ast) \quad \mathcal{L} u = \lambda u^{p-1} \quad \text{on } \mathbb{H}^n; \lambda \text{ constant.}$$

The analogue of stereographic projection is as follows. The Heisenberg group is identified with the boundary of the Siegel upper half space

$$\mathcal{D} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im } w > |z|^2\} \quad \text{by the mapping}$$

\[
(z, t) \mapsto (z, w), \quad w = t + i|z|^2.
\]
Moreover, the domain $\mathcal{D}$ is biholomorphic to the unit ball in $\mathbb{C}^{n+1}$ by the Cayley transform:

$$w = i \left( \frac{\zeta_{n+1}}{1 - \zeta_{n+1}} \right) ; \quad z_k = \zeta_k / (1 + \zeta_{n+1}) \quad k = 1, \ldots, n.$$  

where $\zeta \in \mathbb{C}^{n+1}$, $|\zeta| \leq 1$. When restricted to the boundary this transformation gives what we will call a CR equivalence between $S^{2n+1}$ and $\mathbb{H}^n$.

A CR structure on a manifold $N$ of dimension $2n+1$ is given by a complex $n$-dimensional sub-bundle $T_{1,0}$ of the complexified tangent bundle $\mathcal{T}N$ of $N$, satisfying $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$. We will assume that the CR structure is integrable, i.e. the Lie bracket $[T_{1,0}, T_{1,0}] \subset T_{1,0}$, and that there is a global non-vanishing real 1-form $\theta$ annihilating.
The Levi form is a hermitian form on $T_{1,0}$ given by
$$
\langle z, w \rangle_\theta = -i \Theta(z, \overline{w})
$$
for $z, w \in T_{1,0}$. We will also assume strict pseudoconvexity, i.e., that the Levi form is positive definite. A hypersurface in $\mathbb{C}^{n+1}$ of real codimension 1 has a natural integrable CR structure given by the holomorphic vectors of $\mathbb{C}^{n+1}$ tangent to the hypersurface. In particular $S^{2n+1} \subset \mathbb{C}^{n+1}$ has a natural CR structure. Local biholomorphisms of $\mathbb{C}^{n+1}$ preserve the CR structure.

A pseudohermitian structure on a CR manifold $N$ is a choice of the 1-form $\Theta$. With this choice, $N$ has a natural volume form $\Theta \wedge d\Theta^n$ (non zero because $N$ is strictly pseudoconvex). Dual to the Levi form is an inner product on real 1-forms $\omega$, given locally by
$$
\langle \omega, \omega \rangle_\Theta = \sum_{j=1}^{n} |\omega(W_j)|^2
$$
in which $W_1, \ldots, W_n$ form an orthonormal basis for $T_{1,0}$ with respect to the Levi form. The sublaplacian $\Delta_b$ is defined on functions by
$$
\Delta_b^\circ (\Delta_b u) \wedge \Theta \wedge d\Theta^n = \int_N \langle du, dv \rangle_\Theta \wedge \Theta \wedge d\Theta^n,
$$
for all $v \in C^\infty_0(N)$.

The operator $\Delta_b$ is subelliptic. In fact, it is the real part of the Kohn-Spencer Laplacian $\partial_b$ on functions (cf. [7]).

S. Webster has defined a scalar curvature $R$ associated with a pseudohermitian structure [14]. Under a change of pseudohermitian structure $\Theta = u^{p-2} \Theta$ with $p = 2 + 2/n$, $R$ transforms as calculated in [7]

$$(4) \quad (b_n \Delta_b + R)u = \tilde{R} u^{p-1}, \quad b_n = 2(n+1)/n$$

A key point here is that the value of $p$ is the same as the only possible exponent $p$ in (4). We will comment on this at length in the proof of Theorem 1(a) which contains (4) as a special case.

Hence the problem of finding a pseudohermitian structure with constant scalar curvature on a CR manifold $N$ is equivalent to solving the equation

$$(5) \quad b_n \Delta_b u + Ru = \lambda u^{p-1}$$

for $u \in C^\infty(N), u > 0$ and $\lambda$ constant. Equation (5) is the Euler-Lagrange
equation for the variational problem

\[ (6) \quad \lambda(N,\theta) = \inf \left\{ \int_N \left( b \langle du, du \rangle + R_\theta^2 \right) \omega^n : \int_N |u|^p \omega^n = 1 \right\} \]

In the case of \( \mathbb{H}^n \) we put \( T_{1,0} = \text{span} \{ Z_1, \ldots, Z_n \} \).

The real 1-form \( \theta_0 = dt + \sum_{j=1}^n iz_j dz_j - iz_j dz_j \) is orthogonal to \( T_{1,0} \)

and the vectors \( Z_1, \ldots, Z_n \) form an orthonormal basis with respect to the Levi form for \( \theta_0 \). The operator \( \Delta_\theta \) associated to \( \theta_0 \) is \( \mathcal{L} \) and the scalar curvature for \( \mathbb{H}^n \) with pseudohermitian structure \( \theta_0 \) is zero.

Finally, \( \theta_0 \wedge d\theta_0^n \) is a multiple of the standard volume form. Thus the problem of finding the best constant in \((\alpha)\) is equivalent to the variational problem

\[ (\alpha') \lambda(\mathbb{H}^n, \theta_0) = \inf \left\{ \int_{\mathbb{H}^n} \sum_{j=1}^n |Z_ju|^2 \theta_0 \wedge d\theta_0^n : \int_{\mathbb{H}^n} |u|^p \theta_0 \wedge d\theta_0^n = 1, \, u \text{ real-valued} \right\} \]

Our main theorem is

**Theorem 1** Let \( N \) be a compact, strictly pseudoconvex integrable \( \text{CR} \) manifold of dimension \( 2n+1 \).

a) \( \lambda(N,\theta) \) depends only on the \( \text{CR} \) structure, not the choice of 1-form \( \theta \).

b) \( \lambda(N,\theta) \leq \lambda(S_{2n+1}, \text{standard structure}) = \lambda(\mathbb{H}^n, \theta_0) = \lambda_0 \).

c) If \( \lambda(N,\theta) < \lambda_0 \), then the infimum in \((6)\) is attained by a \( C^\infty \), positive solution to \((5)\). Thus the pseudohermitian structure \( \tilde{\theta} = u^{p-2} \theta \) has constant scalar curvature \( \tilde{R} = \lambda(N,\theta) \).

S.S. Chern and R. Hamilton, while studying contact structure on \( \text{CR} \) manifolds, have independently obtained a result which is equivalent to part (c) in the case \( n = 1 \) and \( \lambda(N,\theta) < 0 \).

Notice that we do not have an analogue to part (d) of Aubin's theorem. The reason is that we do not yet have the complete solution to the problem of identifying extremals in problem \((\alpha')\).

Here is a partial result.

**Theorem 2** The extremals to problem \((\alpha')\) exist, and are positive \( C^\infty \) solutions to \((\alpha')\). Moreover, if we assume in addition that the solution depends
only on $|z|$ and $t$, then it must take the form $c|t+iz|^2 + \alpha^{-n}$ for some $c > 0$ and some $\alpha \in \mathbb{C}$ such that $\text{Im} \alpha > 0$.

We conjecture that these are all the solutions up to left translation on the Heisenberg group.

**Main ideas of the proofs**

Let $t \in C^\infty(M)$, $t > 0$. Under the conformal change $\tilde{g} = t^{q-2}g$, we have

$$
(a_m \tilde{\Delta} + k)\tilde{\varphi} = t^{1-q}(a_m \tilde{\Delta} + K)\varphi \quad \text{for} \quad \tilde{\varphi} = t^{-1}\varphi.
$$

It follows immediately that $\mu(M,g) = \mu(M,\tilde{g})$ in other words part (a) of Aubin's theorem is proved. Likewise on a CR manifold $N$ the transformation law

$$
(b_n \tilde{\Delta}_b + R)u = t^{1-p}(b_n \Delta_b + R)u, \quad \text{with} \quad \tilde{\vartheta} = t^{p-2}\vartheta \quad \text{and} \quad \tilde{u} = t^{-1}u.
$$

Shows that $\lambda(N,\vartheta) = \lambda(N,\tilde{\vartheta})$. This formula follows from transformation laws for $R$ and $\Delta_b$ computed by J.M. Lee in [7]. Indeed, the case $u = t$ is (4).

We first recognized that there must be a transformation law like (7) by approaching it from another point of view. This point of view not only sheds light on the significance of the invariance, but also allows us to see in advance that the critical exponent $p$ of the Sobolev inequality ($\varphi$) will match the exponent of the transformation law (8).

In [41], C. Fefferman constructed a circle bundle $C$ over $N$ and a Lorentz metric $g$ on $C$ such that a change in pseudohermitian structure on $N$ corresponds to a conformal change in $g$. It turns out that the projection of $C$ onto $N$ carries the (Laplace-Beltrami) wave operator to $\Delta_b$ and the scalar curvature of $g$ to a multiple of $R$ (See [7] for proofs). Thus the transformation law (8) follows from the corresponding law (7) on $C$, once we realize that with $m = \dim C = 2n + 2$. The critical exponents $p$ and $q$ are equal.

The remainder of the proof of theorem 1 depends fundamentally on the work of Folland and Stein [5].
Recall that $Z_1, \ldots, Z_n$ are homogeneous of degree $-1$ and that $\theta_o$ is homogeneous of degree 2 with respect to the dilations of $\mathbb{H}^n$.

Folland and Stein normal coordinates can be described as follows:

**Lemma 1** Let $(N, \theta)$ be a pseudohermitian manifold. Every point of $N$ has an open neighborhood $\Omega$ with an orthonormal frame $W_1, \ldots, W_n$ for $T_{1,0} N$ defined on $\Omega$ and a $C^\infty$ mapping $\tilde{\phi}_x : \Omega \to \mathbb{H}^n (x \in \Omega)$ such that $\tilde{\phi}_x(x) = (0,0)$ and

$$(\tilde{\phi}_x)^* W_j = Z_j + (\text{terms of degree } \geq 0).$$

$$(\tilde{\phi}_x)^* u = \theta_o + (\text{terms of degree } \geq 3).$$

We can now prove Theorem 1 b). Denote

$$\mathcal{U}_\delta(v) = \int_N (b < dv, dv > + Rv^2)_{\theta_o} \text{Ad}\theta_{\theta_o}^n \quad \text{and} \quad \mathcal{B}_\delta(v) = \int_N |v|^q \text{Ad}\theta_{\theta_o}^n.$$}

One first shows that for any $\varepsilon > 0$ there is a compactly supported function $u$ (in fact $u \in C^\infty_0(\mathbb{H}^n)$) such that $\mathcal{G}_{\theta_o}^\varepsilon(u) = 1$ and $\mathcal{G}_{\theta_o}^{\varepsilon}(u) < \lambda_o + \varepsilon$. The function $u_{\delta}(x) = \delta^{-n} u(\delta^{-1} x)$ satisfies $\mathcal{G}_{\theta_o}^\varepsilon(u_{\delta}) = \mathcal{G}_{\theta_o}^{\varepsilon}(u) = 1$ and $\mathcal{G}_{\theta_o}^\varepsilon(u_{\delta}) < \lambda_o + \varepsilon$. Moreover the support of $u_{\delta}$ tends to zero as $\delta \to 0$. Lemma 1 shows that $V_\delta(x) = u_{\delta}(\tilde{\phi}_x(x))$ satisfies $\mathcal{G}_{\theta_o}^\varepsilon(V_\delta) \to 1$ as $\delta \to 0$, which proves Theorem 1 b). A refinement of this argument with an explicit formula for the extremal on $\mathbb{H}^n$ should yield an analogue to part d) of Aubin's theorem.

In order to prove Theorem 1 c) we will need to state several regularity theorems of Folland and Stein.

We will keep the notations of Lemma 1 and let $X_j = \text{Re } W_j$, $X_{j+n} = \text{Im } W_j$, $j = 1, \ldots, n$. Denote $X^\alpha = X_{\alpha_1} \ldots X_{\alpha_k}$ where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $1 \leq \alpha_j < 2n$, $\ell(\alpha) = k$. The Folland-Stein Sobolev spaces are defined by

$$S_{\theta_o}^p(\gamma) = \{ f \in L^1_{\text{loc}}(\Omega) : X^\alpha f \in L^p(\Omega) \}.$$}

A natural distance function is defined by

$$\rho(x, y) = |\tilde{\phi}_x(y)|,$$

where $|(z, t)| = (|z|^4 + t^2)^{1/4}$. For any non-integer $\beta > 0$ we let $k$ be the integer such that $k < \beta < k+1$ and denote Folland-
Stein Lipschitz classes by

\[ \Gamma^r_\beta(\Omega) = \{ f : \sup_{x,y\in\Omega} \max_{0<|\alpha|<k} \frac{|x^\alpha f(x) - x^\alpha f(y)|}{\rho(x,y)^{\beta-k}} < \infty \} \]

Finally, ordinary Lipschitz classes (for \( \Omega \subset \mathbb{R}^{2n+1} \)) are defined by

\[ \Lambda^r_\beta(\Omega) = \{ f : \sup_{x,y\in\Omega} \sup_{|\alpha|<k} \frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|}{\|x-y\|^{\beta-k}} < \infty \} \]

With the help of a partition of unity we can, of course, define \( S^p(N), \Gamma^r_\beta(N), \Lambda^r_\beta(N) \).

**Lemma 2** [5] For \( p > 0 \) non-integer, \( k \) an integer \( \geq 1 \), and \( r, 1 < r < \infty \),

\[ S^r_k \subset \Gamma^r_\beta \text{ where } \frac{1}{r} = \frac{k-\beta}{2n+2} \]

\[ \Gamma^r_\beta \subset \Lambda^r_\beta/2 \]

If \( \Delta_b u \in L^r \) then \( u \in S^r_2 \) and if \( \Delta_b u \in \Gamma^r_\beta \) then \( u \in \Gamma^{r+2}_\beta \).

(In order to compare \( \Gamma^r_\beta \) and \( \Lambda^r_\beta/2 \) we have identified \( \Omega \) with its image under \( \phi_X \) for some fixed \( X \in \Omega \).)

**Lemma 3** Suppose that \( f \in L^S(\Omega) \) for some \( s > n + 1 \) that \( u \in S^2_1(\Omega) \), \( u \geq 0 \), and \( \Delta_b u + fu = 0 \) in the weak sense on \( \Omega \), then \( u \) is bounded above and below by positive constants on any compact subset of \( \Omega \).

(This Harnack inequality is analogous to the one used by Trudinger [12]. It is proved, as is Trudinger's, by Moser's technique.)

The basic approach to a variational problem is to take a minimizing sequence and to find a convergent subsequence. The problem is that the exponent \( p \) is exactly the exponent for which the inclusion \( S^2_1(N) \subset L^p(N) \), while true, is not compact. Aubin overcame this difficulty in the Riemannian case with the help of the observation that the best constant in the Sobolev inequality is the same for all compact m-manifolds in the following sense: if \( \mu = \mu(S^m, \text{standend metric}) \) then for any \( M \) and any \( \varepsilon > 0 \)

\[ (\mu-\varepsilon) \left( \int_M |f|^q \, dv \right)^{2/q} < a \int_M \left< df, df \right> \, dv + \sum_{k=1}^n C_{M,k,\varepsilon} \int_M |f|^2 \, dv \]
Subsequently, Aubin's approach has been considerably simplified and extended to a large number of elliptic problems. (See Brezis-Nirenberg [2], P.L. Lions [8] and the surveys [3,9] where many further references can be found.)

This leads to the natural conjecture:

**Conjecture.** If \( \lambda_0 = \lambda(\mathbb{H}^n, \theta_0) \), then for any compact \((2n+1)\)-dimensional pseudohermitian manifold \((N, \theta)\) and any \( \varepsilon > 0 \) there exists \( C = C(N, \varepsilon) \) such that:

\[
(\lambda_0 - \varepsilon)(\int_N |f|^p \Lambda d\theta^n)^{2/p} \leq \int_N (b < df, df> + C |f|^2) \Lambda d\theta^n \quad \text{for all } f \in S^2_1(N).
\]

If true, this conjecture would lead to a much simpler proof of Theorem 1 c).

A partition of unity reduces the conjecture to a local question. Unfortunately, Folland-Stein normal coordinates are not analogous to Riemannian normal coordinates in one important respect: the norm \( \int < df, df > \theta \) inherited from \( N \) is not comparable to the norm \( \int < df, df > \theta_0 \) of \( \mathbb{H}^n \) even in very small neighborhoods of the origin.

Nevertheless, Theorem 1 c) can be proved by a method analogous to a proof of Aubin's theorem due to K. Uhlenbeck [13]. The point is that the regularity estimates in Lemmas 2 and 3 are uniform in a family of (non-comparable) CR structures uniform not only in the base point \( x \) of the coordinate charts \( \theta_x \) but also under dilation.

We now sketch the proof. Consider the variational problems

\[
\lambda_\gamma = \inf \left\{ \int_N (b < du, du >_\theta + R u^2) \Theta \Lambda d\theta^n : \int_N |u|^p \Theta \Lambda d\theta^n = 1 \right\}.
\]

For \( r < p \) one can show a positive, \( C^\infty \) extremal \( u_r \) exists and satisfies \( b \Lambda u_r + Ru_r = \lambda_r u_r^{r-1} \). It is not hard to show that

\[
\lambda_r \to \lambda_p = \lambda(N, \theta) \quad \text{as } r \to p.
\]

**Case 1.** \( < du_j, du_j >_{\gamma \theta} \) is bounded on \( N \) uniformly as \( r \to p \). In this case one can choose a subsequence convergent in \( S^2_1(N) \) (and \( L^s(N) \) for any \( s < \infty \)) norm. Thus the limit function satisfies the equation (5) and the
constraint $\mathcal{J}_\emptyset(u) = 1$. By repeated application of Lemmas 2 and 3, we find that $u \in C^\infty(N), u > 0$.

**Case 2.** $\max < \frac{du}{r}, \frac{du}{\emptyset} \to \infty$ as $r \to p$.

We will show that this contradicts the strict inequality $\lambda_p = \lambda(N,\emptyset) < \lambda(\mathbb{H}^n, \emptyset_0)$.

Choose $x \in N$ such that $\max_{\emptyset} du(r), \frac{du(r)}{\emptyset} = < \frac{du(x)}{r}, \frac{du(x)}{\emptyset}$. We will consider $u$ in local coordinates given by $\frac{\delta}{x}$. Thus $u_x$ is identified with a function in a neighborhood of the origin in $\mathbb{H}^n$ and $x$ is identified with the origin. Define $h_r(z,t) = \delta^{2/(r-2)} u_r(\delta z, \delta^2 t)$ and choose $f = f(r)$ such that $< dh_r(0), dh_r(0) > \emptyset_0 = 1$. Note that $\delta \to 0$ as $r \to p$. Moreover, this change of variable yields an equation for $h_r$ of the form

$$L_r h_r + R_r \frac{\delta^2}{r} h_r = \lambda_r h_r^{r-1}$$

where $L_r$ tends toward $L$ (the sublaplacian of $\mathbb{H}^n$) as $r \to p$ and $R_r$ is the scalar curvature of $(N,\emptyset)$ after suitable change of variable depending on $x$ and $\delta$. The crucial point is that the norm inequalities associated to Lemmas 2 and 3 for the CR structure associated $L_r$ are uniform as $r \to p$. Because $\delta \to 0$, the domain of definition of $h_r$ tends to the entire space $\mathbb{H}^n$ as $r \to p$. Moreover, $< dh_r(x), dh_r(x) > \emptyset < 1$ since the maximum value at the origin is 1. It follows from the uniform versions of Lemmas 2 and 3 that a subsequence of the sequence $h_r$ converges uniformly in, say, $L^{1+\epsilon}$ on compact subsets of $\mathbb{H}^n$ to a limit $h$ satisfying $\mathcal{J}_\emptyset(h) = \lambda_p h^{p-1}$ on all of $\mathbb{H}^n$. One can check that $\mathcal{J}_\emptyset(h) < \infty$ and $\mathcal{E}_\emptyset(h) < 1$. It follows from integration by parts that $h$ gives the bound $\lambda(\mathbb{H}^n, \emptyset_0) \leq \lambda_p$, a contradiction.

We conclude by describing the idea of the proof of Theorem 2. (For Riemannian case, see Obata [10] and Gidas, Ni, Nirenberg [6].)

Given a positive $C^\infty$ solution to $\mathcal{L}u = \lambda u^{p-1}$ depending only on $|z|$ and $t$ we define a collection of functions $h_j$ of $|z|$, $t$ and $u$ and its first and second derivatives and prove a formula of form

$$\sum h_j^2 \frac{\delta^2}{r} = \text{div}(\text{something}).$$
We conclude by integration by parts that each $h_j = 0$. These additional equations are enough to specify $u$.

References:


