

SÉMINAIRE ÉQUATIONS AUX DÉRIVÉES PARTIELLES – ÉCOLE POLYTECHNIQUE

R. PHILLIPS

Moduli space, heights and isospectral sets of metrics

Séminaire Équations aux dérivées partielles (Polytechnique) (1989-1990), exp. n° 2, p. 1-6

<http://www.numdam.org/item?id=SEDP_1989-1990___A2_0>

© Séminaire Équations aux dérivées partielles (Polytechnique)
(École Polytechnique), 1989-1990, tous droits réservés.

L'accès aux archives du séminaire Équations aux dérivées partielles (<http://sedp.cedram.org>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

*CENTRE
DE
MATHEMATIQUES*

Unité de Recherche Associée D 0169

ECOLE POLYTECHNIQUE

F-91128 PALAISEAU Cedex (France)

Tél. (1) 69.41.82.00

Télex ECOLEX 601.596 F

Séminaire 1989-1990

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

MODULI SPACE, HEIGHTS AND ISOSPECTRAL
SETS OF METRICS.

R. PHILLIPS

I would like to report on several papers by Brad Osgood, Peter Sarnak and myself ([1,2,3]). The problem was colorfully stated by Mark Kac as 'can you hear the shape of a drum ? That is given a manifold Σ with metric g and corresponding Laplacian Δ_g with Dirichlet boundary conditions, how many metrics have the same Δ_g -spectrum. In this count we ignore repetitions given by isometric metrics. Little is known about plane domains where this number may well be one. In the case of closed 2-manifolds Vigneras and Sunada have shown that there are arbitrarily large sets of isospectral metrics. Our main result is

Theorem 2.— *An isospectral set of closed Riemannian 2-manifolds is compact in the C^∞ -topology. Likewise an isospectral set M plane domains is also compact in the C^∞ -topology.*

It is known that the spectrum of the Laplacian determines the topology of the manifold so that Σ is fixed. The C^∞ -topology on nonisometric classes of metrics is defined as follows : Let $\mathcal{G}^\infty(\Sigma)$ denote the usual C^∞ -topology on metrics and $D^\infty(\Sigma)$ denote the group of diffeomorphisms in Σ . Let

$$R(\Sigma) = \mathcal{G}^\infty(\Sigma)/D^\infty(\Sigma) .$$

Then $[g]_n \rightarrow [g]$ means that there exist $g_n \in [g]_n$ and $g \in [g]$ such that $g_n \rightarrow g$ in $\mathcal{G}^\infty(\Sigma)$:

An important ingredient in our analysis is the notion of height introduced by Singer and Ray. If $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ denotes the spectrum of $-\Delta_g$, then formally

$$\det' \Delta_g = \prod_{\lambda_j \neq 0} \lambda_j .$$

To make sense of this one need some regularization procedure such as the zeta function :

$$Z(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s} .$$

This can be written in terms of the heat kernel $e^{\Delta_g t}$ as

$$Z(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{tr}'(e^{\Delta_g t}) t^s \frac{dt}{t} ,$$

where

$$\text{tr}'(e^{\Delta_g t}) = \sum_{\lambda_j \neq 0} e^{-\lambda_j t} .$$

Note that

$$Z'(s) = - \sum_{\lambda_j \neq 0} \lambda_j^{-s} \log \lambda_j .$$

It can be shown that $Z(s)$ is meromorphic and regular at $s = 0$. This allows as to define the **height** as

$$h(g) = - \log(\det' \Delta_g) \equiv Z'(0) .$$

It is obvious from this that the height is an isospectral invariant. Further within a conformal class the Polyakov-Alvarez variational formula holds, that is for $g = e^{2\varphi}g_0$

$$*) \quad h(g) = \frac{1}{6\pi} \left\{ \frac{1}{2} \int_{\Sigma} |\nabla_0 \varphi|^2 dA_0 + \int_{\Sigma} K_0 \varphi dA_0 + \int_{\partial\Sigma} k_0 \varphi ds_0 \right\} - \frac{1}{4\pi} \int_{\partial\Sigma} \partial_n \varphi ds_0 + h(g_0);$$

here ∇_0, A_0, s_0 are taken with respect to g_0 and $K =$ Gauss curvature, $k =$ geodesic curvature on $\partial\Sigma$.

Our results hold for 2-manifolds of two kinds : Σ_0^p , closed manifolds of genus p and Σ_n^0 , plane domains of connectivity n . For example



For plane domains $K = 0$ and so for $g = e^{2\varphi}g_0$ φ will be g_0 -harmonic and as a consequence in (*) we will have $\int_{\partial\Sigma} \partial_n \varphi ds_0 = 0$.

Now both area and the length of $\partial\Sigma$ are also isospectral invariants. Hence we can without loss of generality scale the metrics to normalized isospectral sets so that

$$\text{Area}(\Sigma) = 1$$

in the case of closed manifolds and

$$\text{length}(\partial\Sigma) = 1$$

in the case of plane domains.

Note that is $g = \gamma^2 g_0$ then

$$h(g) = \frac{\chi(\Sigma)}{3} \log \gamma + h(g_0),$$

where χ denotes the Euler characteristic. We denote the space of **normalized** classes of metrics by $R_0(\Sigma)$.

Next we introduce the notion of a **uniform** metric. In the case of Σ_0^p a uniform metric has constant Gauss curvature K where for plane domains Σ_n^0 the boundary $\partial\Sigma$ two constant geodesic curvature k . The following are examples of uniform metrics :

- $\Sigma_0^0 = S^2$: standard round metric,
- Σ_0' : flat torus ($K = 0$),
- Σ_0^p : hyperbolic metric ($K < 0$),
- Σ_1^0 : Euclidean metric in the unit disk,
- Σ_2^0 : cylinder ($k = 0$).

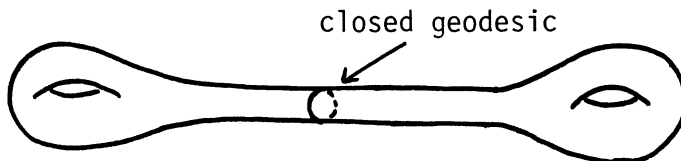
Theorem 2.— *In any conformal class of metrics in $R_0(\Sigma)$ there is a unique uniform metric and it is the unique global minimum of the height within this conformal class.*

The moduli space for Σ consists of conformal structures in $R_0(\Sigma)$.

It follows as a corollary to Theorem 2 that the set of uniform metrics in $R_0(\Sigma)$ represents the moduli space for Σ ; we denote it by $\mathcal{M}_a(\Sigma)$. It can be shown that $\mathcal{M}_a(\Sigma)$ is finite dimensional. Our principal result in $\mathcal{M}_a(\Sigma)$ is

Theorem 3.— *$h(u)$ becomes infinite as u approach the boundary of $\mathcal{M}_a(\Sigma)$.*

In the case of Σ_0^0 and Σ_1^0 where there is only one conformal class there is nothing to prove. For Σ_0^1 (torus) and Σ_2^0 (annulus) the height can be explicitly evaluated in terms of the eta function and theorem 3 proved directly. In the case of Σ_0^p (closed 2-manifolds of genus p) The degeneration of the moduli space can occur in only one way, that is when the length of a the closed geodesic approaches zero :



Theorem 3 was proved in this case by Wolpert.

For plane domains of connectivity $n \geq 3$ the boundary of $\mathcal{M}_a(\Sigma)$ is more complicated and this is reflected in the proof. The ingredients of the proof for $n \geq 3$ are as follows :

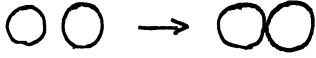
- 1) An explicit description of $\mathcal{M}_a(\Sigma_n^0)$ in terms of **conical** metrics in the complex plane \mathbf{C} :

$$ds = \gamma \prod |z - \tau_j|^{\alpha_j} |dx|$$

when the τ_j are n distinct points, $\alpha_j > -1$ and $\sum \alpha_j = -2$, and

$$\Sigma = \{z : \text{dist}_g(z, \tau_j) > 1 \text{ for all } j\}.$$

The following types of degeneration can occur :

i) 

ii) $\gamma^{(k)} \rightarrow \infty$ or $\gamma^{(k)} \rightarrow 0$;

iii) $\alpha_i^{(k)} \rightarrow -1$;

as well as combinations of the above.

2) To sort these degenerations out we introduced the notion of a **valuation** :

Let $\tau_{ij} = |\tau_i - \tau_j|$: Euclidean distance,

$$\beta(i, r) = \sum_{\tau_{ij} \leq r} \alpha_j ,$$

$$L(i, r) = \gamma r^{1+\beta(i,r)} \prod_{\tau_{ij} > r} \tau_{ij}^{\alpha_j} ,$$

$$\sigma(\tau_i) = \min_j \tau_{ij}, \bar{\sigma}(\tau_i) = \max_j \tau_{ij},$$

$$v(\tau_i) = \inf\{L(i, r), \sigma(\tau_i) < r < \bar{\sigma}(\tau_i)\}$$

$$\bar{v}(\tau_i) = \sup\{L(i, r), \sigma(\tau_i) < r < \bar{\sigma}(\tau_i)\} .$$

The proof is then divided into three cases

i) For some i , $\bar{v}(\tau_i) \leq \text{const}$,

ii) For some i , $v(\tau_i) \leq \text{const}$ but $\bar{v}(\tau_i) \rightarrow \infty$,

iii) For all i , $v(\tau_i) \rightarrow \infty$.

Cases (i) and (ii) are proved inductively by mean of the

Insertion Lemma : If a Jordan curve Γ decomposes Σ into two parts $\Omega_1 \cup \Omega_2$ and Γ is “well separated” from $\partial\Sigma$, then

$$h(\Sigma(g)) \geq h(\Omega_1) + h(\Omega_2) + 0(1),$$

where the error term $0(1)$ depends only in the separation.

In case (iii) we are able to get an explicit approximation for the height and verify theorem 3 directly, to complete the overall induction.

Remark : When $p.n \neq 0$, H. Khuri showed that Theorem 3 is false (Stanford Thesis).

Finally we come back to the problem of isospectral metrics. We represent each g in the isospectral set an terms of the uniform metric in its conformal class : $g = e^{2\varphi} u$. According to Theorem 2,

$$h(u) \leq h(q) = \text{const}.$$

By Theorem 3, u must stay away from $\partial\mathcal{M}_a(\Sigma)$ and hence lies in a compact set of uniform metrics. Using the Polyakov-Alvarez formula we can easily get a bound on the W^1 Sobolev bound in the φ 's.

Next we use the heat invariants on Σ to get a grip in K (for Σ_0^p) or k (for Σ_n^0).

$$\text{tr}'(e^{\Delta_g t}) \sim \frac{1}{t} \sum_0^{\infty} a_j(g) t^j \quad \text{for } \Sigma_0^p,$$

$$\sim \frac{1}{t} \sum_0^{\infty} a_j(g) t^{j/2} \quad \text{for } \Sigma_n^0.$$

The a'_j 's are isospectral invariants. They are universal polynomials in (K, ∇_0) in the Σ_0^p case and in (k, ∂_{s_0}) in the Σ_n^0 case, for which the highest order derivation term dominates. From this one can show that K is C^∞ -compact for Σ_0^p and that k is C^∞ -compact for Σ_n^0 . For Σ_n^0 this result is due to R. Melrose.

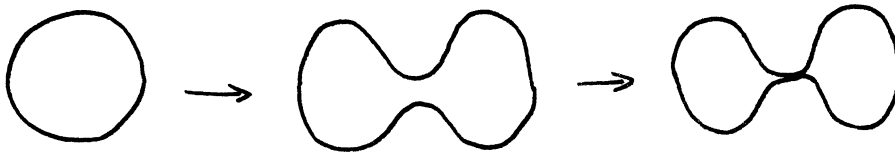
Using the relations

$$K = e^{-2\varphi}(-\Delta_0\varphi + K_0)$$

$$k = e^{-\varphi}(\partial_n\varphi + k_0)$$

together with the above two results, it is easy to show that the φ 's are C^∞ -compact, as need for Theorem 1.

We note that the heat invariants are not by them self enough to give the C^∞ -compactness of the metrics. For example in Σ^0 , we could have



with k remaining uniformly smooth in this degeneration where the metric blows up.

Bibliography

- [1] B. OSGOOD, R. PHILLIPS and P. SARNAK, Extremals of determinante of Laplacians, *J. Funct. Anal.*, 79 (1988) 148-211.
- [2] B. OSGOOD, R. PHILLIPS and P. SARNAK, Compact isospectral sets of surfaces, *J. Funct. Anal.* 79 (1988) 212-239.
- [3] B. OSGOOD, R. PHILLIPS and P. SARNAK, Moduli space, heights and isospectral sets of plane domains. *Annals of Math.*, 129 (1989) 293-362.

S.R. PHILLIPS
105 Peter Coutts Cir,
STANFORD CA 94305
U.S.A.