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Appendice à l'exposé : « Weak bloch property and weight estimates for elliptic operators »

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ÉQUATIONS AUX DÉRIVÉES PARTIELLES

APPENDICE A L'EXPOSE : WEAK BLOCH PROPERTY AND WEIGHT ESTIMATES FOR ELLIPTIC OPERATORS

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On the equality between weak and strong extentions.

by

M.A.Shubin and J.Sjostrand

0. Introduction and statement of the result.

We use the terminology and the results of Shubin [S1]. Let M be a connected Riemannian manifold of bounded geometry, let E,F be vectorbundles on M of bounded geometry (cf [S1]) and let $A: C^\infty(M;E) \to C^\infty(M;F)$ be a uniformly elliptic C^∞-bounded differential operator on M of order m (cf [S1]). Let $a(x,\xi)$ be the principal symbol of A so that $a \in C^\infty(T^*M\setminus 0; \mathrm{Hom}(\pi^*E,\pi^*F))$, is a homogeneous polynomial of degree m in the the fiber variables. Here π^*E, π^*F denote the pull-backs of E,F by means of the natural projection $\pi: T^*M \to M$. As in [S1] we can define the spaces $L^p(M;E)$ and more generally the Sobolev spaces $W_p^s(M;E)$ modelled on the L^p spaces. Here $p\in[1,\infty]$ and $s\in\mathbb{R}$. In the case $p=\infty$ we shall not consider L^∞ but rather the space $\tilde{C}(M;E)$ of continuous sections of E which tend to 0 at infinity. In order to cover all cases we shall put $\mathcal{L}^p = L^p$ when $1\le p<\infty$ and $\mathcal{L}^p=\tilde{C}$ when $p=\infty$. Consider first A as an operator $C_0^\infty(M;E) \to C_0^\infty(M;F)$ and fix some $p\in[1,\infty]$. Then we can define the weak and the strong extensions ${}^wA$ and ${}^sA$ in the following way:

- The domain $\mathcal{D}({}^wA)$ of ${}^wA$ is the set of all $u\in\mathcal{L}^p(M;E)$ such that $Au\in\mathcal{L}^p$.
- The domain of ${}^sA$ is given by $\mathcal{D}({}^sA)=\{u\in\mathcal{L}^p(M;E)$; there exists a sequence $u_j\in C_0^\infty(M;E)$, $j=1,2,..$ with the property that $u_j\to u$ in $\mathcal{L}^p(M;E)$ and $Au_j \to Au$ in $\mathcal{L}^p(M;F)\}$.

In both cases we define ${}^wA: \mathcal{D}({}^wA)\to\mathcal{L}^p$, ${}^sA: \mathcal{D}({}^sA)\to\mathcal{L}^p$ in the sense of distributions. It is then an interesting problem to determine when ${}^wA={}^sA$, that is when $\mathcal{D}({}^wA)=\mathcal{D}({}^sA)$. It is obvious that

(0.1) $\quad \mathcal{D}({}^sA)\subset\mathcal{D}({}^wA)$,

so the problem is to determine when the opposite inclusion is valid.

Kordyukov [Ko2] (see also announcement in [Ko1]) proved that when E=F is an Hermitian vectorbundle and $a(x,\xi)>0$, then ${}^{W}A={}^{S}A$ for all $p\in[1,\infty]$.

In this note we propose to generalize this result, by weakening the assumptions on the E,F,a. Our result is:

Theorem 0.1. Let A be as above. Then for each p in $[1,\infty]$ we have ${}^{W}A={}^{S}A$.

In the case $1<p<\infty$ it is well known that ${}^{W}A={}^{S}A$. Indeed, we have $\mathcal{D}({}^{S}A)\supset W_p^m\supset\mathcal{D}({}^{W}A)$, where the first inclusion is obvious and the second one is a consequence of uniform elliptic regularity. Apart from the result of Kordyukov mentionned above, many authors have obtained the equality between weak and strong extensions or results which imply this in various special cases. Davies [D] obtains such results for second order operators on homogeneous spaces, Lie groups and on some more general manifolds. The work of Strichartz [Str] also treats the second order case on manifolds. Kato [K] studies the Schrödinger operator on $\mathbb{R}^n$ with non smooth potential. Stewart [Ste] studies strongly elliptic operators in the Euclidean case and obtains resolvent estimates in the case $p=1,\infty$. He also refers to some unpublished seminar notes of Masuda.

1. Proof of the theorem.

We introduce a form of the Agmon-Agranovich-Vishik ellipticity condition:

(H) We have E=F and there are constants ρ and C with $|\rho|=1$ and $C>0$, such that $\|(a(x,\xi)-\rho\lambda)^{-1}\|\leq C$ for all $x\in M$, $\|\xi\|=1,\lambda>0$.

Proposition 1.1. Theorem 0.1 holds if we assume (H).

We start by establishing an essentially well known consequence of the assumption (H). (See for instance Browder [B], Agmon [A] in the Euclidean case.)

Proposition 1.2. There exists a constant $\lambda_0>0$, such that for $\lambda>\lambda_0$, the operator $A-\lambda\rho: W_2^{m+\ell}(M)\to W_2^{\ell}(M)$ is bijective for every $\ell\in\mathbb{R}$ with a bounded inverse $(A-\lambda\rho)^{-1}: W_2^{\ell}(M)\to W_2^{m+\ell}(M)$, satisfying the estimate

(1.1)

$$\|(A-\lambda\rho)^{-1}u\|_{m+\ell}+\lambda^{1/m}\|(A-\lambda\rho)^{-1}u\|_{m+\ell-1}+\dots+\lambda\|(A-\lambda\rho)^{-1}u\|_{\ell}\leq C\|u\|_{\ell},$$

for every $u\in W_2^{\ell}(M)$. Here $\|\cdot\|_s$ denotes the norm in $W_2^{s}(M)$, and $C>0$ is a constant which is independent of u and of λ.

Proof. We first notice that it is enough to prove the result with A replaced by $\rho^{-1}A$, which satisfies $|(\rho^{-1}a(x,\xi)-\lambda)^{-1}|\leq C$, $x\in M$, $|\xi|=1$. This is the usual uniform Agmon condition so we can apply the Seeley construction of a local parametrix of $(\rho^{-1}A-\lambda)$ which will satisfy uniform estimates. (See [Se].) We then get a global parametrix by using the uniform partition of unity of [S1]. (Making use of the fact that A is differential, one can give simpler proofs, see for instance [S2].)

□

Let $f\in C^{\infty}(M;\mathbb{R})$ have the property that $\vartheta(x,\partial_x)f$ is a C^{∞} bounded function for every C^{∞} bounded vectorfield ϑ. Then:

(1.2) $e^{f}\circ A\circ e^{-f}=A+B_f$,

where B_f is a C^{∞}-bounded differential operator of order m-1. We then have:

$$e^{f}\circ(A-\lambda\rho)\circ e^{-f}=(A-\lambda\rho)+B_f \quad ,$$

and if we choose $\lambda>\lambda_0$, where λ_0 is given in the proposition, then in the sense of bounded operators from $W_2^{m+\ell}(M)$ to $W_2^{\ell}(M)$, we can write

(1.3) $$e^{f}\circ(A-\lambda\rho)\circ e^{-f}=(I+B_f(A-\lambda\rho)^{-1})\circ(A-\lambda\rho) \quad .$$

If $\lambda>0$ is large enough, (depending only on the bounds on $\partial^{\alpha}f$ for $1\leq|\alpha|\leq m$ in canonical coordinates,) the norm of $B_f(A-\lambda\rho)^{-1}: L^2\to L^2$ is smaller than $\frac{1}{2}$. We conclude that the right hand side of (1.3), viewed as an operator $W_2^m\to L^2$, is bijective with a uniformly bounded inverse when $\lambda>\lambda_1$, and $\lambda_1>0$ is large enough. The identity (1.3) is of course to be understood in the sense of distributions, but we have:

<u>Proposition 1.3.</u> Let f be as above. Then there exists a constant $\lambda_1>0$ depending only on the bounds of $\partial^{\alpha}f$ for $1\leq|\alpha|\leq m$ (in canonical coordinates) such that for $\lambda>\lambda_1$ the uniformly bounded inverse, G_λ of the operator $A-\lambda\rho$: $W_2^m\to L^2$ (which exists according to Proposition 1.1) has the following property: The operator $e^f\circ G_\lambda\circ e^{-f}$ (which a priori maps $L^2\cap\mathcal{E}'$ into $W^m_{2\,loc}$) has a bounded extension $L^2\to W_2^m$, and the norm can be bounded by a constant which is independent of λ and of f.

<u>Proof.</u> If f is a bounded function, then multiplication by $e^{\pm f}$ is a bounded operator on all the spaces W_2^s, and we see that $e^f\circ G_\lambda\circ e^{-f}$ is the inverse of the operator (1.3), and the proposition follows in that case. If f is not a bounded function, we let $\psi(s)$ be a smooth increasing real valued function with $\psi(s)=s$ for $-1\leq s\leq 1$, $\psi(s)=-2$ for $s\leq -3$, $\psi(s)=2$ for $s\geq 3$ and put $\psi_\varepsilon(s)=\varepsilon^{-1}\psi(\varepsilon s)$, for $0<\varepsilon\leq 1$. Notice that $|\partial_s^k\psi_\varepsilon(s)|\leq C_k$ for $k=1,2,\ldots$, where C_k are independent of s and of ε, so that the functions $f_\varepsilon=\psi_\varepsilon\circ f$ satisfy $|\partial^{\alpha}f_\varepsilon(x)|\leq \tilde{C}_\alpha$ for $1\leq|\alpha|\leq m$, with $\tilde{C}_\alpha$ independent of ε. We can then apply the proposition with f replaced by f_ε. We conclude that $e^{f_\varepsilon}\circ G_\lambda\circ e^{-f_\varepsilon}$ is bounded $L^2\to W_2^m$, uniformly with respect to λ and ε. If $u\in L^2\cap\mathcal{E}'$, then for $\varepsilon>0$ small enough, we have $f_\varepsilon=f$ on the support

of u, and if $K \subset\subset M$ is arbitrary, then for $\varepsilon>0$ small enough, we have $e^f G_\lambda e^{-f} u = e^{f_\varepsilon} G_\lambda e^{-f_\varepsilon} u$ on K, hence $\|e^f G_\lambda e^{-f} u\|_{m,K} \leq C\|u\|_0$, with a constant $C>0$ which is independent of u and K. Here $\|\cdot\|_{m,K}$ denotes the W_2^m-norm over K. Since K is arbitrary, we conclude that $e^f G_\lambda e^{-f} u$ belongs to W_2^m and $\|e^f G_\lambda e^{-f} u\|_m \leq C\|u\|_0$. It is then clear that $e^f \circ G_\lambda \circ e^{-f}$ extends to a bounded operator $L^2 \to W_2^m$. □

Notice that the distribution kernel of $e^f \circ G_\lambda \circ e^{-f}$ is of the form $e^{f(x)-f(y)} K_{G_\lambda}(x,y)$, if we denote the distribution kernel of G_λ by $K_{G_\lambda}(x,y)$. Also notice that K_{G_λ} is C^∞ outside the diagonal. We shall apply the above result with $f=f_x(y)=(t+1)\tilde{d}(x,y)$, where $\tilde{d}$ is the function constructed by Kordyukov (see Lemma 2.1 of [S1]). Here x may be an arbitrary point of M, and $t>0$ may be arbitrary but fixed. Then the hypotheses of the last proposition are satisfied uniformly when x varies in M and as in Theorem 2.1 of [S1] we obtain:

Proposition 1.4. Let $t>0$. Then there exists $\lambda(t)>0$ such that for $\lambda \geq \lambda(t)$ we have the the following: For every $\delta>0$ and all multiindices α, β, there exists $C_{\alpha,\beta,\delta}>0$ such that

(1.4) $|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \leq C_{\alpha,\beta,\delta}\, e^{-td(x,y)}$ for all $x,y \in M$ with $d(x,y)>\delta$.

The study of K_{G_λ} in the region $d(x,y)<\delta$ goes through exactly as in section 3 of [S1], and we obtain the following analogue of Theorem 3.2 of that paper:

Theorem 1.5. Let $t>0$. Then there exists $\lambda(t)>0$ such that for $\lambda\geq\lambda(t)$ we have the following: For all multiindices α,β there exists a constant $C_{\alpha,\beta}>0$ such that when $m<n$ and $x\neq y$:

$$(1.5)\quad |\partial_x^\alpha\partial_y^\beta G_\lambda(x,y)|\leq C_{\alpha,\beta}d(x,y)^{m-n-|\alpha|-|\beta|}e^{-td(x,y)},$$

and when $m\geq n$ and $x\neq y$:

$$(1.6)\quad |\partial_x^\alpha\partial_y^\beta G_\lambda(x,y)|\leq C_{\alpha,\beta}(1+d(x,y)^{m-n-|\alpha|-|\beta|}|\log(d(x,y))|)e^{-td(x,y)}.$$

We here also notice that it is well known that the kernel is locally integrable in y for every fixed x and in x for every fixed y.

We have the following result where the only assumption is that M is of bounded geometry:

Lemma 1.6. Let $B(x,r)=\{y\in M;\ d(y,x)<r\}$. There exists a constant $C=C(M)$ such that for all $x\in M$ and $r\geq 0$:

$$(1.7)\quad \mathrm{Vol}(B(x,r))\leq e^{Cr}.$$

A simple proof of the lemma can be obtained by considering coverings by balls of bounded radius, and a more general result due to Bishop, can be found in the book of Gromov [G].

Using the lemma one obtains the following corollary of Theorem 1.4:

Corollary 1.7. There exists $\lambda_0>0$, such if $\lambda>\lambda_0$, then:

$$(1.8)\quad \sup_{x\in M}\int|k_{G_\lambda}(x,y)|dy<+\infty,\quad \sup_{y\in M}\int|k_{G_\lambda}(x,y)|dx<+\infty.$$

Proof. Using (1.5),(1.6), it is easy to see that

$$\sup_{x\in M}\int_{|x-y|\leq\delta}|k_{G_\lambda}(x,y)|dy<+\infty,\quad \sup_{y\in M}\int_{|x-y|\leq\delta}|k_{G_\lambda}(x,y)|dx<+\infty,$$

so we only have to estimate the corresponding integrals over the domain $|x-y|>\delta$, and here we may use (1.4): We get for $\lambda\geq\lambda(t)$

$$\int_{|x-y|>\delta} |K_{G_\lambda}(x,y)|\,dy \le C_0\int_0^{+\infty} e^{-td(x,y)}\,dy = C_0\int_0^{+\infty} e^{-tr}\,dV(r),$$

where $V(r)=\mathrm{Vol}(B(x,r))$. We choose t strictly larger than the constant "C" which appears in Lemma 1.5. Then the last integral is convergent and an integration by parts gives:

$$\int_0^\infty e^{-tr}\,dV(r) = \int_0^\infty te^{-tr}V(r)dr \le \int_0^\infty te^{(C-t)r}dr = t/(t-C).$$

The same estimate is valid for the x-integrals and the corollary follows.

□

From now on we take $\lambda>0$ sufficiently large so that the corollary applies. By Shur's lemma we then know that the restriction of G_λ to C_0^∞ has a unique bounded extension $L^p(M;E)\to L^p(M;E)$, when $1\le p<\infty$. It is also easy to see (using also (1.4)), that G_λ has a unique bounded extension: $\mathfrak{B}^\infty\to\mathfrak{B}^\infty$. Working with some fixed p, we denote this extension $\tilde{G}_\lambda$. For $u\in C_0^\infty$, we have $(A-\lambda\rho)G_\lambda u=u$, and using the continuity of $\tilde{G}_\lambda$ in $\mathfrak{B}^p$ and the continuity of $A-\lambda\rho$ for the weak topology of distributions, we get:

(1.9) $(A-\lambda\rho)\tilde{G}_\lambda=I$ on $\mathfrak{B}^p$.

Let $u\in\mathcal{D}({}^wA)$ so that u and Au belong to $\mathfrak{B}^p$. Then if $\varphi\in C_0^\infty(M;E)$, we get:

(1.10) $(\tilde{G}_\lambda(A-\lambda\rho)u|\varphi)=((A-\lambda\rho)u|G_\lambda^*\varphi)=(u|(A-\lambda\rho)^*G_\lambda^*\varphi)$,

where the scalar products are taken either in $L^2(M;E)$ and $*$ indicates that we take the formal complex adjoint in the sense of distributions. As in [S1], section 4, these manipulations are justified by means of the family of uniformly C^∞ bounded cut-off functions $\chi_N\in C_0^\infty(M;[0,1])$, with $\chi_N=1$ on an exhaustive increasing family of compacts, K_N, $N=1,2,3,\ldots$. (Again we need the estimates (1.4).)

Now $(A-\lambda\rho)^* G_\lambda^* \psi=\psi$, as can be seen by replacing u by a C_0^∞ function ψ in (1.10) and using that $\tilde{G}_\lambda(A-\lambda\rho)\psi=G_\lambda(A-\lambda\rho)\psi=\psi$. Thus (1.10) reduces to:

(1.11) $(\tilde{G}_\lambda(A-\lambda\rho)u|\psi)=(u|\psi)$,

and varying ψ we conclude that:

(1.12) $\tilde{G}_\lambda(A-\lambda\rho)=I$ on $\mathcal{D}(^W A)$.

Thus we have proved that for λ sufficiently large, $(A-\lambda\rho)$ is bijective from $\mathcal{D}(^W A)$ onto $\mathcal{L}^p$ and that the inverse is $\tilde{G}_\lambda$.

We can now end the proof of Proposition 1.1. Let $u\in\mathcal{D}(^W A)$ and put $v=Au$. Let w_j, $j=1,2,..$ be a sequence of C_0^∞-functions converging to $v-\lambda\rho u$ in $\mathcal{L}^p$, and put $u_j=\tilde{G}_\lambda w_j\in\mathcal{L}^p\cap C^\infty$. Then $u_j\rightarrow u$ in $\mathcal{L}^p$ and $Au_j=w_j+\lambda\rho u_j\rightarrow u$ in $\mathcal{L}^p$. It only remains to prove that u_j belongs to $\mathcal{D}(^S A)$. We notice that if $\Omega_j=\text{supp}(w_j)$, then $\sup_x \int_{\Omega_j}(1-\chi_N(x))|K_{G_\lambda}(x,y)|dy$ and $\sup_{y\in\Omega_j}\int(1-\chi_N(x))|K_{G_\lambda}(x,y)|dx$ tend to zero when N tends to infinity, and similarly when $(1-\chi_N(x))K_{G_\lambda}$ is replaced by some x-derivative of the same function. (Indeed, this is proved in the same way as Corollary 1.7.) Hence (still with j fixed) $\chi_N u_j\rightarrow u_j$ and $A(\chi_N u_j)\rightarrow Au_j$ in $\mathcal{L}^p$ when $N\rightarrow\infty$, and the proof is complete. □

<u>Proof of Theorem 0.1.</u> We may assume that E and F are uniformly C^∞-bounded Hermitian vectorbundles. Let A' denote the formal complex adjoint of A, and consider the uniformly elliptic C^∞-bounded formally self adjoint operator: $\mathcal{a}: C^\infty(M;F\oplus E)\rightarrow C^\infty(M;F\oplus E)$ given by the matrix

$$\mathcal{a}=\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$$

We notice that $\mathcal{Q}$ satifies (H) with $p=1$, so we know that $\mathcal{D}({}^{W}\mathcal{Q})=\mathcal{D}({}^{S}\mathcal{Q})$. It is easy to see that $\mathcal{D}({}^{W}\mathcal{Q})=\mathcal{D}({}^{W}A')\oplus\mathcal{D}({}^{W}A)$ and that we have the similar equality for the strong extensions. It follows that ${}^{W}A={}^{S}A$. □

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