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Return method : application to controllability

EQUATIONS AUX DERIVEES PARTIELLES

RETURN METHOD : APPLICATION TO CONTROLLABILITY

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I. Description of the return method.

The goal of this paper is to describe a method called the return method in [Co2] and introduced in [Co1] to solve a stabilization problem, which allows also to prove controllability in some cases.

In order to explain this method, let us consider first the problem of local controllability of a control system in finite dimension. So we consider the control system

$$\dot{x} = f(x, u) \tag{1.1}$$

where $x \in \mathbf{R}^n$ is the state and $u \in \mathbf{R}^m$ is the control ; we assume that f is of class C^∞ and satisfies

$$f(0, 0) = 0. \tag{1.2}$$

There are various possible definitions of local controllability. Here we use the following one.

Definition 1.1.— *System $\dot{x} = f(x, u)$ is locally controllable if for any $T > 0$ there exist ε in $(0, +\infty)$ such that for any x_0 and x_1 of norm less than ε, there exists a bounded measurable function $u : [0, T] \to \mathbf{R}^m$ such that if x is the (maximal) solution of $\dot{x} = f(x, u(t))$ which satisfies $x(0) = x_0$, then $x(T) = x_1$.*

One does not know any interesting necessary and sufficient condition for local controllability but there are many useful necessary conditions and sufficient conditions which have been found during the last twenty years ; for a state of art see for example the survey paper by Kawski [K], two recent research papers by Agrachev-Gamkrelidze [A-G] and by Bianchini-Stefani [B-S], and the references therein. Note that all these conditions rely on Lie bracket and that this geometric tool does not seem to give good results for distributed control systems - in this case x is an infinite dimensional space -. On the other hand for *linear* distributed control systems there are powerful methods to prove controllability - e.g. the H.U.M. method due to J.-L.Lions, see [L1][L2], and the references therein -. The return method consists in reducing the local controllability of a nonlinear control system to the existence of - suitable - periodic trajectories and the controllability of *linear* systems. The idea is the following one : assume that, for any positive real number T, there exists a measurable bounded function $\overline{u} : [0, T] \to \mathbf{R}^m$ such that, if we denote by $\overline{x}$ the (maximal) solution of $\dot{\overline{x}} = f(\overline{x}, \overline{u}(t))$, $\overline{x}(0) = 0$, then

$$\overline{x}(T) = 0 \tag{1.3}$$

and

the linearized control system around $(\overline{x}, \overline{u})$ is controllable on $[0, T]$. $\tag{1.4}$

Then it follows easily from the inverse mapping theorem - see e.g. [So2 ;3.Th.6] - that $\dot{x} = f(x, u)$ is locally controllable.

Let us recall that the linearized control system around $(\overline{x}, \overline{u})$ is the time-varying control system

$$\dot{y} = A(t)y + B(t)v \qquad (1.5)$$

where the state is $y \in \mathbf{R}^n$, the control is $v \in \mathbf{R}^m$ and $A(t) = (\partial f / \partial x)(\overline{x}(t), \overline{u}(t))$, $B(t) = (\partial f / \partial u)(\overline{x}(t), \overline{u}(t))$.

For the linear control system (1.5) controllability on $[0, T]$ means, by definition, that for any y_0 and y_1 in $\mathbf{R}^n$, there exists a bounded measurable function $v : [0, T] \rightarrow \mathbf{R}^m$ such that if $\dot{y} = A(t)y + B(t)v$ and $y(0) = y_0$, then $y(T) = y_1$. There is a well known Kalman-type sufficient condition for the controllability of (1.5) due to Silverman and Meadows [S-M] - see also [So2 ;Cor.3.5.17] -. This is the following one :

Proposition 1.2.— *Assume that for some $\overline{t}$ in $[0, T]$*

$$\mathrm{Span}\left\{ \left(\frac{d}{dt} - A(t)\right)^i B(t)_{|t=\overline{t}}\, v; \ v \in \mathbf{R}^m, \quad i \geq 0 \right\} = \mathbf{R}^n, \qquad (1.6)$$

then system (1.5) is controllable on $[0, T]$. Moreover if A and B are analytic on $[0, T]$ anf if system (1.5) is controllable on $[0, T]$, then (1.6) holds for all $\overline{t}$ in $[0, T]$.

Note that if one takes $\overline{u} \equiv 0$, then the above method just gives the well known fact - see e.g. [So2 ;3.Th.6] - that if the time-invariant linear system $\dot{y} = (\partial f / \partial x)(0, 0)y + (\partial f / \partial u)(0, 0)v$ is controllable, then the nonlinear control system $\dot{x} = f(x, u)$ is locally controllable. But it may happen that (1.4) does not hold for $\overline{u} \equiv 0$, but holds for other choices of $\overline{u}$. Let us give simple examples.

Example 1.3. We take $n = 2$, $m = 1$ and consider the control system

$$\dot{x}_1 = x_2^3 \quad , \quad \dot{x}_2 = u. \qquad (1.7)$$

Let us take $\overline{u} \equiv 0$; then $\overline{x} \equiv 0$ and the linearized control system around $(\overline{x}, \overline{u})$ is

$$\dot{y}_1 = y_1 \quad , \quad \dot{y}_2 = v. \qquad (1.8)$$

which is clearly not controllable. Let us now take $\overline{u} \in C^\infty([0, T]; \mathbf{R})$ such that

$$\int_0^{T/2} \overline{u}(t)dt = 0, \qquad (1.9)$$

$$\overline{u}(T - t) = \overline{u}(t) \quad \forall t \in [0, T]. \qquad (1.10)$$

Then one easily checks that

$$\overline{x}_2(T/2) = 0 \quad , \tag{1.11}$$

$$\overline{x}_2(T - t) = -\overline{x}_2(t) \quad \forall t \in [0, T], \tag{1.12}$$

$$\overline{x}_1(T - t) = x_1(t) \quad \forall t \in [0, T]. \tag{1.13}$$

In particular we have

$$\overline{x}_1(T) = 0 \,, \ \overline{x}_2(T) = 0. \tag{1.14}$$

The linearized control system around $(\overline{x}, \overline{u})$ is

$$\dot{y}_1 = 3\overline{x}_1^2(t)y_2 \,, \ \dot{y}_2 = v \,; \tag{1.15}$$

so

$$A(t) = \begin{pmatrix} 0 & 3\overline{x}_1(t)^2 \\ 0 & 0 \end{pmatrix} \quad , \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.16}$$

and one easily sees that (1.6) holds if and only if

$$\exists i \in \mathbf{N} \quad \text{such that} \quad \frac{d^i \overline{x}_1}{dt^i}(\overline{t}) \neq 0. \tag{1.17}$$

Note that (1.17) holds for at least a $\overline{t}$ in $[0, T]$ if (and only if) $u \not\equiv 0$. So (1.4) holds if (and only if) $u \not\equiv 0$.

Example 1.4. We take $n = 3$, $m = 2$ and the control system is

$$\dot{x}_1 = u_1 \,, \quad \dot{x}_2 = u_2 \,, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \,. \tag{1.18}$$

Again one can check that the linearized control system around $(\overline{x}, \overline{u})$ is controllable on $[0, T]$ if and only if $\overline{u} \not\equiv 0$. Note that for system (1.18) it is easy to achieve the "return condition" (1.3) ; indeed, if

$$\overline{u}(T - t) = -\overline{u}(t) \quad \forall t \in [0, T], \tag{1.19}$$

then

$$\overline{x}(T - t) = \overline{x}(t) \quad \forall t \in [0, T], \tag{1.20}$$

and, in particular,

$$\overline{x}(T) = \overline{x}(0) = 0. \tag{1.21}$$

One may wonder if the local and controllability of $\dot{x} = f(x, u)$ implies the existence of u in $C^\infty([0, T]; \mathbf{R}^m)$ such that (1.3) and (1.4) hold. It has been proved to be true by Sontag in [So1]. Let us also remark that the above examples suggest that

for many choices of $\overline{u}$ then (1.4) holds. This in fact holds in general. More precisely let us assume that

$$\left\{ h(0); h \in \text{Lie} \left\{ \frac{\partial f}{\partial u^\alpha}(\cdot, 0), \quad \alpha \in \mathbf{N}^m \right\} \right\} = \mathbf{R}^n, \tag{1.22}$$

where Lie $\mathcal{F}$ denotes the Lie algebra generated by the vector fields in $\mathcal{F}$; then for generic u in $C^\infty([0, T]; \mathbf{R}^m)$ (1.4) holds ; this is proved in [Co3][Co4], and in [So3] if f is analytic. Let us recall that by a theorem due to Sussmann and Jurdjevic [S-J], (1.22) is a necessary condition for local controllability if f is analytic.

For controllability the return method does not seem to give any new interesting result if x lies in a finite dimensional space ; in particular the local controllability in *Example 1.3* follows from the Hermes condition [He],[Su1],[Su2] and the local controllability in *Example 1.4* follows from Chow's theorem [Ch]. But it gives some new results for distributed control system ; let us mention that if x belongs to an infinite dimensional space then (1.3) and (1.4) do not always imply the local controllability of $\dot{x} = f(x, u)$ on $[0, T]$; for example the phenomenon of "loss of derivatives" may appear ; in some cases this can be solved by using the Nash-Moser process - see e.g. [Ha] -. In section 2 of this paper we sketch a proof, relying on the return method, of the boundary controllability of the Euler equations of incompressible perfect fluids on a bounded connected and simply connected domain of $\mathbf{R}^2$.

For the stabilization problem the return method allows to obtain new results even if x lies in a finite dimensional space. Now given $T > 0$, we want to find $u^* \in C^\infty(\mathbf{R} \times \mathbf{R}; \mathbf{R}^m)$ T-periodic in time vanishing on $\{0\} \times \mathbf{R}$ such that the origin of $\mathbf{R}^n$ is a locally asymptotically stable point of $\dot{x} = f(x, u^*(x, t))$. So now we first try to find $\overline{u} \in C^\infty(\mathbf{R}^n \times \mathbf{R}; \mathbf{R}^m)$ T-periodic in time vanishing on $\{0\} \times \mathbf{R}$ such that

$$(x(0) \quad \text{small and} \quad \dot{x} = f(x, \overline{u}(x, t))) \Longrightarrow x(0) = x(T) \tag{1.23}$$

and, for all $x_0 \in \mathbf{R}^n \setminus \{0\}$ small enough,

the linearized control system around $(x_{x_0}, \overline{u}(x_{x_0}, \cdot))$ is controllable on $[0, T]$ \quad (1.24)

where x_{x_0} denotes the solution of the Cauchy problem

$$\frac{\partial x_{x_0}}{\partial t} = f(x_{x_0}, \overline{u}(x_{x_0}, t)), \quad x_{x_0}(0) = x_0. \tag{1.25}$$

If such a $\overline{u}$ exists one can deduce the existence of u^* with the above properties by pertubing $\overline{u}$ slightly in a suitable way ; let us mention that Pomet has given in [P] - see also [C-P] - a nice method to construct u^* from $\overline{u}$. As an application of the return method we have obtained in [Co1] :

Theorem 1.5.— *Assume that $f(x, u) = \sum\limits_{i=1}^{m} u_i f_i(x)$ and that (1.22) holds. Then, for any positive T, there exists $u^* \in C^\infty(\mathbf{R}^n \times \mathbf{R}; \mathbf{R}^m)$, T-periodic in time, vanishing on $\{0\} \times \mathbf{R}$ such that 0 is a locally asymptotically stable point of $\dot{x} = f(x, u^*(x, t))$.*

In fact in [Co1] we have given a global version of Theorem 1.5 ; but the proofs of [Co1] can be easily adapted to obtain Theorem 1.5.

2. Boundary controllability of the $2 - D$ Euler equations.

Let Ω be a non empty bounded open connected subset of $\mathbf{R}^2$ of class C^∞. Let Γ be the boundary of Ω, let n be the outward unit normal vector field on Γ, and let Γ_0 be a subset of Γ which has a non empty interior in Γ. The problem of the controllability of the Euler equations for (Ω, Γ_0) is the following one : let $T > 0$, y_0 in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$, and y_1 in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$ satisfying

$$\operatorname{div} y_0 = 0 \quad \text{on} \quad \overline{\Omega}\,, \tag{2.1}$$

$$\operatorname{div} y_1 = 0 \quad \text{on} \quad \overline{\Omega}\,, \tag{2.2}$$

$$y_0 \cdot n = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0\,, \tag{2.3}$$

$$y_1 \cdot n = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0\,. \tag{2.4}$$

does there exist $y \in C^\infty(\overline{\Omega} \times [0, T]; \mathbf{R}^2)$ and p in $C^\infty(\overline{\Omega} \times [0, T]; \mathbf{R})$ such that

$$\frac{\partial y}{\partial t} + (y \cdot \nabla)y + \nabla p = 0 \quad \text{on} \quad \overline{\Omega} \times [0, T]\,, \tag{2.5}$$

$$\operatorname{div} y(\cdot, t) = 0 \quad \text{on} \quad \overline{\Omega} \quad \forall t \in [0, T]\,, \tag{2.6}$$

$$y(\cdot, t) \cdot n(\cdot) = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0 \quad \forall t \in [0, T]\,, \tag{2.7}$$

$$y(\cdot, 0) = y_0(\cdot) \quad \text{on} \quad \overline{\Omega}\,, \tag{2.8}$$

$$y(\cdot, T) = y_1(\cdot) \quad \text{on} \quad \overline{\Omega} \ ? \tag{2.9}$$

So now the state space is the set of y in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$ satisfying $\operatorname{div} y = 0$ on $\overline{\Omega}$. The control does not appear in the above formulation ; one can take, for example, $y \cdot n$ on Γ_0 and $\operatorname{curl} y := \partial y^2 / \partial x^1 - \partial y^1 / \partial x^2$ on each point of Γ_0 where $y \cdot n > 0$. If the above problem has a solution (y, p) for all y_0, y_1 in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$ satisfying (2.1) to (2.4) then the Euler equations are, by definition, controllable for (Ω, Γ_0). Then we have proved in [Co5] :

Theorem 2.1.— *If Ω is simply connected, then the Euler equations are controllable for (Ω, Γ_0).*

We briefly sketch the proof of this theorem. First one notices that this theorem is a straightforward corollary of the following proposition :

Proposition 2.2.— *There exists a positive real number ν such that if y_0 is in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$, satisfies (2.1), (2.3), and :*

$$|y_0|_1 := \max\{|\partial^\alpha y_0(x)|; |\alpha| \le 1, x \in \overline{\Omega}\} < \nu,$$

then there exist y in $C^\infty(\overline{\Omega} \times [0,1]; \mathbf{R}^2)$ and p in $C^\infty(\overline{\Omega} \times [0,1]; \mathbf{R})$ satisfying (2.5) to (2.9) with $(T, y_1) = (1, 0)$ and such that

$$\frac{\partial^i y}{\partial t^i}(\cdot, 1) = 0, \quad \frac{\partial^i p}{\partial t^i}(\cdot, 1) = 0 \quad on \quad \overline{\Omega}, \forall i \in \mathbf{N}. \tag{2.10}$$

Indeed one easily checks that if (y, p) satisfies (2.5) to (2.7) and if $\lambda > 0$, then (y_λ, p_λ) defined by $y_\lambda(x, t) = \lambda y(x, \lambda t)$, $p_\lambda(x, t) = \lambda^2 p(x, \lambda t)$ satisfies (2.5) to (2.7) with T/λ instead of T ; similarly (y_-, p_-) defined by $y(x, t) = -y(x, T - t)$, $p(x, t) = p(x, T - t)$ also satisfies (2.5) to (2.7) if (y, p) satisfies (2.5) to (2.7) ; this allows to deduce theorem 2.1 from Proposition 2.2 - see [Co5] for more details -.

In order to prove Proposition 2.2 we use the return method. One first construct a Lipschitzian bounded contractible open subset Ω_1 of $\mathbf{R}^2$ the boundary of which consists of two closed disjoint segments of straight line Γ_{-1} and Γ_1 and of two connected closed disjoint curves Σ' and Σ'' which meet Γ_{-1} and Γ_1 at right angles. We impose also

$$\Omega \subset \Omega_1, \tag{2.11}$$

$$(\partial\Omega \setminus \Gamma_0) \subset \Sigma := \Sigma' \cup \Sigma'', \tag{2.12}$$

and that there exists a neighborhood of Γ_{-1} (resp. Γ_1) such that Ω_1 intersected with this neighborhood is the intersection of one of the open halfspace the boundary of which is the straightline containing Γ_{-1} (resp. Γ_1) with the part of the strip limited by the two straightlines passing through $\partial\Gamma_{-1}$ (resp. $\partial\Gamma_1$) and orthogonal to Γ_{-1} (resp. Γ_1) contained in this neighborhood ; see the figure below.
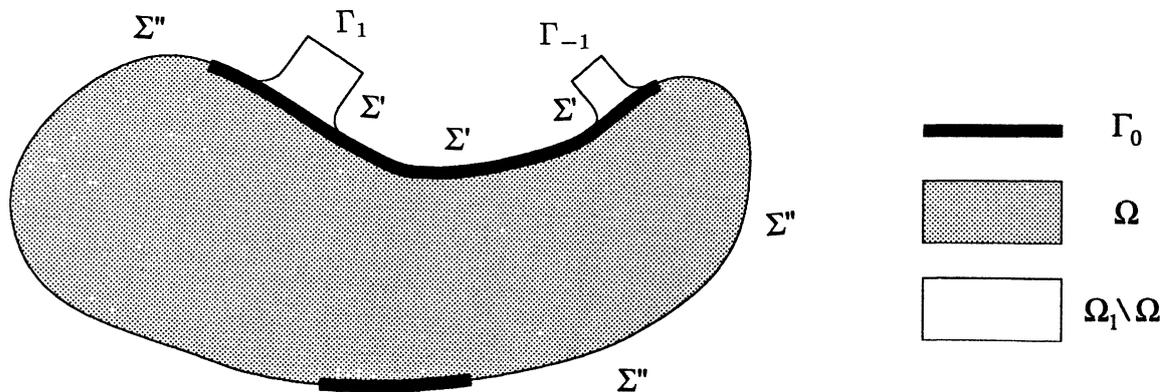
The existence of such a Ω_1 follows from the assumptions on Ω and Γ_0. Clearly there exists $\Theta \in C^\infty(\overline{\Omega}_1 \setminus \partial\Sigma; [-1, 1])$ such that

$$\Delta\Theta = 0 \quad \text{on} \quad \Omega_1 \tag{2.13}$$

$$\Theta = 1 \quad \text{on} \quad \Gamma_1 \setminus \partial\Gamma_1 \tag{2.14}$$

$$\Theta = -1 \quad \text{on} \quad \Gamma_{-1} \setminus \partial\Gamma_{-1} \tag{2.15}$$

$$\partial\Theta/\partial n = 0 \quad \text{on} \quad \Sigma \setminus \partial\Sigma. \tag{2.16}$$

Simple classical arguments relying on extensions by symmetries show that in fact $\Theta \in C^\infty(\overline{\Omega}_1; [-1, 1])$. Let $\gamma \in C^\infty([0, 1]; [0, +\infty])$ be such that

$$\text{Support } \gamma \subset (0, 1), \tag{2.17}$$

$$\gamma \not\equiv 0. \tag{2.18}$$

Let M be a positive real number and let $\overline{y} : \overline{\Omega} \times [0, 1] \to \mathbf{R}^2$ and $\overline{p} : \overline{\Omega} \times [0, 1] \to \mathbf{R}$ be defined by

$$\overline{y}(x, t) = M\,\gamma(t)\,\nabla\Theta(x), \tag{2.19}$$

$$\overline{p}(x, t) = -M\,\dot{\gamma}(t)\Theta(x) - (M^2/2)\gamma(t)^2|\nabla\Theta(x)|^2 \tag{2.20}$$

for all (x, t) in $\overline{\Omega} \times [0, 1]$. Then (2.5) to (2.9) hold with $y = \overline{y}$, $p = \overline{p}$, $T = 1$, $y_0 = 0$, and $y_1 = 0$. Let us study the controllability of the linearized control system around $(\overline{y}, \overline{p})$. This linear control system is :

$$z_t + (\overline{y} \cdot \nabla)z + (z \cdot \nabla)\overline{y} + \nabla\pi = 0 \quad \text{on} \quad \overline{\Omega} \times [0, 1], \tag{2.21}$$

$$\text{div}\,z(\cdot, t) = 0 \quad \text{on} \quad \overline{\Omega} \quad \forall t \in [0, 1], \tag{2.22}$$

$$z(\cdot, t) \cdot n(\cdot) = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0 \quad \forall t \in [0, 1]. \tag{2.23}$$

This linear control system is controllable on [0,1] if given z_0 and z_1 in $C^\infty(\overline{\Omega}; \mathbf{R}^2)$ satisfying

$$\text{div}\,z_0 = \text{div}\,z_1 = 0 \quad \text{on} \quad \overline{\Omega}, \tag{2.24}$$

$$z_0 \cdot n = z_1 \cdot n = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0, \tag{2.25}$$

there exist z in $C^\infty(\overline{\Omega} \times [0, 1]; \mathbf{R}^2)$ and π in $C^\infty(\overline{\Omega} \times [0, 1]; \mathbf{R})$ satisfying (2.21),(2.22),(2.23), and

$$z(\cdot, 0) = z_0(\cdot) \quad \text{on} \quad \overline{\Omega}, \tag{2.26}$$

$$z(\cdot, 1) = z_1(\cdot) \quad \text{on} \quad \overline{\Omega}. \tag{2.27}$$

Taking the curl of (2.21) we get, using (2.13),

$$\frac{\partial \omega}{\partial t} + (\overline{y} \cdot \nabla)\omega = 0 \qquad (2.28)$$

with

$$\omega(\cdot, t) = \operatorname{curl} z(\cdot, t). \qquad (2.29)$$

Let us assume that $\nabla\Theta$ vanishes at some point a in $\overline{\Omega}$. Then, using (2.19) and (2.28), we get that

$$\frac{\partial \omega}{\partial t}(a, \cdot) = 0 \quad \text{on} \quad [0, 1]$$

and so clearly the linear control system (2.21) to (2.23) is not controllable on [0,1]. Conversely if

$$\nabla\Theta(x) \neq 0 \quad \forall x \in \overline{\Omega} \qquad (2.30)$$

and if M is large enough then the linear control system (2.21) to (2.23) is controllable on [0,1]. This can be seen using an extension method similar to the one introduced by Russell in [R]. Let us give the proof. Let Ω_2 be a bounded open subset of $\mathbf{R}^2$ of class C^∞ containing $\overline{\Omega}_1$; let us extend Θ to all of $\overline{\Omega}_2$; this extension, that we will denote also Θ, is required to be of class C^∞ and of compact support in Ω_2. Let $\phi : \overline{\Omega}_2 \times [0, 1] \to \overline{\Omega}_2$ be defined by

$$\frac{\partial \phi}{\partial t} = M\gamma(t)\nabla\Theta(\phi), \quad \phi(x, 0) = x. \qquad (2.31)$$

Note that

$$\frac{\partial}{\partial t}(\Theta(\phi)) = M\gamma(t)|\nabla\Theta(\phi)|^2 \geq 0. \qquad (2.32)$$

Using (2.14), (2.15),(2.16),(2.30), and (2.32), one easily gets that, if M is large enough,

$$\Theta(\phi(x, 1)) > 1 \quad \forall x \in \overline{\Omega}_1. \qquad (2.33)$$

Since $\Theta(\overline{\Omega}_1) \subset [-1, 1]$ we get from (2.33)

$$\phi(x, 1) \notin \overline{\Omega}_1 \quad \forall x \in \overline{\Omega}_1. \qquad (2.34)$$

Let $\mu \in C^\infty([0, 1]; [0, 1])$ be such that $\mu(0) = 1$, $\mu(1) = 0$. We now define $\overline{y}$ on all $\overline{\Omega}_2 \times [0, 1]$ by requiring (2.19) for all (x, t) in $\overline{\Omega}_2 \times [0, 1]$. Let us consider the linear Cauchy problem, where the unknown are $z : \overline{\Omega} \times [0, 1] \to \mathbf{R}^2$ and $\omega : \overline{\Omega}_3 \times [0, 1] \to \mathbf{R}$ and the initial data $\omega_0 : \overline{\Omega}_3 \to \mathbf{R}$,

$$(P) \begin{cases} \frac{\partial \omega}{\partial t} + (\overline{y} \cdot \nabla)\omega = 0 \quad \text{on} \quad \overline{\Omega}_2 \times [0, 1], & (2.35) \\ \omega(\cdot, 0) = \omega_0(\cdot) \quad \text{on} \quad \overline{\Omega}_2, & (2.36) \\ \operatorname{div} z(\cdot, t) = 0 \quad \text{on} \quad \overline{\Omega}, \quad \forall t \in [0, 1], & (2.37) \\ \operatorname{curl} z(\cdot, t) = \omega(\cdot, t) \quad \text{on} \quad \overline{\Omega}, \quad \forall t \in [0, 1], & (2.38) \\ z(x, t) \cdot n(x) = (\mu(t)z_0(x) + (1 - \mu(t))z_1(x)) \cdot n(x), \quad \forall (x, t) \in \partial\Omega \times [0, 1]. & (2.39) \end{cases}$$

Clearly for any ω_0 in $C^\infty(\overline{\Omega}_3; \mathbf{R})$ this Cauchy problem has a (unique) solution (z, ω) in $C^\infty(\overline{\Omega} \times [0,1]; \mathbf{R}^2) \times C^\infty(\overline{\Omega} \times [0,1]; \mathbf{R})$. Note also that, by (2.34), there exist ω_0 in $C^\infty(\overline{\Omega}_2 \times [0,1]; \mathbf{R})$ such that

$$\omega_0(x) = \mathrm{curl}\, z_0(x) \quad \forall\, x \in \overline{\Omega} \tag{2.40}$$

$$\omega_0(x) = \mathrm{curl}\, z_1(\phi(x, 1)) \quad \forall\, x \in \overline{\Omega}_2 \quad \text{such that} \quad \phi(x, 1) \in \overline{\Omega}. \tag{2.41}$$

With such a ω_0 we have, since Ω is simply connected,

$$z(x, 0) = z_0(x) \quad \forall\, x \in \overline{\Omega}, \tag{2.42}$$

$$z(x, 1) = z_1(x) \quad \forall\, x \in \overline{\Omega}. \tag{2.43}$$

Moreover since Ω is simply connected we get from (2.35),(2.37), and (2.38) the existence of π in $C^\infty(\overline{\Omega} \times [0,1]; \mathbf{R})$ such that (2.21) holds. So the linear control system (2.21) to (2.23) is controllable on [0,1]. As mentioned in section 1, this does not imply, since the control system (2.5) to (2.9) is in an infinite dimensional space, that if y_0 is "close" to $\overline{y}(\cdot, 0) \equiv 0$ then there exists (y, p) close to $(\overline{y}, \overline{p})$ such that (2.5) to (2.9) holds with $T = 1$ and $y_1 = 0$. But it is proved in [Co5] this is in fact true - with y_0 "close" to 0 meaning that $|y_0|_1$ is small - and that one can also impose (2.10). The proof relies again on an extension method, similar to the above one.

Finally let us just mention that using the strong maximum principle and Morse theory it is proved in [Co5] that (2.30) holds ; instead of Morse theory one could alternatively used degree theory : indeed using the maximum principle as in [Co5] one can show that $\nabla \Theta$ does not vanish on $\partial \Omega_1$ and that

$$\mathrm{degree}\,(\nabla \Theta, \Omega_1, 0) = 0. \tag{2.44}$$

Since $\partial \Theta / \partial x - i \partial \Theta / \partial y$ is holomorphic in Ω_1 (2.30) follows from (2.44).

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