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Constructible sheaves, Whitney functions and Schwartz’s distributions

EQUATIONS AUX DERIVEES PARTIELLES

CONSTRUCTIBLE SHEAVES, WHITNEY FUNCTIONS
AND SCHWARTZ'S DISTRIBUTIONS

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(joint work with Masaki Kashiwara)
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1 Introduction

Let $f$ be a proper morphism of real analytic manifolds. It is a natural problem to characterize the space of the integrals along $f$ of all $C^\infty$ or distribution densities. Such problems occur in particular when studying correspondences:

$$
\begin{array}{ccc}
S & \xrightarrow{a} & Y \\
\downarrow f & & \\
X & \xrightarrow{g} & \\
\end{array}
$$

as, for example, the Penrose correspondence or else the Radon transform (see [2]).

The simple case of $X = \mathbb{C}(\simeq \mathbb{R}^2)$ and $f : X \to X$ is the map $z \mapsto z^2$ shows that the constructible sheaf $f_*\mathcal{C}_X$, the direct image of the constant sheaf $\mathcal{C}_X$ on $X$ by $f$, plays an essential role in this description.

In [5], Kashiwara has introduced the functor $\mathcal{H}^0_z(\cdot)$ of moderate cohomology. It is defined on the category of $\mathbb{R}$-constructible sheaves on the real analytic manifold $X$, and is characterized by the fact that it is exact and its value on $\mathcal{C}_Z$ for a closed subanalytic subset $Z$ of $\mathbb{R}$ is the sheaf of distributions supported by $Z$.

Here, we shall give another construction of $TH\mathcal{H}_X(\cdot)$ (that we prefer to denote by $THom(\cdot, \mathcal{D}b_X)$) and at the same time, we construct a new functor, dual to the preceding one, the Whitney functor $\mathcal{W}\mathcal{C}_X$. This is again an exact functor characterized by the fact that its value on $\mathcal{C}_U$, for $U$ an open subanalytic subset of $X$, is the subsheaf $\mathcal{T}_{X,U,X}^{\infty}$ of $C^\infty$-functions which vanish to infinite order on $X \setminus U$. If now $X$ is a complex manifold, taking the Dolbeault complexes of the preceding functors, we get the functors $\mathcal{H}om(\cdot, \mathcal{O}_X)$ and $\mathcal{W}\mathcal{O}_X$. The main result of this paper will be the adjunction formulas in section 6.

Section 1 to 6 of this paper are extracted from [7], and section 7 is extracted from [2]. The redaction is due to PS, and does not involve the responsibility of other authors.
2 Construction of functors on $\mathbb{R} - C(X)$

Let $X$ be a real analytic manifold and denote by $\text{Mod}(\mathcal{C}_X)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on $X$, by $\mathbb{R} - C(X)$ the abelian subcategory of $\mathbb{R}$-constructible sheaves and by $\mathbb{R} - C_c(X)$ the subcategory of $\mathbb{R} - C(X)$ of sheaves with compact support. Denote by $\mathcal{S}_X$ the category whose objects are the open subanalytic relatively compact subsets of $X$, the only morphisms being the inclusions $U \subseteq V$. Then $U \mapsto \mathcal{C}_U$ gives a faithful functor $\mathcal{S}_X \to \mathbb{R} - C(X)$. Let $\mathcal{A}$ be an abelian category over $\mathbb{C}$. This means that $\text{Hom}_\mathcal{A}(M, N)$ has a structure of $\mathbb{C}$-vector space for $M, N \in \mathcal{A}$, and the composition of morphisms is $\mathbb{C}$-bilinear. Let $\psi : \mathcal{S}_X \to \mathcal{A}$ be a functor, and consider the conditions:

\[
\psi(\emptyset) = 0. \quad (2.1)
\]

for any $U, V$ in $\mathcal{S}_X$, the sequence

\[
\psi(U \cap V) \to \psi(U) \oplus \psi(V) \to \psi(U \cup V) \to 0 \quad (2.2)
\]

is exact.

for any open inclusion $U \subseteq V$ in $\mathcal{S}_X$, $\psi(U) \to \psi(V)$ is a monomorphism. (2.3)

**Theorem 2.1**

(i) Assume (2.1) and (2.2). Then there is a right exact functor, unique up to isomorphism,

$$
\Psi : \mathbb{R} - C_c(X) \to \mathcal{A}
$$

such that $\Psi(\mathcal{C}_U) \simeq \psi(U)$ functorially in $U \in \mathbb{R} - C_c(X)$.

(ii) Assume (2.1), (2.2) and (2.3). Then $\Psi$ is exact.

(iii) Let $\psi_1$ and $\psi_2$ be two functors from $\mathcal{S}_X$ to $\mathcal{A}$ both satisfying (2.1) and (2.2), and let $\Psi_1$ and $\Psi_2$ be the corresponding functors given in (i). Let $\theta : \psi_1 \to \psi_2$ be a morphism of functors. Then $\theta$ extends uniquely to a morphism of functors

$$
\Theta : \Psi_1 \to \Psi_2.
$$

(iv) In the situation of (i), assume that $\mathcal{A}$ is a subcategory of the category $\text{Mod}(\mathcal{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces on $X$, and that $\mathcal{A}$ is local, that is: an object $F$ of $\text{Mod}(\mathcal{C}_X)$ belongs to $\mathcal{A}$ if for any relatively compact open $U$ there exists $F'$ in $\mathcal{A}$ such that $F'|_U \simeq F'|_U'$. Assume further that $\psi$ is local, that is: $\text{supp}(\psi(U)) \subseteq U$ for any $U \in \mathcal{S}_X$.

Then $\psi$ extends uniquely to $\mathbb{R} - C(X)$ as a right exact functor $\Psi$ which is local, that is, $\Psi(F)|_U \simeq \Psi(F)|_U$ for any $F \in \mathbb{R} - C(X)$ and $U \in \mathcal{S}_X$. Moreover the assertion (ii) remains valid, as well as (iii), provided that both $\psi_1$ and $\psi_2$ are local.
3 The functors $\mathcal{W}C_X^\infty$ and $Thom(\cdot, Db_X)$

On a real analytic manifold $X$, we denote respectively by $A_X, C_X^\infty, Db_X, B_X$ the sheaves of real analytic functions, $C^\infty$-functions, Schwartz’s distributions and Sato’s hyperfunctions. We denote by $A_X^\dagger$ the sheaf of real analytic densities and if $\mathcal{F}$ is an $A_X$-module, we set $\mathcal{F}^\dagger = \mathcal{F} \otimes_{A_X} A_X^\dagger$. We denote by $D_X$ the sheaf of finite order differential operators with coefficients in $A_X$.

**Theorem 3.1** There exist exact functors:

$$\mathcal{W}C_X^\infty : \mathbb{R} - C(X) \longrightarrow \text{Mod}(D_X)$$

$$Thom(\cdot, Db_X) : \mathbb{R} - C(X)^{\text{op}} \longrightarrow (\text{Mod}(D_X)),$$

such that for any $U$ (resp. $Z$) open (resp. closed) subanalytic subset of $X$, one has:

$$C_U \otimes C_X^\infty \simeq I_{X \setminus U, X}^\infty,$$

$$Thom(C_Z, Db_X) \simeq \Gamma_Z(Db_X).$$

(Recall that for $Z$ a closed subset of $X$, $I_{Z, X}^\infty$ denotes the subsheaf of $C_X^\infty$ of functions vanishing to infinite order on $Z$.)

**Proof:** This follows immediately from Theorem 2.1 and the Łojasiewicz’s theorem (see [9]) which asserts that if $Z_1$ and $Z_2$ are closed subanalytic subsets of $X$, then the two sequences:

$$0 \longrightarrow I_{Z_1 \cup Z_2, X}^\infty \longrightarrow I_{Z_1, X}^\infty \oplus I_{Z_2, X}^\infty \longrightarrow I_{Z_1 \cap Z_2, X}^\infty \longrightarrow 0,$$

$$0 \longrightarrow \Gamma_{Z_1 \cap Z_2} Db_X \longrightarrow \Gamma_{Z_1} Db_X \oplus \Gamma_{Z_2} Db_X \longrightarrow \Gamma_{Z_1 \cup Z_2} Db_X \longrightarrow 0$$

are exact. \(q.e.d.\)

**Remark 3.2**

(i) The functor $Thom(\cdot, Db_X)$ has been defined in [5], without using Theorem 2.1.

(ii) The sheaves $F \otimes C_X^\infty$ and $Thom(F, Db_X)$ are sheaves of $C_X^\infty$-modules, hence are soft.

(iii) The vector space $\Gamma(X; F \otimes C_X^\infty)$ may naturally be endowed with a topology of type FN (Fréchet nuclear), the vector space $\Gamma_c(X; Thom(F, Db_X))$ with a topology of type DFN (dual of Fréchet nuclear) and this two spaces are dual to each other. This is proved by reducing to the case where $F = C_Z$, for $Z$ a closed subanalytic subset of $X$. 

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(iv) The functors $\otimes^w_{\mathcal{C}_X} \otimes^w_{\mathcal{C}_X}$ and $\text{Thom}(\cdot, \mathcal{D}b_X)$ being exact, they extend to the derived categories. Hence we obtain functors:

$$\otimes^w_{\mathcal{C}_X} : \mathcal{D}_b^{\mathcal{C}_X}(\mathcal{C}_X) \to \mathcal{D}^b(\mathcal{D}_X)$$

$$\text{Thom}(\cdot, \mathcal{D}b_X) : \mathcal{D}_b^{\mathcal{C}_X}(\mathcal{C}_X)^{\text{op}} \to \mathcal{D}^b(\mathcal{D}_X).$$

(v) Let $F \in \mathcal{D}_b^{\mathcal{C}_X}(\mathcal{C}_X)$ and denote by $D'F$ its dual, $D'F = R\text{Hom}(F, \mathcal{C}_X)$. Then we have a commutative diagram:

$\begin{array}{c}
D'F \otimes \mathcal{A}_X & \longrightarrow & R\text{Hom}(F, \mathcal{A}_X) \\
\downarrow & & \downarrow \\
D'F \otimes \mathcal{C}_X & \longrightarrow & D'F \otimes^w_{\mathcal{C}_X} \longrightarrow R\text{Hom}(F, \mathcal{C}_X) \\
\downarrow & & \downarrow \\
D'F \otimes \mathcal{D}_b X & \longrightarrow & \text{Thom}(F, \mathcal{D}_b X) \longrightarrow R\text{Hom}(F, \mathcal{D}_b X) \\
\downarrow & & \downarrow \\
D'F \otimes \mathcal{B}_X & \longrightarrow & R\text{Hom}(F, \mathcal{B}_X)
\end{array}$

4 Operations on $\otimes^w_{\mathcal{C}_X}$ and $\text{Thom}(\cdot, \mathcal{D}b_X)$

Let $f : Y \to X$ be a morphism of real analytic manifolds. As usual, one denotes by $\mathcal{D}_{Y \to X}$ (resp. $\mathcal{D}_{X \to Y}$) the sheaf $\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{D}_X$ (resp. $\mathcal{A}_Y^\vee \otimes_{f^{-1}\mathcal{A}_X} f^{-1}(\mathcal{D}_X \otimes_{\mathcal{A}_X} \mathcal{A}_X^\vee)$) endowed with its structure of a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$-module (resp. $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$-module). Let $F \in \mathcal{D}_b^{\mathcal{C}_X}(\mathcal{C}_X)$ and let $G \in \mathcal{D}_b^{\mathcal{C}_X}(\mathcal{C}_Y)$, with $f$ proper on supp $G$. There are natural (iso-)morphisms:

$$\mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}(F \otimes^w_{\mathcal{C}_X}) \longrightarrow f^{-1}F \otimes^w_{\mathcal{C}_X} \quad (4.1)$$

$$f^{-1}(F \otimes^w_{\mathcal{C}_X}) \longrightarrow R\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_{Y \to X}, f^{-1}F \otimes^w_{\mathcal{C}_X}) \quad (4.2)$$

$$f_*G \otimes^w_{\mathcal{C}_X} \cong f_*R\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_{Y \to X}, G \otimes^w_{\mathcal{C}_X}) \quad (4.3)$$

Morphism (4.2) is deduced from (4.1) by adjunction. Morphism (4.1) is constructed by using Theorem 2.1 which allows to reduce to the case where $F = \mathcal{C}_U$, $U$ an open subanalytic subset of $X$. Then, setting $Z = X \setminus U$, one has the natural morphism:

$$\mathcal{A}_Y \otimes_{f^{-1}\mathcal{A}_X} f^{-1}\mathcal{T}_{Z, X}^\infty \longrightarrow \mathcal{T}_{f^{-1}(Z), Y}^\infty.$$

When $f$ is a closed embedding, (4.1) is an isomorphism and when $f$ is smooth, (4.2) is an isomorphism. Morphism (4.3) is deduced from (4.2) by adjunction.
prove it is an isomorphism, one can treat separately the case of a closed embedding and that of a smooth map. In the first case, one is reduced to the case where \( Y \) is defined by an equation \( \{ t = 0 \} \) and \( G = \mathbb{C}_U \) for an open subanalytic subset \( U \) of \( Y \). Then (4.3) follows from the isomorphism \( \mathcal{I}^\infty_{X,U,t} \simeq \mathcal{I}^\infty_{Y,U,Y}[[t]] \). If \( f \) is smooth, one reduces to the case where \( Y = X \times \mathbb{R} \), \( f \) is the projection and \( G = \mathbb{C}_Z \) where \( Z \) is a closed subanalytic subset and the fibers of \( f \) on \( Z \) are closed intervals. Then one has to check that the sequence:

\[
0 \longrightarrow \mathcal{I}^\infty_{f(Z),X} \longrightarrow f_* \mathcal{I}^\infty_{Z,Y} \overset{\partial / \partial t}{\longrightarrow} f_* \mathcal{I}^\infty_{Z,Y} \longrightarrow 0
\]

is exact (see [5]).

Similarly, one has a natural isomorphism, assuming \( f \) is proper on \( \text{supp } G \):

\[
f_!(\text{Thom}(G, \mathcal{D}b_Y^\vee) \otimes_{\mathcal{D}Y} \mathcal{D}_{Y \to X}) \simeq \text{Thom}(Rf_!G, \mathcal{D}b_X^\vee), \tag{4.4}
\]

a natural isomorphism, assuming \( f \) is smooth:

\[
R\text{Hom}_{\mathcal{D}Y}(\mathcal{D}_{Y \to X}, \text{Thom}(f^{-1}F, \mathcal{D}b_Y)) \simeq f^{-1}\text{Thom}(F, \mathcal{D}b_X), \tag{4.5}
\]

and a natural isomorphism, assuming \( f \) is a closed embedding:

\[
\text{Thom}(f^{-1}F, \mathcal{D}b_Y) \simeq R\text{Hom}_{\mathcal{D}X}(\mathcal{D}_{X \to Y}, \text{Thom}(F, \mathcal{D}b_X)). \tag{4.6}
\]

5 The functors \( \otimes^w \mathcal{O}_X \) and \( \text{Thom}(-, \mathcal{O}_X) \)

Now assume \( X \) is a complex manifold, denote as usual by \( \overline{X} \) the anti-holomorphic associated complex manifold, and identify \( X \) to the diagonal of \( X \times \overline{X} \).

**Definition 5.1** Let \( F \in \mathcal{D}^b_{\mathbb{R}-c}(\mathbb{C}_X) \). One sets:

\[
F^w \otimes \mathcal{O}_X = R\text{Hom}_{\mathcal{D}X}(\mathcal{O}_{\overline{X}}, F^w \otimes \mathcal{O}_X)
\]

\[
\text{Thom}(F, \mathcal{O}_X) = R\text{Hom}_{\mathcal{D}X}(\mathcal{O}_{\overline{X}}, \text{Thom}(F, \mathcal{D}b_X))
\]

**Example 5.2**

(i) Let \( M \) be a real analytic manifold, \( i : M \hookrightarrow X \) a complexification. Then:

\[
i_* F^w \otimes \mathcal{O}_X \simeq i_*(F^w \otimes \mathcal{O}_M^\infty) \tag{5.1}
\]

\[
\text{Thom}(i_* F, \Omega_X[d_X]) \simeq i_* \text{Thom}(F, \mathcal{D}b_M^\vee) \tag{5.2}
\]

Note that isomorphism (5.2) is a result of Andronikof [1], which extends a previous theorem of Martineau (who treated the case when \( F = \mathcal{O}_M \)).
(ii) Let $Z$ be a closed complex analytic subset of $X$. Then:

$$
\mathcal{C}_Z \otimes \mathcal{O}_X \cong \mathcal{O}_X \mid_Z \quad (5.3)
$$

$$
\text{Thom}(\mathcal{C}_Z, \mathcal{O}_Z) \cong R\Gamma_{|Z|} \mathcal{O}_X \quad (5.4)
$$

Here, $\mathcal{O}_X \mid_Z$ denotes as usual the formal completion of $\mathcal{O}_X$ along $Z$, and $R\Gamma_{|Z|} \mathcal{O}_X$ the algebraic cohomology of $\mathcal{O}_X$ supported by $Z$ (see [4]).

In order to recall the main operations on these functors, we shall follow the notations of [6] for $\mathcal{D}$-modules. In particular if $f : Y \rightarrow X$ is a morphism of complex manifolds and if $\mathcal{M}$ (resp. $\mathcal{N}$) is a $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$)-module, one sets:

$$
\mathcal{L}^{-1} \mathcal{M} = \mathcal{D}_{Y \rightarrow X} \otimes_{\mathcal{L}^{-1} \mathcal{D}_Y} f^{-1} \mathcal{M},
$$

$$
\mathcal{L} \mathcal{N} = Rf_! (\mathcal{D}_{X \rightarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{N}).
$$

Then, using isomorphisms (4.1)-(4.6), we get the following results.

Assuming that $f$ is smooth, we have natural isomorphisms:

$$
f^{-1}(F \otimes \mathcal{O}_X) \cong R\text{Hom}_{\mathcal{D}_Y} (\mathcal{D}_{Y \rightarrow X}, f^{-1}F \otimes \mathcal{O}_Y), \quad (5.5)
$$

$$
R\text{Hom}_{\mathcal{D}_Y} (\mathcal{D}_{Y \rightarrow X}, \text{Thom}(f^{-1}F, \mathcal{O}_Y)) \cong f^{-1} \text{Thom}(F, \mathcal{O}_X). \quad (5.6)
$$

Assuming that $f$ is a closed embedding, we have natural isomorphisms:

$$
f^{-1}(F \otimes \mathcal{O}_X) \cong f^{-1}F \otimes \mathcal{O}_Y, \quad (5.7)
$$

$$
\text{Thom}(f^{-1}F, \mathcal{O}_Y) \cong f^{-1} \text{Thom}(F, \mathcal{O}_X). \quad (5.8)
$$

Assuming that $f$ is proper on $\text{supp} G$, we have natural isomorphisms:

$$
f_! R\text{Hom}_{\mathcal{D}_Y} (\mathcal{D}_{Y \rightarrow X}, G \otimes \mathcal{O}_Y) \cong f_! G \otimes \mathcal{O}_X, \quad (5.9)
$$

$$
\mathcal{L}_! \text{Thom}(G, \Omega_y[dy]) \cong \text{Thom}(Rf_! G, \Omega_X[dx]). \quad (5.10)
$$

Remark 5.3 The functor $\text{Thom}(\cdot, \mathcal{O}_X)$ has been microlocalized by Andronikof ([1]). The specialization of the functor $\otimes \mathcal{O}_X$ is related to the notion of asymptotic expansions. This will be developed elsewhere.

6 Adjunction formulas

Consider a correspondence of complex manifolds:

$$
\begin{array}{c}
S \\
\downarrow f \\
X \\
\downarrow g \\
Y
\end{array}
$$
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let $\mathcal{M} \in D^b_{\text{good}}(\mathcal{D}_X)$, (the triangulated subcategory of $\mathcal{M} \in D^b_{\text{good}}(\mathcal{D}_X)$ generated by the objects whose cohomology groups are coherent and may be endowed with a good filtration on each compact subset of $X$), and let $G \in D^b_{\mathbb{R}-c}(\mathcal{C}_Y)$. Assume:

$f$ is non characteristic for $\mathcal{M}$, \hspace{1cm} (6.1)
f is proper over $g^{-1}(\text{supp } G)$, \hspace{1cm} (6.2)
g is proper over $f^{-1}(\text{supp } \mathcal{M})$. \hspace{1cm} (6.3)

Introduce the notations:

$$\phi_S(G) = Rf_! g^{-1}G [d_S - d_X]$$
$$\phi_S(\mathcal{M}) = q_+ f^{-1} \mathcal{M}.$$

**Theorem 6.1** Assume (6.1)-(6.3). Then there are natural isomorphisms:

$$R\Gamma(X; R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \phi_S(G) \otimes \mathcal{O}_X)) [d_X] \simeq$$
$$R\Gamma(Y; R\text{Hom}_{\mathcal{D}_Y}(\phi_S(\mathcal{M}), G \otimes \mathcal{O}_Y)) [d_Y]$$
$$R\Gamma(X; R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \text{Thom}(\phi_S(G), \mathcal{O}_X))) [d_X] \simeq$$
$$R\Gamma(Y; R\text{Hom}_{\mathcal{D}_Y}(\phi_S(\mathcal{M}), \text{Thom}(G, \mathcal{O}_Y))) [d_Y].$$

**Sketch of proof**

It is enough to treat the case where $S = Y$ and the case where $S = X$. Using the isomorphisms (5.5)-(5.10), the remaining problem is to prove that if $f : Y \rightarrow X$ is a morphism of complex manifolds, if $\mathcal{N} \in D^b_{\text{good}}(\mathcal{D}_Y)$, $f$ is proper over $\text{supp } \mathcal{N}$, and $F \in D^b_{\mathbb{R}-c}(\mathcal{C}_X)$, then there are natural isomorphisms:

$$Rf_1 R\text{Hom}_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}(F \otimes \mathcal{O}_X))[d_Y] \simeq Rf_1 R\text{Hom}_{\mathcal{D}_Y}(f_! \mathcal{N}, F \otimes \mathcal{O}_Y)[d_X],$$
$$Rf_1 \text{Thom}(f^{-1}F, \Omega_X) \otimes_{\mathcal{O}_Y}^L \mathcal{N} \simeq \text{Thom}(F, \Omega_X) \otimes_{\mathcal{O}_X}^L f_! \mathcal{N}.$$

The second isomorphism is deduced from the first one by a duality argument. The first isomorphism is equivalent to:

$$Rf_1 R\text{Hom}_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}(F \otimes \mathcal{O}_X)) \simeq Rf_1 R\text{Hom}_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}(F) \otimes \mathcal{O}_Y).$$

Then the proof follows the main lines of [11]. One reduces to the case where $Y = Z \times X$ and $f$ is the projection, $\mathcal{N} = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{G}$, for $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module, $f$ being proper on $\text{supp } \mathcal{G}$. Then one has to prove the isomorphism:

$$Rf_1 R\text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y) \otimes_{\mathcal{O}_X}^L (F \otimes \mathcal{O}_X) \simeq Rf_1 R\text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f^{-1}F \otimes \mathcal{O}_Y).$$

Since one may represent $Rf_1 R\text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{O}_Y)$ by a complex of FN-free $\mathcal{O}_X$-modules and $F \otimes \mathcal{O}_X$ is an $\mathcal{O}_X$-module of type FN, one may apply Proposition 3.13 of [11], a variant of a theorem of Ramis-Ruget [10].

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7 Application to the Radon transform

As an application of Theorem 6.1, we recall some results of [2].

Let \( P \) be a complex \( n \)-dimensional projective space, \( P^* \) the dual projective space, and \( A \) the hypersurface of \( P \times P^* \) given by the incidence relation. If \( [\xi] = [\xi_0, \ldots, \xi_n] \) is a homogeneous coordinate system on \( P \), \( [\eta] = [\eta_0, \ldots, \eta_n] \) the dual system on \( P^* \), then:

\[
A = \{ (\xi, \eta) \in P \times P^* ; \langle \xi, \eta \rangle = 0 \}.
\]

Let us consider the correspondence:

\[
\begin{array}{ccc}
A & \downarrow f & P \\
\downarrow g & \quad & \quad \\
 & P^* & 
\end{array}
\]

(7.1)

and denote by \( q_1 \) and \( q_2 \) the projections from \( P \times P^* \) to \( P \) and \( P^* \), respectively.

For \( k \in \mathbb{Z} \), we denote by \( \mathcal{O}_P(k) \) the \(-k\)-th power of the tautological line bundle \( \mathcal{O}_P(-1) \), and we set:

\[
\mathcal{D}_P(k) = \mathcal{D}_P \otimes_{\mathcal{O}_P} \mathcal{O}_P(k).
\]

For \( (k, k') \in \mathbb{Z} \times \mathbb{Z} \), we set \( \mathcal{O}_{P \times P^*}(k, k') = \mathcal{O}_{P \times P^*} \otimes_{\mathcal{O}_P} \mathcal{O}_P(k) \otimes_{\mathcal{O}_P} \mathcal{O}_P(k') \). To \( k \in \mathbb{Z} \), we associate:

\[
k^* = -n - 1 - k.
\]

In [8], Leray introduced:

\[
\omega^*(\xi) = \sum_{i=0}^{n} (-1)^i \xi_i d\xi_0 \wedge \ldots \wedge \widehat{d\xi_i} \wedge \ldots \wedge d\xi_n,
\]

\[
s_k(\xi, \eta) = \frac{\omega^*(\xi)}{(\xi, \eta)^{n+1+k}}.
\]

Then \( s_k \) is a well defined section of \( \mathcal{O}_{P \times P^*}^{(n,0)}(-k, k^*) \) on \( P \times P^* \setminus A \). If \( n + 1 + k > 0 \) (i.e. if \( k^* < 0 \)), \( s_k \) has meromorphic singularities on \( A \), and its image via the natural morphism

\[
\Gamma(P \times P^* \setminus A; \mathcal{O}_{P \times P^*}^{(n,0)}(-k, k^*)) \longrightarrow H^1_A(P \times P^*; \mathcal{O}_{P \times P^*}^{(n,0)}(-k, k^*))
\]

defines a section (that we denote by the same symbol):

\[
s_k \in \Gamma(P \times P^*; \mathcal{B}^{(n,0)}_A(-k, k^*)),
\]

where \( \mathcal{B}^{(n,0)}_A(-k, k^*) \) is the sheaf \( \mathcal{B}_A = H^1_A(\mathcal{O}_{P \times P^*}) \) twisted by \( \mathcal{O}_{P \times P^*}^{(n,0)}(-k, k^*) \). One sees easily that the Leray section \( s_k \) defines a \( \mathcal{D}_P \)-linear morphism

\[
\alpha(s_k) : \mathcal{D}_P(-k^*) \longrightarrow \mathcal{D}_A(\mathcal{D}_P(-k))
\]

(7.2)
The main result of [2] is that if $-n - 1 < k < 0$, then $\alpha(s_k)$ is an isomorphism. Applying Theorem 6.1, we find for $F \in D^b_{\mathbb{R}^-}(\mathcal{O}_P)$, the isomorphisms:

\begin{equation}
R\Gamma(P; F^w \mathcal{O}_P(k)) \simeq R\Gamma(P^*; \phi^w(F) \mathcal{O}_P(k^*)) \quad \text{(7.3)}
\end{equation}

\begin{equation}
R\Gamma(P; Thom(F, \mathcal{O}_P(k))) \simeq R\Gamma(P^*; (\phi^w_s(\mathcal{M}), Thom(\phi^w_A(F), \mathcal{O}_P(k^*)))) \quad \text{(7.4)}
\end{equation}

Denote by $P$ and $P^*$ a real projective space of dimension $n > 1$ and its dual, and consider $P$ and $P^*$ as complexifications of $P$ and $P^*$. Let $k \in \mathbb{Z}$, and let $\varepsilon, \bar{\varepsilon} \in \{0, 1\}$ have different parity. We denote by $\mathcal{C}^\infty_P(k, \varepsilon)$ the locally constant sheaf of rank one over $\mathcal{C}^\infty_P$ whose global sections are represented by those functions $f$ on $\mathbb{R}^{n+1} \setminus \{0\}$ satisfying the homogeneity condition:

$$f(\lambda x) = (\text{sgn} \lambda)^\varepsilon \lambda^k f(x), \quad \text{for } \lambda \neq 0.$$

Using explicit integral formulas, Gelfand et al. [3] proved the isomorphisms for $-n - 1 < k < 0$:

$$\Gamma(P; \mathcal{C}^\infty_P(k, \varepsilon)) \simeq \begin{cases} 
\Gamma(P^*; \mathcal{C}^\infty_P(k^*, \bar{\varepsilon})) & \text{for } n \text{ even}, \\
\Gamma(P^*; \mathcal{C}^\infty_P(k^*, \varepsilon)) & \text{for } n \text{ odd}.
\end{cases}$$

We recover here these isomorphisms (and the similar ones with $\mathcal{C}^\infty$ replaced by $\mathcal{D}b$) by applying (7.3) to the case where either $F = \mathcal{C}_P$ or $F = K_P$, the canonical line bundle on $P$. In fact, $\mathcal{C}^\infty_P(k, 0) \simeq \mathcal{C}_P \otimes \mathcal{O}_P(k)$ and $\mathcal{C}^\infty_P(k, 1) \simeq K^w_P \mathcal{O}_P(k)$.

References


Constructible sheaves, Whitney functions and Schwartz's distributions


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