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# GLOBAL THEOREMS ON MANIFOLDS WHICH ADMIT DISTRIBUTIONS

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1. - In this lecture I shall consider two types of problems. The first type concerns the existence of certain affine connexions which are specially related to the distributions. The second type concerns the existence of distributions on homogeneous spaces which are compatible with the homogeneity of the space.

2. - The problems of the first type can be solved by means of the theory of fibre-bundles, as developed for example by Professor EHRESMANN. The solutions that I wish to talk about this afternoon were obtained by A.G. WALKER using classical tensor calculus, and they have one advantage in that an explicit formula is obtained in each case for the connexion coefficients.

Let $M$ be an $n$-dimensional differentiable manifold of class $\infty$ which admits a $C^{\infty}$-distribution of $r$-planes $D'$. Let $D''$ be a complementary distribution of $(n-r)$-planes so that at each point $P$ of $M$ the planes of $D'$ and $D''$ span the tangent space to $M$ at $P$. Let $\underset{\sim}{a}'$, $\underset{\sim}{a}''$ be projection tensors associated with $D'$ and $D''$, i.e. $\underset{\sim}{a}'(x)$, $\underset{\sim}{a}''(x)$ are mixed tensor fields defined globally over $M$ such that

$$\underset{\sim}{a}'.\underset{\sim}{a}' = \underset{\sim}{a}' \; ; \quad \underset{\sim}{a}''.\underset{\sim}{a}'' = \underset{\sim}{a}'' \; ; \quad \underset{\sim}{a}'.\underset{\sim}{a}'' = \underset{\sim}{a}''.\underset{\sim}{a}' = \underset{\sim}{0} \; ; \quad \underset{\sim}{a}' + \underset{\sim}{a}'' = \underset{\sim}{I} \; .$$

A contravariant vector $\underset{\sim}{u}$ at $P$ can be projected into its components $\underset{\sim}{u}'$, $\underset{\sim}{u}''$ in $D'$, $D''$ respectively so that

$$\underset{\sim}{u} = \underset{\sim}{u}' + \underset{\sim}{u}''$$

where

$$\underset{\sim}{u}' = \underset{\sim}{a'u} \, , \quad \underset{\sim}{u}'' = \underset{\sim}{a''u} \; .$$

In terms of components we shall write

$$u^{i'} = a'^{i}{}_{j}\, u^{j}\,, \quad u^{i''} = a''^{i}{}_{j}\, u^{j}\,, \quad u^{i} = u^{i'} + u^{i''} \quad .$$

A covariant vector $\underset{\sim}{v}$ with components $v_i$ is projected into $\underset{\sim}{v}'$ and $\underset{\sim}{v}''$ where

$$v_{i'} = a'^{j}{}_{i}\, v_{j}\,, \quad v_{i''} = a''^{j}{}_{i}\, v_{j}\,, \quad v_{i} = v_{i'} + v_{i''} \,.$$

A tensor with components $T^{i}{}_{jk}$ can be projected completely or partially into invariant sub-spaces associated with the projection tensors $\underset{\sim}{a}'$ , $\underset{\sim}{a}''$ . For example,

$$T^{i'}{}_{j''k} = a'^{i}{}_{m}\, a''^{n}{}_{j}\, T^{m}{}_{nk} \quad .$$

One advantage of this notation is that the summation convention can be used regardless of primes e.g.

$$v_{i'}\, u^{i} = v_{i}\, u^{i'} = v_{i'}\, u^{i'} \quad .$$

This follows because $v_{i'}\, u^{i''} = v_{i''}\, u^{i'} = 0$ .

The notation can be used in conjunction with covariant differentiation with respect to an affine connexion provided that by

$$u^{i'}{}_{|j}$$

we mean $a'^{i}{}_{k}\, u^{k}{}_{|j}$ and NOT $(a'^{i}{}_{k}\, u^{k})_{|j}$ , i.e. we differentiate first and the multiply by the projection tensor.

3. Identities satisfied by $\underset{\sim}{a}'$ , $\underset{\sim}{a}''$ and their covariant derivatives.

It is easy to prove the identities

$$a'^{i}{}_{j''|k} = a'^{i''}{}_{j|k}\,, \quad a'^{i}{}_{j''|h} = a'^{i'}{}_{j|h} \quad ,$$

where $|$ denotes covariant différentiation with respect to some affine connexion L . It will be convenient to change the notation slightly and write $\underset{\sim}{a}$ instead of $\underset{\sim}{a}'$ .

Then the above identities become

(3.1) $$a^{i}{}_{j''|k} = a^{i''}{}_{j|k}\ , \quad a^{i}{}_{j''|h} = a^{i'}{}_{j|h}$$

and these imply

(3.2) $$a^{i'}{}_{j'|k} = 0\ , \quad a^{i''}{}_{j''|k} = 0\ .$$

When differentiation is taken with respect to a symmetric connexion it will be convenient to replace $a^{i}{}_{j|k}$ by $a^{i}{}_{jk}$ .

4. Properties of $D'$ , $D''$ in terms of $a'$ , $a''$ .

Condition of integrability of $D'$

(4.1) $$a^{i}{}_{j'k'} = a^{i}{}_{k'j'}\ .$$

Condition of integrability of $D''$

(4.2) $$a^{i}{}_{j''k''} = a^{i}{}_{k''j''}\ .$$

Condition of parallelism of $D'$

(4.3) $$a^{i}{}_{j'|k} = 0$$

Condition of parallelism of $D''$

(4.4) $$a^{i}{}_{j''|k} = 0$$

Condition of parallelism of both $D'$ and $D''$

(4.5) $$a^{i}{}_{j|k} = 0$$

5. PROBLEM 1. - To find an affine connexion $(L^{i}{}_{jk})$ with the property that $D'$ is parallel with respect to $L$ and is symmetric when $D'$ is integrable.

Write $L^{i}{}_{jk} = \Gamma^{i}{}_{jk} + X^{i}{}_{jk}$ where $\Gamma^{i}{}_{jk}$ is symmetric. Then $L$ is symmetric if and only if $X$ is symmetric in $j$ and $k$ . We have

$$a^{i}{}_{j|k} = a^{i}{}_{jk} + X^{i}{}_{j'k} - X^{i'}{}_{jk}\ ,$$

so $a^{i}{}_{j'|k} = a^{i}{}_{j'k} + X^{i}{}_{j'k} - X^{i'}{}_{j'k} = a^{i}{}_{j'k} + X^{i''}{}_{j'k}$ .

The condition for parallelism of D' given by (4.3) is satisfied if

$$X^{i''}{}_{j'k} = - a^{i}{}_{j'k} \ .$$

This is satisfied by $X = T$ where

$$T^{i}{}_{jk} = - a^{i}{}_{j'k} - a^{i}{}_{k'j} + a^{i}{}_{k'j'} \ . \tag{5.1}$$

Condition (4.1) shows that T is symmetric when D' is integrable.

PROBLEM 2. - To find an affine connexion $(L^{i}{}_{jk})$ with the property that D' , D" are both parallel with respect to L , and L is symmetric when D' , D" are integrable.

It is easily vérified that a similar analysis leads to a suitable connexion L where

$$L = \Gamma + S$$

and

$$S^{i}{}_{jk} = - a^{i}{}_{j'k} - a^{i}{}_{k'j''} + a^{i}{}_{j''k} + a^{i}{}_{k''j} \ .$$

PROBLEM 3. - Given two supplementary distributions D' , D" , to find a positive definite Riemannian metric with respect to which D' and D" are orthogonal.

The condition of orthogonality is

$$g_{ij}\, u^{i'}\, v^{j''} = 0 \quad \text{for all } u^{i'} \text{ and } v^{j''} \ ,$$

i.e.

$$g_{i'j''} = 0 \ .$$

If $h_{ij}$ is any positive definite metric defined globally over $M_n$ , write

$$g_{ij} = h_{i'j'} + h_{i''j''} \ .$$

Then $g_{ij}$ obviously has the required properties.

PROBLEM 4. - For any complementary distributions D' , D" , orthogonal with respect to a metric tensor $g_{ij}$ , to find a global connexion L such that D' , D" are parallel with respect to L , and also $g_{ij|k} = 0$ .

Write $L = \Gamma + X$ where $\Gamma$ is now the Christoffel connexion associated with the metric tensor $g_{ij}$ . Then it is easily verified that

$$X = W = a^i_{j''k} - a^i_{j'k}$$

satisfies the required conditions.

Note that the solutions given to problems 1, 2, 3, 4 are by no means unique, and in particular further geometric conditions may be imposed in the case of problem 4. However, we cannot impose the additional condition that L must be symmetric, for this would imply $L = \Gamma$ and hence $X = 0$ , and thus

$$a^i_{jk} = 0 .$$

It is easy to prove that

<u>If a compact orientable $M_n$ admits a distribution D' of r-dimensions, parallel with respect to a positive definite Riemannian metric, then necessarily</u> $b_r > 0$ , <u>where</u> $b_r$ <u>is the</u> r-th <u>Betti number</u>.

It follows that $S_3$ , which certainly admits D' in the form of a vector field, cannot admit a positive definite metric with respect to which D' is parallel. For this would imply $b_1 > 0$ , whereas for $S_3$ we have $b_1 = 0$ . Moreover, corresponding to $S_3$ , one can construct for any n a manifold $M_n$ which admits a distribution of r-planes but which cannot be given a Riemannian metric with respect to which the distribution is parallel. This raises the following.

PROBLEM 5. - To find a set of <u>necessary and sufficient conditions</u> in order that a manifold $M_n$ which admits a distribution D' can be given a riemannian structure with respect to which D' is parallel.

This appears to be an open problem, so we proceed with problems of type 2.

## 6. <u>Group Manifolds.</u>

Let M be the underlying manifold of a Lie group G , whose Lie algebra is $A$ . Then evidently any linear subspace V of the tangent space $T_e$ at the identity may be carried by left translations over the whole of M to give a distribution $D_l$. Similarly by right translations V gives rise to another distribution $D_R$. Then we have

THEOREM 6.1. - $D_l$, $D_R$ coincide if and only if V is an ideal in A.

This follows from the condition $d\,\ell_x V = dr_x V$, i.e. $(d\,\ell_x \cdot dr_{x^{-1}})V = V$, i.e. V is invariant under the infinitesimal adjoint group. If $\eta \in A$, $\xi \in V$, this condition gives $\eta \times \xi \in V$ for all $\eta \in A$ so that V is an ideal.

The following results are well known.

THEOREM 6.2. - $D_l$ integrable $\Longleftrightarrow$ V is a subalgebra of A.

THEOREM 6.3. - $D_R$ integrable $\Longleftrightarrow$ V is a subalgebra of A.

THEOREM 6.4. - $D_l$ integrable $\Longleftrightarrow$ $D_R$ integrable.

THEOREM 6.5. - $D_l$ parallel with respect to the 0-connexion $\Longleftrightarrow$ V ideal in A.

7. Homogeneous spaces.

$M_n$ is a manifold on which a Lie group G acts as a topological transformation group. If H is the isotropy subgroup of G, i.e. the subgroup of G which sends the fixed point O into itself, we may identify $M_n$ with the coset-space G/H.

The distribution D is homogeneous over M if it is compatible with the homogeneous structure of M, i.e. if $D \longrightarrow D$ under all transformations of G. The following result is easily proved :

THEOREM 7.1. - If $D_O$ is a subspace of the tangent space to M at O, then $D_O$ generates a homogeneous distribution over M if and only if $D_O$ is invariant under the linear isotropy group at O.

THEOREM 7.2. - No sphere $S^n$ admits a distribution homogeneous with respect to SO(n). Write $S^n = SO(n+1)/SO(n)$. Then if O is any point of $S^n$, the isotropy group SO(n) will rotate a given vector at O into any prescribed direction in the tangent space at O. It follows that the only invariant space under the isotropy group is the whole tangent space at O, so no sphere admits a non-trivial homogeneous distribution.

It is not difficult to express the conditions of theorem 7.1 in terms of the Lie Algebra of $G$ and $H$. We find

THEOREM 7.3. - If $V$ is a linear subspace of $A$, disjoint from $H$, then $V$ generates a homogeneous distribution over $M$ if and only if

$$H \times V \subset H + V .$$

Suppose that $V$ is a linear subspace of $A$ which is not disjoint from $H$, and suppose that this condition is satisfied. Then if we define $V' \subset V$ by $H + V' = H + V$, $V' \cap H = \emptyset$ we see that $V'$ generates a smaller distribution over $M$.

THEOREM 7.4. - The homogeneous distribution generated by $V$ is integrable if and only if $H + V$ is a subalgebra of $A$.

Incidentally, if $V$ is an ideal in $A$, then we have

$$(H + V) \times V \subset V .$$

This, together with $H \times H \subset H$, implies that $H + V$ is a subalgebra, and hence the distribution generated by $V$ is integrable. It follows that in order to find an integrable distribution over $M$ it is merely necessary to find a subalgebra of $A$ which contains $H$. If $H'$ is such a subalgebra, we can write

$$H' = H + V , \quad H \cap V = \emptyset ,$$

and the subspace $V$ thus determined will generate an integrable homogeneous distribution.

In conclusion, I wish to acknowledge with gratitude the assitance I have received from Professor A.G. WALKER during many conversations about these topics.