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Norihiko Kazamaki

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A REMARK ON A PROBLEM OF GIRSANOV

N.KAZAMAKI

We always consider on a complete probability space (Ω, F, P) with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all null sets. In this note we deal only with continuous local martingales M over (F_t) such that $M_0 = 0$, and let us denote by $\langle M \rangle$ the continuous non-decreasing process such that $M^2 - \langle M \rangle$ is a local martingale. M is said to be a BMO-martingale if $\|M\|^2_{BMO} = \sup_t \operatorname{ess.sup}_\omega E[\langle M \rangle_\infty - \langle M \rangle_t | F_t]$ is finite. If $\|M\|^2_{BMO} < 1$, then we have

$$E[e^{(\langle M \rangle_\infty - \langle M \rangle_t)} | F_t] \leq \frac{1}{1 - \|M\|^2_{BMO}} .$$

We call it the John-Nirenberg type inequality.

Our aim is to prove the following :

<u>THEOREM.</u> If M is a BMO-martingale, then the process Z defined by $Z_t = \exp(M_t - \frac{1}{2}\langle M \rangle_t)$, $t \geq 0$, is a uniformly integrable martingale.

The process Z is a positive local martingale and, as is well-known, it is not always a martingale. The problem of finding sufficient conditions for Z to be a martingale, which was proposed by I.V.Girsanov [1], is important in some questions concerning the absolute continuity of measures of diffusion processes.

If M is a BMO-martingale, so is aM for every real number a. Thus we get :

COROLLARY. If M is a BMO-martingale, then for any real number a the process $Z^{(a)}$ defined by $Z_t^{(a)} = \exp(aM_t - \frac{1}{2}a^2\langle M\rangle_t)$, $t \geq 0$, is a uniformly integrable martingale.

Before proving the theorem, we state the following two lemmas.

LEMMA 1. Let δ be a number > 0, and set $r = \frac{(1+2\delta)^2}{1+4\delta} > 1$. Then we have

$$\| Z_t \|_r \leq \left\| e^{(\frac{1}{2}+\delta)M_t} \right\|_1^{\frac{4\delta}{(1+2\delta)^2}}, \quad t \geq 0.$$

PROOF. Set $p = 1+4\delta > 1$. The exponent conjugate to p is $q = \frac{1+4\delta}{4\delta}$. Then, by the Hölder inequality we get

$$E[Z_t^r] = E[\, e^{\sqrt{\frac{r}{p}}M_t - \frac{r}{2}\langle M\rangle_t} \, e^{(r-\sqrt{\frac{r}{p}})M_t}\,]$$

$$\leq E[\, e^{(\sqrt{pr}M_t - \frac{pr}{2}\langle M\rangle_t)}\,]^{\frac{1}{p}} \, E[\, e^{(r-\sqrt{\frac{r}{p}})q M_t}\,]^{\frac{1}{q}} .$$

The first term on the right side is bounded by 1, because the process $\{ \exp(\sqrt{pr}M_t - \frac{pr}{2}\langle M\rangle_t), F_t \}$ is a positive local martingale. By a simple calculation we have $(r-\sqrt{\frac{r}{p}})q = \frac{1}{2}+\delta$ and $rq = \frac{(1+2\delta)^2}{4\delta}$. Thus the lemma is proved.

Consequently, if there exists a constant $\delta > 0$ such that $\exp((\frac{1}{2}+\delta)M_t) \in L^1$ for every t, then Z is a martingale.

LEMMA 2. If $\|M\|_{BMO} < \sqrt{2}$, then Z is a uniformly integrable martingale.

PROOF. Let c be a number > 0. Then applying the Schwarz inequality

$$E[\, e^{cM_t} \,] = E[\, e^{(cM_t - c^2\langle M\rangle_t)}\, e^{c^2\langle M\rangle_t} \,]$$

$$\leq E[\, e^{(2cM_t - 2c^2\langle M\rangle_t)} \,]^{\frac{1}{2}}\, E[\, e^{2c^2\langle M\rangle_t} \,]^{\frac{1}{2}} .$$

By the supermartingale inequality the first term on the right side is smaller than 1, so that we have $E[\, \exp(cM_t) \,] \leq E[\, \exp(2c^2\langle M\rangle_t) \,]^{1/2}$.
Now let us take $\delta > 0$ such that $(\frac{1}{2} + \delta + \delta^2)\, \|M\|^2_{BMO} < 1$. Then by John-Nirenberg type inequality we get

$$E[\, e^{((\frac{1}{2}+\delta)M_t)} \,] \leq E[\, e^{(\frac{1}{2}+\delta+\delta^2)\langle M\rangle_t} \,]^{\frac{1}{2}}$$

$$\leq \frac{1}{(\, 1 - (\frac{1}{2}+\delta+\delta^2)\|M\|^2_{BMO} \,)^{1/2}} .$$

Namely, $\sup_t E[\, \exp((\frac{1}{2}+\delta)M_t) \,] < \infty$ and so from Lemma 1, Z is a uniformly integrable martingale. This completes the proof.

<u>PROOF OF THEOREM.</u> We may assume that $0 < \|M\|^2_{BMO} < \infty$. Let us choose a number a such that $0 < a < \mathrm{Min}(\, 1,\, 2/\|M\|^2_{BMO} \,)$. Then, as $\|aM\|^2_{BMO} < 2$, it follows from Lemma 2 that the process $Z^{(a)}$ is a uniformly integrable martingale. Therefore, for any stopping time T

$$1 = E[\, \frac{Z^{(a)}_\infty}{Z^{(a)}_T} \mid F_T]$$

$$= E[\, e^{(\, a(M_\infty - M_T) - \frac{a}{2}(\langle M\rangle_\infty - \langle M\rangle_T) \,)}\, e^{\frac{a}{2}(1-a)(\langle M\rangle_\infty - \langle M\rangle_T)} \mid F_T \,] .$$

Now, applying the Hölder inequality with exponents $\frac{1}{a}$ and $\frac{1}{1-a}$ to the right side we can obtain :

$$1 \leqslant E\left[\frac{Z_\infty}{Z_T}\Big|F_T\right] E\left[e^{\frac{a}{2}(\langle M\rangle_\infty - \langle M\rangle_T)}\Big|F_T\right]^{\frac{1-a}{a}}.$$

By the John-Nirenberg type inequality the second term on the right side is smaller than

$$\frac{1}{(1-\frac{a}{2}\|M\|_{BMO}^2)^{\frac{1-a}{a}}} = \left\{(1-\frac{a}{2}\|M\|_{BMO}^2)^{-2/a\|M\|_{BMO}^2}\right\}^{(1-a)\|M\|_{BMO}^2/2},$$

which converges to $\exp(\frac{1}{2}\|M\|_{BMO}^2)$ as $a \longrightarrow 0$. Therefore we have

$$Z_T \leqslant E[Z_\infty | F_T]\, e^{\frac{1}{2}\|M\|_{BMO}^2}.$$

This implies that Z is a uniformly integrable martingale.

In a forthcoming paper [2] we shall give another results on the relation between the processes M and Z.

REFERENCES

[1]. I.V.Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Teor.Veroyatnost i.Primen 5, 314-330 (1960)

[2]. N.Kazamaki and T.Sekiguchi, A property of BMO-martingales, (in preparation).