BERNARD MAISONNEUVE
On Lévy’s downcrossing theorem and various extensions
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Our aim is to show that the results of [7] can be extended to regenerative systems as taken in a weak sense which will be made precise. Such a generality is motivated by Lévy's downcrossing theorem, which does not fit to the framework of [7] due to a lack of homogeneity of the processes involved. The first six sections are devoted to this result.

1. FIRST NOTATIONS.

Let $X = (\Omega, \mathcal{F}, \mathbb{P}, \epsilon, X_t, \theta_t, P)$ denote the canonical one dimensional brownian motion started at the origin: $\Omega$ is the set of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$; $(X_t)_{t \geq 0}$ is the process of the coordinates; $(\theta_t)_{t \geq 0}$ is the process of the shifts; the progression $(\mathbb{F}_t)_{t \geq 0}$ is the $P$-completion of the natural progression $(\mathbb{F}_t^0)$ of the process $(X_t)$; finally $P[X_0 = 0] = 1$.

Now let us introduce some basic notations for our problem: for each $t \geq 0$ we put

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2. LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon > 0$, $t \geq 0$ let $d_t(\varepsilon)$ denote the number of down-crossings of the process $Y$ over the interval $(0, \varepsilon]$ by time $t$. Lévy's downcrossing theorem asserts that

\[
\tag{2.1} P\left[ \lim_{\varepsilon \to 0} \varepsilon d_t(\varepsilon) = C_t, t \in \mathbb{R}_+ \right] = 1. 
\]

\[
\tag{2.2} \text{HISTORICAL REMARK. The result (2.1) was only conjectured by P. Lévy. The first proof can be found in ITO, McKEAN [4], including some gaps that were filled by CHUNG and DURRETT [1]. Another complete proof was given simultaneously by GETOOR [2] in a much more general context. Finally a short proof was discovered by Williams [8], [9], but his proof remains much more complicated than that of the similar result of Lévy's involving the length of the excursions, namely that there exists $\lambda \in (0, \infty)$ such that
\]

\[
\tag{2.3} P\left[ \lim_{\varepsilon \to 0} \varepsilon \delta_t(\varepsilon) = \lambda C_t, t \in \mathbb{R}_+ \right] = 1, 
\]
where $\delta_t(\varepsilon)$ denotes the number of contiguous intervals of length $>\varepsilon$ contained in $[0,t]$. The term "contiguous" means maximal in the complement of $M$. Our proof (adapted from [7]) will follow Lévy's very simple method for proving (2.3) and will apply to much more general situations.

(2.4) MATHEMATICAL REMARK. (2.1) shows that the processes $(C_t)$ and $(X_t)$ are $(Y_t)$-adapted up to null sets. (2.3) even shows that $(C_t)$ is adapted to the smallest complete progression which makes $M$ progressive. This can be viewed in many other ways.

3. A REGENERATIVE SYSTEM.

Let us introduce new shifts $(\eta_t)$:

\[ \eta_t = \theta_t - X_t = X_{t+} - X_t. \]

With these shifts the strong Markov property of the process $X$ can be stated as follows: for each stopping time $T$ and each $f \in \mathcal{B}_F$

\[ P \left[ f \circ \eta_T \mid \mathcal{F}_T \right] = P(f) \quad \text{on } \{ T < \infty \}. \]

Furthermore it is immediate to check that the following $M$-homogeneity holds for the processes $(Y_t)$ and $(M_t)$: for each $s, t \geq 0$

\[ Y_{t+s} = Y_{s \circ \eta_t} \quad \text{on } \{ t \in M \}, \]
We shall sum up these properties by saying that the collection
\((\Omega, \mathcal{F}, \mathbb{P}, \mathbf{X}, \mathbf{Y}, \eta, M, \mathbb{P})\) is a regenerative system (see §8 for a more
formal definition).

4. EXCURSIONS OF THE PROCESS \(Y\).

Let \(\Omega^0\) be the set of all functions from \(\mathbb{R}^+\) to \(\mathbb{R}^+\) which
remain in 0 after their first hitting of 0. On \(\Omega^0\) we define the
process of the coordinates \((X^0_s)\) and the \(\sigma\)-field \(\mathbb{F}^0\) generated by
the \(X^0_s, s \geq 0\). For \(\omega \in \Omega, t \geq 0\) let \(i_t^\omega\) be the element of \(\Omega^0\)
such that for each \(s \geq 0\)

\[
X^0_s(i_t^\omega) = \begin{cases} 
Y_{t+s}(\omega) & \text{if } t+s < \inf\{u > t : u \in M(\omega)\}, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \(G\) be the random set of the left-end-points in \((0, \infty)\) of the
\(M\)-contiguous intervals. Both the \(\Omega\)-valued process \((i_t^\omega)\) and the
random set \(G\) are \(M\)-homogeneous and it follows immediately that
for each \(A \in \mathbb{F}^0\) the increasing process

\[
N^A_t = \sum_{s \in G \cap (0, t]} I_{A \cap i_s^\omega},
\]

\(t \geq 0\),
is an \(M\)-additive (non adapted) functional, that is,

\[
N^A_{t+s} = N^A_t + N^A_{s \cap i_s^\omega},
\]
on \(\{t \in M\}\).
The random collection \{i_t, t \in G\} is called the collection of the \textit{excursions} of Y; \(N^A_t\) is the number of excursions of type A which occur by time \(t\).

5. TIME CHANGED EXCURSIONS.

The process \((C_t)\) increases exactly on \(M\) and is \(M\)-additive with respect to the shifts \(\eta_t\). Therefore its right continuous inverse \((S_t)\), defined by

\[
S_t = \inf\{s : C_s > t\}, \quad t \geq 0,
\]

satisfies the following additivity property: for all \(s,t \geq 0\)

\[
S_{t+s} = S_t + S_{s \cap S_t} \quad \text{on} \{S_t < \infty\};
\]

in fact \(S_t \in M\) on \(\{S_t < \infty\}\) and \(C_{S_t} = t\) on \(\{S_t < \infty\}\), due to the continuity of \((C_t)\).

(4.3) and (5.2) further imply that for each \(A \in \mathbb{F}_t^0\) the process \(\nu^A_t = N^A_{S_t}\) satisfies

\[
\nu^A_{t+s} = \nu^A_t + \nu^A_{s \cap S_t} \quad \text{on} \{S_t < \infty\}.
\]

But \(S_t < \infty\) a.s. since \(\lim_{t \to \infty} C_t = +\infty\) a.s.. Hence \((S_t)\) is a subordinator, due to (5.2) and to (3.2) applied with \(T = S_t\), and whenever the process \((\nu^A_t)\) is a.s. finite, it has independent and homogeneous increments, due to (5.3) and (3.2); it is even a Poisson process, since it increases by unit jumps. In the
same manner, let $A_1, \ldots, A_n$ be $n$ pairwise disjoint sets in $\mathbb{R}^0$ such that the processes $(\nu_{t}^{A_i})$ are a.s. finite; then the $n$-dimensional process $(\nu_{t}^{A_1}, \ldots, \nu_{t}^{A_n})$ has independent and homogeneous increments and its components $(\nu_{t}^{A_1}), \ldots, (\nu_{t}^{A_n})$ are Poisson processes which pairwise have no common time of jump; therefore, due to a classical result of Lévy, these processes are independent. We have just extended to the present situation Itô's excursion theory [3] and this will allow us to proceed as in [7].

6. PROOF OF LEVY'S DOWNCROSSING THEOREM.

For $\varepsilon \in (0, \infty]$ let $A_{\varepsilon} = \{\sup_{s} X_{0}^{0} > \varepsilon\}$. For $0 < \varepsilon < \varepsilon' < \infty$ the process $(\nu_{t}^{A_{\varepsilon} \setminus A_{\varepsilon'}})$, which is a.s. finite, is a Poisson process by previous considerations. If $0 < \varepsilon_1 < \ldots < \varepsilon_n \leq \infty$ the processes $(\nu_{t}^{A_{\varepsilon_i - A_{\varepsilon_{i+1}}}}), i = 1, \ldots, n-1$ are further independent. But

$$\nu_{t}^{A_{\varepsilon_i} \setminus A_{\varepsilon_{i+1}}} = \nu_{t}^{A_{\varepsilon_i}} - \nu_{t}^{A_{\varepsilon_{i+1}}}$$

and therefore the process $\varepsilon \rightarrow \nu_{t}^{A_{\varepsilon}}$ is a process with independent (non-homogeneous) increments for each fixed $t$. The strong law of large numbers applies to this process as $\varepsilon \rightarrow 0$ and yields

$$(6.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{\nu_{t}^{A_{\varepsilon}}}{P[\nu_{t}^{A_{\varepsilon}}]} = 1 \quad \text{a.s.}$$

But we shall see that the denominator in (6.1) equals $t/\varepsilon$; hence (6.1) becomes
Due to the monotonicity in $t$ of $\nu^{A_t}_\varepsilon$ and $t$, the null set in (6.2) can be chosen independently of $t$; therefore one has

\[ P \left[ \lim_{\varepsilon \to 0} \nu^{A_t}_\varepsilon = C_t, \ t \in \mathbb{R}_+ \right] = 1 \]

and since $\nu^{A_t}_C = N^{A_t}_t$, we get

\[ P \left[ \lim_{\varepsilon \to 0} \nu^{A_t}_N = C_t, \ t \in \mathbb{R}_+ \right] = 1. \]

Lévy's downcrossing theorem follows from the fact that $|d_t(e) - N^{A_t}_t| \leq 1$ for each $t$.

It remains to prove that $P \left[ \nu^{A_t}_e \right] = t/e$. Put $T_\varepsilon = \inf \{ s : Y_s > \varepsilon \}$ From the equality $Y_{T_\varepsilon} = e$ a.s. and from the martingale property of $X$, one immediately checks that $P \left[ C^{A_t}_{T_\varepsilon} \right] = \varepsilon$. On the other hand, $C^{A_t}_{T_\varepsilon}$ is the time of the first jump of the process $(\nu^{A_t}_e)$, which is Poisson; therefore

\[ P(\nu^{A_t}_e) = t/P(C^{A_t}_{T_\varepsilon}) = t/e. \]

7. OTHER LIMIT RESULTS FOR THE PROCESS $(C_t)$.

(7.1) THEOREM. Let $\alpha \in (0, \infty]$ and let $\{ A_\varepsilon, 0<\varepsilon<\alpha \}$ be a decreasing right continuous family of elements of $\mathbb{F}_0^\alpha$. Set
(7.2) \[ T_{A_c} = \inf\{t \in G: \inf_{\tau \in A_c} = \inf\{t: N_{A_c}^A > 0\} \]

and suppose that

(7.3) \[ P \left[ 0 < T_{A_c} < \infty, \, c \in (0, \alpha]; \lim_{c \to 0} T_{A_c} \right] = 1 \]

Then, with the notation (4.2), one has

(7.4) \[ P \left[ \lim_{c \to 0} P \left[ C_{T_{A_c}} \right] N_{T_{A_c}}^A = C_t, \, t \in \mathbb{R}^+ \right] = 1 \]

The proof is similar to the proof of Lévy's downcrossing theorem. For more details we refer to the proof of theorem 2 of [7] and to the appendix.

(7.6) REMARK. Theorem (7.1) unifies the results (2.1) and (2.3):
for (2.1) choose \( A_c = \{\sup X^0_s > \varepsilon\} \), for (2.3) choose \( s \) rational \( A_c = \{X^0_c > 0\} \).

8. EXTENSIONS TO REGENERATIVE SYSTEMS.

Let us consider a regenerative system \((\Omega, \mathcal{F}, \mathcal{F}_t, Y_t, \eta_t, M, P)\) in the sense of [5], except that the homogeneity properties are only required on \( M \). More precisely \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) is a stochastic basis with usual conditions, \((Y_t)\) is a progressive process (with state space \((\mathbb{E}, \mathbb{F})\)), \((\eta_t)\) is a measurable process with values in \((\Omega, \mathcal{F})\), \( M \) is a right closed progressive random set. We further assume the following properties:
(8.1) **M-homogeneity:** for $s, t \geq 0$

\[
Y_s \circ \eta_t = Y_{t+s} \quad \text{on } \{t \in M\},
\]

\[
M_s \circ \eta_t = M_{t+s} \quad \text{on } \{t \in M\},
\]

where $M_t = I\{t \in M\}$;

(8.2) **Regeneration:** For each stopping time $T$ and each $f \in bF$

\[
P\left[ f \circ \eta_T \mid F_T \right] = P[f] \quad \text{on } \{T \in M\},
\]

(8.3) **REMARK.** This weak notion of regenerative system was already introduced in [6], in order to time change a Markov process by using the inverse of a non-continuous additive functional.

Throughout this section let us assume that the random set $M$ is perfect, unbounded, with an empty interior a.s. and that $(C_t)$ is a local time of $M$, that is $(C_t)$ is a continuous adapted $M$-additive functional which increases exactly on $\overline{M}$ (the closure of $M$).
Then all considerations of Sections 4, 5, 7 extend to the present framework, with the following differences: in the definition (4.1) of \( i_t \omega \) we set

\[ X_s^0(i_t \omega) = \delta \quad \text{if } t + s \geq \inf \{ u > t : u \in M(\omega) \}, \]

where \( \delta \) is a distinguished point in \( E \) which is a.s. ignored by the process \( Y \) and such that \( \{ \delta \} \in \mathcal{E} \); in the definition (4.2) of \( N_t^A \), we assume that \( A \) is a subset of the space \( \Omega^0 \) of all mappings from \( \mathbb{R}_+ \) to \( E \) with life time and that \( A \) further belongs to the \( \sigma \)-field \( \mathcal{E}^0 \) generated by the coordinates of \( \Omega^0 \).

Finally under the assumptions (7.2) and (7.3) we can state the following constructive result, which is the analog of theorem 2' of [7]:

\[ (8.4) \quad \text{THEOREM. There exists a local time } C'_t \text{ such that} \]

\[ P \left[ \lim_{\varepsilon \to 0} p(\varepsilon) N^A_{t \varepsilon} = C'_t, \ t \in \mathbb{R}_+ \right] = 1, \]

where we set \( p(\varepsilon) = P \left[ \gamma_{A_{t \varepsilon}} = T_{A_{t \varepsilon}} \right] \).

9. APPENDIX.

This appendix is devoted to fixing the proof of theorem 2 of [7], which is incomplete. We shall do this in the framework of theorem (7.1) of the present paper. For \( A \in \mathcal{E}^0 \), set
\( 0(A) = P[\nu_1^A] \) and for \( \varepsilon \in (0, \alpha] \) set \( q(\varepsilon) = \Omega(A_{\varepsilon}). \) Let \( p \) (resp. \( \overline{p} \)) be the right (resp. left) continuous inverse of \( q: \)

\[
p(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) > u\}, \quad u \geq 0,
\]

\[
\overline{p}(u) = \sup\{\varepsilon \in (0, \alpha] : q(\varepsilon) \geq u\}, \quad u \geq 0.
\]

Let us fix \( t > 0 \) and define the processes \( Z, \overline{Z} \) by setting

\[
Z_u = \nu_t^p(u), \quad \overline{Z}_u = \nu_t^{\overline{p}}(u), \quad u \geq 0.
\]

It was claimed in [7] that the restriction to the set \( T = q((0, \alpha]) \) of the process \( Z \) is left continuous. Here is a proof of this fact. Let \( D \) be the set of all points \( u \) in \( T \) which are not isolated from the left and which are such that \( p(u) \neq \overline{p}(u). \) For each \( u \in D \) one has \( q(p(u)) = q(\overline{p}(u)). \)

Therefore the set

\[
B = \bigcup_{u \in D} (A_p(u) \setminus A_{\overline{p}}(u))
\]

is null for the measure \( \Omega \) and the variable \( \nu_t^B \) vanishes a.s.

This implies that

\[
P[ Z_u = \overline{Z}_u, \; u \in D ] = 1 \]
and the a.s. left continuity of the process \((Z_u)_{u \in T}\) now follows from the left continuity of \(Z\) \((u_n \uparrow u \Rightarrow \overline{p}(u_n) \downarrow \overline{p}(u) \Rightarrow \nu_t \overline{p}(u_n) \uparrow \nu_t \overline{p}(u))\).

The proof ends like in [7]. Basically one applies the strong law of large numbers to the process \((Z_u)_{u \in T}\): this process has independent increments and for \(u, v \in T, u \leq v\), \(Z_v - Z_u\) is Poisson distributed with parameter \(t(v-u)\), since \(q(p(u)) = u\) for each \(u \in T\). Since we have not been able to find a reference for the version of the strong law of large numbers which is needed here, we state and prove it as a

(9.1) **Lemma.** Let \(T\) be a left (resp. right) closed unbounded subset of \(\mathbb{R}^+_+\) and let \((Z_t)_{t \in T}\) be a left (resp. right) continuous integrable process with independent increment defined on \((\Omega, F, P)\). Assume that there exists a convolution semi-group \((\mu_s)_{s \in (0, \infty)}\) of probability measures on \(\mathbb{R}\) such that \(Z_v - Z_u\) has the distribution \(\nu_{v-u}\) for all \(u, v \in T, u \leq v\). Then one has

\[
\lim_{t \to \infty} \frac{Z_t}{t} = \int x \mu_1(dx) \quad \text{P-a.s.}
\]

(9.2)

(9.3) **Remark.** The result is well known if \(T = \mathbb{R}^+_+\): See Doob [10] p. 364. The proof given below follows the martingale method indicated by Doob [10] p. 365.

**Proof.** We can restrict ourselves to the case where \(0 \in T\), \(Z_0 = 0\). Consider, on some auxiliary space \((W, G, Q)\) a right contin-
uous process \( (Y_s)_{s \in \mathbb{R}^+} \) such that \( Y_0 = 0 \) and such that \( Y_v - Y_u \) has the distribution \( u \sim v \) for all \( u, v \in \mathbb{R}^+ \), \( u < v \). One checks easily that for \( k, \ell \in \mathbb{N} \) with \( k \leq \ell \)

\[
\frac{Y_{\ell/2^n}}{\ell} = \mathbb{Q} \left[ \frac{Y_{k/2^n}}{k} \mid Y_u, u \geq \ell/2^n \right],
\]

which implies that for \( s, t \in \mathbb{R}^+ \), with \( s \leq t \)

\[
\frac{Y_t}{t} = \mathbb{Q} \left[ \frac{Y_s}{s} \mid Y_u, u \geq t \right].
\]

Since the process \( (Z_t)_{t \in T} \) has the same distribution as the process \( (Y_t)_{t \in T} \) (both are markovian relative to the same semi-group), one has also for \( s, t \in T \), with \( s \leq t \)

\[
\frac{Z_t}{t} = \mathbb{P} \left[ \frac{Z_s}{s} \mid Z_u, u \geq t \right].
\]

Fix \( s > 0 \) in \( T \) and let \( t \to \infty \) in \( T \). By the backward martingale convergence theorem, \( \frac{Z_t}{t} \) converges a.s. The limit has to be constant by the 0.1 law and equal to \( \mathbb{P} \left[ \frac{Z_s}{s} \right] = \int x\mu_1(dx) \) by uniform integrability.

* with independent increments
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B. Maisonneuve
Université de Grenoble II
I.M.S.S.
47X-38040 Grenoble Cedex, France