EDWIN A. PERKINS

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LOCAL TIME AND PATHWISE UNIQUENESS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

By Edwin Perkins

In this note we use the notion of local time of a semimartingale to obtain pathwise uniqueness results for one-dimensional Itô equations. The first such use of local time (to our knowledge) seems to be in [1] (see Th. II.3.1). Since the well-known results of Yamada and Watanabe [4, Th.1] involve an approximation of $f(x) = |x|$ by C^2 functions and an application of Itô's lemma, it is not surprising that this approach simplifies matters somewhat. Although our main result (Cor. 2, Th. 4) is slightly more general than Th. 1 of [4] (since it also applies to certain diffusion coefficients which need not be Hölder continuous of any order (see Examples 3)), it is the simplicity of the proofs that we wish to stress.

We first consider equations with random diffusion coefficients and no drift, and then use a result of Zvonkin [5] to handle bounded measurable drifts.

Assume $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ satisfies the usual hypotheses and B_t is a $\{\mathcal{F}_t\}$ - Brownian motion (in the usual sense). Unless otherwise indicated, we assume throughout this work that X_0 is an $\mathcal{F}_0$-measurable random variable, $\sigma : [0,\infty)\times \mathbb{R}\times \Omega \to \mathbb{R}$ is jointly measurable, and $\sigma(s,\cdot,\cdot)$ is Borel $\times\, \mathcal{F}_s$-measurable for all s. We say that $X(t,\omega)$ is a solution of

$$(1)\quad X(t) = X_0 + \int_0^t \sigma(s,X(s),\omega)\,dB(s)$$

with lifetime ρ if $T_n(X) = \inf\{t \mid |X(t)| > n\} \wedge n$ satisfy $\lim_{n\to\infty} T_n(X) = \rho > 0$ a.s. and X satisfies (1) on $[[0,\rho[[$ (we may set $X = \Delta$ on $[[\rho,\infty[[$). We say that pathwise uniqueness holds in (1) if whenever X_1 and X_2 are solutions of (1) with lifetimes ρ_1 and ρ_2, respectively, then $\rho_1 = \rho_2$ and $X_1(t) = X_2(t)$ for all $t < \rho_1$ a.s.

If Y is a semimartingale, $L_t^x(Y)$ denotes its local time (see Meyer [2, p.365]).

Theorem 1. Suppose there is a solution, X_1, of (1), and measurable mappings $\delta : \Omega \to (0,\infty)$ and $\rho : [0,\infty)\times\Omega \to [0,\infty]$ such that

(2) $\int_{0+} \rho^{-1}(z)dz = \infty$ a.s. (i.e. $\infty > \int_{\varepsilon}^{1} \rho^{-1}(z)dz$ increases to ∞ a.s. as $\varepsilon \to 0^+$)

(3) If $f(s,x,\omega) = \sup_{0<y<\delta(\omega)} \rho^{-1}(y)(\sigma(s,x+y,\omega) - \sigma(s,x,\omega))^2$, then

$$\int_0^{T_m(X_1)} f(s,X_1(s),\omega)ds < \infty \text{ for all } m \in \mathbb{N} \text{ a.s.}$$

Then pathwise uniqueness holds in (1).

Proof. If X_2 is another solution of (1), let $T_m = T_m(X_1) \wedge T_m(X_2)$ and $X_i^m(t) = X_i(t \wedge T_m)$. Define $\varepsilon_n(\omega)$ by $\int_{\varepsilon_n}^{n^{-1}} \rho^{-1}(z)dz = n$. Using the continuity of local time we see that w.p.1,

$$\begin{aligned} L^0_{T_m}(X_2^m - X_1^m) &= \lim_{n\to\infty} n^{-1} \int_{\varepsilon_n}^{n^{-1}} \rho^{-1}(x) L^x_{T_m}(X_2^m - X_1^m)dx \\ &\le \limsup_{n\to\infty} n^{-1} \int_0^{T_m} \rho^{-1}(X_2^m(s) - X_1^m(s))\, I(X_2^m(s) - X_1^m(s) \in (\varepsilon_n, n^{-1})) \\ &\qquad (\sigma(s,X_2^m(s)) - \sigma(s,X_1^m(s)))^2 ds \\ &\le \limsup_{n\to\infty} n^{-1} \int_0^{T_m} f(s,X_1(s))ds = 0 . \end{aligned}$$

Therefore

$$E(|X_2^m(t) - X_1^m(t)|) = E(L^0_{T_m \wedge t}(X_2^m - X_1^m)) = 0 ,$$

and pathwise uniqueness follows.□

Corollary 2. Suppose there are measurable mappings $\delta : \Omega \to (0,\infty)$, $\rho : [0,\infty)\times\Omega \to [0,\infty]$, $c_m : [0,\infty)\times\Omega \to [0,\infty]$, and $g_m : \mathbb{R}\times\Omega \to [0,\infty]$ $(m \in \mathbb{N})$ such that in addition to (2) we have (w.p.1)

(4) $(\sigma(s,x+y,\omega) - \sigma(s,x,\omega))^2 \le \rho(y,\omega)(c_m(s,\omega) + \sigma(s,x,\omega)^2 g_m(x,\omega))$ for all $0 < y < \delta(\omega)$, $s \le m$ and $|x| \le m$.

(5) $c_m(\cdot,\omega)$ and $g_m(\cdot,\omega)$ are Lebesgue integrable over compacts.

Then pathwise uniqueness holds in (1) .

Proof. If f is as in Theorem 1, then for $s \le m$ and $|x| \le m$,

$$f(s,x,\omega) \le c_m(s,\omega) + g_m(x,\omega)\ \sigma(s,x,\omega)^2 .$$

Therefore if X is a solution of (1) ,

$$\begin{aligned}\int_0^{T_m(X)} f(s,X(s),\omega)\,ds &\le \int_0^m c_m(s,\omega)\,ds + \int_0^{T_m} g_m(X(s),\omega)\ \sigma^2(s,X(s),\omega)\,ds \\ &= \int_0^m c_m(s,\omega)\,ds + \int_{-m}^m g_m(x,\omega)\,L_x^{T_m}(X)\,dx \\ &< \infty \quad \text{a.s.}\end{aligned}$$

The result follows from Theorem 1. □

Remarks. 1) By being just a little more careful in the proofs of the previous two results one can weaken the hypotheses of Corollary 2 to the following:

Suppose there is a sequence of measurable functions $h_n : [0,\infty) \times \Omega \to [0,\infty)$ that integrate to one a.s. and satisfy $\{x | h_n(x) \ne 0\} \subset (0,\delta_n)$ for some random variables δ_n , decreasing to zero a.s., and for each $m \in \mathbb{N}$ there are sequences of functions $\{c_m^n | n \in \mathbb{N}\}$ and $\{g_m^n | n \in \mathbb{N}\}$, uniformly (Lebesgue) integrable on $[0,m]$ and $[-m,m]$, respectively, that satisfy

$$(\sigma(s,x+y,\omega) - \sigma(s,x,\omega))^2 \le h_n(y)^{-1}(c_m^n(s,\omega) + \sigma(s,x,\omega)^2\ g_m^n(x,\omega))$$

for all $s \le m$, $|x| \le m$ and $0 < y < \delta_n(\omega)$.

Note that the hypotheses of Corollary 2 imply the above conditions by taking $h_n(y) = \rho^{-1}(y)\, I(\varepsilon_n < y < n^{-1})$, where $\varepsilon_n(\omega)$ is defined by $\int_{\varepsilon_n}^{n^{-1}} \rho^{-1}(y)dy = 1$. An analagous modification may also be made in Theorem 1.

2) The above results also hold if B is replaced by a continuous local martingale M provided we assume slightly stronger measurability conditions on σ (such as $(s,\omega,x) \to \sigma(s,x,\omega)$ is $\mathcal{O} \times$ Borel-measurable, where $\mathcal{O}$ is the optional σ-field) to ensure that (1) makes sense, and replace ds with $d[M,M]_s$ in the integrability conditions on f and c_m (i.e. in (3) and (5)).

3) If we take $g_m = 0$ in Corollary 2 we obtain (essentially) Theorem 1 of Yamada and Watanabe [4]. It is interesting to note, however, that by setting $c_m = 0$ one can obtain pathwise uniqueness in cases not handled by that theorem, as the following examples show:

Examples 3. (i) $\sigma(x) = 1 + |x|^p$ for $p > 0$.

The mean value theorem shows that for some $c > 0$, and for all $0 < y < 1$,

$$(\sigma(x+y) - \sigma(x))^2 \le y(c \max(1,|x|^{p-1})) .$$

Since $\max(1,|x|^{p-1})$ is integrable over compacts, pathwise uniqueness follows from Corollary 2.

(ii) $\sigma(x) = 1 + [\log(|x|^{-1} \vee 2)]^{-p}$, for $p > 0$.

Note that σ is not Hölder continuous of any order at zero. Again, the mean value theorem shows that (4) holds for all $0 < y < 1/4$, with $\rho(y) = y|\log y|$, $g_m(x) = c|x|^{-1}(\log |x|^{-1})^{-(2p+1)} I_{[-3/4,1/2]}(x)$ and $c_m = 0$. Hence pathwise uniqueness holds.

Of course in the above examples, pathwise uniqueness also follows from a result of Nakao [3], stating that pathwise uniqueness holds in (1) if $0 < \varepsilon \le \sigma(s,x,\omega) = \sigma(x) \le M < \infty$ and σ is of bounded variation on compacts. By adding a non-negative function, $g(x)$, of unbounded variation on every interval of positive length, and satisfying Lévy's modulus of continuity for Brownian motion (for example, the absolute value of any "typical" Brownian path) to the above σ's one obtains a coefficient satisfying the hypotheses of Corollary 2 but neither the hypotheses of Nakao nor Yamada and Watanabe. □

<u>Theorem 4</u>. Suppose that $\sigma, b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(6) σ and σ^{-1} are bounded and continuous

(7) b is bounded and measurable

(8) There exists a non-decreasing $\rho : [0,\infty) \to [0,\infty]$ satisfying (2), functions $g : \mathbb{R} \to [0,\infty]$ and $c : [0,\infty) \to [0,\infty]$ that are integrable on compacts, and $\delta > 0$ such that for all $|y| < \delta$, and (s,x),

$$(\sigma(s,x+y) - \sigma(s,x))^2 \le \rho(|y|)(g(x) + c(s)) .$$

Then pathwise uniqueness holds in

$$(9) \qquad X(t) = X_0 + \int \sigma(s,X(s))dB(s) + \int b(s,X(s))ds .$$

Moreover there is a solution of (1) (with infinite lifetime) adapted to the filtration generated by X_0 and $B(t)$.

<u>Proof</u>. The last assertion follows from the first by means of Corollary 3 of [4].

Let X_1 and X_2 be solutions of (9) (with necessarily infinite lifetimes). By Theorems 2 and 3 of Zvonkin [5] there exist $T > 0$, depending only on the bounds of b, σ and σ^{-1} , and a mapping $u: [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

(10) For each fixed t , $u(t,\cdot)$ is a bijection of $\mathbb{R}$ with inverse $v(t,\cdot)$.

(11) Both $u_x(t,x)$ and $v_x(t,x)$ are bounded and Hölder continuous of order α for each $\alpha < 1$.

(12) $Y_i(t) = u(t \wedge T, X_i(t \wedge T))$ $(i = 1,2)$ are solutions of

$$Y_i(t) = u(0,X_0) + \int_0^t \hat{\sigma}(s,Y_i(s))dB_s ,$$

where

$$\hat{\sigma}(s,x) = \begin{cases} u_x(s,v(s,x))\ \sigma(s,v(s,x)) & s \le T \\ 0 & s > T \end{cases} .$$

We claim that $\hat{\sigma}$ satisfies the hypotheses of Theorem 1. Note that for $s \le T$ and $0 < y < \delta$ there are positive constants $c_1, c_2, \ldots$ such that

$$\begin{aligned}(\hat{\sigma}(s,x+y) - \hat{\sigma}(s,x))^2 &\le c_1((u_x(s,v(s,x+y)) - u_x(s,v(s,x)))^2 \\ &\quad + (\sigma(s,v(s,x+y)) - \sigma(s,v(s,x))^2) \\ &\le c_2(y + \rho(c_3 y)(g(v(s,x)) + c(s))) \qquad ((8) \text{ and } (11)) \\ &\le (g(v(s,x)) + c(s) + 1)\ \tilde{\rho}(y) ,\end{aligned}$$

where $\tilde{\rho}(y) = c_4(y + \rho(c_3 y))$. It is easy to check that $\int_{0+} \tilde{\rho}(y)^{-1}dy = \infty$ (recall that ρ is non-decreasing) . Moreover,

$$\int_0^T g(v(s,Y_1(s)) + c(s) + 1\, ds = \int_0^T g(X_1(s))ds + \int_0^T (c(s)+1)ds$$

$$\le c_5 \int_0^T g(X_1(s))\sigma^2(s,X_1(s))ds + \int_0^T c(s)ds + T$$

$$\le c_5 \int_{-\infty}^{\infty} g(x) L_T^x(X_1)dx + \int_0^T c(s)ds + T$$

$$< \infty \quad \text{a.s.}$$

since $L_T^x(X_1)$ has compact support. Theorem 1 implies that $Y_1 = Y_2$, whence $X_1 = X_2$ on $[0,T]$ a.s. Finally we may prove that $X_1 = X_2$ by applying the above argument on each of the intervals $[iT,(i+1)T]$ as in the proof of Theorem 4 in [5]. □

Remark. An obvious stopping time argument shows that theorem remains valid if, instead of (8), there exist functions g_m, c_m, each integrable on compacts such that for all $|y|< \delta$, $s \le m$ and $|x| \le m$,

$$(\sigma(s,x+y) - \sigma(s,x))^2 \le \rho(|y|)(g_m(x) + c_m(s)) .$$

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Department of Mathematics
University of British Columbia
Vancouver, B.C.
Canada V6T 1Y4