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$L(B_t, t)$ is not a semimartingale

by

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Let B be a one-dimensional Brownian motion, with $B_0 = 0$, and let $L(a, t)$, $a \in \mathbf{R}$, $t \geq 0$ be a continuous version of its local time. We shall show that the process Y , defined by $Y_t = L(B_t, t)$, is not a semimartingale. The essence of the proof is the remark that whereas the paths of a continuous semimartingale satisfy a Holder condition of order $\frac{1}{2} - \varepsilon$ almost everywhere, for any $\varepsilon > 0$, the paths of Y just fail to satisfy a Holder condition of order $\frac{1}{4}$.

For a process or function X set

$$D^\alpha(X) = \{t \geq 0 : \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/\alpha} |X_{t+\varepsilon} - X_t| > 0\} .$$

LEMMA Let $\alpha > 1$, and $f : \mathbf{R}_+ \rightarrow \mathbf{R}$ be a function such that

$D^\alpha(f) = \emptyset$. Let $\tau(t)$ be an increasing function, and $g(t) = f(\tau(t))$.

Then $|D^\alpha(g)| = 0$.

Proof By Lebesgue's density theorem, $\tau'(t)$ exists and is finite almost everywhere. For such a t

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/\alpha} |g(t+\varepsilon) - g(t)| \\ &= \lim_{\delta \rightarrow 0} (\tau'(t))^{1/\alpha} \delta^{-1/\alpha} |f(\tau(t) + \delta) - f(\tau(t))| \\ &= 0 , \end{aligned}$$

so that $t \notin D^\alpha(g)$.

PROPOSITION Let X be a continuous semimartingale. Then for $\alpha > 2$, $|D^\alpha(X)| = 0$ a.s.

Proof Let $X = M + A^+ - A^-$ be the decomposition of X into the sum of a martingale and the difference of two increasing processes. It is plain that $D^\alpha(X) \subset D^\alpha(M) \cup D^\alpha(A^+) \cup D^\alpha(A^-)$. By the lemma, setting $f(t) = t$ and $\tau(t) = A_t^+$ or A_t^- , we have $|D^\alpha(A^+)| = |D^\alpha(A^-)| = 0$.

Now let τ_t be the right-continuous inverse of $\langle M \rangle$, and $U_t = M_{\tau_t}$. Then U is a Brownian motion, and $M_t = U_{\langle M \rangle_t}$. By Lévy's Hölder condition on the variation of Brownian paths, for $\alpha > 2$, $D^\alpha(U) = \phi$ a.s., and thus, by the lemma, $|D^\alpha(M)| = 0$ a.s.

THEOREM (i) For each $t > 0$, $B_t \in D^2(L(\cdot, t))$ a.s.

(ii) $D^4(Y)$ is of full Lebesgue measure a.s.

(iii) Y is not a semimartingale.

Proof From the results of Ray [1] on Brownian local time,

$0 \in D^2(L(\cdot, t))$ a.s. Let t be fixed, and $\tilde{B}_s = B_t - B_{t-s}$ for $0 \leq s \leq t$. Then \tilde{B} is a Brownian motion, and if \tilde{L} denotes its local time, $\tilde{L}(a, t) = L(B_t - a, t)$, so that $B_t \in D^2(L(\cdot, t))$ whenever $0 \in D^2(\tilde{L}(\cdot, t))$, establishing (i).

We may restate (i) as follows: there exist \mathcal{B}_t -measurable random variables A_n and C with $|A_n - B_t| < 1/n$, and $C > 0$ a.s., such that

$$|L(A_n, t) - L(B_t, t)| \geq |A_n - B_t|^{\frac{1}{2}} \cdot C \text{ for all } n.$$

If (a_n) is a sequence converging to 0, and $T_n = \inf\{t \geq 0; B_t = a_n\}$, then $P(T_n < a_n^2) = k > 0$, for some

constant k . Thus $P(T_n < a_n^2 \text{ for infinitely many } n) = 1$ by the Borel-Cantelli lemmas, and the Blumenthal 01 law.

Now let $S_n = \inf\{u > t: B_u = A_n\}$. By the preceding argument, and the Markov property of B at t ,

$$S_n - t < (A_n - B_t)^2 \text{ for infinitely many } n, \text{ a.s.}$$

Thus

$$\begin{aligned} \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |Y_{S_n} - Y_t| \\ &= \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |L(A_n, t) - L(B_t, t)| \\ &\geq \limsup_{n \rightarrow \infty} (S_n - t)^{-\frac{1}{2}} |A_n - B_t|^{\frac{1}{2}} C \\ &\geq C \qquad \qquad \qquad \text{a.s.} \\ &> 0 \qquad \qquad \qquad \text{a.s.} \end{aligned}$$

Therefore $t \in D^2(Y)$ a.s., and (ii) follows by a Fubini argument.

(iii) is an immediate consequence of (ii) and the proposition.

Reference

1. D.B. Ray : Sojourn times of a diffusion process. Illinois J. Math. 7; 615-630. (1963).

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