Séminaire de probabilités (Strasbourg)

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Séminaire de probabilités (Strasbourg), tome 16 (1982), p. 459-468
<http://www.numdam.org/item?id=SPS_1982__16__459_0>

STOCHASTIC DIFFERENTIAL EQUATIONS WITH FEEDBACK IN THE DIFFERENTIALS

by

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ABSTRACT

Existence, unicity, and stability of solutions of stochastic differential equations of the type $Z = M + FZ \cdot Y + GZ \cdot HZ$ are established. M and Y are semimartingales with continuous paths. The novelty here is that instantaneous feedback in the driving term is allowed.

1. INTRODUCTION

The theory of stochastic differential equations with semimartingale differentials is now well developed (see [3], [4], or [7]). It is always assumed, however, that one is given a coefficient F, a driving term Y, and an exogenous term M to yield an equation: $Z = M + FZ \cdot Y$. We consider here instead equations of the type:

$$Z = M + FZ \cdot Y + GZ \cdot HZ, \tag{E}$$

where H is a given operator on semimartingales. The solution is permitted to feedback instantaneously into one of the differentials. In the deterministic case this corresponds to certain types of singular equations.

We prove in Theorem 3.1 that a solution of (E) exists and is unique under appropriate restrictions on G and H. We also show that equations of the type (E) are stable in the semimartingale topology (Theorem 3.4).

The solutions here are strong solutions in the sense that they are defined on the same space that M, Y, F, G, and H are defined on. The

*Supported in part by NSF Grant #0464-50-13955.

semimartingales are always assumed to have continuous paths. A different approach to this genre of problems is considered in [5].

2. PRELIMINARIES

We assume the reader is familiar with the semimartingale calculus and its standard notations (cf [4], [7], or [8]). In particular, $D\cdot X$ denotes $\int_0^t D_s dX_s$.

(2.1) DEFINITION. For $K > 0$, an operator F is in $\mathrm{Lip}(K)$ if

(i) $X^{T-} = Y^{T-}$ implies $(FX)^{T-} = (FY)^{T-}$

(ii) $(FX-FY)^* \leq K(X-Y)^*$ as processes where $X_t^* = \sup_{s\leq t} |X_s|$.

We will be concerned here only with <u>continuous</u> semimartingales. For a given continuous semimartingale X, let $X = M + A$ be its unique decomposition into a local martingale M and a process A with paths of bounded variation on compacts.

(2.2) DEFINITION. For a continuous semimartingale $X = M + A$ and p, $1 \leq p \leq \infty$, define:

$$||X||_{\mathcal{H}^p} = ||[M,M]_\infty^{1/2} + \int_0^\infty |dA_s|\,||_{L^p}$$

$$||X||_{\mathcal{S}^p} = ||X_\infty^*||_{L^p}$$

As a consequence of the Burkholder, Davis, and Gundy inequalities we have

(2.3) $$||X||_{\mathcal{S}^p} \leq C_p ||X||_{\mathcal{H}^p}, \quad 1 \leq p < \infty$$

for universal constants C_p. Dellacherie and Meyer [1, p.304] have shown

(2.4) $$C_1 \leq 4.$$

Emery has shown the following:

(2.5) $$||D\cdot X||_{\mathcal{H}^r} \leqq ||D||_{\mathcal{S}^p} ||X||_{\mathcal{H}^q}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 \leqq p, q \leqq \infty$. See Meyer [9] for an exposition and extension of (2.5).

(2.6) DEFINITION. An operator H mapping continuous semimartingales into continuous semimartingales will be said to be in $\mathcal{S}_c(K)$ if

$$||HX-HY||_{\mathcal{H}^1} \leqq K||X-Y||_{\mathcal{H}^1}$$

for any continuous semimartingales X and Y.

Emery [2] has developed a topology for semimartingales, which was inspired by a study of the stability of solutions of stochastic differential equations. (Métivier and Pellaumail [7] independently developed the same topology.) Here is a characterisation: continuous semimartingales (X^n) converge to X in the semimartingale topology if for any subsequence (n') one can extract a sub-subsequence (n") such that $X^{n''}$ converges locally in $\mathcal{H}^1$ (or $\mathcal{H}^p$, $p \geqq 1$) to X. (By "converges locally" we mean that there exist stopping times T^k tending to ∞ a.s. such that $||(X^n-X)^{T^k}||_{\mathcal{H}^1}$ tends to 0.)

3. THEOREMS AND PROOFS

Recall the equation:

(E) $$Z = M + FZ\cdot Y + GZ\cdot HZ.$$

We consider only the case where M and Y (and hence Z) have continuous paths. If one is willing to specify the operator H, one can handle jumps with a modification of the usual techniques, taking care to avoid impossible requirements on the jumps (such as $\Delta Z = 2\Delta Z$, etc.).

(3.1) THEOREM. *Let* M, Y *be continuous semimartingales. Let* $F \in \mathrm{Lip}(K_1)$, $G \in \mathrm{Lip}(K_2)$, $G0=0$, *and* $H \in \mathcal{S}_c(K_3)$ *with* $\|HX\|_{\mathcal{H}^\infty} \leqq a$ *for any continuous* _semimartingale_ X. _If_ $K_2 a < 1/c_1$ $(c_1 \leqq 4)$, _then there exists a unique non-exploding solution of_ (E).

(3.2) COMMENTS. (i) One can trivially replace the condition $||HX||_{\mathcal{H}^\infty} \leqq a$ with $||HX||_{\mathcal{H}^\omega} \leqq a$, using the $\mathcal{H}^\omega$ norm of Meyer [9] for semimartingales, which is a generalization of the BMO norm for martingales. The $\mathcal{H}^\omega$ norm is slightly weaker than the $\mathcal{H}^\infty$ norm, but for most examples the $\mathcal{H}^\infty$ norm is simpler and suffices.

(ii) By considering the deterministic example $M_t = t$, $FZ=0$, $GZ=2Z$, and $HZ = Z$, we get $Z(t) = (1\pm\sqrt{1-4t})/2$, for $t \leqq 1/4$; thus these equations are closely related to singular ODE's, and one sees that some sort of condition like $K_2 a < 1/c_1$ is necessary.

We begin the proof of Theorem (3.1) with a lemma.

(3.3) LEMMA. _Assume the hypotheses of Theorem_ (3.1). _Suppose in addition that_:

(i) $c_1 K_1 y + K_3 \gamma + c_1 K_2 a < 1$

(ii) $||Y||_{\mathcal{H}^\infty} \leqq y$

(iii) $||GX||_{\underline{S}^\infty} \leqq \gamma$ _for any continuous semimartingale_ X

(iv) $||M||_{\mathcal{H}^1} < \infty$.

Then there exists a unique nonexploding solution of (E).

Proof: Set $X^0 = M$, and set

$$X^{n+1} = M + FX^n \cdot Y + GX^n \cdot HX^n .$$

Since

$$X^{n+1} - X^n = (FX^n - FX^{n-1}) \cdot Y + (GX^n - GX^{n-1}) \cdot HX^n + GX^{n-1} \cdot (HX^n - HX^{n-1}) ,$$

using the inequalities (2.5), we have

$$||X^{n+1}-X^n||_{\mathcal{H}^1} \leqq \{c_1K_1y+c_1K_2a+K_3\gamma\}||X^n-X^{n-1}||_{\mathcal{H}^1}$$

$$= r||X^n-X^{n-1}||_{\mathcal{H}^1}$$

where $r < 1$. Since M and each X^n is in $\mathcal{H}^1$ we have that (X^n) is a Cauchy sequence in $\mathcal{H}^1$. Let X be the limit of X^n. One easily checks that X is a solution of (E), and the uniqueness of limits in $\mathcal{H}^1$ is used to show X is a unique solution. □

Proof of Theorem (3.1): To complete the proof of Theorem (3.1), it remains only to remove the supplementary hypotheses (i) through (iv) of Lemma (3.3).

Step 1: We remove hypothesis (iv): $||M||_{\mathcal{H}^1} < \infty$. Given a continuous semimartingale M, there exists a sequence of stopping times $(T^k)_{k\geqq 1}$ increasing to ∞ a.s. such that $M^{T^k} \in \mathcal{H}^1$. Let X^k be the solution of

$$Z = M^{T^k} + FZ\cdot Y + GZ\cdot HZ.$$

Then if $\ell \geqq k$, $X^\ell = X^k$ on $[0,T^k]$ by the uniqueness of solutions; hence we can define a solution X of (E) on $[0,\infty[$ by $X = X^k$ on $[0,T^k]$, each $k \geqq 1$.

Step 2: We remove hypothesis (iii) that $||GX||_{\mathcal{S}^\infty} \leqq \gamma$ for any continuous semimartingale X. We define a new operator G^1 by:

$$G^1J_t = \begin{cases} GJ_t & \text{if} \quad |GJ_t| \leqq \gamma/2 \\ (\text{sign } GJ_t)\gamma/2 & \text{if} \quad |GJ_t| > \gamma/2 . \end{cases}$$

Let Z^1 be the unique solution for (E) with G^1 replacing G. Define

$$T^1 = \inf\{t: |GZ_t^1| \geqq \gamma/2\}.$$

Inductively assume $T^1,\ldots,T^{n-1}$ are defined. Define G^n by:

$$G^nJ = \begin{cases} GJ - GJ^{T^{n-1}} & \text{if} \quad |GJ-GJ^{T^{n-1}}| \leqq \gamma/2 \\ \text{Sign}\,(GJ-GJ^{T^{n-1}})\gamma/2 & \text{if} \quad |GJ-GJ^{T^{n-1}}| > \gamma/2 \end{cases}$$

Let Z^n be the unique solution of:

$$Z^n = (Z^{n-1})^{T^{n-1}} + M - M^{T^{n-1}} + FZ^n(Y-Y^{T^{n-1}}) + G^nZ^n\cdot(HZ^n-(HZ^n)^{T^{n-1}}).$$

Define $T^n = \inf\{t: |GZ_t-GZ_t^{T^{n-1}}| \geqq \gamma/2\}$. Letting $T = \sup T^n = \lim T^n$, we can define a unique solution Z on $[0,T[$. It remains to show $T = \infty$ a.s. But stopping M at a time R^k so that $||M^{R^k}||_{\mathcal{H}^1} \leqq m(k) < \infty$, we have $||Z^{R^k\wedge T^n}||_{\mathcal{H}^1} \leqq \frac{1}{1-r}\left\{ y\|FM\|_{\mathcal{S}^1} + a\|GM\|_{\mathcal{S}^1}\right\} < \infty$ where $r = c_1K_1y + c_1K_2a < 1$. Thus $||Z^{R(k)\wedge T}||_{\mathcal{H}^1} < \infty$ and hence $\lim\limits_{t\uparrow\uparrow T\wedge R(k)} Z_t = Z_{T\wedge R(k)}$ exists and is finite a.s. But on $\{T<\infty\}$, GZ must have an oscillatory discontinuity or an explosion which cannot happen. Thus $T = \infty$ a.s.

Step 3: We remove hypothesis (ii) that $||Y||_{\mathcal{H}^\infty} \leqq y$. Given a $y > 0$, since Y is continuous there exists a sequence of stopping times (T^k) increasing to ∞ a.s. such that $||Y^n||_{\mathcal{H}^\infty} \leqq y$, where $Y^n = Y^{T^n} - Y^{T^{n-1}}$. Define $M^n = M^{T^n} - M^{T^{n-1}}$, and H^n by $H^nJ = (HJ)^{T^n} - (HJ)^{T^{n-1}}$. Inductively suppose Z^{n-1} is the (unique) solution on $[0,T^{n-1}]$. Then let Z^n be the solution of:

$$Z^n = (Z^{n-1})^{T^{n-1}} + M^n + FZ^n\cdot Y^n + GZ^n\cdot H^nZ^n.$$

We know Z^n exists and clearly $(Z^n)^{T^{n-1}} = (Z^{n-1})^{T^{n-1}}$; thus we can set $Z = Z^n$

on $[0,T^n]$, and we have a solution on $[0,\infty[$. This completes the proof of Theorem (3.1). □

We now wish to consider the question of the stability of equations of this type. The natural framework is the semimartingale topology developed by Emery [2] and independently by Métivier and Pellaumail [7]. See also [1]. Under appropriate hypotheses on M^n, F^n, Y^n, G^n, and H^n, we want solutions Z^n of (E_n) below to converge to a solution Z of (E).

$$(E_n) \qquad Z^n = M^n + F^n Z^n \cdot Y^n + G^n Z^n \cdot H^n Z^n .$$

(3.4) THEOREM. Let $(M^n)_{n\geq 1}$, M, $(Y^n)_{n\geq 1}$, Y be continuous semimartingales. Let $(F^n)_{n\geq 1}$, F be in $\mathrm{Lip}(K_1)$, $(G^n)_{n\geq 1}$, G in $\mathrm{Lip}(K_2)$, $(H^n)_{n\geq 1}$, H in $\mathcal{S}_c(K_3)$ with $\|H^n X\|_{\mathcal{H}^\infty}$, $\|HX\|_{\mathcal{H}^\infty} \leq a$ for any continuous semimartingale X. Assume $K_2 a < 1/c_1$. Assume $M^n \to M$, $Y^n \to Y$, and $H^n Z \to HZ$ in the semimartingale topology, where Z is a solution of (E). Assume further that $F^n Z \to FZ$, $G^n 0 = G0 = 0$ and $G^n Z \to GZ$ locally in $\mathcal{S}^1$. Then $Z^n \to Z$ in the semimartingale topology, where Z^n is the solution of (E_n).

Proof: By considering a subsequence if necessary and by stopping at a stopping time, we may assume without loss of generality:

(i) $M^n \to M$, $Y^n \to Y$, $H^n Z \to HZ$ in $\mathcal{H}^1$

(ii) $F^n Z \to FZ$, $G^n Z \to GZ$ in $\mathcal{S}^1$.

Let us make three temporary additional hypotheses:

(iii) $\|Y\|_{\mathcal{H}^\infty} \leq y$

(iv) $\|G^n Z\|_{\mathcal{S}^\infty} \leq \gamma$

(v) $\|F^n Z^n\|_{\mathcal{S}^\infty} \leq C < \infty$, all $n \geq 1$, where $y > 0$, $\gamma > 0$, are taken so that $c_1K_1y + \gamma K_3 + c_1K_2a = r < 1$. (Recall $c_1K_2a < 1$.) One easily deduces under (i) through (v):

$$||Z-Z^n||_{\mathcal{H}^1} \leqq \alpha(n) + r||Z-Z^n||_{\mathcal{H}^1}$$

where $\alpha(n) \to 0$ as $n \to \infty$ and $r < 1$.

To remove hypothesis (iv), we note that we are assuming $G^n0 = 0$ and $G^n \in \mathrm{Lip}(K_2)$. Set

$$T^1 = \inf\{t: \ |Z_t| \geqq \gamma/K_2\}$$
$$\vdots$$
$$T^n = \inf\{t: \ |Z_t - Z_t^{T^{n-1}}| \geqq \gamma/K_2\}\ .$$

Define $G^{n(k)}$ by:

$$G^{n(k)}J = G^nJ^{T^k} - G^nJ^{T^{k-1}};$$

then $||G^{n(k)}Z||_{\mathcal{S}^\infty} \leqq K_2||Z^{T^k} - Z^{T^{k-1}}||_{\mathcal{S}^\infty}$

$$\leqq \gamma\ .$$

Thus if $Z^{n(k)}$ solves, on $[0,T^k]$, the equation:

$$Z^{n(k)} = (Z^{n(k-1)})^{T^{k-1}} + M^n - (M^n)^{T^{k-1}}$$
$$+ F^nZ^{n(k)} \cdot \{Y^n - (Y^n)^{T^{k-1}}\}$$
$$+ G^{n(k)}Z^{n(k)} \cdot \{H^nZ^{n(k)} - (H^nZ^{n(k)})^{T^{k-1}}\}$$

we have that $Z^{n(k)} \to Z^{(k)}$ in $\mathcal{H}^1$, using the inductive hypothesis that $Z^{n(k-1)} \to Z^{(k-1)}$ in $\mathcal{H}^1$. Since the sequence (T^k) was defined in terms of Z, we have T^k increases to ∞ a.s.

To remove hypothesis (iii) that $||Y||_{\mathcal{H}^\infty} \leqq y$, we proceed as in Step 3 of the proof of Theorem (3.1). Let (T^k) be stopping times increasing to ∞ a.s. such that $Y^{(k)} = Y^{T^k} - Y^{T^{k-1}}$ and $||Y^{(k)}||_{\mathcal{H}^\infty} \leqq y$. Define $M^{n(k)} = (M^n)^{T^k} - (M^n)^{T^{k-1}}$, and $Y^{n(k)}$, $H^{n(k)}$, analogously. Then let $Z^{n(k)}$ solve:

$$Z^{n(k)} = (Z^{n(k-1)})^{T^{k-1}} + M^{n(k)} + F^n Z^{n(k)} \cdot Y^{n(k)}$$
$$+ G^n Z^{n(k)} H^{n(k)} Z^{n(k)} ,$$

and inductively $Z^{n(k-1)} \to Z^{(k-1)}$ locally in $\mathcal{H}^1$ gives that $Z^{n(k)} \to Z^{(k)}$ locally in $\mathcal{H}^1$.

Finally, the removal of hypothesis (v) follows exactly as in the stability theory without instantaneous feedback. The reader can find the details in Protter [10, pp. 343-4], so we do not bother to recopy them here. □

(3.5). COMMENTS. (i) It is clear from the proofs that these theorems hold as well for systems of equations.

(ii) By using a localisation technique of Lenglart [6] (see Emery [3, pp. 291-2] for details) one can obtain the same results for F^n, F, G^n, G, H^n, H all Lipschitz with random, finite-valued Lipschitz constants.

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