Séminaire de probabilités (Strasbourg)

Michel Métivier

Pathwise differentiability with respect to a parameter of solutions of stochastic differential equations

Séminaire de probabilités (Strasbourg), tome 16 (1982), p. 490-502
<http://www.numdam.org/item?id=SPS_1982__16__490_0>

© Springer-Verlag, Berlin Heidelberg New York, 1982, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

PATHWISE DIFFERENTIABILITY WITH RESPECT TO A PARAMETER
OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

by

Michel METIVIER

Ecole Polytechnique - Palaiseau - France.

Abstract

We consider a stochastic differential equation

$$X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

where S *is a semimartingale and* q *a random measure and where the "coefficients" depend on a parameter* u *. We prove under suitable differentiability-conditions that the solution* $X^u(t,\omega)$ *can be choosen for each* u *in such a way that the mapping* $u \rightsquigarrow X^u(t,\omega)$ *is continuously differentiable for every* (t,ω) *.*

I - INTRODUCTION

The goal of this paper is to prove that under sufficient differentiability conditions on the coefficients, stochastic differential equations of the type

$$(1.1)\qquad X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

where S is a semimartingale, q a random measure with zero dual predictable projection and u a parameter taking its values in a bounded open subset G of $\mathbb{R}^d$, admit for each u a solution which can be determined in such a way that P.a.s. the functions $u \rightsquigarrow X^u(t,\omega)$ are for every t continuously differentiable.

This is a concept of differentiability different from the one considered by Sikhmann (see [3] and [4]), who studied the differentiability of the mapping $u \rightsquigarrow X^u_t(.)$ as a mapping from G into $L^p(\Omega)$ for some p and in the framework of Ito-equations. Recently Bichteler took the same point of view and considered equations of the type (1.1) with $q = u$ and S and X^u possibly infinite dimensional. J. Jacod in [6] considered differentiability "in probability".

Pathwise differentiability was considered by P. Malliavin and M. Bismut for the solutions of Ito-Stratonovitch equation as functions of the initial conditions (see [2] and [8]). In [7] H. Kunita proved pathwise differentiability with respect to the initial conditions for the solutions of an equation driven by a continuous martingale. In [11] P.A. Meyer proved the same result for equations driven by a semimartingale (equations of Doleans-Dade-Protter type).

We consider here equations of type (1.1) and of a more general type with coefficients depending on a parameter u .

In section II we recall a few facts on the type of equations which are studied here. In section III we give sufficient conditions for the continuity of solutions with respect to u and in section IV we deal with differentiability.

II - THE EQUATION UNDER CONSIDERATION

2.1. - Inequalities for stochastic integrals

We assume that the random measure q in (1.1) is of the form $\mu(\omega;ds;du) - \nu(\omega;ds;du)$ where $\mu(\omega;]0,t],du)$ is for each ω and t a borelian measure in an open subset $\mathbb{E}$ of $\mathbb{R}^m - \{0\}$ such that for some $\alpha > 0$ $\int \frac{|x|^\alpha}{1+|x|^\alpha} |\mu|(\omega;]0,t],du) < \infty$ ($|\mu|$ denotes the variation of μ and α does not depend on ω and t) and where ν is the dual predictable projection of μ).

$\mathbb{H}$ denotes a separable Hilbert space. We have shown in [9] (see also J. Jacod [5]) the existence of an increasing positive adapted process b and of a process $\{\mathring{q}(\omega,s,.) : (\omega,s) \in \Omega \times \mathbb{R}^+\}$ the values of which are measures on $\mathbb{E} \times \mathbb{E}$ such that :

i) For each $\mathbb{H}$-valued function h on $\mathbb{E}$ such that $\langle h(x),h(y)\rangle_{\mathbb{H}}$ is $\mathring{q}(\omega,s,dx \otimes dy)$ integrable, the integral $\int \langle h(x),h(y)\rangle_{\mathbb{H}} \mathring{q}(\omega,s,dx \otimes dy)$ defines a positive optional process ;

ii) If Y is an $\mathbb{H}$-valued $\mathscr{P}\otimes\mathscr{B}_{\mathbb{E}}$ measurable(*) function on $\mathbb{R}^+\times\Omega\times\mathbb{E}$ and if we denote by $\lambda_s(Y)$ the $\mathbb{H}$-valued positive random variable

$$\lambda_s(Y) := \int \langle Y(s,\ ,x),\ Y(s,\ ,y)\rangle_{\mathbb{H}}\ \overset{\circ}{q}(.,s,dx\otimes dy)$$

(set to be equal to $+\infty$ when the integral does not exist) and

iii) the following inequality holds for every stopping time τ

$$(2.1)\qquad E\Big(\sup_{t<\tau}\Big\|\int_{]0,t]\times\mathbb{E}} Y(s,.x)q(.,ds,dx)\Big\|^2\Big) \leq 4\ E\Big(\int_{[0,\tau[}\lambda_s(Y)db_s\Big)$$

where $\left(\int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)\right)_{t\geq 0}$ is the stochastic integral process of Y with respect to q which is defined as soon as the process $\left(\int_{]0,t]}\lambda_s(Y)db(s)\right)_{t\geq 0}$ is finite.

If S is a $\mathbb{K}$-valued ($\mathbb{K}$: separable Hilbert space) right continuous semimartingale we know that there exist two positive increasing adapted processes a and $\tilde{a}$ such that for every $\mathcal{L}(\mathbb{K};\mathbb{H})$-valued locally bounded predictable process $\{f(s,\omega);(s,\omega)\in\mathbb{R}^+\times\Omega\}$ and every stopping time τ :

$$(2.2)\qquad E\Big(\sup_{t<\tau}\Big\|\int f(s,.)dS_s\Big\|^2\Big) \leq E\Big(\tilde{a}_{\tau-}.\int_{[0,\tau[}\|f(s)\|^2da(s)\Big)$$

To simplify the writing we shall call Z_t the process $Z_t := (S_t,q(.,]0,t],dx))$ which takes its values in $(\mathcal{L}(\mathbb{K};\mathbb{H})\times\mathcal{M}^\alpha)$ where $\mathcal{M}^\alpha$ is the space of borelian measures ν on $\mathbb{E}$ such that $\int_{\mathbb{E}}\frac{|x|^\alpha}{1+|x|^\alpha}|\nu|(du) < \infty$.

Setting $A_t := b(t) + a(t)$ $\tilde{A}_t := 8 + 2\tilde{a}_t$ $\Phi := (f,Y)$

$$(2.3)\qquad \int_{]0,t]}\Phi(s)dZ_s := \int_{]0,t]} f(s,.)dS_s + \int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)$$

and

$$(2.4)\qquad \lambda_s(\Phi) := \|f(s,.)\|^2 + \lambda_s(Y)$$

the following inequality holds for every stopping time

$$(2.5)\qquad E\Big(\sup_{t<\tau}\Big\|\int_{]0,t]}\Phi(s)dZ_s\Big\|^2\Big) \leq E\Big(\tilde{A}_{\tau-}.\int_{]0,\tau]}\lambda_s(\Phi)dA_s\Big)$$

(*) $\mathscr{P}$ is the σ-algebra of predictable subsets of $\mathbb{R}^+\times\Omega$ and $\mathscr{B}_{\mathbb{E}}$ of Borel subsets of $\mathbb{E}$.

Extending a classical argument on martingales (see [13]) it is also easy to see that for every $p \geq 2$ exists an increasing positive adapted process $(\tilde{A}^p_t)_{t\geq 0}$ such that for every stopping τ

$$(2.6)\qquad E\left(\sup_{t<\tau}\left\|\int_{]0,t]}\Phi(s)dZ_s\right\|^p\right) \leq E\left(\tilde{A}^p_{\tau^-}\cdot\int_{[0,\tau[}\left(\lambda_s(\Phi)\right)^{p/2} dA_s\right)$$

2.2. - Hypothesis on equation (1.1)

The space of parameters u is an open bounded subset G of $\mathbb{R}^d$.

In equation (1.1) σ is a mapping from $(G \times \mathbb{R}^+ \times \Omega \times \mathbb{H})$ into $\mathcal{L}(\mathbb{K};\mathbb{H})$ which is continuous on $\mathbb{H}$ and such that for every $h \in \mathbb{H}$ and $u \in G$ the process $\{\sigma(u,s,\omega,h) : (s,\omega) \in \mathbb{R}^+ \times \Omega\}$ is predictable. f is a mapping of $(G\times\mathbb{R}^+\times\Omega\times\mathbb{H}, \mathbb{E})$ into $\mathbb{H}$ which if continuous on $\mathbb{H}$ and such that for every $u \in G$, $h \in \mathbb{H}$ the mapping $(s,\omega,x) \rightsquigarrow f(u,s,\omega,h,x)$ is $\mathscr{P}\otimes\mathscr{B}_{\mathbb{E}}$ measurable

In the sequel we shall call g the couple (σ,f) and according to the notations of (2.1) the equation (1.1) will be written in the *abreviated form* :

$$(2.7)\qquad X^u(t) = V^u(t) + \int_0^t g(u,s,X^u_{s^-})dZ_s$$

Here V^u is for each $u \in G$ a given $\mathbb{H}$-valued adapted cad-lag process.

III - CONTINUITY OF THE SOLUTIONS WITH RESPECT TO u .

3.1. - Hypothesis

L is an increasing positive adapted process and p is a positive real number with $p \geq d + \varepsilon$ for some $\varepsilon > 0$.

If ξ is a cad-lag $\mathbb{H}$-valued adapted process we write $g(u,\xi)$ for the process $(t,\omega) \rightsquigarrow g(u,s,\omega,\xi_{s^-}(\omega))$ and $\lambda_s \circ g(u,\xi)$ for the positive functional of this process defined by formula (2.4).

With these notations we formulate the following hypotheses :

(H$_1$) $\sup_{s\leq t} \|V_s^u - V_s^v\| \leq L_t \|u-v\|$ for all t, u and $v \in G$

and

$$\sup_{u\in G\ s<t} \|V_t^u\| < \infty$$

(H$_2$) (Lipschitz hypotheses) :

$$\forall t\in \mathbb{R}^+ \quad \int_{]0,t]} [\lambda_s o(g(u,\xi)-g(u,\xi'))]^{p/2} dA_s \leq \int_{]0,t]} \sup_{r\leq s} \|\xi_r - \xi'_r\|^p dL_s$$

for every couple (ξ,ξ') of $\mathbb{H}$-valued adapted cad-lag processes, $\mathbb{P}$.a.s.

(H$_3$) $$\int_{]0,t]} [\lambda_s \circ g(u,\xi)]^{p/2} dA_s \leq \int_{]0,t]} (1 + \sup_{r\leq s} \|\xi_s\|^p) dL_s$$

for every $u \in G$ every $\mathbb{H}$-valued adapted cal-lag ξ, P.a.s.
(Note that (H$_3$) is implied by (H$_2$) in most classical cases).

(H$_4$) Ψ being a given positive increasing (possibly constant) function on $\mathbb{R}^+$, for every stopping time τ the following inequality holds for every $\mathbb{H}$-valued cad-lag adapted ξ every u and v in G :

$$E\left(\sup_{t<\tau^-}\left[\lambda_t \circ [g(u,\xi)-g(v,\xi)]\right]^{p/2}\right) \leq \|u-v\|^{d+\varepsilon}\Psi\left(E(\sup_{t<\tau}\|\xi_t\|^p)\right)$$

3.2. - Theorem

1°) Under the above hypotheses (H$_1$) *to* (H$_4$) *, the equation* (2.7) *has for each* u *a unique strong solution* X^u *on* $\mathbb{R}^+$ *and the random function* $(t,\omega,u) \rightsquigarrow X_t^u(\omega)$ *can be determined in such a way that* $u \rightsquigarrow X_t^u(\omega)$ *is continuous on* G *for every* t *and* ω *while the mapping* $t \rightsquigarrow X_t^{(.)}(\omega)$ *is for each* ω *cad-lag from* $\mathbb{R}^+$ *into the set* $C_b^{\mathbb{H}}(G)$ *of bounded continuous* $\mathbb{H}$*-valued functions on* G *endowed with the uniform topology.*

2°) There exists an increasing sequence (σ_n) *of stopping times and constants* $K(\Psi,n,p,Z)$ *such that*

a) $\lim_n P\{\sigma_n < T\} = 0$ *for every* $T > 0$

b) $E\left(\sup_{t<\sigma_n} \|X^u(t) - X^v(t)\|^p\right) \leq K(Y,n,p,Z)\ \|u-v\|^p$

Proof.

The stopping times σ_n are defined as follows :

$$\sigma_n := \inf \{t : \widetilde{A}^p_t \vee L_t \vee \sup_{\substack{u\in G \\ s\leq t}} \|V^u_t\|^p \vee A_t > n\}$$

Next we have the following lemmas

3.3. - Lemma 1

$$E\left(\sup_{t<\sigma_n} \|X^u_t\|^p\right) \leq 2^p(n+n^2) \sum_{j=0}^{2^P n^2} (2^P n^2)^j$$

Proof of Lemma 1

We remark that $A^p_{\sigma_n^-} \leq n, L_{\sigma_n^-} \leq n, \sup_{t<\sigma_n} \sup_u \|V^u_t\|^p \leq n$

We then apply inequality (2.6) to the second member of (2.7) and get

$$E\left(\sup_{t<\sigma_n} \|\mathbf{X}^u_t\|^p\right) \leq 2^{(P-1)} n+2^{(P-1)} E\left(\widetilde{A}^p_{\sigma_n^-} \int_{]0,\sigma_n[} [\lambda_s \circ g(u,\mathbf{X}^u)]^{P/2} dA_s\right)$$

and property (H_3) gives for every stopping time $\tau \leq \sigma_n$

$$E\left(\sup_{t<\sigma_n} \|\mathbf{X}^u_t\|^p\right) \leq 2^{(P-1)}(n+n^2) + 2^{(P-1)} n\, E\left(\int_{]0,\tau[} (\sup_{s<t} \|\mathbf{X}^u_s\|^P) dL_s\right)$$

Applying the "Gronwall stochastic Lemma" as in [10] section 7.1 we get the inequality of the lemma.

3.4. - Lemma 2

There exist constants $K(\Psi,n,p,A,\widetilde{A}^p)$ such that

$$\forall\, u,v \quad E\left(\sup_{t<\sigma_n} \|X^u_t - X^v_t\|^p\right) \leq K(\Psi,n,p,A,\widetilde{A}^p) \|u-v\|^p$$

Proof of Lemma 2

Applying again inequality (2.6) to the stochastic integrals

$\int_{]0,t]} (g_s(u,X^u_{s^-}) - g_s(v,X^u_{s^-}))dZ_s$ and

$\int_{]0,t]} [g_s(v,X^u_{s^-}) - g_s(v,X^v_{s^-})]dZ_s$

and using properties (H_1), (H_2) and (H_4) we can write for every stopping time $\tau \leq \sigma_n$:

$$E\left(\sup_{s<\tau} \| X^u(s)-X^v(s) \|^p\right) \leq 3^{P-1} n^P \| u-v \|^p + 3^{(P-1)} n\, \Psi\left(E\left(\sup_{s<\tau} \| X^u_s \|^p\right)\right)$$
$$+ 3^{(P-1)} n\, E\left(\int_{]0,\tau[} (\sup_{t<s} \| X^u(s)-X^v(s) \|^P)\, dL_s\right)$$

Applying as above the same "Gronwall-inequality" we obtain the lemma.

Theorem 3.2 is now a direct consequence of the following lemma which is a straight-forward extension of a lemma as stated by Neveu in [12] (see also P. Priouret [13] chap. 3. lemme 13 :

3.5 - Lemma 3

Let $\{Y(t,\omega,u) : t \in \mathbb{R}^+, \omega \in \Omega, u \in G\}$ *an* $\mathbb{H}$*-valued random function such that for every* $u : t \rightsquigarrow Y(t,\omega,u)$ *is a.s. cad-lag and such that for every* t *:*

$$E\left(\sup_{s\leq t} \| Y_{s,u} - Y_{s,v} \|^p\right) \leq a_{t,p} \| u-v \|^{d+\varepsilon}$$

Then there exists a mapping $Y^* : (t,\omega,u) \rightsquigarrow Y^*(t,\omega,u) \in \mathbb{H}$ *such that*

a) $u \rightsquigarrow Y^*(t,\omega,u)$ *is continuous*

b) $\forall u \in G$, $Y(t,u,.) = Y^*(t,u,.)$ *for all* t *a.s.*

b) $t \rightsquigarrow Y^*(t,.,\omega)$ *is for* P*-almost all* ω *a cad-lag mapping from* $\mathbb{R}^+$ *into* $C_b^{\mathbb{H}}(G)$ *endowed with the uniform topology.*

Proof.

We omit the proof which is pretty similar to the one given in [13].

This finishes the proof of theorem 3.2. ■

IV - PATHWISE DIFFERENTIABILITY

4.1. - Hypothesis

We consider the same equation (1.1) or in abreviated notation : (2.7).

For a couple $g := (\sigma, f)$ of "coefficients" as in (1.1) we write to simplify :

$$\| g(u,s,\omega,h,.)\|_\Lambda := \left[\|\sigma(u,s,\omega,h)\|^2_{\mathcal{L}(\mathbb{K};\,\mathbb{H})} + \int_{\mathbb{E}\times\mathbb{E}} \langle f(u,s,\omega,h,x), f(u,s,\omega,h,y)\rangle_{\mathbb{H}} \, \overset{o}{q}(\omega,s,dx\otimes dy)\right]^{\frac{1}{2}}$$

We set $v^*_t := \sup_{u\in G}\sup_{s<t} \|D_u V^u_s\| + \|V^u_s\| + \|D^2_{u^2} V^u_s\|$

were $D_u\Phi$ denotes the first order derivative and $D^2_{u^2}\Phi$ the second order derivative with respect to u of a function Φ on u.

In the hypothesis below C is a constant and $(K_t)_{t\geq 0}$ is an increasing positive process.

[D_1] For all t and ω the derivatives $D_u V^u(t,\omega)$ and $D^2_{u^2} V^u(t,\omega)$ exist and $v^*_t < \infty$

[D_2] The derivatives $D_u g(s,u,x)$ D_u $g(s,u,x)$ $D_{u,x} g(s,u,x)$ and $D_x g(s,u,x)$ exist and

$$\sup_{u,s,x} (\|D_u g(s,u,x)\|_\Lambda + \|D^2_u g(s,u,x)\|_\Lambda + \|D^2_{ux} g(s,u,x)\|_\Lambda + \|D_x g(s,u,x)\|_\Lambda) \leq C$$

[D_3] For all x,y u and v :

$$\|D_x g(s,u,x) - D_x g(s,v,y)\|_\Lambda \leq C(\|y-x\| + \|u-v\|)$$

4.2. - Theorem

Under the above hypothesis [D_1] *to* [D_3] *equation* (2.7) *has a unique (up to indistinguability) solution* X^u *on* $\mathbb{R}^+$ *and there exists a version* $(\omega,t,u) \rightsquigarrow X^u_t(\omega)$ *of this random function such that for* P-*almost all* ω :

a) $u \rightsquigarrow X^u_t(\omega)$ *is continuously differentiable for every* t

b) $t \rightsquigarrow X^{(.)}_t(\omega)$ *and* $t \rightsquigarrow D_u X^{(.)}_t(\omega)$ *are cad-lag for the uniform norm on* $C_b(G;\mathbb{H})$ *and* $C_b(G;\mathcal{L}(G;\mathbb{H}))$ *respectively.*

c) For every u *the stochastic process* $(D_u X^u_t)_{t\geq 0}$ *is a strong solution of the following stochastic equation (where* X^u *is the process solution of* 2.7 *as in theorem* 3.2*)* :

$$(4.1) \quad Y^u(t) = D_u V^u_t + \int_{]0,t]} \left(D_u g(s,u,X^u_{s-}) + D_x g(s,u,X^u_{s-}) \circ Y^u_s \right) dZ_s$$

Proof.

The proof is in several steps corresponding to lemmas 4 and 5 and section 4.5 bellow :

4.3. - Lemma 4

Under hypothesis $[D_1]$, $[D_2]$, $[D_3]$, *equations* (2.7) *and* (4.1) *satisfy the conditions* $[H_1]$ *to* $[H_4]$ *of section* 3.1 *for every* $p \geq 2$ *on any interval* $]0,\sigma_n]$ *as defined in theorem* 1.

Proof.

Let us first consider equation (2.7). (H_1) is trivially implied by $[D_1]$. $[D_2]$ implies also the Lipschitz property (H_2) and conditions (H_3) and (H_4) which is here expressed in the much stronger form $\|g(s,u,x)-g(s,v,x)\|_\Lambda \leq C\,\|u-v\|$.

We turn now to equation (4.1). The only condition (H_i) which is not immediately implied by the hypothesis of the lemma is condition (H_4). We write

$$\|D_u g(s,v,X^v_{t^-}) - D_u g(s,u,X^u_{t^-}) + D_x g(s,v,X^v_{t^-}) \circ \xi_t - D_x g(s,u,X^u_{t^-}) \circ \xi_{t^-}\|^p_\Lambda$$

$$\begin{aligned}
&\leq 4^{p-1}\,\{\|D_u g(s,v,X^v_{t^-}) - D_u g(s,u,X^v_{t^-})\|^p_\Lambda\} + \\
&\quad + 4^{p-1}\,\{\|D_u g(s,u,X^v_{t^-}) - D_u g(s,u,X^u_{t^-})\|^p_\Lambda\} \\
&\quad + 4^{p-1}\,\{\|[D_x g(s,v,X^v_{t^-}) - D_x g(s,u,X^v_{t^-})] \circ \xi_{t^-}\|^p_\Lambda\} \\
&\quad + 4^{p-1}\,\{\|[D_x g(s,u,X^v_{t^-}) - D_x g(s,u,X^u_{t^-})] \circ \xi_{t^-}\|^p_\Lambda\} \\
&\leq 4^{p-1}\, C^p (\|u-v\|^p + \|X^v_{t^-} - X^u_{t^-}\|^p + \\
&\quad + 4^{p-1}\, C^p \|u-v\|^p\, \|\xi_{t^-}\|^p + 4^{p-1}\, C^p \|(X^v_{t^-} - X^u_{t^-}) \circ \xi_{t^-}\|^p
\end{aligned}$$

One knows from proposition 2 that there exists an increasing sequence (σ_n) of stopping times and constants C_n such that

$$E \sup_{s<\sigma_n} \|Y^u(s) - Y^v(s)\|^{2p} \leq C_n \|u-v\|^{2p}$$

If we write for every stopping time τ

$$E\Big(\sup_{t<\tau\wedge\sigma_n} \|(X^v_t - X^u_t) \circ \xi_{t^-}\|^p\Big) \leq$$

$$\left[E\left(\sup_{t<\tau\wedge\sigma_n} \| X_t^v - X_t^u \|^{2p}\right)\right]^{\frac{1}{2}} \left[E\left(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\frac{2p}{2p-1}}\right)\right]^{\frac{2p-1}{2}}$$

$$\leq C_n^{\frac{1}{2}} \| u-v \|^p \; E\left(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\alpha}\right)^{p/\alpha}$$

with $\alpha = \frac{2p}{2p-1}$

Therefore

$$E\left(\sup_{s<\tau\wedge\sigma_n} \| g(s,u,\xi_{s^-}) - g(s,v,\xi_{s^-}) \|_{\Lambda}^{p}\right) \leq 4^{p,1} \, c^p \| u-v \|^p [1+C_n + E(\sup_{s<\tau\wedge\sigma_n} \| \xi_s \|^p)]$$

$$+ C_n^{\frac{1}{2}} [E(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\alpha})]^{p/\alpha}$$

If we remark that $E(\sup_{t<\tau\wedge\sigma_n} \| \xi_{t^-} \|^p) \geq [E(\sup_{t<\tau\wedge\sigma_n} \| \xi_{t^-} \|^{\alpha})]^{p/\alpha}$

we see that property (H_4) holds with

$$\Psi(\rho) = 1 + C_n + (1 + C_n^{\frac{1}{2}})\rho$$

4.4. - Lemma 2

If we define

$$\Phi_t(e,u,\lambda) = \frac{1}{\lambda} [X_t^{u+\lambda e} - X_t^u - \lambda Y_t^u \circ e]$$

there exists an increasing sequence (τ_n) *of stopping times such that* $\lim_n P\{\tau_u < T\} = 0$ *and a sequence* C_n *of constants such that*

$$E\left\{\sup_{t<\tau_n} \| \Phi_t(e,.,\lambda) \|^2_{L^2(G)}\right\} \leq C_n \lambda^2$$

Proof.

For each u the process $(\Phi_t(e,u,\lambda))_{t\leq T}$ is solution of

$$(4.2) \quad \Phi_t(e,u,\lambda) = \frac{1}{\lambda} (V_t^{u+\lambda e} - V_t^u - \lambda D_e V_t^u) +$$

$$+ \int_{]0,t]} \frac{1}{\lambda} \Big[g(s,u+\lambda e, X_{s^-}^{u+\lambda e}) - g(s,u,X_{s^-}^u) -$$

$$\lambda D_e g(s,u,X_{s^-}^u) - \lambda D_x g(s,u,X_{s^-}^u) \circ Y_{s^-}^u \circ e \Big] \, dS_s$$

We may write for $x,y \in \mathbb{H}$ and $\eta \in \mathcal{L}(\mathbb{H};\mathbb{H})$

$$(4.3)\quad g(s,u+\lambda e,y) - g(s,u,x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e =$$

$$\lambda D_e g(s,u,y) + D_x g(s,u,x) \circ (y-x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e +$$

$$+ h(s,u,x,y,\eta,\lambda,e)$$

$$= D_x g(s,u,x) \circ (y-x-\lambda\eta \circ e) + \tilde{h}(s,u,x,y,\eta,\lambda)$$

with

$$(4.4)\quad \|\tilde{h}(s,u,x,y,\eta,\lambda)_\Lambda \leq |\lambda|\, K\, (\|y-x\| + |\lambda|)$$

for some constant K

The equation (4.2) can therefore be written

$$(4.5)\quad \Phi_t(e,u\lambda) = H_t(u,\lambda,e) + \int_{]0,t]} D_x g(s,u,X^u_{s^-}) \circ \Phi_{s^-}(e,u,\lambda)\, dZ_s$$

where the process $H(u,\lambda,e)$ satisfies

$$(4.6)\quad \|H_t(u,\lambda,e)\|_{\mathbb{H}} \leq |\lambda|\ \ v^k_t + \left\|\int_{]0,t]} \frac{1}{\lambda}\, h(s,u,X^{u+\lambda e}_{s^-}, X^u_{s^-}, Y^u_{s^-} \circ e)\, dZ_s\right\|$$

Using (4.5) we obtain from (4.6) for every stopping time σ :

$$E\left(\sup_{t<\sigma} \|H_t(u,\lambda,e)\|^2\right) \leq 2\,\lambda^2\, v^*_{\sigma^-} + E\left(\tilde{A}_{\tau^-} \cdot \int_{]0,\tau[} [\lambda^2 + C^2\|X^{u+\lambda e}_{s^-} - X^u_s\|^2]\, dA_s\right)$$

Using then theorem we see that there exists a sequence (σ_n) of stopping times and a sequence of constants (K_n) such that

$$(4.7)\quad \sup_{s<\sigma_n} (\tilde{A}_s \vee A_s) \leq n \quad \text{and}$$

$$(4.8)\quad E(\sup_{t<\sigma_n} \|H_t(u,\lambda,e)\|^2) \leq K_n\, \lambda^2$$ (use a standard stopping procedure for processes $v^*, \tilde{A}$ and A).

This implies

$$(4.9)\quad E\left(\sup_{t<\sigma_n}\int_G \|H_t(u,\lambda,e)\|^2\, du\right) \leq \int_G K_n\, \lambda^2\, du \leq \tilde{K}_n\, \lambda^2$$

We next consider the $L^2(G)$-valued process $(\Phi_t(e,.,\lambda))_{t\leq T}$

As $D_x g$ is bounded by some constant C, inequality (4.6) shows that the $L^2(G)$-valued process Φ_t satisfies an inequality of the following type for every stopping time $\tau \leq \sigma_n$

$$E\left\{\sup_{t<\tau}\|\Phi_t(e,.,\lambda)\|_{L^2(G)}\right\} \leq 2\,\tilde{K}_n\,\lambda^2 + 2\,E\left(\tilde{A}_\tau - \int_{[0,\tau[} C^2 \sup_{s<t}\|\Phi_s(e,.,\lambda)\|^2_{L^2(G)}\, dA_s\right)$$

$$\leq 2\,\tilde{K}_n\,\lambda^2 + 2n\,C^2\int_{[0,\tau[}\sup_{s<t}\|\Phi_s(e,.,\lambda)\|_{L^2(G)}\, dA_s$$

The already used "Gronwall inequality" of [10] shows immediately the existence of a constant C_n as in the lemma.

4.5. - End of the proof of the theorem

We make use of the following easily proved property : let $f \in L^2_{\mathbb{H}}(\bar{G})$ let $f \in L^2(\mathbb{G}\,;\mathbb{H}) \cap C(G\,;\mathbb{H})$ and $\bar{f} \in L^2(G\,;\mathcal{L}(\mathbb{H}\,;\mathbb{H})) \cap C(G\,;\mathcal{L}(\mathbb{H}\,;\mathbb{H}))$ such that for all $e \in \mathbb{R}^d$, all $u \in \mathbb{R}^d$ and some decreasing sequence $\lambda_k \downarrow 0$:

$$\lim_{k\to\infty}\|f(u+\lambda_k e) - f(u) - \lambda_k\,\bar{f}(u)\circ e\|_{L^2(G\,;\,\mathbb{H})} = 0$$

then $\bar{f}$ is the derivative of f in the sense of distributions and therefore in the ordinary sense in every point $u \in G$. Let us consider for each ω and n a P-negligeable set Ω_n and a sequence λ_k such that $\lambda_k \downarrow 0$ and

$$\lim_{k\to\infty}\sup_{t<\tau_n(\omega)}\|\Phi_t(e,.,\omega,\lambda_k)\|_{L^2(G)} = 0 \quad \text{for every } \omega \notin \Omega_n$$

The above property shows that for every $\omega \notin \Omega_n$ and $t < \tau_n(\omega)$ $Y^u_t(\omega)$ is the derivative of $u \rightsquigarrow X^u_t(\omega)$ at point u. Therefore $Y^u_t(\omega)$ is the derivative of $u \rightsquigarrow X^u_t(\omega)$ for all $t<\tau_n(\omega)$ and $\omega \notin (\bigcup_n \Omega_n)$.

This proves the theorem. ∎

BIBLIOGRAPHY

[1] S. BICHTELER
Stochastic Integrations with Stationary Independant increments
(To appear in Z. Wahr. verw. Geb.)

[2] M. BISMUT
A generalized formula of Ito and some other properties of stochastic flows
Z. Wahr. verw. Geb. 55, 1981, pp. 331-350.

[3] I.I. GIKHMAN
On the theory of differential equations of random processes
Uhr. Mat. Zb. 2, n° 4, 1950, pp. 37-63.

[4] I.I. GIKHMAN and A.V. SKOROKHOD
Stochastic Differential equations
Springer-Verlag, 1972.

[5] J. JACOD
Calcul stochastique et problèmes de martingales
Lecture Notes Math. 714, Springer-Verlag, New York, 1979.

[6] J. JACOD
Equations différentielles stochastiques : continuité et dérivabilité en probabilité
(Preprint)

[7] H. KUNITA
On the decomposition of solutions of stochastic differential equations.
Proc. of the L.M.S. Symposium on Stoch. Diff. Equations, Durham, juillet 1980, Lecture Notes in Math. Springer-Verlag, 1981.

[8] P. MALLIAVIN
Stochastic Calculus of variations and Hypoelliptic operators.
Proc. of the Intern. Symposium on Stochastic Differential Equations of Kyoto, 1976, pp. 195-263. Tokyo, Kinokuniya and New York, Wiley, 1978.

[9] M. METIVIER
Stability theorems for stochastic Integral Equations driven by random measures and semimartingales
J. of Integral Equations, 1980 (to appear).

[10] M. METIVIER and J. PELLAUMAIL
Stochastic Integration
Acad. Press. New York, 1980.

[11] P.A. MEYER
Flot d'une équation différentielle stochastique
Séminaire de Probabilité XV. Lecture Notes in Math. 850, Springer-Verlag, 1981.

[12] J. NEVEU
Intégrales stochastiques et applications
Cours de 3e Cycle. Univ. de Paris VI, 1971-1972.

[13] P. PRIOURET
Processus de diffusion et équations différentielles stochastiques
Ecole d'Eté de Prob. de St-Flour. Lecture Notes in Math. 390, Springer-Verlag, 1974.