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Abstract

Four examples are presented which show that stochastic integral processes with anticipating integrands can have very different sample path behaviour from those with adapted ones. For example, Skorohod integral processes need not be semimartingales, but can still have smooth occupation densities. Moreover, even if they are continuous, and have finite quadratic variation, this may still be essentially bigger than expected for "smooth" integrands.

Key words and phrases: Skorohod integral; occupation densities; semimartingales; quadratic variation.

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1 Introduction

There has recently been considerable interest in the study of stochastic differential equations (SDEs) with the Wiener process W as driving noise, but with unusual initial or boundary data. For example, the initial condition may depend on the whole information present in W, or one may impose coupled, for example periodic, boundary conditions on the parameter interval. In these cases it is necessary to use an extension of the Itô integral, due to Skorohod, which can deal with anticipating integrands. The resulting integral processes (which we will call Skorohod integral processes) have been used in a great number of papers to describe solutions of SDEs of the kind indicated. Apart from existence and uniqueness results the main property of the solutions to have been studied is the Markov property. Among many other references see Nualart, Pardoux [9], [10], [11], Ocone, Pardoux [12], [13], Donati-Martin [5], Buckdahn [2], [3], [4], Pardoux, Protter [14].

It is therefore of interest to study the sample path behaviour of Skorohod integral processes. If the integrand is smooth (in the sense of the Malliavin derivative) then it is well known that these processes have properties very similar to those of Itô integral processes:

for example Nualart and Pardoux [9] prove that Skorohod integral processes with smooth integrands are continuous, and have a quadratic variation equal to the Lebesgue integral of the squared integrands.

However, up to now, relatively little is known about the sample path properties of general Skorohod integral processes (see [6]), and this paper arose from the following question: "Can the sample paths of a general Skorohod integral process behave in an essentially different way than those of a semimartingale?"

As the easiest way to destroy semimartingale-type behaviour is to find discontinuities of the second kind, we looked at a surprising example given by Buckdahn [2]. He considered the simple linear SDE

$$X_t = \eta + \int_0^t X_s dW_s, \quad \eta = \operatorname{sgn}(W_1), \quad 0 \leq t \leq 1,$$

and discovered that the Skorohod integral process $X_t - \eta$ is able to jump. After a little thought it became clear that this example could to be modified so as to to destroy the semimartingale property of the corresponding integral process. Indeed it turns out that playing with η in this equation gives a whole variety of Skorohod integral processes with sample path properties very different from those of their Itô counterparts. In this paper we present four of them.

Example 1 takes up Buckdahn's example in [2] and compares the Skorohod solution with a corresponding Stratonovitch solution and with the solution given by an Itô integral process with respect to the Wiener filtration enlarged by the information present in W_1. Though the proofs are straightforward the difference remains rather mysterious at an intuitive level and all we can do is to ask questions.

In Example 2 we exhibit a random variable η which makes the solution X jump so erratically that the Skorohod integral process $X - \eta$ is not a semimartingale. However, it still possesses a continuous occupation density. This does not come as a complete surprise, since sample path irregularity tends to lead to regularity of local times.

Example 3 shows that even worse behaviour is possible, and that a Skorohod integral process can have unbounded oscillations on every interval, and an analytic occupation density.

In Example 4 we show that even continuous Skorohod integral processes can have essentially more "energy" than expected for smooth integrands. By taking η to be a function of W_1 varying as quickly as a Brownian path, we show that the quadratic variation of the corresponding process $\int_0^\cdot X_s dW_s$ is essentially bigger than the expected quantity $\int_0^\cdot X_s^2 ds$.

2 Notations and preliminaries

Our basic process is the Wiener process W, indexed by the unit interval, on the corresponding canonical Wiener space $(\Omega, \mathbf{F}, P)$. If S denotes the dense subset of $L^2(\Omega)$ consisting of all random variables of the form

$$F = f(W_{t_1}, \ldots, W_{t_n}),$$

where $t_1, ..., t_n \in [0,1]$, $f \in C_0^\infty(\mathbf{R}^n)$, we define the Malliavin derivative on S by

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, ..., W_{t_n}) 1_{[0,t_i]}(t), \quad t \in [0,1].$$

D is an unbounded closable operator from $L^2(\Omega)$ to $L^2(\Omega \times [0,1])$, and its extension to the closure $\mathbf{D}_{2,1}$ of S with respect to the norm

$$\| F \|_{2,1} = \| F \|_2 + \| DF \|_2,$$

is denoted by D as well. The Skorohod integral is the adjoint operator of D. More precisely, $u \in L^2(\Omega \times [0,1])$ is called Skorohod integrable with integral $\int_0^1 udW$, if

$$F \mapsto E(\int_0^1 u_t D_t F dt)$$

is continuous with respect to $\| \, . \, \|_2$. Then we have the duality relation

$$E(\int_0^1 u_t D_t F dt) = E(\int_0^1 udWF), \quad F \in \mathbf{D}_{2,1}.$$

If for all $t \in [0,1]$, $1_{[0,t]}u$ is Skorohod integrable, then the Skorohod integral process of u is given by the family of random variables $\int_0^1 1_{[0,t]}udW$, also denoted by $\int_0^t u_s dW_s$. Note that this process is unique only up to modifications. So by the statement "$\int_0^\cdot u_s dW_s$ is continuous" we mean that this process has a continuous modification.

A sequence of partitions $(\mathbf{J}_n)_{n \in \mathbf{N}}$ of $[0,1]$ by intervals $J = [s^J, t^J]$ is called a 0-sequence, if it increases with respect to fineness and its mesh converges to 0. For the increment of a function f over an interval J we write $\Delta_J f$. We use λ for Lebesgue measure.

3 The examples

For some $p > 2$ let $\eta \in L^p(\Omega, \mathbf{F}, P)$ and consider the stochastic integral equation

$$X_t = \eta + \int_0^t X_s dW_s, \quad t \in [0,1], \tag{1}$$

where the stochastic integral is taken in the Skorohod sense. Though they are given in a more general setting in Buckdahn [2], let us briefly review the arguments which identify the solution of (1).

Proposition 1 *For $t \in [0,1]$ let*

$$Y_t = \exp(W_t - \frac{1}{2}t), \quad Z_t = \eta(W_. - t \wedge .), \quad X_t = Y_t Z_t.$$

If $\eta \in L^p(\Omega, \mathbf{F}, P)$ for some $p > 2$ then $1_{[0,t]}X$ is Skorohod integrable and (1) is satisfied for any $t \in [0,1]$.

Proof. By the choice of η, $X_t - \eta$ is square integrable for any $t \in [0,1]$. Hence the duality of the Skorohod integral and the Malliavin derivative implies that it is sufficient to verify that

$$E(\int_0^t X_s D_s F ds) = E((X_t - \eta)F), \quad t \in [0,1], \quad F \in \mathbf{D}_{2,1}. \tag{2}$$

Further as S is dense in $D_{2,1}$ it is enough to establish (2) for any simple random variable of the form

$$F = f(W_{t_1}, ..., W_{t_n}),$$

where $f \in C_0^\infty(\mathbf{R}^n)$, $t_1, ..., t_n \in [0,1]$, $n \in \mathbf{N}$.

The key observation is that for each $t \in [0,1]$ the "partially drifted" process

$$W - t \wedge .$$

is, by Girsanov's theorem, a Brownian motion under the new probability measure $Y_t.P$. Hence we have

$$E((X_t - \eta)F) = E\big(\eta(W)(F(W + t \wedge .) - F(W))\big).$$

Now by the choice of f, for any $\omega \in \Omega, t \in [0,1]$,

$$\begin{aligned}
&(F(W + t \wedge .) - F(W))(\omega) \\
&= f(\omega_{t_1} + t \wedge t_1, ..., \omega_{t_n} + t \wedge t_n) - f(\omega_{t_1}, ..., \omega_{t_n}) \\
&= \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\omega_{t_1} + s \wedge t_1, ..., \omega_{t_n} + s \wedge t_n) 1_{[0,t_i]}(s) ds \\
&= \int_0^t D_s F(\omega + s \wedge .) ds.
\end{aligned}$$

Hence another application of Girsanov's theorem yields

$$\begin{aligned}
E\big(\eta(W)(F(W + t \wedge .) - F(W))\big) &= E(\int_0^t \eta(W) D_s F(W + s \wedge .) ds) \\
&= \int_0^t E(\eta(W - s \wedge .) Y_s D_s F(W)) ds \\
&= E(\int_0^t X_s D_s F ds).
\end{aligned}$$

This finally implies (2) and we are done.

The most important consequence for what follows is that we have an explicit formula for the Skorohod integral process of X, given by:

$$U_t = \int_0^t X_s dW_s = X_t - \eta, \quad t \in [0,1].$$

A striking difference between the behaviour of solutions of (1) and that of Itô integral processes was discovered by Buckdahn [2] in the following simple case.

Example 1 (Buckdahn [2]): Let $\eta = \text{sgn}(W_1)$. Then

$$X_t = \text{sgn}(W_1 - t)\, Y_t\,,$$

where, as before,

$$Y_t = \exp(W_t - \tfrac{1}{2}t), \quad t \in [0,1].$$

Since Y is strictly positive, we encounter the rather surprising fact that X and thus U not only changes sign when t reaches W_1 in the case $0 < W_1 < 1$, but that even more strangely it jumps at that time. This behaviour has no counterpart either for classical differential equations or for ordinary SDEs. Let us briefly compare the two cases. Suppose first that $(V_t)_{t\in[0,1]}$ is a process of bounded variation such that $V_0 = 0$, and instead of (1) we consider the stochastic integral equation

$$X_t = \eta + \int_0^t X_s dV_s, \quad \eta = \text{sgn}(V_1). \tag{3}$$

Then there is a canonical choice of integral in (3), a pathwise Riemann- Stieltjes integral, and the unique solution, given by

$$X_t = \eta \exp(V_t), \quad t \in [0,1],$$

is continuous and keeps the sign prescribed by the initial condition η.

As a second contrast, let us consider various other ways of interpreting the SDE (1) in the case $\eta = \text{sgn}(W_1)$. Aware of the fact that there is no canonical choice of stochastic integral, let us this time enlarge the filtration and work with the closest relative of the Skorohod integral, namely the Itô integral. More precisely, let

$$\mathbf{G}_t = \sigma(W_s : s \le t) \vee \sigma(W_1),$$

completed by the P – null sets of $\mathbf{F}$, $t \in [0,1]$. Considering (1) as an Itô SDE with respect to the new filtration still makes sense: the driving process W is now a semimartingale (see Jeulin [7], p. 46), and the (pathwise unique) solution is given by

$$X_t = \text{sgn}(W_1)\, Y_t$$

(see Protter [15], p. 77). Once again, the solution keeps the sign prescribed by its initial value.

We can also use the following alternative argument. Replace η in (1) by a deterministic initial value $\text{sgn}(x)$ for $x \in \mathbf{R}$ and solve the resulting SDE to get the adapted process $\text{sgn}(x)Y_t$, $t \in [0,1]$. Now evaluate x at W_1 to get the Itô solution in the enlarged filtration described above. The difference between the Skorohod solution of (1) and the Itô solution in the enlarged filtration is thus just contained in the generalization of the formula of Proposition 4.12 of Nualart, Pardoux [9]. Apart from this purely formal difference, there seems to be no intuitively appealing explanation for this strange difference of behaviour.

As a final contrast, let us couple X to the driving noise in (1) differently. Instead of the Skorohod integral with respect to W let us take the Stratonovitch integral: that is we consider the integral equation

$$X_t = \eta + \int_0^t X_t \circ dW_s, \quad t \in [0,1], \tag{4}$$

where we use the extended Stratonovitch integral in the sense of Nualart, Pardoux [9]. To solve (4), we again may replace η in the first step by a deterministic initial value $x \in \mathbf{R}$, to get a solution process $\phi_t(x) = x\exp(W_t)$, $t \in [0,1]$. In the second step we use the property for which the Stratonovitch integral is famous, which makes its calculus "pseudoclassical": given an integrand, say $u(.,x)$, sufficiently smoothly parametrized by $x \in \mathbf{R}$, the Stratonovitch integral $\int u(s,x) \circ dW_s$, evaluated at some random variable η, is the same as $\int u(s,\eta) \circ dW_s$ (see Nualart, Pardoux [9], p. 570). Here we take $u(.,x) = \phi_t(x)$, to be evaluated at $\eta = \operatorname{sgn}(W_1)$. Hence (4) has the solution

$$X_t = \operatorname{sgn}(W_1)\exp(W_t),$$

$t \in [0,1]$, which, just as the two versions before, keeps the sign prescribed by η.

For other choices of η, the jump in the preceding example can be replaced by even more serious consequences, as will be shown now.

Example 2: In this example, we will show that η in Proposition 1 can be chosen such that the corresponding Skorohod integral process is not a semimartingale, but still possesses a continuous occupation density.

Let A be a Cantor set in $[0,1]$ which is totally disconnected and has positive Lebesgue measure. We set

$$B = \bigcup_{n \in \mathbf{Z}} (A+n):$$

note that $\lambda((x-B)\cap[0,1]) = \lambda(A)$ for all $x \in \mathbf{R}$. Let

$$\eta = (1_B - 1_{B^c})(W_1):$$

clearly $\eta \in L^p(\Omega, \mathbf{F}, P)$ for any $p \geq 1$. It is easy to see that the corresponding solution X of (1) and thus

$$U_. = \int_0^{\cdot} X_s dW_s = X_. - \eta$$

is not a semimartingale in any filtration. Indeed, choose an arbitrary $\omega \in \Omega$. Then

$$c = \inf_{t\in[0,1]} Y_t(\omega) > 0.$$

Now $Z(\omega)$ has oscillatory discontinuities of size 2 at every point in the uncountable set

$$(\omega_1 - B)\cap[0,1],$$

which, by the choice of A, has positive Lebesgue measure. Consequently, $X(\omega)$ has jumps of size $\geq 2c$ on this set. So X is not cadlag, and further no modification of X is cadlag: it follows that no modification of X is a semimartingale.

Let us now examine the occupation density of the process X. The behaviour of the Skorohod integral process may be described by "wildly jumping between two branches of semimartingale type behaviour". The semimartingale branches, the processes Y and $-Y$, each have quadratic variation equivalent to Lebesgue measure on the unit interval, so it is reasonable to measure occupation time by Lebesgue measure. Hence, for $\omega \in \Omega, F \in \mathbf{B}(\mathbf{R})$ let

$$\mu(F,\omega) = \int_0^1 1_F(X_s(\omega))ds.$$

The two branches do not interfere, since in one of them X is positive, and in the other one negative. We may therefore define separate occupation times

$$\begin{aligned}\mu_1(F,\omega) &= \int_0^1 1_{F\cap]0,\infty[}(X_s(\omega))ds \\ &= \int_{(\omega_1-B)\cap[0,1]} 1_{F\cap]0,\infty[}(Y_s(\omega))ds,\end{aligned}$$

$$\begin{aligned}\mu_2(F,\omega) &= \int_0^1 1_{F\cap]-\infty,0[}(X_s(\omega))ds \\ &= \int_{(\omega_1-B^c)\cap[0,1]} 1_{F\cap]-\infty,0[}(-Y_s(\omega))ds,\end{aligned}$$

where $F \in \mathbf{B}(\mathbf{R})$. Then obviously

$$\mu = \mu_1 + \mu_2.$$

Moreover, since Y is a semimartingale with quadratic variation equivalent with respect to Lebesgue measure, μ_1 and μ_2 are absolutely continuous with respect to λ. Hence so is μ and we have established that X possesses an occupation density which, in addition, is continuous. Since $U = X - \eta$, so does U.

For the the remaining two examples it will be helpful to use an independent auxiliary Gaussian process. So let $(\Xi, \mathbf{G}, Q)$ be another probability space, carrying a Gaussian process $(B_t, t \in \mathbf{R})$ satisfying $E(B_t) = 0$, and with covariance function

$$E(B_sB_t) = \sigma(s,t), \quad \text{where } \sigma(t,t) \leq K(1\vee \mid t \mid)$$

for some $K < \infty$.

For each $\xi \in \Xi$ let $\eta_\xi(\omega) = B_{\omega_1}(\xi)$, and consider the SDE (1) in the probability space $(\Omega, \mathbf{F}, P)$. We indicate expectation with respect to P, Q, by E_P, E_Q respectively. We must first check that η_ξ satisfies the integrability condition of Proposition 1: however

$$\begin{aligned}E_Q E_P \exp(\eta_\xi) &= E_P(E_Q(\exp(B_{\omega_1})) \\ &= E_P(\exp(\tfrac{1}{2}\sigma(W_1, W_1)) \\ &\leq E_P(\exp(\tfrac{1}{2}K(1+ \mid W_1 \mid))) < \infty.\end{aligned}$$

Thus if $\Xi_0 = \{\xi : E_P \exp(\eta_\xi) < \infty\}$, then $Q(\Xi_0) = 1$, and for each $\xi \in \Xi_0$ we may apply Proposition 1 to deduce that the solution to (1) is

$$X_t(\omega,\xi) = Y_t(\omega)B_{\omega_1-t}(\xi), \quad 0 \leq t \leq 1.$$

Example 3. Let $\sigma(s,t) = g(t-s)$, where g satisfies, for some $\delta > 0$,

$$g(0) - g(t) \geq (\log \mid t \mid^{-1})^{-1} \text{ for } \mid t \mid < \delta. \tag{5}$$

Then Berman, [1], p. 298, proves that B has sample paths of unbounded variation on every interval, and also has an analytic local time. As Y is continuous and strictly positive, the paths of X have similar properties to those of B, and are unbounded on every interval.

We now show that in addition X has ($P \times Q$ a.s.) an analytic local time. Fix $\omega \in \Omega$, and consider the process $X_.(\omega, .)$ under the law Q: this is a mean zero Gaussian process with covariance given by

$$\begin{aligned} \sigma_\omega(s,t) &= E_Q X_s(\omega,.)X_t(\omega,.) \\ &= Y_s Y_t E_Q(B_{\omega_1 - t} B_{\omega_1 - s}) = Y_s Y_t g(t-s). \end{aligned}$$

Berman's proof extends easily to non-stationary Gaussian processes, and shows that such a process has analytic local time provided its covariance function $\tau(s,t)$ satisfies

$$\tau(t,t) - 2\tau(s,t) + \tau(s,s) \geq c(\log \mid t-s \mid^{-1})^{-1} \tag{6}$$

for some $c > 0$. However we have

$$\begin{aligned} & Y_t^2 g(0) - 2Y_t Y_s g(t-s) + Y_s^2 g(0) \\ \geq\ & 2Y_t Y_s (g(0) - g(t-s)) + (Y_t - Y_s)^2 g(0) \\ \geq\ & 2Y_t Y_s (g(0) - g(t-s)) \\ \geq\ & 2Y_t Y_s (\log \mid t-s \mid^{-1})^{-1}. \end{aligned}$$

As Y is strictly positive, it follows from (5) that σ_ω satisfies (6) for ω fixed, and therefore by a Fubini type argument that X (and so U) has an analytic local time.

One might be tempted to attribute the strange behaviour of U in Examples 2 and 3 to the fact that it is able to jump. This is wrong. We will now exhibit another family of Skorohod integral processes which are continuous, but which fail to have other properties expected by a well-behaved Itô integral process. It is shown in Nualart, Pardoux [9], p. 558, that if u is "smooth enough", $\int_0^\cdot u_s dW_s$ has a quadratic variation which is given by $\int_0^\cdot u_s^2 ds$. In our next example, the quadratic variation of the integral processes is essentially bigger.

Example 4. We continue with the notation set out above, but now take

$$\begin{aligned} \sigma(s,t) &= \mid s \mid \wedge \mid t \mid \qquad \text{if } \operatorname{sgn}(s) = \operatorname{sgn}(t) \\ &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

Thus $(B_t)_{t\geq 0}$ and $(B_{-t})_{t \geq 0}$ are independent Wiener processes. As before we take $\eta_\xi(\omega) = B_{\omega_1}(\xi)$, so if

$$X_t(\omega, \xi) = Y_t(\omega) B_{\omega_1 - t}(\xi)$$

then for $\xi \in \Xi_0$ the process $X_.(., \xi)$ is a solution of (1).

We wish to study the quadratic variation of X for fixed ξ: to do so we will first study the process $X_t(.,.)$ on the product space $(\Omega \times \Xi, \mathbf{F} \times \mathbf{G}, P \times Q)$. (We extend random variables defined on the spaces $(\Omega, \mathbf{F}, P)$ and $(\Xi, \mathbf{G}, Q)$ to the product space in the obvious way). For simplicity, we will work on the whole unit interval and only remark that this could be done on any interval in $[0,1]$. It will be helpful to change the multiplicative decomposition of X in order to be able to argue by independence. For $t \geq 0$ let

$$\begin{aligned} Y'_t &= \exp(W_t - tW_1), \\ Y''_t &= \exp(t(W_1 - \tfrac{1}{2})), \\ Z'_t &= Y''_t B_{\omega_1 - t}. \end{aligned}$$

Then

$$X = Y'Z'$$

and, moreover, (as $(W_t - tW_1, t \in [0,1])$ is a Brownian bridge independent of W_1), Y' and Z' are independent.

Now let $(\mathbf{J}_n)_{n\in\mathbf{N}}$ be any 0-sequence of partitions of $[0,1]$. For $n \in \mathbf{N}$ we have

$$\begin{aligned} \sum_{J\in\mathbf{J}_n} (\Delta_J X)^2 &= \sum_{J\in\mathbf{J}_n} (\Delta_J Z')^2 Y'^2_{t_J} + \sum_{J\in\mathbf{J}_n} (\Delta_J Y')^2 Z'^2_{s_J} \\ &\quad + 2 \sum_{J\in\mathbf{J}_n} Y'_{t_J} Z'_{s_J} \Delta_J Y' \Delta_J Z'. \end{aligned} \tag{7}$$

Let us investigate the limits of the sequences on the right hand side of (7) separately. Since W_1 and B are independent, the quadratic variation of the process Z' is given by $\int_0^1 Y''^2_t dt$. Hence, using that Y' is in addition independent of Z', we get

$$\sum_{J\in\mathbf{J}_n} (\Delta_J Z')^2 Y'^2_{t_J} \to \int_0^1 Y'^2_t Y''^2_t dt = \int_0^1 Y_t^2 dt. \tag{8}$$

in $L^2(\Omega \times \Xi, \mathbf{F} \times \mathbf{G}, P \times Q)$. Next, observe that Y' is a semimartingale with quadratic variation

$$\int_0^1 Y'^2_t dt.$$

A similar argument as for the first term now yields that for the second one we get

$$\sum_{J\in\mathbf{J}_n} (\Delta_J Y')^2 Z'^2_{s_J} \to \int_0^1 Z'^2_t Y'^2_t dt = \int_0^1 X_t^2 dt. \tag{9}$$

Again independence of Y' and Z', as well as the fact that Z' is continuous and Y' is a semimartingale forces the last term on the right hand side of (7) to converge to 0 in $L^2(\Omega \times \Xi, \mathbf{F} \times \mathbf{G}, P \times Q)$ as $n \to \infty$. Hence, (8) and (9) give

$$\sum_{J\in\mathbf{J}_n} (\Delta_J X)^2 \to \int_0^1 (X_t^2 + Y_t^2) dt$$

in $L^2(\Omega \times \Xi, \mathbf{F} \times \mathbf{G}, P \times Q)$ for any 0-sequence of partitions of $[0,1]$. Thus

$$\sum_{J\in\mathbf{J}_n} (\Delta_J U)^2 \to \int_0^1 (X_t^2 + Y_t^2) dt$$

as $n \to \infty$ in $L^2(\Omega \times \Xi, \mathbf{F} \times \mathbf{G}, P \times Q)$. In particular, we may fix $\xi \in \Xi$ on a set of Q measure 1, and find a subsequence $(\mathbf{K}_n)_{n\in\mathbf{N}}$ of $(\mathbf{J}_n)_{n\in\mathbf{N}}$ such that

$$\sum_{J\in\mathbf{K}_n} (\Delta_J U)^2(.,\xi) \to \int_0^1 (X_t^2(.,\xi) + Y_t^2)dt \tag{10}$$

in $L^2(\Omega, \mathbf{F}, P)$. So even though X is continuous the limit in (10) is not equal to the quantity one obtains for Skorohod integral processes with smooth integrands (cf. Nualart, Pardoux [9], p. 558), namely

$$\int_0^1 X_t^2(.,\xi)dt.$$

Remarks. 1. It is clear that many other examples of this kind are possible. In particular, by choosing a suitable stationary Gaussian process B_t with a covariance function $\sigma(s,t) = g(t-s)$, with $(g(0) - g(t)) \sim \mid t \mid^p$ as $\mid t \mid \to 0$ for some $0 < p < 1$, we obtain a continuous Skorohod integral process X with infinite quadratic variation but with finite q^{-1} - th order variation for each $q > p$. (See [8]).

2. It is not possible to obtain from Proposition 1 examples of Skorohod integral processes violating the local property:

$$u_t = 0 \quad \lambda \times P\text{-a.e. on } [0,1] \times A \quad \text{implies } \int_0^1 u_t dW_t = 0 \text{ on } A, \tag{11}$$

and indeed Nualart and Pardoux [9] conjecture that (11) holds in general. However, we can find processes which violate the local property in the following weaker sense. Take

$$\eta = 1_{\{W_1 < 0\}}, \quad T = 0 \vee (W_1 \wedge 1), \quad A = \{0 \le W_1 \le 1\}.$$

Then T is a nontrivial random time, but not a stopping time, A a nontrivial subset of Ω and we have

$$u_t = X_t = 1_{\{W_1 - t \le 0\}} Y_t = 0$$

on the set $[0, T[\times A$, whereas

$$X_T = \int_0^T u_s dW_s = Y_T > 0$$

on this set. Hence the integral of the process u jumps to a nonzero value at T, even though u vanishes before T.

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