YUKUANG CHIU

From an example of Lévy’s

Séminaire de probabilités (Strasbourg), tome 29 (1995), p. 162-165

<http://www.numdam.org/item?id=SPS_1995__29__162_0>
The motive of this paper is to prove completely an assertion of P. Lévy [3], who claimed that for each positive integer \( n \), there exists a polynomial \( F_n \) of degree \( n \) such that the Wiener integral with respect to Brownian motion \( \{B(u); 0 \leq u\} \)

\[
X(t) = \int_0^t F_n \left( \frac{u}{t} \right) dB(u)
\]
is again a Brownian motion and

\[
B(X, t) \subset B(B, t).
\]

Here \( B(X, t) \) is the \( \sigma \)-field generated by \( \{X(s); s \leq t\} \).

Over the last few years, the non-canonical representation of Brownian motion of this kind related has been of interest to many authors, in particular Th. Jeulin & M. Yor [4, 5, 6, 7] and M. Hitsuda [2].

Let \( X_0(t) = B(t), (t \geq 0) \) be a standard Brownian motion. In this paper, our precise purpose is to construct a sequence of Brownian motion \( \{X_n(t); t \geq 0\} (n \geq 0) \) such that \( X_n(t) \) can be represented as a Wiener integral

\[
X_n(t) = \int_0^t F_n \left( \frac{u}{t} \right) dB(u), \quad (n \geq 1).
\]

Here \( F_n(t) \) is a polynomial of degree \( n \) in \( t \). And if \( M(X_n; t) \) is a linear span of \( \{X_n(u); 0 \leq u \leq t\} \) in \( L^2(\Omega, P) \), then for all \( t > 0 \),

\[
M(X_{n+1}; t) \not\subset M(X_n; t), \quad (n \geq 0),
\]

and further, for all \( t > 0 \) and \( n \geq 1 \),

\[
\int_0^t F_n \left( \frac{u}{t} \right) u^k du = 0, \quad k = 1, 2, \ldots, n.
\]

Let \( F_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \). We know by calculating the covariance that \( X_n(t) \) is a Brownian motion if and only if the coefficients of \( F_n(t) \) satisfy the following equations

\[
\begin{align*}
\frac{a_0 a_0}{1} + \frac{a_0 a_1}{2} + \cdots + \frac{a_0 a_n}{n+1} &= 1, \\
\frac{a_n a_0}{2} + \frac{a_n a_1}{3} + \cdots + \frac{a_n a_n}{n+2} &= 0, \\
&\quad \vdots \\
\frac{a_n a_{n+1}}{n+1} + \frac{a_n a_{n+2}}{n+2} + \cdots + \frac{a_n a_{2n+1}}{2n+1} &= 0.
\end{align*}
\]

In the simplest case

\[
\begin{cases}
a_n = \frac{2n+1}{n}, a_0 = -\frac{n+1}{n}, \\
a_1 = a_2 = \cdots = a_{n-1} = 0.
\end{cases}
\]
is a solution of equation (3).
Theorem 1  If \( P_n(t) = \frac{2n+1}{n} t^n - \frac{n+1}{n} \) and if \( F_n(t) \) are defined by the following recursive formula

\[
\begin{align*}
F_1(t) &= P_1(t) \\
F_n\left(\frac{u}{t}\right) &= F_{n-1}\left(\frac{u}{t}\right) - \int_0^t F_{n-1}\left(\frac{\tau}{t}\right) \frac{\partial}{\partial \tau} P_n\left(\frac{\tau}{t}\right) d\tau, \quad (n \geq 2),
\end{align*}
\]

then \( F_n(t) \) satisfies (2), the coefficients of \( F_n(t) \) are given by

\[
a_k = (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n}, \quad k = 0, 1, \ldots, n
\]

and

\[
X_n(t) := \int_0^t F_n\left(\frac{u}{t}\right) dB(u), \quad (n \geq 1)
\]

are Brownian motions satisfying condition (1). Further, \( X_n(t) \) and \( X_{n+1}(t) \) are related by

\[
X_{n+1}(t) = \int_0^t P_{n+1}\left(\frac{u}{t}\right) dX_n(u), \quad (n \geq 0).
\]

In order to prove the theorem, we prepare the following lemma.

Lemma 1  If \( s < n \), we have

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{s} = 0.
\]

To prove this, we note

\[
\frac{1}{s!} \left( \frac{d}{dx} \right)^s (1 + x)^n x^n = \frac{1}{s!} \left( \frac{d}{dx} \right)^s \left( \sum_{k=0}^{n} \binom{n}{k} x^{n+k} \right) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{s} x^{n+k-s}.
\]

The result follows by letting \( x = -1 \).

The validity of the coefficients of \( F_n(t) \) can be established by mathematical induction. The assertion is trivial for \( n = 1 \). Suppose the assertion holds for \( n \). Now using lemma 1 and then noting

\[
\binom{n}{k} \binom{n+1+k}{n+1} = \frac{2(n+1)+1}{n+1-k},
\]

we see that

\[
F_{n+1}\left(\frac{u}{t}\right) = F_n\left(\frac{u}{t}\right) - \int_0^t F_n\left(\frac{\tau}{t}\right) \frac{\partial}{\partial \tau} P_{n+1}\left(\frac{\tau}{t}\right) d\tau
\]

\[
= \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n} \left( 1 - \frac{2(n+1)+1}{n+1-k} \right) \binom{u}{t}^k
\]

\[
+ \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+1+k}{n} \frac{2(n+1)+1}{n+1-k} \binom{u}{t}^{n+1}
\]
= \sum_{k=0}^{n} (-1)^{n+1+k} \binom{n+1}{k} \binom{n+2+k}{n+1} \left(\frac{u}{t}\right)^k + (2n+3) \binom{n}{n+1} \left(\frac{u}{t}\right)^{n+1}
= \sum_{k=0}^{n+1} (-1)^{n+1+k} \binom{n+1}{k} \binom{n+2+k}{n+1} \left(\frac{u}{t}\right)^k .

which shows the assertion holds for coefficients of $F_{n+1}(t)$.

We next show (2). By the recursive formula of $F_n$, we obtain

$$\int_0^t F_n \left(\frac{u}{t}\right) u^k du = \int_0^t F_{n-1} \left(\frac{u}{t}\right) u^k du - \int_0^t \int_0^t F_{n-1} \left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} F_n \left(\frac{\tau}{t}\right) u^k d\tau du$$

$$= \int_0^t F_{n-1} \left(\frac{u}{t}\right) u^k du - \int_0^t \int_0^{\tau} \frac{\partial}{\partial \tau} F_{n-1} \left(\frac{\tau}{t}\right) \int_0^\tau F_{n-1} \left(\frac{u}{\tau}\right) u^k d\tau du$$

This equals zero if $k < n$ by induction; and when $k = n$, this becomes

$$\int_0^t F_{n-1} \left(\frac{u}{t}\right) \left[ u^n \right]_0^\tau d\tau = \int_0^t P_n \left(\frac{\tau}{t}\right) \left[ u^n \right]_0^\tau d\tau = 0 .$$

which is what we needed to prove.

Again we easily verify, by mathematical induction, that

$$\int_0^1 F_n(u) du = \frac{(-1)^n}{n+1} .$$

Thus we have proved, in combination with the previous equation, that the coefficients of $F_n$ are another solution to equation (3).

Now if we write

$$X_n(t) = \int_0^t F_n \left(\frac{u}{t}\right) dB(u) ,$$

then by the above argument, $X_n(t)$ is again a Brownian motion. The differential of $X_n(t)$, by Itô’s formula [1], is seen to be

$$dX_n(u) = dB(u) + \int_0^u \frac{\partial}{\partial u} F_n \left(\frac{\tau}{u}\right) dB(\tau) du .$$

Therefore

$$\int_0^t P_{n+1} \left(\frac{u}{t}\right) dX_n(u)$$

$$= \int_0^t P_{n+1} \left(\frac{u}{t}\right) dB(u) + \int_0^t \left\{ P_{n+1} \left(\frac{u}{t}\right) \int_0^u \frac{\partial}{\partial u} F_n \left(\frac{\tau}{u}\right) dB(\tau) \right\} du$$

$$= \int_0^t P_{n+1} \left(\frac{u}{t}\right) dB(u) + \int_0^t dB(\tau) \left\{ F_n \left(\frac{\tau}{t}\right) P_{n+1} \left(\frac{u}{t}\right) \right\} \left|_u^\tau \right. - \int_\tau^t F_n \left(\frac{\tau}{u}\right) \frac{\partial}{\partial u} P_{n+1} \left(\frac{u}{t}\right) du \right\}$$

$$= \int_0^t F_n \left(\frac{u}{t}\right) dB(u) - \int_0^t \left\{ \int_0^\tau F_n \left(\frac{u}{\tau}\right) \frac{\partial}{\partial \tau} P_{n+1} \left(\frac{\tau}{t}\right) d\tau \right\} dB(u)$$

$$= X_{n+1}(t) .$$
This establishes (4).

To show (1), let us fix $t_0 > 0$ and let $z = \int_0^{t_0} u^{n+1} dX_n(u)$. Now $z \in M(X_n; t)$ and note that for all $t$ such that $0 < t \leq t_0$,

$$E[X_{n+1}(t) \cdot z] = \int_0^t P_{n+1} \left( \frac{u}{t} \right) u^{n+1} du = 0.$$

This verifies (1). The proof of theorem is thus completed.

**Remark 1** This construction was suggested by P. Lévy in his book [3] and $F_1(t)$ was given there.

**Remark 2** Although we have (1), for all $n > 0$, we notice,

(5) \[ M(B; \infty) = M(X_n; \infty). \]

This equation has the following interpretation. For each finite time $t$, as we have already seen, $B(X_n; t)$ contains less information than $B(B; t)$. Nevertheless, $B(X_n; t)$ will “catch up” with $B(B; t)$ by increasing time to infinity.

**Acknowledgements**

I sincerely thank M. Yor for his suggestions and preprints which have greatly enriched my knowledge on this topic.

**References**


