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AN ADDENDUM TO A REMARK ON SLUTSKY'S THEOREM

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In [D], I gave a counter-example to the following statement. If X_n is a sequence of measurable functions taking values in a Polish space E, and converging almost surely to a measurable function X, then for every Borel function h defined on E, $h(X_n)$ converges a.s. to $h(X)$. In the case of convergence in probability, the statement holds provided the image measures (or distributions) form a relatively weakly compact sequence (Slutsky's theorem). After the paper was printed, I discovered in the paper by Dellacherie, Feyel and Mokobodzki, [DFM], that the counterexample was already known. In fact there the authors show:

Theorem. *If $(X_n)_{n\geq 1}$ is a sequence of measurable mappings and if X is a measurable function, all defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in the Polish Space $(E, \mathcal{T})$, then the following are equivalent*

(1) *for each real valued Borel function $h: E \to \mathbb{R}$ we have that $h(X_n)$ tends to $h(X)$ almost surely*
(2) *X_n tends to X almost surely in a stationary way, i.e. for almost every $\omega \in \Omega$ there is n_0 (depending on ω) such that for all $n \geq n_0$ we have $X_n(\omega) = X(\omega)$*

The aim of this addendum is not only to give credit to Dellacherie, Feyel and Mokobodzki but also to give some extra background information. As an example we will see that the Slutsky result can be stated in a different way. To fix notation, let $(E, \mathcal{T}, \mathcal{E})$ be a Polish space equipped with its topology $\mathcal{T}$ and its Borel structure $\mathcal{E}$. Let μ_n be a sequence of probability measures on $(E, \mathcal{E})$. We recall that the sequence μ_n tends weak* to the probability μ if for every $\mathcal{T}$-continuous bounded function f on E we have $\int f\, d\mu_n$ tends to $\int f\, d\mu$. We say that μ_n tends to μ weakly if for every Borel function f on E we have that $\int f\, d\mu_n$ tends to $\int f\, d\mu$. In this case we have that μ_n tends to μ weakly in the sense of the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$. By a result of Grothendieck, to have weak convergence, it is sufficient to ask that for every open set $O \in \mathcal{T}$ we have that

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$\mu_n(O)$ tends to $\mu(O)$. But for Polish spaces there are many topologies that give the same Borel structure, in fact every finer topology $\mathcal{T}'$, which is still Polish, gives, by Blackwell's theorem, the same Borel sets $\mathcal{E}$ on E. The relation between the weak* and the weak convergence becomes clearer thanks to the following result, see [S] page 91–93.

Theorem. *If A_n is a sequence of Borel sets in E, then there exists a finer topology $\mathcal{T}'$ on E, still Polish and such that each A_n is an open-closed set in $\mathcal{T}'$.*

Corollary. *If f_n is a sequence of Borel functions $f_n: E \to \mathbb{R}$, then there is a finer, still Polish, topology $\mathcal{T}'$ on E such that each f_n is continuous.*

This result gives us the following theorem.

Theorem. *For a sequence of probability measures on $(E, \mathcal{E})$, the following two properties are equivalent*

(1) *the sequence μ_n converges weakly to μ,*
(2) *for every Polish topology $\mathcal{T}'$, finer than $\mathcal{T}$, the sequence μ_n converges weak* to μ.*

The result of Dellacherie, Feyel and Mokobodzki can now be rephrased as

Theorem. *For a sequence of measurable mappings defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Polish space $(E, \mathcal{T})$, the following are equivalent*

(1) *the sequence X_n tends to X is a stationary way (as above)*
(2) *for every Polish topology $\mathcal{T}'$ finer than $\mathcal{T}$, we have that X_n tends to X almost surely.*

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