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A teachers’ note on no-arbitrage criteria

Yuri Kabanov and Christophe Stricker
UMR 6623, Laboratoire de Mathématiques,
Université de Franche-Comté
16 Route de Gray, F-25030 Besancon Cedex, FRANCE

Abstract

We give a new proof of the classical Dalang-Morton-Willinger theorem.

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1. Introduction. The Dalang-Morton-Willinger theorem asserts, for a discrete-time model of security market, that there is no arbitrage if and only if the price process is a martingale with respect to an equivalent probability measure. This remarkable result sometimes is referred to as the First Fundamental Theorem of mathematical finance, [9]. A simple statement suggests a simple proof and many attempts were made to find such one, cf. [1], [10], [8], [6], [7], [4], [2]. Various aspects were investigated in details and the theorem was augmented by additional equivalent conditions revealing its profound difference from the Harrison-Pliska theorem [3] which is the same criterion but for the model with finite Ω. Unfortunately, all existing proofs are too cumbersome for lecture courses. This note is a new attempt to provide a concise proof which uses only results from the standard syllabus.

2. No-arbitrage criteria. Let (Ω, F, P) be a probability space equipped with a finite discrete-time filtration (Ft), t = 0, ..., T, FT = F, and let S = (St) be an adapted d-dimensional process. Let RT := {ξ : ξ = H · ST, H ∈ P} where P is the set of all predictable d-dimensional processes (i.e. Ht is Ft-1-measurable) and

\[ H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t, \quad \Delta S_t := S_t - S_{t-1}. \]

Put AT := RT − L+; \( \bar{A}_T \) is the closure of AT in probability, \( L^+_T \) is the set of non-negative random variables.

Theorem 1 The following conditions are equivalent:

(a) \( A_T \cap L^+_T = \{0\} \);
(b) \( A_T \cap L^+_T = \{0\} \) and \( A_T = \bar{A}_T \);
(c) \( \bar{A}_T \cap L^+_T = \{0\} \);
(d) there is a probability \( \tilde{P} \sim P \) with \( d\tilde{P}/dP \in L^\infty \) such that S is a \( \tilde{P} \)-martingale.
In the context of mathematical finance this model corresponds to the case where the "numéraire" is a traded security, $S$ describes the evolution of prices of risky assets, and $H \cdot S_T$ is the terminal value of a self-financing portfolio. Condition (a) is interpreted as the absence of arbitrage; it can be written in the obviously equivalent form $R_T \cap L^0 = \{0\}$ (or $H \cdot S_T \geq 0 \Rightarrow H \cdot S_T = 0$). We include in the formulation only the basic equivalences: various other ones known in the literature can easily be deduced from the listed above.

If $\Omega$ is finite then $A_T$ is closed being a polyhedral cone in a finite-dimensional space. For infinite $\Omega$ the set $A_1$ may be not closed, see an example in [8], while $R_T$ is always closed (this can be checked in a similar way as the implication $(a) \Rightarrow (b)$ in the proof below).

3. Auxiliary results. The following observation is elementary.

**Lemma 2** Let $\eta^n \in L^0(\mathbb{R}^d)$ be such that $\eta := \liminf n \to \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbb{R}^d)$ such that for all $\omega$ the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.

**Proof.** Let $\tau_0 := 0$. Define the random variables $\tau_k := \inf\{n > \tau_{k-1} : |\eta^n| - \eta| \leq k^{-1}\}$. Then $\tilde{\eta}_0 := \eta^{-k}$ is in $L^0(\mathbb{R}^d)$ and $\sup_k |\tilde{\eta}_0^k| < \infty$. Working further with the sequence of $\tilde{\eta}_0^k$ we construct, applying the above procedure to the first component, a sequence of $\tilde{\eta}_1^k$ with convergent first component and such that for all $\omega$ the sequence of $\tilde{\eta}_1^k(\omega)$ is a subsequence of the sequence of $\tilde{\eta}_0^k(\omega)$. Passing on each step to the newly created sequence of random variables and to the next component we arrive to a sequence with the desired properties. \hfill \Box

**Remark.** The above claim can be formulated as follows: there exists an increasing sequence of integer-valued random variables $\sigma_k$ such that $\eta^{\sigma_k}$ converges a.s.

For the sake of completeness, we recall the proof of the well-known result due to Kreps and Yan, [5], [11].

**Lemma 3** Let $K \supseteq -L^1_+$ be a closed convex cone in $L^1$ such that $K \cap L^1 = \{0\}$. Then there is a probability $P \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $E\tilde{\xi} \leq 0$ for all $\xi \in K$.

**Proof.** By the Hahn–Banach theorem for any $x \in L^1_+$, $x \neq 0$, there is $z_x \in L^\infty$ such that $Ez_x \xi < Ez_x x$ for all $\xi \in K$. It follows, since $K$ is a cone, that $Ez_x \xi \leq 0$ for all $\xi \in K$. Since $K$ contains all negative random variables, $z_x \geq 0$ and $Ez_x x > 0$.

Normalizing, we assume that $z_x \leq 1$. The Halmos–Savage theorem asserts that the family of measures $\{z_x P\}$ contains a countable equivalent subfamily $\{z_{x_i} P, \ x_i \in \mathbb{N}\}$ (i.e., both vanish on the same sets). Put $\rho := \sum 2^{-i}z_{x_i}$ and $\tilde{x} := I_{\{\rho=0\}}$. Then $Ez_{x_i} \tilde{x} = 0$ for all $i$ and, hence, $Ez_{x_i} \tilde{x} = 0$ for all $x \in L^1_+$. Thus, $\tilde{x} = 0$ (otherwise we would have $Ez_{x_i} \tilde{x} > 0$) and the measure $\tilde{P} := c\rho P$ with $c = 1/E\rho$ meets the requirements. \hfill \Box

**Remark.** The Halmos–Savage theorem is simple and the reference can be replaced by its proof which is as follows. Consider the larger family $\{y P\}$ where $y$ are convex combinations of $z_x$. Then $\esssup_i I_{\{y > 0\}}$ can be attained on an increasing sequence of $I_{\{y_k > 0\}}$. Clearly, $\{y_k P\}$ is a countable equivalent subfamily of $\{y P\}$ and it is a convex envelope of a countable family $\{z_{x_i} P\}$ we are looking for.

4. Proof of Theorem 1. $(a) \Rightarrow (b)$ To show that $A_T$ is closed we proceed by induction. Let $T = 1$. Suppose that $H^n_1 \Delta S_1 - r^n \to \xi$ a.s. where $H^n_1$ is $\mathcal{F}_0$-measurable.
and \( r^n \in L_+^0 \). It is sufficient to find \( \mathcal{F}_0 \)-measurable random variables \( \tilde{H}_1^k \) which are convergent a.s. and \( \tilde{r}^k \in L_+^0 \) such that \( \tilde{H}_1^k \Delta S_1 - \tilde{r}^k \rightarrow \zeta \) a.s. convergent.

Let \( \Omega_1 \in \mathcal{F}_0 \) form a finite partition of \( \Omega \). Obviously, we may argue on each \( \Omega_i \) separately as on an autonomous measure space (considering the restrictions of random variables and traces of \( \sigma \)-algebras).

Let \( H_1 := \lim \inf |H_1^n| \). On the set \( \Omega_1 := \{ H_1 < \infty \} \) we can take, using Lemma 2, \( \mathcal{F}_0 \)-measurable \( \tilde{H}_1^k \) such that \( \tilde{H}_1^k(\omega) \) is a convergent subsequence of \( H_1^n(\omega) \) for every \( \omega \); \( \tilde{r}^k \) are defined correspondingly. Thus, if \( \Omega_1 \) is of full measure, the goal is achieved.

On \( \Omega_2 := \{ H_1 = \infty \} \) we put \( G_1^n := H_1^n / |H_1^n| \) and \( h^n_1 := r^n_1 / |H_1^n| \) and observe that \( G_1^n \Delta S_1 - h^n_1 \rightarrow 0 \) a.s. By Lemma 2 we find \( \mathcal{F}_0 \)-measurable \( \tilde{G}_1^k \) such that \( \tilde{G}_1^k(\omega) \) is a convergent subsequence of \( G_1^n(\omega) \) for every \( \omega \). Denoting the limit by \( \tilde{G}_1 \), we obtain that \( \tilde{G}_1 \Delta S_1 = h_1 \) where \( h_1 \) is non-negative, hence, in virtue of (a), \( \tilde{G}_1 \Delta S_1 = 0 \).

As \( \tilde{G}_1(\omega) \neq 0 \), there exists a partition of \( \Omega_2 \) into \( d \) disjoint subsets \( \Omega_2 \in \mathcal{F}_0 \) such that \( \tilde{G}_1 \neq 0 \) on \( \Omega_2 \). Define \( \tilde{H}_1^n := H_1^n - \beta^n \tilde{G}_1 \) where \( \beta^n := H_1^n / \tilde{G}_1^n \) on \( \Omega_2 \). Then \( \tilde{H}_1^n \Delta S_1 = H_1^n \Delta S_1 \) on \( \Omega_2 \). We repeat the entire procedure on each \( \Omega_2 \) with the sequence \( \tilde{H}_1^n \) knowing that \( \tilde{H}_1^n = 0 \) for all \( n \). Apparently, after a finite number of steps we construct the desired sequence.

Let the claim be true for \( T - 1 \) and let \( \sum_{t=1}^T H_t^n \Delta S_t - r^n \rightarrow \zeta \) a.s. where \( H_t^n \) are \( \mathcal{F}_t \)-measurable and \( r^n \in L_+^0 \). By the same arguments based on the elimination of non-zero components of the sequence \( H_t^n \) and using the induction hypothesis we replace \( H_t^n \) and \( r^n \) by \( \tilde{H}_t^k \) and \( \tilde{r}^k \) such that \( \tilde{H}_t^k \) converges a.s. This means that the problem is reduced to the one with \( T - 1 \) steps.

(b) \( \Rightarrow \) (c). Trivial.

c) \( \Rightarrow \) (d). Notice that for any random variable \( \eta \) there is an equivalent probability \( P' \) with bounded density such that \( \eta \in L^1(P') \) (e.g., one can take \( P' = C e^{-|\eta|} P \)). Property (c) (as well as (a) and (b)) is invariant under equivalent change of probability. This consideration allows us to assume that all \( S_t \) are integrable. The convex set \( A_t^1 := A_t \cap L^1 \) is closed in \( L^1 \). Since \( A_t^1 \cap L_+^0 = \{ 0 \} \), Lemma 3 ensures the existence of \( \tilde{P} \sim P \) with bounded density and such that \( \tilde{E} \xi \leq 0 \) for all \( \xi \in A_t^1 \), in particular, for \( \xi = \pm H_t \Delta S_t \) where \( H_t \) is bounded and \( \mathcal{F}_{t-1} \)-measurable. Thus, \( \tilde{E}(\Delta S_t | \mathcal{F}_{t-1}) = 0 \).

(d) \( \Rightarrow \) (a). Let \( \xi \in A_T \cap L_+^0 \), i.e. \( 0 \leq \xi \leq H \cdot S_T \). As \( \tilde{E}(H_t \Delta S_t | \mathcal{F}_{t-1}) = 0 \), we obtain by conditioning that \( \tilde{E} H \cdot S_T = 0 \). Thus, \( \xi = 0 \). \( \square \)

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References


