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DUALITY AND QUASI–CONTINUITY

FOR SUPERMARTINGALES

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Abstract: Considering the ordered convex cone $\mathcal{S}$ of all positive right continuous supermartingales, we give a complete description of its dual. Also we study quasi-boundedness, quasi-continuity and subtractivity in $\mathcal{S}$, proving that: the universally quasi-bounded potentials are exactly the potentials of class (D), the quasi-continuous potentials are exactly the regular potentials of class (D), and the subtractible elements are exactly the local martingales.

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0. Introduction

Given a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, the convex cone of supermartingales arises. Because of their many applications, the right continuous (and positive) supermartingales are already well studied in the theory of general processes (especially under the usual conditions). In [1], Cornea and Licea considered the positive (without any integrability condition) right continuous supermartingales as excessive elements with respect to a given resolvent of kernels on the σ-lattice cone of positive optional processes. They show that the convex cone $\mathcal{S}$ of all positive, right continuous supermartingales is an H-cone (this also follows from [2] and our section 3) and they obtain several interesting results concerning duality.

The aim of this paper is to study the ordered convex cone $\mathcal{S}$. In order to eliminate some technical complications, we restrict ourselves from start to the supermartingales (X_t) such that each random variable X_t is finite and integrable, that is, the familiar supermartingales considered by the theory of martingales, which we use as our main tool. In fact, we intensely use the results from the general theory of processes as exposed in [2]. More precisely, we first give a result concerning duality for a large class of optional processes (Theorem 1.1), from which we then obtain the complete description of the dual of $\mathcal{S}$ (Theorem 1.2) and even more. Further, we describe the universally quasi–bounded potentials (Prop. 2.1), the quasi-continuous potentials (Theorem 2.2), the subtractible elements of $\mathcal{S}$ (Prop. 2.4).

1. Throughout this paper $(\Omega, \mathcal{F}, P)$ is a probability space endowed with a filtration $(\mathcal{F}_t)$ such that $\mathcal{F}_0$ contains all negligible sets and $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$. We denote by $\mathcal{S}$ the cone of all right continuous, positive supermartingales. In fact, the elements of $\mathcal{S}$ are classes of real (and optional) processes with respect to the equivalence relation associated with the preorder relation on the set of all real processes: $X \leq Y$ iff the set $\{(t,\omega) : X_t(\omega) \not\leq Y_t(\omega)\}$ is evanescent, that is, $P\{\omega : \exists t \geq 0$ such that $X_t(\omega) \not\leq Y_t(\omega)\} = 0$. The order on $\mathcal{S}$ is the canonical order relation associated with "$\leq$" and will be expressed by the same symbol. We denote by $\vee$ (resp. $\wedge$) the least upper bound (resp. the greatest lower bound). We also identify a supermartingale X to the class which contains it, when no confusion is possible. It follows then (see [3]) that for any increasing and dominated (resp. decreasing) family $F \subset \mathcal{S}$, there exists an increasing (resp. decreasing) sequence $(X_n) \subset F$ such that $\vee F = \bigvee_n X_n$ (resp. $\wedge F = \bigwedge_n X_n$). In fact, by the celebrated theorem of Doob, it is easy to see that $\bigvee_n X_n = \sup_n X_n$, where "sup" denotes the usual supremum of the real functions X_n defined on $R_+ \times \Omega$. When X_n decreases, we have (by semianalogy) $\bigwedge_n X_n = (\inf_n X_n)_+$, where for any real process X we denote by X_+ the process of right-hand limits (if they exist). Here, the right-hand (and even left-hand) limits exist at every $t \geq 0$ since $\inf X_n$ is still an (optional) *strong* supermartingale ([2], App., 1,4).

Following [1], an H-cone is an ordered convex cone $\mathcal{S}$ such that the following axioms hold:

H1) For any increasing and dominated family $F \subset \mathcal{S}$ there exists $\vee F$ and we have $s + \vee F = \vee(s + F)$ for all $s \in \mathcal{S}$.

H2) For any non-empty family $F \subset \mathcal{S}$ there exists $\wedge F$ and we have $s + \wedge F = \wedge(s + F)$ for all $s \in \mathcal{S}$.

H3) $\mathcal{S}$ satisfies the Riesz decomposition property, i.e., for any $s, s_1, s_2 \in \mathcal{S}$ such that $s \leq s_1 + s_2$ there exists $t_1, t_2 \in \mathcal{S}$ satisfying $s = t_1 + t_2$, $t_1 \leq s_1$, $t_2 \leq s_2$.

It is proved in [1] that $\mathcal{S}$ is an H-cone. The above discussion precises obviously H1), and also H2) by considering the decreasing family $F' = \{s_1 \wedge s_2 \wedge \ldots \wedge s_n;$ $s_i \in F$ for $i = 1, 2, \ldots, n$, and $n \in N$ is arbitrary $\}$. Axiom H3) will also follow easily from our section 3.

Definition. The dual of $\mathcal{S}$ is the set all mappings $\mu : \mathcal{S} \longrightarrow \bar{R}_+$ that are additive, increasing and *σ-continuous* in order for below (that is, $X_n \nearrow X \Rightarrow \mu(X_n) \nearrow \mu(X)$), and such that for each $X \in \mathcal{S}$ there exists a sequence $(X_n) \subset \mathcal{S}$, increasing to X such that $\mu(X_n) < \infty$ for any $n \in N$.

Further, we remark that for any stopping time T, the stochastic interval $[0, T)$ is an element of $\mathcal{S}$ (it is a potential if $T < \infty$ a.s.) which we denote by $1_{[0,T)}$. Moreover, it is not difficult to check that for any $X \in \mathcal{S}$, the product $1_{[0,T)} \cdot X$ still belongs to $\mathcal{S}$, and we denote it by ${}_TX$. We stress the distinction between this "cancellation from the moment T" and the stopping at the moment T, that is, the supermartingale X^T defined by $X_t^T \stackrel{\text{def}}{=} X_{t \wedge T}, t \geq 0$.

Definition. Let Λ be a set of processes containing all positive bounded martingales, and all stochastic intervals $1_{[0,T)}$, where T ranges over the set of all stopping times. A mapping $\lambda : \Lambda \longrightarrow R_+$ is called *separately* σ-continuous in order from

a) for any positive martingale $M \in \Lambda$ and *for any* sequence of positive martingales M_n increasing to M, we have $\lambda(M_n) \longrightarrow \lambda(M)$

b) for any *predictable* stopping time T, *there exists* a sequence of stopping times (T_n) foretelling T, such that $\lambda(1_{[0,T_n)}) \longrightarrow \lambda(1_{[0,T)})$.

Remark. The aim of above definition is to mark a weaker form of σ-continuity in order from below, that deals only with two special kinds of supermartingales: the martingales and the stochastic intervals.

In the sequel we use the abbreviation r.c.l.l for a real process, to mean that its trajectories are right continuous with left limits, a.s.

Following [2, VI, 51], an *increasing process* (A_t) is an adapted process with values in $[0,\infty)$ such that its trajectories are increasing and right continuous a.s. We put $A_{0-} = 0$. We say that (A_t) is *integrable* if $E[A_\infty] < \infty$. $(A_\infty = \lim_{t\to\infty} A_t)$

Theorem 1.1. Let $\mathcal{C}$ be a minstable convex cone of positive processes, containing the constants and satisfying the following properties:

1. Every $X \in \mathcal{C}$ is bounded, adapted, r.c.l.l. with limit at infinity $X_{\infty-}$.

2. $\mathcal{C}$ contains all positive bounded martingales, and all stochastic intervals $1_{[0,T)}$, where T ranges over the set of all stopping times.

Then, for every mapping $\lambda : \mathcal{C} \longrightarrow R_+$ which is positive homogenous, increasing and separately σ-continuous in order from below, there exists a unique (up to evanescence) integrable increasing (adapted, r.c.l.l.) positive process (A_t), such that one has

$$\lambda(X) = E\Big[\int_{[0,\infty)} X_t dA_t\Big], \quad \text{for all } X \in \mathcal{C}.$$

(We recall that the order relation on $\mathcal{C}$ is that considered above and again $\mathcal{C}$ may be considered as a set of classes of processes.)

Proof. Let us consider the linear space $\mathcal{D} = \mathcal{C} - \mathcal{C}$. We extend naturally the mapping λ to a linear form on D, still denoted by λ, which is obviously positive on the convex cone $\mathcal{D}_+$, since λ is increasing on $\mathcal{C}$. Let us consider a decreasing sequence (X^n) of elements of $\mathcal{D}_+$ such that

$$\lim_n (X^n)^* = 0 \quad \text{a.s.} \tag{1}$$

(We recall that for any real process X, we denote by X^* the positive function $\sup_{t\geq 0} |X_t|$). We remark first that for any $n \in N$, it follows that $(X^n)^*$ is measurable, since X^n is r.c.l.l. We can therefore consider, for each $n \in N$, a right continuous (with left limits) martingale M^n, closed to the right by $(X^n)^*$, that is,

$$M_t^n = E[(X^n)^* | \mathcal{F}_t], \quad \text{for any } t \geq 0. \tag{2}$$

Since we have obviously

$$X_t^n \leq M_t^n \ \text{ a.s., } \ \text{for any } t \geq 0 \tag{3}$$

and since both X^n and M^n are right continuous, the above inequality holds except on an evanescent set. If we consider the martingales $P^n = M^1 - M^n$, it follows from (1), (2) and Lebesgue's convergence theorem, that the sequence (P^n) increases

to M^1 in $\mathcal{C}$. By our hypoteses it follows then that $\lambda(P^n) \nearrow \lambda(M^1)$ and hence $\lambda(M^n) \searrow 0$. Using (3), we conclude that (1) implies the relation

$$\lim_n \lambda(X^n) = 0$$

We can now apply the stochastic variant of Daniell's theorem from ([2], VII, 2,3a)). There exist therefore two integrable increasing (positive) processes: A^+ optional which may jump at 0 but not at infinity; A^- predictable, which may jump at infinity but not at 0, such that we have for all $X \in D$:

$$\lambda(X) = E\Big[\int_{(0,\infty]} X_{s-} dA_s^- + \int_{[0,\infty)} X_s dA_s^+\Big] \tag{4}$$

Moreover we can require A^- to be purely discontinuous, and this representation is then unique. In order to get the desired result, we show that $A_\cdot^- \equiv 0$ on $[0,\infty]$, a.s. At this moment, we use the second part of the separate σ-continuity in order from below property of λ; since A^- is purely discontinuous and predictable, it follows from the predictable cross section theorem ([2], IV, 85, 86) that $A_\cdot^-$ is null on $[0,\infty)$ a.s., iff $\Delta A_T^- = 0$ a.s. on $\{T < \infty\}$ for any predictable stopping time T. By our hypothesis, there exists a sequence of stopping times (T_n) foretelling T and such that $\lambda(1_{[0,T_n)}) \nearrow \lambda(1_{[0,T)})$ (that is, (T_n) increases to T and $T_n < T$ on $\{T > 0\}$, for any $n \in N$).

We apply (4) to the processes $X^n = 1_{[0,T_n)})$ and $X = 1_{[0,T)}$. Since $X^n \nearrow X$, it follows immediatelly from Lebesgue's monotone convergence theorem that

$$\int_{[0,\infty)} X_s^n dA_s^+ \quad \nearrow \quad \int_{[0,\infty)} X_s dA_s^+ \tag{5}$$

On the other hand we have that $X_-^n = 1_{[0,T_n]}$ on $\{T_n > 0\}$ and $X_- = 1_{[0,T]}$ on $\{T > 0\}$. Hence the trajectory $X_-^n(\omega)$ increases to $1_{[0,T(\omega))}$ if $T(\omega) > 0$ and is null if $T(\omega) = 0$. Again by Lebesgue's monotone convergence theorem, it follows that

$$E\Big[\int_{(0,T]} dA_s^-\Big] = E\Big[\int_{(0,T)} dA_s^-\Big]$$

from (4), (5), and the choice of T_n. Therefore

$$E[\Delta A_T^-; T > 0] = 0.$$

Since A^- does not jump at 0 we have in fact

$$E[\Delta A_T^-] = 0.$$

As T is arbitrary, this implies that A^- is null on $[0,\infty)$ (from the cross-section theorem) and moreover if we take $T \equiv \infty$, we see that the jump at infinity of A^- is null.

We have proved that A^- is null except on an evanescent set, hence (4), restricted of course for $X \in \mathcal{C}$, gives the desired result, and the proof is finished.

Definition. An element u of $\mathcal{S}$ is called *weak unit* if the following relation holds:

$$s = \bigvee_n (s \wedge nu), \quad \text{for every } s \in \mathcal{S}.$$

In order to apply the above result to $\mathcal{S}$, we recall first a simple fact concerning duality for lattice convex cones with weak unit: an additive, increasing and σ-continuous in order from below mapping $\mu : \mathcal{S} \longrightarrow \bar{R}_+$ belongs to the dual iff

there exists some weak unit u of $\mathcal{S}$ such that $\mu(u) < \infty$ (the implication "$\Longleftarrow$" is clear and for the opposite implication we consider a weak unit u_0 and an increasing sequence $(u_n) \subset \mathcal{S}$, $u_n \nearrow u_0$ such that $\mu(u_n) < \infty$ for all $n \in N$. Then the element $v = \sum_{n=1}^{\infty} \frac{u_n}{2^n(1+\mu(u_n))}$ is a weak unit and $\mu(v) < \infty$).

Obviously $\mathcal{S}$ has weak unit (the constant process equal to 1) and its weak units are exactly those supermartingales X such that $X > 0$ except on an evanescent set.

Coming back to processes, we remark that any increasing real process (A_t) is *left locally integrable*, that is, there exists an increasing sequence of stopping times (T_n) such that $\lim_n T_n = \infty$ a.s. and $E[A_{T_n-}] < \infty$ for any $n \in N$. We note the distinction from the usual definition of local integrability which requires that $E[B_{T_n}] < \infty$, where B is the process defined by $B_t = A_t - A_0$ for all $t \geq 0$.

The elements of the dual of $\mathcal{S}$, denoted by $\mathcal{S}^*$, are called *H-integrals.*

Theorem 1.2. For any increasing process $A = (A_t)$ the mapping

$$\mathcal{S} \ni X \longrightarrow E\Big[\int_{[0,\infty)} X_t dA_t\Big]$$

is an H-integral and conversely, for any H-integral μ on $\mathcal{S}$ there exists a unique increasing process $A = (A_t)$ such that

$$\mu(X) = E\Big[\int_{[0,\infty)} X_t dA_t\Big], \quad \text{for all} \quad X \in \mathcal{S}.$$

Proof. Let (A_t) be an increasing process and let (T_n) be an increasing sequence of stopping times such that $\lim_n T_n = \infty$ a.s. and $E[A_{T_n-}] < \infty$ for all $n \in N$. If we consider the sequence $(u_n) = (1_{[0,T_n)})$ and the mapping defined on $\mathcal{S}$ by $\mu(X) = E[\int_{[0,\infty)} X_t dA_t]$, then μ is obviously additive, increasing, σ-continuous in order from below and moreover $\mu(u_n) < \infty$ for all $n \in N$. If we consider the element u of $\mathcal{S}$ defined by $u = \sum_{n=1}^{\infty} \frac{u_n}{2^n(1+\mu(u_n))}$, then u is a weak unit and $\mu(u) < \infty$. Therefore μ is an H-integral on $\mathcal{S}$.

Conversely, let μ be an H-integral on $\mathcal{S}$. There exists a weak unit u of $\mathcal{S}$ such that $\mu(u) < \infty$; so u is a supermartingale such that $u > 0$ except on an evanescent set. We consider the sequence of stopping times (T_n) defined for all $n \in N$ by

$$T_n = \inf\Big\{t : u \leq \frac{1}{n}\Big\}.$$

Then (T_n) is obviously increasing and from ([2], VI, 17) it follows that $\lim_n T_n = \infty$ a.s., since $u > 0$ except on an evanescent set.

We now fix $n \in N$ and we consider the mapping

$$\mathcal{S} \ni X \longrightarrow \mu(X \cdot 1_{[0,T_n)})$$

which we denote by μ_n. Obviously μ_n is additive, increasing, σ-continuous in order from below. Moreover we have

$$\mu_n(1) = \mu(1_{[0,T_n)}) \leq n\mu(u) < \infty \tag{1}$$

since $u > \frac{1}{n}$ on $[0, T_n)$ and therefore $1_{[0,T_n)} \leq nu$. It follows that the restriction of μ_n to the convex cone $\mathcal{C} = b\mathcal{S}$ consisting of all *bounded* elements of $\mathcal{S}$ satisfies

the conditions from theorem 1.1, which applies, and there exists therefore a unique integrable increasing process $A^n = (A_t^n)$ such that we have:

$$\mu(X \cdot 1_{[0,T_n)}) = \mu_n(X) = E\Big[\int_{[0,\infty)} X_t dA_t^n\Big] \tag{2}$$

for all $X \in b\mathcal{S}$. By uniqueness, it then follows that $A_t^{n+1} = A_t^n$ for all $t \in [0, T_n(\omega))$, for almost all $\omega \in \Omega$. Hence there exists a unique (up to evanescence) increasing process $A = (A_t)$ such that $A = A^n$ on $[0, T_n)$ for each $n \in N$. From (2) we have

$$\mu(X) = \lim_n \mu(X \cdot 1_{[0,T_n)}) = \lim_n E\Big[\int_{[0,T_n)} X_t dA_t\Big] = E\Big[\int_{[0,\infty)} X_t dA_t\Big] \tag{3}$$

for all $X \in b\mathcal{S}$ and then for all $X \in \mathcal{S}$ by σ-continuity in order from below. The existence of the desired representation being now clear, let $B = (B_t)$ be another left locally integrable increasing process representing the H-integral μ in the sense of (3). If (S_n) is an increasing sequence of stopping times such that $E[B_{S_n-}] < \infty$ for all $n \in N$, we consider the sequence of stopping times (U_n) defined by $U_n = S_n \wedge T_n$. Then the uniqueness assertion of theorem 1.1 implies that

$$E\Big[\int_{[0,U_n)} X_t dA_t\Big] = E\Big[\int_{[0,U_n)} X_t dB_t\Big]$$

for any $X \in \mathcal{S}$. By taking limits when $n \to \infty$ we get the desired uniqueness of the representation and the proof is finished. □

We describe now the order relation on $\mathcal{S}^*$, defined for $\mu, \nu \in \mathcal{S}^*$ by $\mu \leq \nu$ iff $\mu(X) \leq \nu(X)$ for all $X \in \mathcal{S}$. For this purpose (and the sequel) we shall consider the supermartingales of class (D). We recall that a measurable real process X is of class (D) iff the set of random variables $\{|X_T| \cdot 1_{\{T<\infty\}} : T \text{ stopping time }\}$ is uniformly integrable ([2], VI, 20). If we denote by $\mathcal{S}_D$ the convex subcone of $\mathcal{S}$ consisting of all elements of class (D), it is obvious that $\mathcal{S}_D$ is *solid* in $\mathcal{S}$ and increasingly dense (since $b\mathcal{S} \subset \mathcal{S}_D$). We also recall the Doob-Meyer decomposition: an element X of $\mathcal{S}$ belongs to $\mathcal{S}_D$ iff there exists an integrable increasing predictable process A, indexed by $[0, \infty]$, which may jump at ∞, but without jump at 0 such that

$$X_t = E[A_\infty - A_t | \mathcal{F}_t] \text{ a.s., for all } t \geq 0,$$

and moreover the process $A = (A_t)$ is unique up to evanescence with these properties.

From now on we identify by theorem 1.2 the dual $\mathcal{S}^*$ of $\mathcal{S}$ with the set of all increasing processes (A_t), as convex cones.

Proposition 1.3. Let $\mu, \nu \in \mathcal{S}^*$. Then $\mu \leq \nu$ iff $A_t^1 \leq A_t^2$ a.s. for all $t \geq 0$, where A^1 and A^2 are increasing processes representing respectively μ and ν.

Proof. Since $\mathcal{S}_D$ is increasingly dense in $\mathcal{S}$, we have the equivalence

$$\mu \leq \nu \quad \text{iff} \quad \mu(X) \leq \nu(X) \text{ for all } X \in \mathcal{S}_D$$

We fix now $X \in \mathcal{S}_D$ and we consider the predictable proces $B = (B_t)$ given by the Doob-Meyer decomposition of X. We have the relation:

$$\begin{aligned} &\mu(X) \leq \nu(X) \Longleftrightarrow \\ &E\Big[\int_{[0,\infty)} (E[B_\infty|\mathcal{F}_t] - B_t) dA_t^1\Big] \leq E\Big[\int_{[0,\infty)} (E[B_\infty|\mathcal{F}_t] - B_t) dA_t^2\Big]. \end{aligned} \tag{1}$$

Replacing X by $X \cdot 1_{[0,T)}$ for a suitable stopping time T, we may suppose that both A^1 and A^2 are integrable. Further, as $E[B_\infty|\mathcal{F}_T]$ is the optional projection of the process B_∞ constant in time ([2], VI, 43,45), it follows from ([2], VI, 57) that the above inequality if equivalent to the following one:

$$E\Big[\int_{[0,\infty)} (B_\infty - B_t)dA_t^1\Big] \leq E\Big[\int_{[0,\infty)} (B_\infty - B_t)dA_t^2\Big]. \tag{2}$$

Since A_1 and A_2 do not jump at infinity, from the integration by parts formula applied to each trajectory ([2] VI, 90), the above inequality is equivalent to:

$$E\Big[\int_{[0,\infty]} A_{t-}^1 dB_t\Big] \leq E\Big[\int_{[0,\infty]} A_{t-}^2 dB_t\Big]. \tag{3}$$

We let now X vary in $\mathcal{S}_D$ and therefore (3) holds for any integrable increasing predictable process $B = (B_t)$. If we fix $t \geq 0$, then for any positive, integrable and $\mathcal{F}_{t-}$ measurable random variable H, the increasing process $B_t = H \cdot 1_{[t,\infty)}$ is predictable. Applying (3) to this (B_t) we get

$$E[A_{t-}^1 \cdot H] \leq E[A_{t-}^2 \cdot H] \tag{4}$$

Since A_{t-}^1 and A_{t-}^2 are $\mathcal{F}_{t-}$ measurable, it follows from (4) that $A_{t-}^1 \leq A_{t-}^2$ a.s., and since $A^i = (A_-^i)_+$ for $i = 1, 2$, we get the desired relation

$$A_t^1 \leq A_t^2 \quad \text{a.s.} \tag{5}$$

The converse is easy: if (5) holds for all $t \geq 0$, then for almost all $\omega \in \Omega$ we have that $A_t^1(\omega) \leq A_t^2(\omega)$ for every $t \geq 0$, hence (3), (2), (1) hold obviously, that is, $\mu \leq \nu$ and the proof is finished. □

Remark. The proof of above proposition involves the computation of $\mu(X)$ when $X \in \mathcal{S}_D$ is represented by $B = (B_t)$ and μ is represented by $A = (A_t)$:

$$\mu(X) = E\Big[\int_{[0,\infty]} A_{t-} dB_t\Big] \tag{E}$$

Corollary 1.4. If $\mu, \nu \in \mathcal{S}^*$ are represented by $A = (A_t)$ and $B = (B_t)$ respectively, then $\mu \vee \nu$ and $\mu \wedge \nu$ are represented by $A_t \vee B_t$ and $A_t \wedge B_t$, where "$\vee$" and "$\wedge$" denote first the supremum (infimum) in the lattice $\mathcal{S}^*$, and then the pointwise supremum (infimum) on the set of random variables.

We finally remark the striking analogy between the representation given by theorem 1.2 and the mutual energy of two excessive functions from classical Potential Theory. Here, the correspondent of the second excessive function (coexcessive) is necessarily a "potential" whose "mass" is the increasing process (A_t), that is, the space $\Omega \times R_+$ is "saturated" with respect to our structure. If moreover $X \in \mathcal{S}$ is a potential of class (D) represented by the "mass" (B_t) (Doob-Meyer decomposition), then formula (E) considered above corresponds to the classical mutual energy of two masses.

2. We describe in this section some remarquable classes of elements of the H-cone $\mathcal{S}$ by following the analogy with classical Potential Theory. First, we recall from ([2], VII, 17) that the elements of class (D) of $\mathcal{S}$ are exactly the (infinite) sums of bounded elements of $\mathcal{S}$. We refine this result for *potentials*:

Proposition 2.1. Let $u \in \mathcal{S}$ be a weak unit and $X \in \mathcal{S}_D$ a potential. Then there exists a sequence of elements $(X^n) \subset \mathcal{S}$ such that $X^n \leq u$ for all $n \in N$, and $X = \sum_{n=1}^{\infty} X^n$.

Proof. We define the sequence of stopping times (T_n) by $T_n = \inf\{t : X_t \geq nu_t\}$. Then (T_n) is obviously increasing. Moreover, we have that $\lim_n T_n = \infty$ a.s. from ([2], VI, 17), since $u > 0$ except on an evanescent set, and the supermartingale X possesses a (finite) left limit at any $t \geq 0$ on almost all trajectories ([2], VI, 3). If $A = (A_t)$ is the predictable increasing process from the Doob-Meyer decomposition of X, then (A_t) does not jump at infinity (since X is a *potential*) and if we consider the predictable increasing processes $A^n = (A_t^n)$ defined for each $n \geq 0$ by:

$$A_t^n = A_{t \wedge T_n},$$

then for each $n \in N$ the predictable process $B^n = A^n - A^{n-1}$ is increasing. It is easy to see that the corresponding sequence of potentials $(\overline{X}^n)$ defined by

$$\overline{X}_t^n = E[B_\infty^n - B_t^n | \mathcal{F}_t], \quad \text{for any } t \geq 0,$$

satisfies the properties: $\overline{X}^n \leq nu$ for all $n \in N$ and $X = \sum_{n=1}^{\infty} \overline{X}^n$. It is then trivial to derive from $(\overline{X}^n)$ the sequence (of potentials) (X^n) possesing the desired properties. □

Remark. An element s of $\mathcal{S}$ is called *universally quasi-bounded* if for every weak unit u of $\mathcal{S}$ there exists a sequence $(s_n) \subset \mathcal{S}$ such that $s_n \leq u$ for all $n \in N$ and $s = \sum_{n=1}^{\infty} s_n$. From the preceding facts it follows that the potentials of class (D) are exactly the universally quasi-bounded potentials elements of $\mathcal{S}$.

We say that an element s of $\mathcal{S}$ is *quasi-continuous* if for every family $(s_i)_{i \in I}$ increasing to s, one has $\bigwedge_{i \in I} R(s - s_i) = 0$, where $R(s - t) \stackrel{\text{def}}{=} \wedge\{u \in \mathcal{S} : u + t \geq s\}$ denotes the "réduite" of $s - t$ in the $\mathcal{S}$ (see our section 3 and [3]). One can check that quasi-continuity implies universal quasi-boundedness. Since $X - R(X - X^i) \in \mathcal{S}$ (see section 3), for $I = N$ the above relation is equivalent to $\inf_{i \in N} R(X - X^i) = 0$.

Finally, we recall that an optional process X of class (D) is called *regular* ([2], VI, 50) if for all increasing and uniformly bounded sequences of stopping times (T_n), $E[X_{\lim_n T_n}] = \lim_n E[X_{T_n}]$.

Theorem 2.2. Let $X \in \mathcal{S}$ be a potential of class (D). Then X is quasi-continuous in $\mathcal{S}$ iff X is a regular process.

Proof. "$\Longrightarrow$". Let T be a stopping time and T_n be a sequence of stopping times increasing to T. The process X^n defined by

$$X^n = \begin{cases} X & \text{on } [0, T_n) \\ E[X_T \mid \mathcal{F}_t] & \text{on } [T_n, T) \\ X & \text{on } [T, \infty) \end{cases}$$

is a supermartingale such that $X^n \nearrow X$. By quasi-continuity of X, $\bigwedge_n R(X-X^n) = 0$. Putting $Y^n = R(X - X^n) \geq X - X^n$, one has $Y^n_{T_n} \geq X_{T_n} - X^n_{T_n}$ and

$$0 \leq E[X_{T_n}] - E[X_T] = E[X_{T_n} - E[X_T \mid \mathcal{F}_{T_n}]] \leq E[Y^n_{T_n}] \leq E[Y^n_0] \to 0.$$

"$\Longleftarrow$" Let now $X \in \mathcal{S}$ be a *bounded* regular potential and $(X^n) \subset \mathcal{S}$ be a sequence increasing to X. We want to prove that

$$\bigwedge_n R(X - X^n) = 0 \tag{6}$$

For this purpose, we invoque (the proof of) ([2], VII, 20) which tells us that under the above conditions we have:

$$\lim_n (X - X^n)^* = 0 \quad \text{in probability.}$$

Hence the martingales $M^n_t = E[(X - X^n)^* \mid \mathcal{F}_t]$ (verifying the relation $M^n \geq R(X - X^n)$) are decreasing to 0 in $\mathcal{S}$ (by Lebesgue's dominated convergence theorem). Therefore $\bigwedge_n R(X - X_n) = 0$, that is, X is quasi-continuous.

If we consider now an arbitrary regular potential X of class (D), it follows from [2, VII, 17a] that there exists a sequence (X_n) of *bounded* regular potentials such that $X_n \nearrow X$, and $X - X_n \in \mathcal{S}$ for all $n \in N$. From above, each X_n is quasi-continuous. The fact that X is quasi-continuous follows from the following relations (and proposition below). Let $X^i \nearrow X$, put $X^i_n = X^i \wedge X_n$ ($\nearrow X_n$ for fixed n) and $Y_n = X - X_n$. We have for all $i \in I$ and $n \in N$:

$$R(X - X^i) \leq R(X_n - X^i) + Y_n = R(X_n - X^i_n) + Y_n$$

and we vary first i and then n. □

Remark. In the course of above proof we met the following situation: we have a decreasing sequence (X^i) of elements of $\mathcal{S}$ such that $\bigwedge_i X^i = \inf X^i = 0$ and we used from these informations only the fact that $\lim_i E[X^i_0] = 0$. It seems that we have not used the whole information, since only the values at 0 of the processes X^i interfere in the above limit.

It is not the case, as follows from the following "remark".

Proposition 2.3. Let (X^i) be a decreasing sequence of elements of $\mathcal{S}$ such that $\lim_i E[X^i_0] = 0$. Then we have

$$\bigwedge_i X^i = \inf X^i = 0.$$

Proof. Let X be the (optional) strong supermartingale $\inf_i X^i$. It follows that for every stopping time T we have

$$X_T \leq X^i_T \text{ a.s. on } \Omega, \text{ for all } i \in N.$$

Using now the strong supermartingale property, we have

$$E[X_T] \leq \lim_i E[X^i_T] \leq \lim_i E[X^i_0] = 0.$$

Therefore $X_T = 0$ a.s. for every stopping time T, and $X = 0$ by the optional cross section theorem. □

We now recall that a r.c.l.l. process X is called a *local martingale* ([4], VI, 27) if there exists a sequence (T_n) of stopping times such that $\lim_n T_n = \infty$ a.s., and for each $n \in N$ the process $X^{T_n} 1_{\{T_n > 0\}}$ is a uniformly integrable martingale (X^{T_n} is the process obtained by stopping the process X at the time T_n); one says that T_n reduces X.

We say that an element s of $\mathcal{S}$ is *subtractible* if for every $t \in \mathcal{S}$ such that $s \leq t$, s is specifically dominated by t.

Proposition 2.4. Let $X \in \mathcal{S}$. Then if X is subtractible in $\mathcal{S}$, the process X is a local martingale.

Proof. (The converse is just ([2], VI, 36).) From [2, VII, 13], we deduce that there exist an integrable increasing, predictable process $B = (B_t)$ null at 0, with no jump at infinity, and a positive local martingale $N = (N_t)$, such that

$$X_t = E[B_\infty - B_t | \mathcal{F}_t] + N_t \text{ a.s., for all } t \geq 0.$$

If we consider the element $Y \in \mathcal{S}$ defined by

$$Y_t = E[B_\infty | \mathcal{F}_t] + N_t$$

then obviously $X \leq Y$. By hypothesis it follows that the process

$$Y_t - X_t = B_t$$

is a supermartingale, and this forces the increasing process (B_t) to be null (up to evanescence). Hence $X = N$ is a local martingale. $\square$.

3. The aim of this closing section is to show an alternative way to [1] to deduce the "cone of potentials" property of $\mathcal{S}$.

We denote by $\mathcal{J}$ the set of all stopping times. For each positive, optional process Y such that Y_T is integrable for all $T \in \mathcal{J}$, we consider the random variables $Z'_T, T \in \mathcal{J}$ defined by

$$Z'_T = \operatorname*{ess\,sup}_{\substack{S \in J \\ S \geq T}} E[Y_S | \mathcal{F}_T].$$

If $\int Z'_T dP < \infty$ for all $T \in \mathcal{J}$, then there exists a unique (up to evanescence) optional strong supermartingale Z such that

$$(S) \qquad Z_T = Z'_T \quad \text{a.s., for all } T \in \mathcal{J}.$$

More precisely, Z is the Snell envelope of Y considered in ([2] App. 22, 23b)). We denote by $\widetilde{\mathcal{S}}$ the positive ordered convex cone of all positive, optional strong supermartingales; of course $\mathcal{S} \subset \widetilde{\mathcal{S}}$. (The usual conditions are in force.)

If $X^1, X^2 \in \widetilde{\mathcal{S}}$, then the positive optional process $Y = (X^2 - X^1) \vee 0$ satisfies the above integrability conditions and the Snell envelope of $(X^2 - X^1) \vee 0$ will be denoted by $R(X^2 - X^1)$, the *réduite* of $X^2 - X^1$ in $\widetilde{\mathcal{S}}$.

Proposition 3.1. For all $X^1, X^2 \in \widetilde{\mathcal{S}}$, the process $X^2 - R(X^2 - X^1)$ belongs to $\widetilde{\mathcal{S}}$.

Proof. We may suppose that $X^1 \leq X^2$, by replacing X^1 by $X^1 \wedge X^2$ if necessary ($\widetilde{\mathcal{S}}$ is minstable). Using the above defining property (S) of the Snell envelope, we have for every $T \in \mathcal{J}$ the (a.s.) relation:

$$(1) \qquad (R(X^2 - X^1))_T = \operatorname*{ess\,sup}_{\substack{S \in J \\ S \geq T}} E[X^2_S - X^1_S | \mathcal{F}_T]$$

Let now $T, T' \in \mathcal{J}, T \leq T'$ be arbitrary fixed. Since the process $X^2 - R(X^2 - X^1)$ is obviously optional and strong integrable, it suffices to prove that

$$(X^2 - R(X^2 - X^1))_T \geq E[(X^2 - R(X^2 - X^1))_{T'} | \mathcal{F}_T],$$

or equivalently

$$(R(X^2 - X^1))_T \leq X_T^2 - E[X_{T'}^2|\mathcal{F}_T] + E[(R(X^2 - X^1))_{T'}|\mathcal{F}_T].$$

According to (1), it suffices to prove that for all $S \in \mathcal{J}, S \geq T$, we have

$$E[X_S^2 - X_S^1|\mathcal{F}_T] \leq X_T^2 - E[X_{T'}^2|\mathcal{F}_T] + E[(X_{S\vee T'}^2 - X_{S\vee T'}^1)|\mathcal{F}_T] \tag{2}$$

since obviously $S \vee T' \geq T'$, and a well known property of conditional expectations holds ($\mathcal{F}_T \subseteq \mathcal{F}_{T'}$).

But from the strong supermartingale property for X^1 we have

$$X_S^1 \geq E[X_{S\vee T'}^1|\mathcal{F}_S],$$

and hence

$$E[X_S^1|\mathcal{F}_T] \geq E[E[X_{S\vee T'}^1|\mathcal{F}_S]|\mathcal{F}_T] = E[X_{S\vee T'}^1|\mathcal{F}_T]. \tag{3}$$

since $\mathcal{F}_T \subset \mathcal{F}_S$. Owing to (3), inequality (2) will follow quickly once it is proved that

$$E[(X_S^2 + X_{T'}^2)|\mathcal{F}_T] \leq E\big[E[X_{S\vee T'}^2|\mathcal{F}_T]\big] + X_T^2. \tag{4}$$

But $T \leq S, T \leq T'$, and therefore $T \leq S \wedge T'$. Hence, and from the strong supermartingale property for X^2, we have

$$X_T^2 \geq E[X_{S\wedge T'}^2|\mathcal{F}_T]. \tag{5}$$

Since the following identity is obvious:

$$X_S^2 + X_{T'}^2 = X_{S\vee T'}^2 + X_{S\wedge T'}^2\ , \tag{6}$$

the desired (4) follows immediately by taking the conditional expectation and summing to (5). The proof is finished. □

Remark. The above result holds without the usual conditions.

We return now to the usual conditions and right continuous supermartingales, and we consider $\mathcal{S}$ as an ordered convex subcone of $\widetilde{\mathcal{S}}$. If $X^1, X^2 \in \mathcal{S}$, it follows from ([2], App., 23 c)) that $R(X^1 - X^2)$ is r.c.l.l., and hence it belongs to $\mathcal{S}$ (the same is true then for $X^1 - R(X^1 - X^2)$). Hence, we have obviously

$$R(X^1 - X^2) = \wedge\{Y \in \mathcal{S} : Y + X^2 \geq X^1\}.$$

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