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AN $F_\sigma$ SEMIGROUP OF ZERO MEASURE WHICH CONTAINS A TRANSLATE OF EVERY COUNTABLE SET

by John A. HAIGHT (*)

In 1942, PICCARD [10] gave an example of a set of real numbers whose sum set has zero Lebesgue measure but whose difference set contains an interval. About thirty years later, various authors (CONNOLLY, JACKSON, WILLIAMSON and WOODALL) in a series of papers constructed $F_\sigma$ sets $E \subset \mathbb{R}$ such that $E - E$ contains an interval while $\mu(kE) = 0$ for progressively larger values of $k$, where

$$(k)E = \{x_1 + x_2 + \ldots + x_k; \ x_i \in E, \ 1 \leq i \leq k\}.$$ 

These authors' interest was in an approach to the construction of asymmetric Raikov systems, [5], defined as follows.

If $G$ is a locally compact abelian group, a Raikov system is a family $\mathcal{F}$ of $F_\sigma$ subsets satisfying the following conditions:

(a) If $F_1, F_2, \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} F_n \in \mathcal{F}$. 
(b) If $F_1 \subset F_2 \in \mathcal{F}$ and $F_1$ is $F_\sigma$ then $F_1 \in \mathcal{F}$. 
(c) If $F_1, F_2 \in \mathcal{F}$ then $F_1 + F_2 \in \mathcal{F}$.

A Raikov system is said to be asymmetric if $\Lambda \in \mathcal{F}$ does not necessarily imply $-\Lambda \in \mathcal{F}$.

CONNOLLY and WILLIAMSON [3] noted that the existence in $\mathbb{R}$ of an asymmetric Raikov system which was maximal among proper Raikov systems was equivalent to the existence of an $F_\sigma$ semigroup of zero Lebesgue measure which is not contained in any proper subgroup of $\mathbb{R}$, which in turn is equivalent to the existence of an $F_\sigma$ set $E$ such that $E - E = \mathbb{R}$, but $\mathbb{m}(k)E = 0$ for $k = 1, 2, \ldots$. I was able to solve this problem, although unfortunately the central idea was rather obscured by technical details. Recently, however, BROWN and MORAN [1] have simplified my proof. The results of this paper are a generalization of this simplification.

If $\mathcal{R}$ is a ring and $\alpha, \beta \in \mathbb{Q}$ and $E, F \subset \mathcal{R}$, we write

$$\alpha \cdot E + \beta \cdot F = \{\alpha x + \beta y; \ x \in E, \ y \in F\};$$

if $E$ is finite, $|E|$ denotes the number of elements in $E$. In this notation, the statement $E - E = \mathbb{R}$ is equivalent to the statement that, for every $F \subset \mathcal{R}$ such

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that \( |F| \leq 2 \), there is a \( c \in \mathbb{R} \) such that \( F + \{ c \} \subset \mathbb{E} \). (From now on, we shall write \( "c" \) instead of \( \{ c \} \). This leads to the question: If \( (k) E \) is "small", how "large" is the family of sets \( F \) that can be translated into \( \mathbb{E} \)?

For any \( n \in \mathbb{N} \), we write \( I(n) = \{ 0, \ldots, n - 1 \} \) and \( \mathcal{Z}(n) \) for the integers modulo \( n \).

**Theorem 1.** - For all \( j, n \in \mathbb{N} \) and \( \varepsilon > 0 \), there is an \( N = N(j, n, \varepsilon) \) such that, for any \( N \geq N \), there is a subset \( A \) of \( \mathcal{Z}(N) \) such that

(a) \( F \subset \mathcal{Z}(n) \), \( |F| \leq n \), there is a \( c = c(F) \) such that \( F + c \subset A \).

(b) \( |(j) A| \leq n \).

If \( F \subset I(n)^n \), \( i = 1, 2, \ldots \), for some \( n > 1 \), we shall say that the set \( C = \{ x = (x_1, \ldots, x_n) : x_i \in F_i \} \) is a Cantor set. If \( |F| \leq r \), we shall say that \( C \subset b(n, r) \) if \( F_i = F, i = 1, 2, \ldots \), then we shall say that \( C \subset b(n, |F|) \) is self-similar.

**Theorem 2.** - For any \( n, r, j \in \mathbb{N} \) and \( \varepsilon > 0 \) there is an \( N = N(j, n, \varepsilon) \) and a self-similar Cantor set \( K \subset \mathbb{R}^n \) such that

(a) For any \( F \in b(n, r) \), there is a \( d = d(F) \) such that \( F + d \subset K \).

(b) \( (j) K \in b(n, \varepsilon N) \).

We note that (a) implies that if \( F \) is any finite set containing not more than \( r \) points, then there is a \( c \) such that \( F + c \subset K \).

**Theorem 3.** - There is an \( F \subset \mathbb{R}^n \) such that

(a) If \( F \subset \mathbb{R}^n \) is a countable set, there is a \( c = c(F) \) such that \( F + c \subset E \).

(b) For any \( k \in \mathbb{N} \), \( m(k) E = 0 \).

Cassels [2] proved that if \( \lambda_1, \ldots, \lambda_r \) are real numbers, there is a number \( \omega \) such that \( \| (\omega + \lambda_i) u \| > \alpha / u \), \( u \in \mathbb{N} \), \( i = 1, \ldots, r \) \( (\|x\| \) denotes the distance of \( x \) from the nearest integer to \( x \), \( \alpha = \alpha(r) \).

In our notation \( \{ \lambda_1, \ldots, \lambda_r \} + \alpha \in \mathbb{R}^r \)

\[ B_r = \{ x : \|x_u\| > \alpha \} \quad u, u \in \mathbb{N} \] .

Let \( B = \bigcup_{i \in \mathbb{N}} B_r \), then \( B \) is the set of "badly approximable numbers". It is well-known that \( B = \bigcup_{i \in \mathbb{N}} F(n) \) where \( F(n) \) is the set of numbers whose continued fraction expansions have partial quotients \( \leq n \) and that \( m(B) = 0 \). However, (2) \( B = \mathbb{R} \). Indeed K. Hall [7] proved that (2) \( F(4) = \mathbb{R} \) (more recently H. A. H. A showed (4) \( F(2) = \mathbb{R} \)). Davenport and Schmidt [4], [11] extended Cassels' result in various ways. In particular, Schmidt's theorem implies that, for every countable \( \mathbb{R}^n \), there is an \( \omega \in \mathbb{R}^n \) (actually many such \( \omega \)) such that \( F + \omega \subset B \) where \( B \) in \( \mathbb{R}^n \) is defined as

\[ \{ x = (x_1, \ldots, x_n) : \max \{ \|ux_1\|, \ldots, \|ux_n\| \} < c(x) \} \].
Again \( n(B) = 0 \), and \((2) B = \mathbb{R}^n \).

Proof of Theorems.

**Lemma 1.** For all \( j, n \in \mathbb{N} \) and \( \epsilon > 0 \), there is an \( N = N(j, n, \epsilon) \) and a subset \( A \) of \( \mathbb{Z}(n) \) such that

(a) If \( |F| \leq n \), there is a \( c = c(F) \) such that \( F + c \subseteq A \).

(b) \(|(j)A| N^{-1} < \epsilon \).

If \( x, y \in \mathbb{R}^n \) for some \( n \geq 1 \), write

\[
x \cdot y = x_1 y_1 + \cdots + x_n y_n.
\]

**Lemma 2.** Given \( \mathbb{N}, j, n \in \mathbb{N} \) and \( \epsilon > 0 \), there is an \( N = N(n, j, n, \epsilon) \) such that for each \( j \in I(n)^n \), there is \( c = c(j) \in I(n) \), such that if \( c = (c_1, \ldots, c_n) \) and

\[
E = E(h, \mathbb{N}) = \{ x; x \in I(n), x = g(j + c^*) \text{ mod } N, g \in I(h)^n, j \in I(h)^n \},
\]

then \(|(j)E| N^{-1} < \epsilon \).

If \( h \geq 2 \) this is stronger than lemma 1. For if \( F = \{ j_1, \ldots, j_m \} \), then \( j = (j_1, \ldots, j_m) \) and, taking \( g(i) = (0, \ldots, 1 \text{th place}, \ldots, 0) \), we have \( g(i) \in I(n)^n \), and so \( g(i)(j + c^*) \in E \), that is \( j_1 + c \in E \), \( i = 1, \ldots, n \).

**Proof of Lemma 2.** Fix \( n \). We use induction on \( j \). Assume the lemma holds for some \( j \) and for all \( n, \epsilon \).

Given \( \mathbb{N}, \epsilon \), let \( T = N(2n, j, n, \epsilon/2) \). Let

\[
G = \{ g; g : F \to I(h)^n \ \setminus \ \{(0, \ldots, 0)\}, \ F \in I(T)^n, 0 < |F| \leq j + 1 \}.
\]

Let \( \mathcal{L} = |G| \), and let \( \phi : I(\mathcal{L}) \to G \) be a bijection.

Now choose primes \( P_0 < P_1 < \cdots < P_{j-1} \) such that

(a) \( P_0 > nMT \),

(b) \( \sum_{0 \leq i \leq j-1} \frac{1}{P_i} < \epsilon/2 \).

Let \( P = P_0 \times \cdots \times P_{j-1} \) and let \( S = P^j \).

For \( j \in I(S)^n \), let \( j_1 \in I(T)^n \), where \( j_1 = j \text{ mod } T \), we choose \( c(j) \) to satisfy (1) and (2)

(1) \( c(j) = c(j_1) \text{ mod } T \)

(2) \( \phi(r)(j_1)(j + c^*(j)) = 0 \text{ mod } P_r \)

if \( j_1 \in \text{ dom } \phi(r) \), otherwise let \( c(j) = 0 \text{ mod } P_r \), \( r = 0, \ldots, j - 1 \).

We need to show that (2) is possible. If \( j_1 \in \text{ dom } \phi(r) \), then \( \phi(r)(j_1) = \lambda \) say, where \( \lambda \in I(h)^n \ \setminus \ \{(0, \ldots, 0)\} \) so
Now \(0 < \lambda_1 + \ldots + \lambda_n \leq n^k\) so condition (b) ensures that we can find \(c(j)\) so that the above equation is congruent to zero mod \(P_j\).

Now if \(x \in (j+1)E(\mathbb{H},S)\),
\[
x = \sum_{i=1}^{j+1} \lambda(i)(j(i) + c(j(i))) \pmod{S}
\]
for some \(\lambda(i) \in \mathbb{I}(\mathbb{H})^n\) and \(j(i) \in \mathbb{I}(\mathbb{H})^n\), \(i = 1, \ldots, j+1\).

We have two cases:

**Case 1.** \(j(s) = j(t)\), for some \(s \neq t\). Then
\[
x' = \sum_{i=1}^{j+1} \lambda(i)(j(i) + c(j(i)))
\]
\[
= \sum_{1 \leq i \leq j+1, i \neq s,t} \lambda(i)(j(i) + c(j(i))) + (\lambda(s) + \lambda(t))(j(s) + c(j(s)))
\]
so \(x \in (j)E(2\mathbb{H},T) + \mathbb{Z}\).

**Case 2.** \(j(s) = j(t)\) only if \(s = t\). Then there is an \(r, 0 \leq r < j-1\), such that
\[
\phi(r)(j(i)) = \lambda(i)\text{ for } i = 1, \ldots, j+1.
\]
So
\[
x = \sum_{i=1}^{j+1} \phi(r)(j(i))(j(i) + c(j(i))) \in P_j\mathbb{Z}
\]
Thus we have
\[
(j+1)E(\mathbb{H},S) + S \cdot \mathbb{Z} = (\bigcup_{r=0}^{j-1} P_r \cdot \mathbb{Z}) \cup ((j)E(2\mathbb{H},T) + T \cdot \mathbb{Z}) + S \cdot \mathbb{Z},
\]
so \(|(j+1)E(\mathbb{H},S)| S^{-1} < \varepsilon/2 + \varepsilon/2 = \varepsilon\).

We modify the argument for \(j = 0\): take \(T = 1\) and \(G = \mathbb{I}(\mathbb{H})^n \setminus \{(0, \ldots, 0)\}\) and use (2) (only) to define \(<(j)\). Then if \(x \in E(\mathbb{H},S)\),
\[
x = \lambda(j + c^*(j)) = \phi(r)(j + c^*(j))\text{ for some } r.
\]
So \(x \in \bigcup_{0 \leq r < j-1} P_r \cdot \mathbb{Z}\).

**LEMMA 3.** For any \(j, n \in \mathbb{N}\) and \(\varepsilon > 0\), there is an \(N \in \mathbb{N}\) and an \(E \subseteq T\) (where \(T\) is \(R\) modulo 1) such that

1. \(E\) consists of a finite union of closed intervals.
2. If \(G \subseteq T\) and \(|G| \leq n\), there is a \(d = d(G)\) such that \(G + d \subseteq E\).
3. \(\mu((j)E + r^{-1}(0,5j)) < \varepsilon\) for all \(r > N\).

The difference between Lemma 1 and Theorem 1 is that the modulus of Lemma 1 has to be the product of very large primes. We need Lemma 3 to show that, rather surprisingly, any sufficiently large modulus will do.
Proof of Lemma 3 (assuming Lemma 1). - Let $T = \lambda(j, n) \frac{2}{6j}$, where $N$ is as in Lemma 1. Let $E = N^{-1} A + (0, N^{-1})$. We show that $E$ has the required properties.

$$(j) E + r^{-1}(0, 5j)$$

$$(j) N^{-1} A + (0, jN^{-1}) + (0, 5jN^{-1}) \subset (j) N^{-1} A + (0, 6jN^{-1})$$

so $n((j) E + r^{-1}(0, 5j)) \leq |(j) A| \times 6j/N < c$.

Now $G$ can be covered by at most $n$ intervals of the form $(i/N, (i + 1)/N)$, $i \in \mathbb{Z}$, so

$$G \subset N^{-1} F + (0, N^{-1}) \text{ for some } F \subset \mathbb{Z}(N) \text{ where } |F| \leq n.$$ 

If $c = c(F)$, we have

$$G + c/N \subset N^{-1}(F + c) + (0, N^{-1}) \subset N^{-1} A + (0, N^{-1}) = E,$$

which gives (2), taking $d = c/N$.

Proof of Theorem 1 (assuming Lemma 3). - Since $E$ is a finite union of closed intervals,

$$E = \bigcup_{i=1}^{k} (x_i + (0, \delta_i)) \text{ for some } x_1, \ldots, x_k \text{ and } \delta_1, \ldots, \delta_k.$$ 

If $M \geq N$, then

$$E + (0, N^{-1}) = \bigcup_{i=1}^{k} (x_i + (0, \delta_i)) + (0, N^{-1})$$

$$= \bigcup_{i=1}^{k} (x_i M^{-1} + (0, \delta_i)) \text{ for } y_i = (Mx_i + 1), i = 1, \ldots, k$$

$$= \bigcup_{i=1}^{k} (y_i M^{-1} + (0, z_i N^{-1})) \text{ where } z_i = [N \delta_i], i = 1, \ldots, k$$

$$= \bigcup_{i=1}^{k} (y_i M^{-1} + M^{-1} I(z_i) + (0, M^{-1}))$$

$$= M^{-1} B + (0, M^{-1}) \quad \text{(A)}$$

where $B = \bigcup_{i=1}^{k} (y_i + I(z_i))$.

Now

$$M^{-1} B + (0, 2/M) = \bigcup_{i=1}^{k} (y_i M^{-1} + M^{-1}(0, z_i)) + (0, 2/M)$$

$$= \bigcup_{i=1}^{k} ([y_i M^{-1}, (y_i + 1) M^{-1}] + M^{-1}(0, z_i + 1))$$

$$= \bigcup_{i=1}^{k} (x_i + 1/M + [0, \delta_i])$$

$$= E + 1/M \quad \text{(B)}$$
Now suppose $F \subseteq \mathbb{Z}(M)$, $|F| \leq n$. Then there is, by lemma 3, a $d$ such that $M^{-1} F + d \subseteq \mathbb{Z}$. So

$$E + (0, 2/M) \supseteq M^{-1} F + d + (0, 2/M)$$

$$\supseteq M^{-1} F + (d, d + 1/M) + (0, 1/M)$$

$$\supseteq M^{-1} F + c/M + (0, 1/M)$$

where $c = (6d + 1)$. But by (B),

$$M^{-1} E + (0, 5/M) \supseteq E + 1/M + (0, 2/M)$$

$$\supseteq M^{-1}(F + c + 1) + (0, 1/M)$$

This is equivalent to

$$M^{-1}(E + I(5)) + (0, 1/M) \supseteq M^{-1}(F + c + 1) + (0, 1/M)$$

which implies that $F + c \subseteq B + I(5) - 1$.

We take $A = B + I(5) - 1$ (mod $M$). Then

$$|\{(j) A| M^{-1} = \eta((j) M^{-1} A + (0, M^{-1}))$$

$$= \eta((j)(M^{-1} B + M^{-1} I(5)) + (0, M^{-1}))$$

$$< \eta((j)(E + (0, M^{-1}) + M^{-1} I(5)))$$

$$= \eta((j) E + M^{-1} (0, 5j)) \eta $, by lemma 3.$

Proof of Theorem 2. We work in $\mathbb{Z}^n$ (mod $n$ modulo 1), for convenience. Choose $N(j, r, c/2j)$ to be the same as in Theorem 1. If $C \in B(M, r)$ then, from the definition

$$C = \sum_{i=1}^{\infty} N^{-1} F_i$$

for some $F_1, F_2, \ldots \in I(N)^n$.

Write $G_i = N^{-1}(F_i + N \mathbb{Z}^n) + N^{-1} (0, 1)^n$. Then $C = \bigcap_{i=1}^{\infty} G_i$. Also, since

$$|F_i| < r,$$

$F_i \subseteq G_i^{(1)} \times \ldots \times G_i^{(n)}$ for some $G_i^{(k)} \subseteq I(N)^n$, $|G_i^{(k)}| < r$, $1 \leq k \leq n$.

Applying Theorem 1 for each $G_i^{(k)}$, we have a $c_i^{(k)} \subseteq I(N)$ such that

$$G_i^{(k)} + c_i^{(k)} + N \mathbb{Z} \subseteq A + N \mathbb{Z}.$$

So if $c_i = c_i^{(1)}(F_i) = (c_i^{(1)}(1), \ldots, c_i^{(n)}(n))$,

$$F_i + c_i + N \mathbb{Z} \subseteq A + N \mathbb{Z}.$$

Let $d = \sum_{i=1}^{\infty} c_i N^{-1}$. Then
So, if \( B = A + \{0, 1\} \mod N, \) and \( K = \{\sum_{i=1}^{\infty} x_i n^{-i}; x_i \in B^n\}, \) we have \( C = d \subset K. \) Since \( K \) is a closed perfect self-similar Cantor set, we need only show (b). But

\[
K = \bigcap_{i=1}^{\infty} (N^{-1}(B + n_i \mathbb{Z})^n + N^{-1}(0, 1)^n)
\]

so

\[
(j) K = \bigcap_{i=1}^{\infty} (N^{-1}((j) B + n_i \mathbb{Z})^n + N^{-1}(0, j)^n)
\]

\[
= \bigcap_{i=1}^{\infty} (N^{-1}((j) B + I(j) + n_i \mathbb{Z})^n + N^{-1}(0, j)^n).
\]

Now

\[
(j) B + I(j) = (j) A + I(2j), \mod N
\]

and

\[
(j) A + I(2j) \leq 2j (j) A < \epsilon.
\]

**Proof of Theorem 2.** Again we work in \( \mathbb{T}^n. \) If \( F \subset \mathbb{T}^n \) is any countable set, let \( x_1, x_2, \ldots \) be an enumeration of \( F. \) In the notation of Theorem 1, let \( n_i = N(i, i, i^{-1}) \), \( A_i = A(n_i) \) (so that \( |(i) A_i| \leq i^{-1} n_i \)) and let

\[
M_i = n_1 \times n_2 \times \ldots \times n_i.
\]

Assuming that we have chosen \( c_k \in I(n_k)^n, \) \( 0 \leq k \leq i - 1, \) we choose \( C_i \) as follows.

There is an \( F_i \subset I(n_i)^n, \) \( |F_i| \leq i \) such that

\[
M_i-1 \{x_1, \ldots, x_i\} + \mathbb{Z}^n \subset n_i^{-1} F_i + n_i^{-1}(0, 1)^n + \mathbb{Z}^n
\]

(this is just a way of saying that we need at most \( i \) intervals to contain \( i \) points).

As in the proof of Theorem 2, we may choose \( c_i \) so that

\[
P_i + c_i + n_i \mathbb{Z}^n \subset A_i^n + n_i \mathbb{Z}^n,
\]

so

\[
(1) \{x_1, \ldots, x_i\} + c_i n_i^{-1} + n_i^{-1} \mathbb{Z}^n \subset n_i^{-1} A_i^n + n_i^{-1}(0, 1)^n + n_i^{-1} \mathbb{Z}^n \ldots
\]

For \( k \in \mathbb{N}, \ E \subset \mathbb{T}^n, \ u(k^{-1} E + k^{-1} \mathbb{Z}^n) = u(E), \) so if

\[
c_i = M_i^{-1} A_i^n + M_i^{-1} \mathbb{Z}^n + M_i^{-1}(0, 2)^n,
\]
and so

\[(2) \quad \mu((i) \mathbf{c}_i) = \mu((i) (\mathbf{m}_i^{-1} A_i^n + \mathbf{u}_i^{-1} (0, 2)^n)) \leq |(i) A_i^n| \times 2^n/\mu_i^n < 2^n/\mu_i^n.\]

Let \( c = \sum_{i=1}^{\infty} c_i M_i^{-1} \), and suppose \( x_i \) is any member of \( F \). Then, if \( k \geq i \),

\[x_i + c = x_i + \sum_{j=1}^{k-1} c_j M_j^{-1} + \sum_{j=k}^{\infty} c_j M_j^{-1}\]

\[\leq x_i + \sum_{j=1}^{k-1} c_j M_j^{-1} + n^{-1} Z \]

\[= x_i + c_k N_k^{-1} + n^{-1} Z + \sum_{j=k+1}^{\infty} c_j N_j^{-1}\]

\[\subseteq \{x_1, \ldots, x_k, c_k N_k^{-1} + n^{-1} Z + c_k (0, 1)^n\}\]

\[= \mathbf{c}_k \quad \text{(from (1)).}\]

So \( x_i + c \in \bigcap_{k=i}^{\infty} \mathbf{c}_k \). If we put \( K = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} \mathbf{c}_k \), \( K \) is \( F \) and \( F + c \subseteq K \).

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