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ON BLASCHKE MANIFOLDS AND HARMONIC MANIFOLDS

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0. — A compact riemannian manifold $M$ is called a Blaschke manifold if the diameter of $M$ and the injectivity radius of $M$ coincide. It is known that if $M$ is a Blaschke manifold, then $M$ is diffeomorphic to $S^n$ or $\mathbb{RP}^n$, or $\pi_1(M) = \{0\}$ and $H^*(M,\mathbb{Z}) \cong \mathbb{Z}$. The $\mathbb{Z}$-cohomology ring of $CP^n$, $HP^n$, $CaP^2$.

The main problem about Blaschke manifolds is to know if the following conjecture, the Blaschke conjecture, is true or not: if $M$ is a Blaschke manifold, then it would be a compact rank one symmetric space.

There are classes of riemannian manifolds related to Blaschke manifolds. A riemannian manifold $M$ is called a globally harmonic manifold if the determinant of $d(\exp_p)_x : T_pM \to T_{\exp_p x}M$ ($p \in M$, $x \in T_pM$) depends only on the norm $|x|$. A compact riemannian manifold is called a $C_1$-manifold if all of its geodesics are closed and have the same length 1. The relation is as follows:

compact, simply connected, globally harmonic $\implies$ Blaschke $\implies$ $C_1$.

The following results are known:

1. (Green, Berger et al.). — If $(S^n,g)$ is a Blaschke manifold, then it is isometric to the standard one.

2. (Green, Berger et al.). — If $(\mathbb{RP}^n,g)$ is a $C_1$-manifold, then it is isometric to the standard one.

3. (Kiyohara). — Let $P$ be one of the projective spaces $CP^1$, $HP^n$ ($n \geq 2$), $CaP^2$, and let $(P,g)$ be a $C_1$-manifold. If the metric $g$ is sufficiently close to the standard $C_1$-metric $g_0$, then $(P,g)$ is isometric to the standard one $(P,g_0)$.

4. (Zoll, Weinstein). — There are non-standard $C_1$-manifolds $(S^n,g)$ for any dimension $n \geq 2$. 

1. From now on we assume $M$ is a Blaschke manifold, $\pi_1(M) = \{0\}$, $H^*(M, \mathbb{Z}) \cong H^*(CP^n, \mathbb{Z})$ (dim $M = 2n$, $n \geq 2$), and the diameter of $M$ is $\pi/2$. The followings are known about $M$:

1) For any $p \in M$ and any $q \in \text{Cut}(p)$ (the cut locus of $p$), the distance $d(p, q) = \pi/2$.

2) Every cut locus is a submanifold of codimension 2.

3) Let $\rho$ be the bundle projection $TM \to M$, and let $\{\zeta_i\}$ be the geodesic flow on $SM$. Then $\rho \circ \zeta_{\pi/2} : S_p(M) \to \text{Cut}(p)$ is a fibre bundle whose fibres are great circles on $S_p M$.

4) For $p, q \in M$ with $d(p, q) = \pi/2$, we denote by $\Sigma(p, q)$ the union of geodesic orbits through $p$ and $q$. Then $\Sigma(p, q)$ is a 2-dimensional submanifold diffeomorphic to $S^2$.

Now we define a mapping $I : SM \to SM$ as follows: since $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$, we fix a positive generator. Then on each $\Sigma(p, q)$ the orientation is determined. Hence we have an orientation on each fibre $S^1$ of the fibre bundle $\rho \circ \zeta_{\pi/2} : S_p M \to \text{Cut}(p)$, because the fibre $S^1$ over $q \in \text{Cut}(p)$ is nothing but the unit sphere of $T_p \Sigma(p, q)$. So $I : SM \to SM$ is defined by the conditions:

1) If $v \in S_p M$, then $Iv \in S_p M$ and $\rho(\zeta_{\pi/2}v) = \rho(\zeta_{\pi/2}Iv)$.

2) $\langle v, Iv \rangle = 0$.

3) $\{v, Iv\}$ is positive in this order.

We extend the mapping $I$ to $TM \setminus \{0\}$ homogeneously, and let $I_v : T_{p(v)}M \to T_{p(v)}M$ be the differential of $I|T_{p(v)}M \setminus \{0\}$ at $v$. From the definition the mapping $I$ satisfies $I \circ I = (-1)$ identity. So it looks like an almost complex structure, and we have the following

**Proposition A.** Assume $I_v^2 + 1 = 0$ for all $v \in SM$. Then $I : T_p M \setminus \{0\} \to T_p M \setminus \{0\}$ can be extended to a linear mapping on $T_p M$ for every $p \in M$, i.e. $I$ is an almost complex structure and it is integrable. Therefore $(M, I)$ is a hermitian manifold. Moreover each cut locus is a complex submanifold and is holomorphically isomorphic to $CP^{n-1}$.

**Proposition B.** Assume $\dim M = 4$. If $I_v^2 + 1 = 0$ for all $v \in SM$ and if every cut locus is minimal, then $M$ is isometric to $(CP^2, g_0)$.

**Lemma C.** If $M$ is moreover globally harmonic, then $(I_v^2 + 1)^{n-1} = 0$ for every $v \in SM$ and every cut locus is minimal (dim $M = 2n$).

**Corollary D.** If $\dim M = 4$ and $M$ is globally harmonic, then $M$ is isometric to $(CP^2, g_0)$.
Remarque. — This corollary is already known by a different method. See [1].

2. — For the proof of propositions we need some lemmas.

**Lemma 1.** There is a Jacobi field $Y(t)$ along the geodesic $\gamma_v(t) = \rho(\zeta_t v)$ such that

$$
\begin{bmatrix}
Y(0) \\
Y'(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
I_v
\end{bmatrix}, \quad \begin{bmatrix}
Y(\pi/2) \\
Y'(\pi/2)
\end{bmatrix} = \begin{bmatrix}
0 \\
-I_{\overline{v}}
\end{bmatrix}, \quad \overline{v} = \zeta_{\pi/2} v.
$$

Moreover if a Jacobi field $X(t)$ along $\gamma_v(t)$ satisfies $X(0) = X(\pi/2) = 0$ then $X(t)$ is a constant multiple of $Y(t)$.

For $X, Y \in T_p M, \neq 0$, we put $\nabla_X I \cdot Y = \nabla_{\partial/\partial t}(IY_t)_{t=0}$, where we take a curve $c(t)$ in $M$ such that $c'(0) = X$, and $Y_t$ is the parallel displacement of $Y$ along $c(t)$. $\nabla_X I \cdot Y$ is linear in $X$, but not necessarily in $Y$.

**Lemma 2.** Let $Y(t)$ be a periodic Jacobi field along the geodesic $\gamma_v(t)$, $v \in SM$. Then we have a periodic Jacobi field $Z(t)$ along the geodesic $\gamma_{e^{sI}v}(t)$ ($e^{sI}v = v \cos s + I v \sin s$) such that

$$
\begin{bmatrix}
Z(0) \\
Z'(0)
\end{bmatrix} = \begin{bmatrix}
Y(0) \\
(\cos s + \sin s I_v)v + \sin s(\nabla I \cdot v)Y(0)
\end{bmatrix},
\begin{bmatrix}
Z(\pi/2) \\
Z'((\pi/2)
\end{bmatrix} = \begin{bmatrix}
Y(\pi/2) \\
(\cos s + \sin s I_v)v - \sin s(\nabla I \cdot \overline{v})Y(\pi/2)
\end{bmatrix},
$$

where $(\nabla I \cdot v)Y(0) = \nabla Y(0)I \cdot v$, etc.

**Lemma 3.**

1) There are Jacobi fields $Y_1(t), Y_2(t)$ along $\gamma_v(t)$ such that

$$
\begin{bmatrix}
Y_1(0) \\
Y_1'(0)
\end{bmatrix} = \begin{bmatrix}
I_v \\
-I_{\overline{v}}
\end{bmatrix}, \quad \begin{bmatrix}
Y_1(\pi/2) \\
Y_1'(\pi/2)
\end{bmatrix} = \begin{bmatrix}
-I_{\overline{v}} \\
\nabla_{\overline{v}} I_{\overline{v}}
\end{bmatrix}
$$

$$
\begin{bmatrix}
Y_2(0) \\
Y_2'(0)
\end{bmatrix} = \begin{bmatrix}
2\nabla_v I \cdot v \\
R(I_v, v) - \nabla_v^2 I \cdot v
\end{bmatrix}, \quad \begin{bmatrix}
Y_2(\pi/2) \\
Y_2'(\pi/2)
\end{bmatrix} = \begin{bmatrix}
-2\nabla_v I \cdot \overline{v} \\
-R(I_{\overline{v}}, \overline{v}) + \nabla_{\overline{v}} I \cdot \overline{v}
\end{bmatrix}.
$$

2) $\nabla_{e^{sI}v} I \cdot d^s I v = \nabla v I \cdot v$.

**Lemma 4.** Let $Y(t)$ be a periodic Jacobi field along $\gamma_v(t)$. Then there is a periodic Jacobi field $Z(t)$ along $\gamma_v(t)$ such that

$$
\begin{bmatrix}
Z(0) \\
Z'(0)
\end{bmatrix} = \begin{bmatrix}
I_v Y(v) + (\nabla I \cdot v - \nabla I \cdot v + Y(0) + (\nabla I \cdot v + Y'(0)I_v))
\end{bmatrix},
\begin{bmatrix}
Z(\pi/2) \\
Z'(\pi/2)
\end{bmatrix} = \begin{bmatrix}
I_v Y(\pi/2) \\
-I_{\overline{v}} - Y(\pi/2) - (\nabla I_{\overline{v}} - \nabla I_{\overline{v}})Y(\pi/2) + Y'(0)I_v + Y'(\pi/2)I_{\overline{v}}
\end{bmatrix}.
$$

3. Proof of Proposition A. — Fix $p \in M$ and consider the $S^1$-principal bundle $p \circ \zeta_{\pi/2} : S_p M \rightarrow \text{Cut}(p)$, where the $S^1$-action is given by $e^{sI}, 0 \leq s \leq 2\pi$. 

We define a 1-form \( \omega \) on \( S_p M \) by
\[
\omega(X) = \langle X, I_* v \rangle, \quad X \in T_v(S_p M) = \{ Y \in T_p M | \langle v, Y \rangle = 0 \}.
\]
As is easily seen, \( \omega \) is a connection form, \textit{i.e.} invariant under the \( S^1 \)-action. We have
\[
d\omega(X,Y) = \langle (I_* v - t I_* v) X, Y \rangle.
\]
So there is a unique closed 2-form \( \Omega \) on \( \text{Cut}(p) \) such that \( (\rho \circ \zeta_{\pi/2})^* \Omega = d\omega \). We can see that \([1/(2\pi)\Omega]\) is a generator of \( H^2(\text{Cut}(p), \mathbb{Z}) \cong \mathbb{Z} \). Therefore
\[
\frac{1}{(2\pi)^{n-1}} \int_{\text{Cut}(p)} \Omega^{n-1} = 1
\]
under a proper orientation of \( \text{Cut}(p) \), and thus
\[
\int_{S_p M} \omega \wedge (d\omega)^{n-1} = (2\pi)^n.
\]
Now put \( J_v = I_* v - t I_* v \), \( S_v = I_* v + t I_* v \). Then \( 2I_* v = J_v + S_v \) and
\[
I_* v + 1 = 0 \iff J_v^2 + S_v^2 + 4 + J_v S_v + S_v J_v = 0 \quad \text{(*)}
\]
Let \( e_1, \ldots, e_{2n-2} \) be an orthonormal basis of the orthogonal complement to \( R_v + R_I v \) in \( T_p M \) such that \( J_v e_{2i-1} = \lambda_i e_{2i} \), \( J_v e_{2i} = -\lambda_i e_{2i-1} \), \( \lambda_i > 0 \), \( i = 1, \ldots, n - 1 \). By (*) we have
\[
-\lambda_i^2 + |S_v e_{2i}|^2 + 4 = 0.
\]
Hence \( \lambda_i \geq 2 \), and \( \lambda_i = 2 \) for every \( i \) if and only if \( S_v = 0 \). Then
\[
(\omega \wedge (d\omega)^{n-1})(I_* v, e_1, \ldots, e_{2n-2}) = (n - 1)! \prod_{i=1}^{n-1} \lambda_i \geq 2^{n-1}(n - 1)!,
\]
and the equality holds if and only if \( S_v = I_* v + t I_* v = 0 \). Therefore we have
\[
(2\pi)^n = \int_{S_p M} \omega \wedge (d\omega)^{n-1} \geq 2^{n-1}(n - 1)! \text{vol}(S_p M).
\]
But \( \text{vol}(S_p M) \) is just \( 2\pi^n/(n - 1)! \). So the equality holds in the above inequality. Hence we have \( S_v = I_* v + t I_* v = 0 \) for any \( v \in SM \). Since \( I_* v + 1 = 0 \), it follows that \( I_* v I_* v = 1 \). This implies that the mapping \( I : S_p M \to S_p M \) is an isometry, and therefore the restriction of a linear orthogonal transformation of \( T_p M \). Hence \( I \) is extended as a tensor field of type \((1,1)\) with \( I^2 = -1 \), \textit{i.e.} an almost complex structure on \( M \), and \((M, I)\) is an almost hermitian manifold.

By using the square of the endomorphisms on the space of Jacobi fields in Lemma 4, one gets
\[
\{ [I(I \cdot v - t I \cdot v) - (I \cdot v - t I \cdot v)I] X, Y \} = 0, \quad X, Y \perp v, I_* v.
\]
Moreover Lemma 3 (2) gives
\[
\{ [I(I \cdot v + t I \cdot v) - (I \cdot v + t I \cdot v)I] X, Y \} = 0, \quad X, Y \perp v, I_* v.
\]
These formula gives
\[ \nabla_{IX}I = I\nabla_XI \]
for any vector \( X \).

By this it is easy to see that the Nijenhuis's tensor vanishes, and \((M, I)\) turns out to be a hermitian manifold.

It is now clear that each cut locus is a complex submanifold of \( M \) and the \( S^1 \)-fibration \( \rho o \zeta_{\pi/2} : S_p M \to \text{Cut}(p) \) is nothing but the standard Hopf fibration: \( S^{2n-1} \to \mathbb{C}P^{n-1} \). Hence the last statement of the proposition follows.

Proof of Proposition B. — For \( v \in SM \) we define the symmetric endomorphism \( \Phi_v \) of \( T_{\rho(v)}M \) by \( \Phi_v v = \Phi_v I v = 0 \) and
\[ \langle \Phi_v X, Y \rangle = -(h(X, Y), v) \quad X, Y \in T_{\rho(v)}M \ , \ X, Y \perp v, IV \ , \]
where \( h \) is the second fundamental form of \( \text{Cut}(\rho \zeta_{\pi/2} v) \) in \( M \) at \( \rho(v) \). If we take a curve \( c(t) \) in \( \text{Cut}(\rho \zeta_{\pi/2} v) \) such that \( c'(0) = X \), and a normal vector field \( v_t \) to \( \text{Cut}(\rho \zeta_{\pi/2} v) \) along \( c(t) \), we have
\[ \langle \Phi_v X, Y \rangle = \langle \nabla_{\partial/\partial t} v_t |_{t=0}, Y \rangle . \]

So the following lemma is clear.

**Lemma 5.** — \( \Phi_I v X = I\Phi_v X + (\nabla_X I)v \ , \ X \in T_{\rho(v)}M \ , \ X \perp v, IV \ . \)

Since every cut locus is minimal, it follows that \( \text{tr} \Phi_v = 0 \) for any \( v \in Sm \), \( \text{tr} \) being the trace. Hence in view of Lemma 5 one gets
\[ \text{tr} \langle \nabla I \rangle v = 0 . \]

This together with the formula \( \nabla_{IX}I = I\nabla_XI \), shown in the proof of Proposition A, implies that \( \nabla I = 0 \), i.e. \((M, I)\) is kählerian.

By applying Lemma 1 to the Jacobi field \( Y_2 \) in Lemma 3,
\[ R(Iv, v)v = c(v) Iv \ , \ v \in SM \ , \]
where \( c \) is a function on \( SM \) satisfying \( c(\zeta_{\pi/2} v) = c(v) \). As is easily seen, \( c(v) \) is pointwise constant, i.e. if \( v_1 \) and \( v_2 \) are based at the same point on \( M \), then \( c(v_1) = c(v_2) \). Using the fact that for any two points \( p \) and \( q \) on \( M \), there is a point \( m \) such that \( d(p, m) = d(q, m) = \pi/2 \), we see the constancy of \( c(v) \).

Since \((M, I)\) is kählerian and has constant holomorphic sectional curvature, it must be holomorphically isometric to \((\mathbb{C}P^2, g_0)\).

Lemma C is an immediate consequence of Lemma 4.

**Reference**