Some topics in spectral geometry


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SOME TOPICS IN SPECTRAL GEOMETRY

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The first two theorems of this note express a quasisymmetry relation between the positive and the negative part of the eigenfunctions of the Laplace operator on a Riemannian manifold.

Let $M$ be a two-dimensional compact real analytic Riemannian manifold, $u_1, u_2, \ldots$ the eigenfunctions of the Laplace operator on $M$,
\[ \Delta u_i = \lambda_i u_i. \]

**THEOREM 1.** — There exists a positive constant $C$ which depends on $M$ such that, for every $i = 1, 2, \ldots$
\[ \text{vol}\{x \in M, \ u_i(x) > 0\} > C. \]

**PROBLEM 1.** — Is the analytic condition in theorem 1 essential? Is it possible to prove theorem 1 for $n$-dimensional manifolds with $n > 2$, for example in the case $M = S^3$ with the standard Riemannian metric?

Let $M$ be an $n$-dimensional compact smooth Riemannian manifold, $u_1, u_2, \ldots$, the eigenfunctions of the Laplace operator on $M$.

**THEOREM 2.** — There exists a positive constant $C$ which depends only on $n$, an integer $N$ which depends on $M$, such that, for all $i > N$,
\[ \frac{1}{C} < \frac{\sup_M u_i}{\left| \inf_M u_i \right|} < C. \]

In the article [1] we proved

**THEOREM 3.** — The multiplicity of the first non zero frequency of a bounded and plane simply connected membrane with free boundary is not more than 3.

In [1] we also proved that the condition that the membrane is simply connected in theorem 3 is essential. We built an example of membrane with three holes for which the multiplicity of the first non zero frequency is equal to 3.
PROBLEM 2. — What is the sharp estimate for the multiplicity of the first non-zero frequency of a plane membrane with free boundary which has one or two holes?

Let us consider the problem of the vibrations of an elastic beam. The energy of the deformation of the homogeneous elastic beam will be written in the form

$$\int_0^1 |u''|^2 \, dx .$$

The energy of the deformation of a nonhomogeneous elastic beam will be written in the form

$$\int_0^1 |a(x)u''(x)|^2 \, dx ,$$

where $a(x) > 0$. So, the main frequency of the vibrations of the nonhomogeneous elastic beam can be represented in a variational form as,

$$\inf_{u \in W_2^2(0,1)} \frac{\int_0^1 |a(x)u''(x)|^2 \, dx}{\int_0^1 u^2(x) \, dx} . \quad (1)$$

THEOREM 4. — The first eigenfunction $u_1$ of the vibrations of a nonhomogeneous elastic beam with fixed ends has a constant sign on $(0,1)$ and, hence, the corresponding frequency is simple.

Proof. — Let us assume that the contrary holds. Thus we assume that there exists points $0 < x_1 < x_2 < x_3 < 1$ such that $u_1(x_1) > 0, u_1(x_2) = 0, u_1(x_3) < 0$. Denote

$$\tilde{u} = \begin{cases} u_1(x), & x \in [0,x_2], \\ -u_1(x), & x \in [x_2,1]. \end{cases}$$

Let us take the common tangent line to the part of the graph of the function $\tilde{u}$ which is above the axis, on the segment $[0,x_2]$ and to the part of the graph which is above the axis on the segment $[x_2,1]$, (see the pictures below):
We built a new function $v$ from the two parts of the graph of the function $u$ and the segment of the tangent line between them. It is clear that:

$$\int_0^1 |a(x)u''(x)|^2 dx > \int_0^1 |a(x)v''(x)|^2 dx,$$

$$\int_0^1 u_1^2(x)dx < \int_0^1 v^2(x)dx$$

and thus, $u_1$ cannot be a solution of the variational problem (1).

**Problem 3.** — *Is it possible to have a Sturm oscillating theory for the elastic nonhomogeneous beam similar to the one used for the string?*

Let us consider the problem of vibration of a plane plate with fixed boundaries,

$$\Delta \Delta u = \lambda u \text{ in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain.

The main frequency of a plate can be not simple. At this point the vibration of the plate is different from the vibration of a membrane with a fixed boundary.

**Theorem 5.** — *There exists a bounded domain $\Omega \subset \mathbb{R}^2$ such, that the first eigenvalue of the problem (2) has multiplicity two.*

**Problem 4.** — *Does there exist an a-priori estimate for the multiplicity of the first eigenvalue of the problem (2) which does not depend on the geometry of the domain $\Omega$?*

Let $L$ be a selfadjoint differential operator on $[0,1]$ of order $2n$ with constant coefficients. Let us consider the spectral problem:

$$Lu = \lambda u \text{ on } [0,1],$$

$$\frac{\partial u^{(i)}}{\partial x^i}(0) = \frac{\partial u^{(i)}}{\partial x^i}(1) = 0, i = 0, 1, \ldots, n - 1$$

It is clear that the multiplicity of the eigenvalues of the problem (3) is $\leq n$.

**Problem 5.** — *Is it possible to improve the last estimate?*
Bibliography