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Some aspects of the De Rham complex on non-compact manifolds

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In my talk I gave a report on some aspects of a recent joint work with Jochen Brüning.

The purpose is to develop a general framework for the investigation of elliptic complexes on non-compact manifolds and to illustrate at some examples the phenomena that occur.

I consider a Riemannian manifold M, hermitian vector bundles E_i over M, $0 \leqslant i \leqslant N$ and a family of differential operators, $d_i : C_0^\infty(E_i) \longrightarrow C_0^\infty(E_{i+1})$, such that

$$0 \longrightarrow C_0^\infty(E_0) \xrightarrow{d_0} C_0^\infty(E_1) \xrightarrow{d_1} \ldots \xrightarrow{d_{N-1}} C_0^\infty(E_N) \longrightarrow 0 \tag{1}$$

is an elliptic complex. Such a complex will be abbreviated by $(C_0^\infty(E), d)$. For convenience, we may think of the De Rham complex $(\Omega_0(M), d)$ of exterior differential forms. If M is compact, the operators d_i have a unique closed extension in $L^2(E_i)$. Of course, in general this is not the case for non-compact M. I am interested in closed extensions that preserve the complex structure. A choice of closed extensions $D_i \supset d_i$ with domains $\mathcal{D}_i \subset L^2(E_i) =: H_i$ such that we have a complex

$$(\mathcal{D}, D) \quad 0 \longrightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \ldots \xrightarrow{D_{N-1}} \mathcal{D}_N \longrightarrow 0 \tag{2}$$

is called an ideal boundary condition. The complex property means that $\operatorname{im} D_{i-1} \subset \mathcal{D}_i$ and $D_i \circ D_{i-1} = 0$.

Example. — The absolute and relative boundary conditions are defined as

$$D_i^a := d_{i,\max} := (\delta_i)^* \ ; \ D_i^r := d_{i,\min} := \overline{d_i}, \tag{3}$$

where δ_i denotes the formal adjoint of d_i.

The notions absolute and relative boundary condition are adapted from the De Rham complex on compact manifolds with boundary, since in this case they coincide with the usual absolute and relative elliptic boundary conditions for the Gauss-Bonnet operator (*cf.* [G] § 4.1).

(2) is an example of a Hilbert complex. Such complexes are investigated systematically in [BL]. I only mention a few properties. The homology of $(\mathcal{D}, D)$ are the spaces

$$\mathcal{H}_i = \operatorname{Ker} D_i / \operatorname{im} D_{i-1} \tag{4}$$

and the complex is called Fredholm if $\dim \mathcal{H}_i < \infty$. In this case one defines the index of the complex as

$$\operatorname{ind}(\mathcal{D}, D) := \sum_{i \geqslant 0} (-1)^i \dim \mathcal{H}_i \ . \tag{5}$$

The generalized harmonics are

$$\widehat{\mathcal{H}}_i := \operatorname{Ker} D_i \cap \operatorname{Ker} D_{i-1}^* \; . \tag{6}$$

PROPOSITION. — *Let $(\mathcal{D}, D)$ be a Hilbert complex.*

(1) One has the orthogonal decomposition (weak Hodge decomposition)

$$H_i = \widehat{\mathcal{H}}_i \oplus \overline{\operatorname{im} D_{i-1}} \oplus \overline{\operatorname{im} D_i^*} \; . \tag{7}$$

(2) If $(\mathcal{D}, D)$ is Fredholm the strong Hodge decomposition

$$H_i = \widehat{\mathcal{H}}_i \oplus \operatorname{im} D_{i-1} \oplus \operatorname{im} D_i^* \tag{8}$$

holds.

For the absolute boundary condition of the De Rham complex the homology is just the L^2-cohomology of M which has been investigated by several authors ([C], [Z]). In this case the weak Hodge decomposition is due to Kodaira.

The domains $\mathcal{D}_i$ are in general very complicated. Therefore it is an interesting question whether the homology of an ideal boundary condition of an elliptic complex can be computed with C^∞-sections. The answer is affirmative in general.

THEOREM. — *Let $(\mathcal{D}, D)$ be an ideal boundary condition of the elliptic complex $(C_0^\infty(E), d)$. Put*

$$\mathcal{E}_i := C^\infty(E_i) \cap \mathcal{D}_i \; . \tag{9}$$

Then the inclusion $(\mathcal{E}, D|_{\mathcal{E}}) \hookrightarrow (\mathcal{D}, D)$ induces an isomorphism on homology, i.e.

$$\mathcal{H}_i(\mathcal{D}, D) \cong \operatorname{Ker} D_i|_{\mathcal{E}_i} / \operatorname{im} D_{i-1}|_{\mathcal{E}_{i-1}} \tag{10}$$

For the absolute boundary condition of the De Rham complex, this theorem is due to Cheeger [C], who used a regularizing argument of De Rham [R]. But in this generality the proof requires completely different methods [BL], Theorem 3.5.

Finally, I am going to discuss a special aspect of the De Rham complex, namely Poincaré duality. For the formal adjoint δ_k of the exterior derivative d_k on an oriented Riemannian manifold M, one finds

$$\delta_k = (-1)^{Nk+1} *_{N-k} d_{N-k-1} *_{k+1} \; . \tag{11}$$

The question is, whether there is an ideal boundary condition $(\mathcal{D}, D)$ such that we have the equality

$$D_k^* = (-1)^{Nk+1} *_{N-k} D_{N-k-1} *_{k+1} \tag{12}$$

in the strong operator sense. This would imply that $*$ induces an isomorphism $\widehat{\mathcal{H}}_k \longrightarrow \widehat{\mathcal{H}}_{N-k}$, called Poincaré duality.

LEMMA. — *If* $N \equiv 0 \pmod 2$ *there is always an ideal boundary condition with* (12).

If $N \equiv 1 \pmod 2$ *there exists an ideal boundary condition* $(\mathcal{D}, D)$ *with* (12) *if and only if the symmetric operator*

$$D = \begin{cases} *_{(N+1)/2} d_{(N-1)/2}, & N \equiv 3 (\mathrm{mod}\, 4) \\ \dfrac{1}{\sqrt{-1}} *_{(N+1)/2}\, d_{(N-1)/2}, & N \equiv 1 (\mathrm{mod}\, 4) \end{cases} \tag{13}$$

has self-adjoint extensions.

So, the question is reduced to the question for self-adjoint extensions of a symmetric operator. If $N \equiv 3 \pmod 4$, D is a real differential operator, for which it is well known that it has self-adjoint extensions. For the remaining case $N \equiv 1 \pmod 4$ I have a very surprising example. But first, I recall the notion of deficiency index. For a symmetric operator D the deficiency indices are defined as

$$n_\pm = \dim \left(\mathrm{Ker}(D^* \mp \sqrt{-1} I) \right) \tag{14}$$

and it is a well known theorem of functional analysis that D has self-adjoint extensions if and only if $n_+ = n_-$. So $n_+ - n_-$ is the obstruction for having self-adjoint extensions.

THEOREM. — *Let* N *be a compact oriented Riemannian* $4k$*-manifold and put* $M := (0, \infty) \times N$ *with metric* $g = dx^2 + x^2 g_N$. *Then for the operator* D *in* (13) *the deficiency indices* $n_\pm$ *are finite and one has*

$$n_+ - n_- = \mathrm{sign}\, N\ .$$

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