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Typical surfaces and random graphs


<http://www.numdam.org/item?id=TSG_1998-1999__17__99_0>
In this talk, we describe an approach to the problem: What does a typical Riemann surface of large genus look like geometrically? In large part, this is joint work with Eran Makover.

As various parts of this program have been described elsewhere ([PS], [SGB], [LFE], [RCRS]), we will take the present occasion to describe some of the motivating ideas behind the program. See [FERS] for an announcement of results in this direction.

A central problem, which we have attacked from a number of points of view, is to come to some geometrical understanding of the following theorem, due to Selberg:

**Theorem 1 ([Sel]).** Let \( \Gamma = \text{PSL}(2, \mathbb{Z}) \), and let

\[
\Gamma_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{k} \right\}.
\]

Then the first eigenvalue \( \lambda_1(\mathbb{H}^2/\Gamma_k) \) satisfies

\[
\lambda_1(\mathbb{H}^2/\Gamma_k) \geq 3/16.
\]

The number 3/16 has been improved by Luo, Rudnick, and Sarnak [LRS], but we will not be interested here in precise constants. Rather, we will say that \( \lambda_1 \) of a Riemann surface is large if it is bounded below by a positive constant independent of the genus.

A natural question arising from Selberg's Theorem is whether the phenomenon of large first eigenvalue is something which is special for arithmetically defined surfaces, or whether it is a property enjoyed by "typical" Riemann surfaces, of which such arithmetically defined surfaces just happen to be good examples.

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To understand this question, we may perform the following thought experiment: let $R_g$ be a Riemann surface whose geometric description is like our usual picture of a Riemann surface:

![Figure 1: The surface $R_g$](image)

We have drawn on $R_g$ a curve which divides it into two pieces.

Instead of trying to visualize the first eigenvalue, we instead consider the Cheeger constant

$$h(R) = \inf_{C} \frac{\text{length}(C)}{\min(\text{area}(A), \text{area}(B))},$$

where $C$ is a (possibly disconnected) curve which splits $R$ into two parts $A$ and $B$.

It is then easy to see that as $g$ gets large, the Cheeger constant $h(R_g)$ tends to 0, as the surface is divided into two pieces of equal size by a curve such as the curve in Figure 1, whose length is fixed independent of the genus.

Now let us divide the surface in half, as in Figure 2 below, and then glue the legs of the top half randomly to legs in the bottom half. It is easy to convince oneself that for a suitably random gluing of the legs, there is no longer any convenient way to divide the surface in half by a relatively short curve.

One would like to believe that a typical Riemann surface looks more like one of the random gluings than like $R_g$ itself. The problem in making this precise is two-fold:

1. First of all, it would seem to be difficult to describe processes such as the random gluings in terms of, say, Fenchel-Nielsen coordinates. In general, it would seem to be difficult to use Fenchel-Nielsen coordinates to control the spectral geometry of a surface of large genus.
(ii) Secondly, the gluing process described above seems to rest on a combinatorial structure which would seem to be absent in a typical Riemann surface. How can one describe a typical Riemann surface in a way which reflects a combinatorial structure analogous to this?

Both of these difficulties are met by the following construction: let $G$ be a finite trivalent graph, and $\mathcal{O}$ an orientation of $G$—i.e., for each vertex $v$ of $G$, $\mathcal{O}$ gives a cyclic ordering of the vertices emanating from $v$.

We may then associate to the pair $(G, \mathcal{O})$ two Riemann surfaces $S^\mathcal{O}(G, \mathcal{O})$ and $S^\mathcal{C}(G, \mathcal{O})$, as follows: $S^\mathcal{O}(G, \mathcal{O})$ is constructed from $G$ by pasting one hyperbolic ideal triangle for each vertex, and gluing triangles together according to the graph and orientation, see [TS] for details. $S^\mathcal{C}(G, \mathcal{O})$ is then a finite-area Riemann surface, whose geometry is well-controlled by the pair $(G, \mathcal{O})$. $S^\mathcal{C}(G, \mathcal{O})$ is then the conformal compactification of $S^\mathcal{O}(G, \mathcal{O})$.

The two problems mentioned above can be rephrased in the following way:

**Question 1.** To what extent can we transfer the good geometric control that we have on the surfaces $S^\mathcal{O}(G, \mathcal{O})$ to the surfaces $S^\mathcal{C}(G, \mathcal{O})$?

**Question 2.** To what extent are the surfaces $S^\mathcal{C}(G, \mathcal{O})$ typical Riemann surfaces?

Question 2 is answered by the following theorem, which is an easy consequence of the Belyi Theorem [Be]:

**Theorem 2.** If $S$ is any compact Riemann surface, then for any $\varepsilon$, there is a surface of the form $S^\mathcal{C}(G, \mathcal{O})$ is $\varepsilon$-close to $S$. 

*Figure 2: A random gluing*
Here, "\( \varepsilon \)-close" may be taken in any convenient metric on moduli space, for instance the Teichmüller metric. Thus, the surfaces \( S^C(G, \mathcal{O}) \) are a dense set of surfaces in the moduli space of all surfaces.

The answer to Question 1 is somewhat more complicated. It is not hard to see that the surfaces \( S^O(G, \mathcal{O}) \) and \( S^C(G, \mathcal{O}) \) might be quite different geometrically. For instance, \( S^O(G, \mathcal{O}) \) always carries a complete hyperbolic metric, but \( S^C(G, \mathcal{O}) \) might be a sphere. However, the theorem of [PS] guarantees that this cannot happen when the cusps are large:

**Theorem 3 ([PS]).** — For any \( \varepsilon \), there exists an \( L \) with the following property: if \( S^O \) is a finite-area Riemann surface, all of whose cusps have length \( \geq L \), then outside of cusp neighborhoods, depending only on \( L \), the hyperbolic metrics \( ds^2_O \) on \( S^O \) and \( ds^2_C \) on its conformal compactification \( S^C \) satisfy:

\[
\frac{1}{1 + \varepsilon} \leq ds^2_C \leq (1 + \varepsilon) ds^2_O.
\]

The proof is an application of the Ahlfors-Schwarz Lemma [A], together with playing with differential inequalities.

When the condition of large cusps is satisfied, Theorem 3 can be used to show that the geometric control one has over \( S^O(G, \mathcal{O}) \) transfers to control over \( S^C(G, \mathcal{O}) \). Furthermore, the large cusps condition has a simple graph-theoretic interpretation which is easily studied.

In [SGB] and [RCRS], we use the Bollobas model of random regular graphs [Bo1], [Bo2] to study the large cusps condition. Let \( \mathcal{G}_k \) denote the finite set of 3-regular graphs on \( 2k \) vertices, and \( \mathcal{G}_k^* \) the finite set of oriented 3-regular graphs on \( 2k \) vertices. Then:

**Theorem 4 ([SGB]).** — With probability \( \to 1 \) as \( k \to \infty \), a graph selected randomly from \( \mathcal{G}_k \) carries an orientation \( \mathcal{O} \) such that all the cusps of \( S^O(G, \mathcal{O}) \) are large.

**Theorem 5 ([RCRS]).** — There is a positive constant \( C_1 \) independent of \( k \) such that, for a pair \( (G, \mathcal{O}) \) randomly chosen from \( \mathcal{G}_k^* \), \( S^O(G, \mathcal{O}) \) has large cusps with probability at least \( C_1 \).

Theorems 4 and 5 can be used to construct compact surfaces which enjoy properties enjoyed by random 3-regular graphs. In particular, Theorem 5 shows that there is a constants \( C_2 \) such that a randomly chosen surface \( S^C(G, \mathcal{O}) \) satisfies

\[
\lambda_1(S^C(G, \mathcal{O})) \geq C_2
\]

with probability at least \( C_1 \).
REFERENCES


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