

THE CONNES-KASPAROV CONJECTURE FOR ALMOST CONNECTED GROUPS AND FOR LINEAR p -ADIC GROUPS

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Dedicated to the memory of Peter Slodowy

ABSTRACT

Let G be a locally compact group with cocompact connected component. We prove that the assembly map from the topological K -theory of G to the K -theory of the reduced C^* -algebra of G is an isomorphism. The same is shown for the groups of k -rational points of any linear algebraic group over a local field k of characteristic zero.

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1. Introduction and statement of results

The main result of this paper is a proof of the Connes-Kasparov conjecture as formulated in [3,29]. In addition, an analogue result is obtained for all linear algebraic groups over local fields of characteristic zero. To be more precise, we prove the following theorem:

Theorem 1.1. — *Let G be a second countable almost connected group (i.e., G/G_0 is compact, where G_0 denotes the connected component of G) or the group of k -rational points of a linear algebraic group over a local field k of characteristic zero (i.e., $k = \mathbf{R}$ or \mathbf{C} – in which cases G is also almost connected – or k is a finite extension of some p -adic number field \mathbf{Q}_p). Then G satisfies the Baum-Connes conjecture with trivial coefficient \mathbf{C} , i.e., if $K_*^{\text{top}}(G)$ denotes the topological K -theory of G , then the Baum-Connes assembly map*

$$\mu : K_*^{\text{top}}(G) \rightarrow K_*(C_r^*(G))$$

is an isomorphism.

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It is well known (see [3] and the discussion in §7 below) that the original Connes-Kasparov conjecture is equivalent to the Baum-Connes conjecture with trivial coefficient for almost connected groups. So the above theorem gives a complete solution to the Connes-Kasparov conjecture and to its natural extension to linear algebraic groups over local fields of characteristic zero. For almost connected groups, Kasparov has shown in [31] that the result is true if the group is also amenable. In fact, by a more recent result of Higson and Kasparov, we know that the Baum-Connes conjecture with arbitrary coefficients holds for any amenable group. By work of A. Wassermann [54], extending earlier work of Valette [53], we also know that Theorem 1.1 is true for all connected reductive linear Lie groups. More recently, Lafforgue used quite different methods to give a proof in the case where G is a real reductive group whose semi-simple part has finite centre (see [38, §4.2] – we should point out that this does not cover all real reductive groups) or where G is a reductive p -adic group, which then also extends a result of Baum, Higson and Plymen on p -adic $\mathrm{GL}(n)$ (see [4]). The main idea of the proofs of our general results is to use the Mackey-machine approach, as outlined in [12], in order to reduce to the reductive case. The strategy for doing this relies heavily on some ideas presented in Pukánszky's recent book [49] where he reports on his deep analysis of the representation theory of connected groups. In particular the methods of his proof that locally algebraic connected real Lie groups are type I, presented on the first four pages of his book, were most enlightening.

The result on almost connected groups in Theorem 1.1 is actually a special case of a more general result which we shall explain below. If G is a second countable locally compact group, then by a G -algebra A we shall always understand a C^* -algebra A equipped with a strongly continuous action of G by $*$ -automorphisms of A . Let $\mathcal{E}(G)$ denote a locally compact universal proper G -space in the sense of [33] (we refer to [13] for a discussion about the relation to the notion of universal proper G -space as introduced by Baum, Connes and Higson in [3]). If A is a G -algebra, the topological K -theory of G with coefficient A is defined as

$$K_*^{\mathrm{top}}(G; A) = \lim_{\mathbf{X}} \mathrm{KK}_*^G(C_0(\mathbf{X}), A),$$

where \mathbf{X} runs through the G -compact subspaces of $\mathcal{E}(G)$ (i.e., \mathbf{X}/G is compact) ordered by inclusion, and $\mathrm{KK}_*^G(C_0(\mathbf{X}), A)$ denotes Kasparov's equivariant KK -theory. If $A = \mathbf{C}$, we simply write $K_*^{\mathrm{top}}(G)$ for $K_*^{\mathrm{top}}(G; \mathbf{C})$.

The construction of Baum, Connes and Higson presented in [3, §9] produces a homomorphism

$$\mu_A : K_*^{\mathrm{top}}(G; A) \rightarrow K_*(A \rtimes_r G),$$

usually called the *assembly map*. We say that G satisfies BC for A (i.e., G satisfies the Baum-Connes conjecture for the coefficient algebra A), if μ_A is an isomorphism. The result on almost connected groups in Theorem 1.1 is then a special case of

Theorem 1.2. — *Suppose that G is any second countable locally compact group such that G/G_0 satisfies BC for arbitrary coefficients, where G_0 denotes the connected component of G . (By the results of Higson and Kasparov [26] this is in particular true if G/G_0 is amenable or, more general, if G/G_0 satisfies the Haagerup property.) Then G satisfies BC for $\mathcal{K}(H)$, H a separable Hilbert space, with respect to any action of G on $\mathcal{K}(H)$.*

It is well known that in case of almost connected groups, the topological K-theory $K_*^{\text{top}}(G; A)$ has a very nice description in terms of the maximal compact subgroup L of G . In fact, under some mild extra conditions on G , the group $K_*^{\text{top}}(G; A)$ can be computed by means of the K-theory of the crossed product $A \rtimes L$. We give a brief discussion of these relations in §7 below. As was already pointed out in [50], our results have interesting applications to the study of square-integrable representations. In fact, combining our results with [50, Theorem 4.6] gives

Corollary 1.3 (cf [50, Corollary 4.7]). — *Let G be a connected unimodular Lie group. Then all square-integrable factor representations of G are type I. Moreover, G has no square-integrable factor representations if $\dim(G/L)$ is odd, where L denotes the maximal compact subgroup of G .*

The paper is outlined as follows: In our preliminary section, §2, we recall the main results from [11, 12] (which have been extended more recently in [14]) on the permanence properties of the Baum-Connes conjecture which are needed in this work. We will also use these results to perform some first reductions of the problem. In §3 we prove a result on continuous fields of actions, showing under some mild conditions on the group G and the base space X of the field, that G satisfies the Baum-Connes conjecture for the algebra of C_0 -sections of the field if it satisfies the conjecture for all fibres. This result will be another basic tool for the proof of our main theorem.

In §4 we are concerned with the conjecture for reductive groups. Using (and slightly extending) some recent results of Lafforgue [38], we will show that the results on continuous fields obtained in §3 imply that the conjecture (with trivial coefficient) holds for all reductive real groups without any extra conditions. In §5 we will then use Pukánszky’s methods in combination with an extensive use of the permanence properties for BC to give the final steps for the proof of Theorem 1.1 in case of almost connected groups. In §6 we shall give the proof of Theorem 1.1 for linear algebraic groups over local fields k which are finite extensions over \mathbf{Q}_p . Since algebraic groups over \mathbf{R} or \mathbf{C} are almost connected, this will complete the proof of Theorem 1.1. Finally, in §7 we shall discuss the connection between the K-theory of the reduced group algebra of an almost connected group G with the representation ring of the maximal compact subgroup L of G as given in the original formulation of the Connes-Kasparov conjecture.

This paper has been circulated as a preprint since fall 2001. The only difference (up to some minor modifications) between the preprint and the present paper is the inclusion of the case of linear p -adic groups into this paper.

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2. Some preliminaries and first reductions

Let us collect some general facts which were presented in [12] – for the definitions of twisted actions and twisted equivariant KK-theory we refer to [11]. Assume that G is a second countable group and let B be a G - C^* -algebra. We say that G satisfies BC with coefficient B if the assembly map

$$\mu_B : K_*^{\text{top}}(G; B) \rightarrow K_*(B \rtimes_r G)$$

is an isomorphism. If N is a closed normal subgroup of G , there exists a twisted action of (G, N) on $B \rtimes_r N$ such that the twisted crossed product $(B \rtimes_r N) \rtimes_r (G, N)$ is canonically isomorphic to $B \rtimes_r G$. Moreover, we can use the twisted equivariant KK-theory of [11] to define the topological K-theory $K_*^{\text{top}}(G/N; B \rtimes_r N)$ with respect to the twisted action of (G, N) on $B \rtimes_r N$, and a twisted version of the assembly map

$$\mu_{B \rtimes_r N} : K_*^{\text{top}}(G/N; B \rtimes_r N) \rightarrow K_*((B \rtimes_r N) \rtimes_r (G, N)).$$

In [11] we constructed a partial assembly map

$$\mu_{N,B}^G : K_*^{\text{top}}(G; B) \rightarrow K_*^{\text{top}}(G/N; B \rtimes_r N)$$

such that the following diagram commutes

$$\begin{array}{ccc} K_*^{\text{top}}(G; B) & \xrightarrow{\mu_{N,B}^G} & K_*^{\text{top}}(G/N; B \rtimes_r N) \\ \mu_B \downarrow & & \downarrow \mu_{B \rtimes_r N} \\ K_*(B \rtimes_r G) & \xrightarrow{\cong} & K_*((B \rtimes_r N) \rtimes_r (G, N)). \end{array}$$

Extending earlier results of [11,12,46], the following theorem is shown in [14, Theorem 2.1]:

Theorem 2.1. — Let N be a closed normal subgroup of G and let B be a separable G -algebra. Assume that the following condition (A) holds:

(A) If $N \subseteq L$ is a closed subgroup of G and L/N is compact, then L satisfies BC for B .

Then G satisfies BC for B if and only if (G, N) satisfies BC for $B \rtimes_r N$.

Remark 2.2. — We know from [26] that all groups having the Haagerup property (in particular all amenable groups) satisfy BC for all coefficients. Hence, it follows from [15, Example 6.1.6] that condition (A) is automatically satisfied if N has the Haagerup property.

In order to avoid the use of twisted actions we may use the version of the Packer-Raeburn stabilization trick as given in [47, 21]:

Proposition 2.3 (cf [47, Theorem 3.4] and [21, Corollary 1]). — Assume that G is a second countable group and let N be a closed normal subgroup of G . Let (α, τ) be a twisted action of (G, N) on the separable C^* -algebra A . Then there exists an ordinary action $\beta : G/N \rightarrow \text{Aut}(A \otimes \mathcal{K})$, $\mathcal{K} = \mathcal{K}(l^2(\mathbf{N}))$, such that β is stably exterior equivalent (and hence Morita equivalent) to (α, τ) .

Note that BC is invariant under passing to Morita equivalent actions. Thus, in order to conclude that (G, N) satisfies BC for $B \rtimes_r N$, it is enough to show that G/N satisfies BC for $(B \rtimes_r N) \otimes \mathcal{K}$ with respect to an appropriate action of G/N on $(B \rtimes_r N) \otimes \mathcal{K}$. In particular, if G/N is amenable, it follows that $\mu_{B \rtimes_r N} : K_*^{\text{top}}(G/N; B \rtimes_r N) \rightarrow K_*((B \rtimes_r N) \rtimes_r (G, N))$ is always an isomorphism.

In what follows we need to study the following special situation: Assume that $\alpha : G \rightarrow \text{Aut}(\mathcal{H})$ is an action of G on $\mathcal{H} = \mathcal{K}(H)$ for some separable Hilbert space H . Since $\text{Aut}(\mathcal{H}) \cong \text{PU}(H) = \text{U}(H)/\mathbf{T}1$, we can choose a Borel map $V : G \rightarrow \text{U}(H)$ such that $\alpha_s = \text{Ad} V_s$ for all $s \in G$. Since α is a homomorphism, we see that there exists a Borel cocycle $\omega \in Z^2(G, \mathbf{T})$ such that

$$V_s V_t = \omega(s, t) V_{st} \quad \text{for all } s, t \in G.$$

The class $[\omega] \in H^2(G, \mathbf{T})$ is called the *Mackey obstruction* for α being unitary. Let

$$1 \mapsto \mathbf{T} \rightarrow G_\omega \rightarrow G \rightarrow 1$$

be the central extension of G by \mathbf{T} corresponding to ω , i.e., we have $G_\omega = G \times \mathbf{T}$ with multiplication given by

$$(g, z)(g', z') = (gg', \omega(g, g')zz'),$$

and the unique locally compact group topology which generates the product Borel structure on $G \times \mathbf{T}$ (see [44]). Then the following is true

Lemma 2.4. — For each $n \in \mathbf{Z}$ let $\chi_n : \mathbf{T} \rightarrow \mathbf{T}$; $\chi_n(z) = z^n$. Let $\alpha : G \rightarrow \text{Aut}(\mathcal{H})$ and G_ω be as above. Then α is Morita equivalent to the twisted action (id, χ_1) of (G_ω, \mathbf{T}) on \mathbf{C} .

Proof. — Let $V : G \rightarrow U(H)$ be as in the discussion above, i.e., $\alpha_s = \text{Ad } V_s$ and $V_s V_t = \omega(s, t) V_{st}$ for all $s, t \in G$. Then it is easy to check that $\tilde{V} : G_\omega \rightarrow U(H)$ defined by $\tilde{V}_{(s,z)} = z V_s$ is a homomorphism which implements the desired equivalence on the $\mathcal{K} - \mathbf{C}$ bimodule H (we refer to [21] for an extensive discussion of Morita equivalence for twisted actions). \square

Another important result is the continuity of the Baum-Connes conjecture with respect to inductive limits of the coefficients, at least if G is exact. For this we need

Lemma 2.5. — *Assume that $(B_i)_{i \in I}$ is an inductive system of G -algebras and let $B = \lim_i B_i$ be the C^* -algebraic inductive limit. Assume further that one of the following conditions is satisfied:*

- (i) *All connecting maps $B_i \rightarrow B_j$, $i \leq j \in I$ are injective, or*
- (ii) *G is exact.*

Then $B \rtimes_r G = \lim_i (B_i \rtimes_r G)$ with respect to the obvious connecting homomorphisms.

Proof. — If all connecting maps are injective, we may regard each B_i as a subalgebra of B . But this implies that we also have $B_i \rtimes_r G$ as subalgebras of $B \rtimes_r G$, and hence the inductive limit $\lim_i (B_i \rtimes_r G) = \overline{\cup \{B_i \rtimes_r G : i \in I\}}$ sits inside $B \rtimes_r G$. But it is easy to check that $\cup \{C_c(G, B_i) : i \in I\} \subseteq \lim_i (B_i \rtimes_r G)$ is dense in $B \rtimes_r G$.

Suppose now that G is exact. In this situation we want to reduce the proof to situation (i). Consider the canonical homomorphisms $\phi_i : B_i \rightarrow B$. Let $I_i = \ker \phi_i$ and let $I_{ij} = \ker \phi_{ij}$, where the $\phi_{ij} : B_i \rightarrow B_j$ denote the connecting homomorphisms for $j \geq i$. Of course, if $i \leq j \leq j'$ then $I_{ij} \subseteq I_{ij'}$, so for each $i \in I$ the system $(I_{ij})_{j \geq i}$ is an inductive system with injective connecting maps. It follows directly from the definition of the inductive limit that $I_i = \overline{\cup \{I_{ij} : j \geq i\}} = \lim_{j \geq i} I_{ij}$, and hence it follows from (i) that $I_i \rtimes_r G = \lim_{j \geq i} (I_{ij} \rtimes_r G)$. By exactness of G it follows that $I_i \rtimes_r G$ is the kernel of $\phi_i \rtimes_r G : B_i \rtimes_r G \rightarrow B \rtimes_r G$. By the previous discussion it follows that $I_i \rtimes_r G = \lim_{j \geq i} (I_{ij} \rtimes_r G)$ is also the kernel of the canonical homomorphism $B_i \rtimes_r G \rightarrow \lim (B_j \rtimes_r G)$. Thus, dividing out the kernels, i.e., by considering the system $(B'_i)_{i \in I}$ with $B'_i = B_i / I_i$ we conclude from another use of (i) that

$$B \rtimes_r G = \lim (B'_i \rtimes_r G) = \lim (B_i \rtimes_r G). \quad \square$$

As a direct consequence we obtain

Proposition 2.6. — *Assume that the G -algebra B is an inductive limit of the G -algebras B_i , $i \in I$, such that G satisfies BC for all B_i . Assume further that G is exact or that all connecting homomorphisms $B_i \rightarrow B_j$ are injective. Then G satisfies BC for B .*

Proof. — It follows from Lemma 2.5 and the continuity of K -theory that $K_*(B \rtimes_r G) = \lim_i K_*(B_i \rtimes_r G)$. On the other side, it is shown in [12, Proposition 7.1]

that $K_*^{\text{top}}(G; B) = \lim_i K_*^{\text{top}}(G; B_i)$. Since by assumption $K_*^{\text{top}}(G; B_i) \cong K_*(B_i \rtimes_r G)$ via the assembly map, and since the assembly map commutes with the K-theory maps induced by the G-equivariant homomorphism $B_i \rightarrow B_j$, the result follows. \square

As a first application we get

Proposition 2.7. — *Let G be a separable locally compact group such that G/G_0 satisfies BC for arbitrary coefficients. Then the following are equivalent:*

- (1) *For every central extension $1 \rightarrow \mathbf{T} \rightarrow \bar{G} \rightarrow G \rightarrow 1$ the group \bar{G} satisfies BC for \mathbf{C} .*
- (2) *G satisfies BC for the compact operators $\mathcal{K} \cong \mathcal{K}(\mathbf{H})$ for all separable Hilbert spaces H and with respect to all possible actions of G on \mathcal{K} .*

Proof. — Assume that (1) holds. Let $\alpha : G \rightarrow \text{Aut}(\mathcal{K})$ be any action of G on \mathcal{K} and let $[\omega] \in H^2(G, \mathbf{T})$ denote the Mackey obstruction for this action. Let

$$1 \rightarrow \mathbf{T} \rightarrow G_\omega \rightarrow G \rightarrow 1$$

denote the central extension determined by ω . It follows from Lemma 2.4 that α is Morita equivalent to the twisted action (id, χ_1) of (G_ω, \mathbf{T}) on \mathbf{C} . By assumption, we know that G_ω satisfies BC for \mathbf{C} . It follows from Theorem 2.1 that (G_ω, \mathbf{T}) satisfies BC for $C^*(\mathbf{T}) \cong C_0(\mathbf{Z})$, or, equivalently, that G satisfies BC for $C_0(\mathbf{Z}, \mathcal{K})$ with respect to the appropriate action of G (use Proposition 2.3).

Writing $C_0(\mathbf{Z}) = \bigoplus_{n \in \mathbf{Z}} \mathbf{C}$, the twisted action of (G_ω, \mathbf{T}) is given by the twisted action (id, χ_n) of (G_ω, \mathbf{T}) on the n'th summand. Let $q_1 : C_0(\mathbf{Z}) \rightarrow \mathbf{C}$ be the projection on the summand corresponding to $1 \in \mathbf{Z}$. Consider the diagram

$$\begin{CD} K_*^{\text{top}}(G; C_0(\mathbf{Z})) @>\mu_{C_0(\mathbf{Z})}>> K_*(C_0(\mathbf{Z}) \rtimes_r (G_\omega, \mathbf{T})) \\ @Vq_{1,*}VV @VVq_{1,*}V \\ K_*^{\text{top}}(G; \mathbf{C}) @>\mu_{\mathbf{C}}>> K_*(\mathbf{C} \rtimes_r (G_\omega, \mathbf{T})) \end{CD}$$

(Here the topological K-theory $K_*^{\text{top}}(G; \mathbf{C})$ is computed with respect to the twisted action (id, χ_1) of $G \cong G_\omega/\mathbf{T}$ and $\mu_{\mathbf{C}}$ denotes the twisted assembly map!) Since the vertical arrows are split-surjective and the upper horizontal arrow is bijective, it follows that the lower horizontal arrow is also bijective. Thus we see that (G_ω, \mathbf{T}) satisfies BC for \mathbf{C} with respect to the twisted action (id, χ_1) . By Morita equivalence this implies that G satisfies BC for \mathcal{K} with respect to α .

For the opposite direction assume that (2) holds. Let $1 \rightarrow \mathbf{T} \rightarrow \bar{G} \rightarrow G \rightarrow 1$ be as in (1). From Theorem 2.1, it follows that \bar{G} satisfies BC for \mathbf{C} if (\bar{G}, \mathbf{T}) satisfies BC for $C^*(\mathbf{T}) = C_0(\mathbf{Z})$. Using the stabilization trick, the latter is true if G satisfies BC for $C_0(\mathbf{Z}, \mathcal{K})$ with respect to an appropriate action of G on $C_0(\mathbf{Z}, \mathcal{K})$ which fixes the base \mathbf{Z} . Using continuity of BC, this follows easily from the fact that G satisfies BC for arbitrary actions on \mathcal{K} . \square

We also need a result on induced algebras as obtained in [12]. For this recall that if H is a closed subgroup of G and A is an H -algebra, then the induced algebra $\text{Ind}_H^G A$ is defined as

$$\text{Ind}_H^G A = \left\{ f \in C_b(G, A) : f(sh) = h^{-1}(f(s)) \text{ and } (sH \mapsto \|f(s)\|) \in C_0(G/H) \right\}.$$

Equipped with the pointwise operations and the supremum-norm, $\text{Ind}_H^G A$ becomes a C^* -algebra with G -action defined by

$$s \cdot f(t) = f(s^{-1}t).$$

The following result follows from [12, Theorem 2.2]:

Theorem 2.8. — *Let G, H, A and $\text{Ind}_H^G A$ be as above. Then G satisfies BC for $\text{Ind}_H^G A$ if and only if H satisfies BC for A .*

The result becomes most valuable for us when combined with the following result of [20]:

Proposition 2.9. — *Suppose that H is a closed subgroup of G and B is a G -algebra. Let \widehat{B} denote the set of equivalence classes of irreducible representations of B equipped with the usual G -action defined by $s \cdot \pi(b) = \pi(s^{-1} \cdot b)$. Then B is isomorphic (as a G -algebra) to $\text{Ind}_H^G A$ for some H -algebra A if and only if there exists a G -equivariant continuous map $\varphi : \widehat{B} \rightarrow G/H$. Moreover, if $\varphi : \widehat{B} \rightarrow G/H$ is such a map, then A can be chosen to be B/I , with $I = \cap \{\ker \pi : \varphi(\pi) = eH\}$ equipped with the obvious H -action.*

As a corollary of Theorem 2.8 and Proposition 2.9 we get in particular:

Corollary 2.10. — *Suppose that G is a locally compact group and B is a G -algebra which is type I and such that G acts transitively on \widehat{B} . Let $\pi \in \widehat{B}$ and let G_π denote the stabilizer of π for the action of G on \widehat{B} . Then G satisfies BC for B if and only if G_π satisfies BC for $B/\ker \pi \cong \mathcal{K}(H_\pi)$, where H_π denotes the Hilbert space of π .*

Proof. — Since there is only one orbit for the G -action on \widehat{B} , it follows from results of Glimm [24], that \widehat{B} is homeomorphic to G/G_π via $sG_\pi \mapsto s \cdot \pi$. In particular, it follows that \widehat{B} is Hausdorff, which implies that $B/\ker \pi \cong \pi(B) = \mathcal{K}(H_\pi)$. The inverse of the above map is clearly a continuous G -equivariant map of \widehat{B} to G/G_π , and Proposition 2.9 then implies that $B \cong \text{Ind}_{G_\pi}^G (B/\ker \pi)$. The result then follows from Theorem 2.8. \square

We now give a short outline of the proof of Theorem 1.1. The main work is required for proving the following proposition:

Proposition 2.11. — Assume that G is a Lie group with finitely many components and let $\alpha : G \rightarrow \text{Aut}(\mathcal{K})$ be an action of G on the compact operators on some separable Hilbert space H . Then G satisfies BC for \mathcal{K} .

The body of this paper is devoted to give a proof of this proposition by using induction on the dimension of G . From Proposition 2.11, we deduce the following statement which implies Theorem 1.2 by an easy application of Theorem 2.1.

Proposition 2.12. — Let G be any almost connected group and let $\alpha : G \rightarrow \text{Aut}(\mathcal{K})$ be any action of G on the compact operators on some separable Hilbert space H . Then G satisfies BC with coefficient \mathcal{K} .

Proof. — By the structure theory of almost connected groups (e.g. see [45]) we can find a compact normal subgroup $C \subseteq G$ such that G/C is a Lie group with finitely many components. Using Theorem 2.1 we see that G satisfies BC for \mathcal{K} if and only if G/C satisfies BC for $\mathcal{K} \rtimes C$ (with respect to an appropriate twisted action). Since C is compact, it follows that $X := (\mathcal{K} \rtimes C)^\wedge$ is discrete, and (after stabilizing if necessary) $\mathcal{K} \rtimes C \cong C_0(X, \mathcal{K})$. Let $\tilde{G} := G/C$ and let X/\tilde{G} denote the space of \tilde{G} -orbits in X . Since X is discrete, the same is true for X/\tilde{G} , and we get a decomposition $C_0(X, \mathcal{K}) \cong \bigoplus_{\tilde{G}(x) \in X/\tilde{G}} C_0(\tilde{G}(x), \mathcal{K})$. By continuity of BC (see Proposition 2.6), we conclude that \tilde{G} satisfies BC for $C_0(X, \mathcal{K})$ if and only if \tilde{G} satisfies BC for $C_0(\tilde{G}(x), \mathcal{K})$ for all $x \in X$. Using Corollary 2.10, this will follow if all stabilizers $\tilde{G}_x \subseteq \tilde{G}$ satisfy BC for \mathcal{K} . But since X is discrete, it follows that each stabilizer \tilde{G}_x contains the connected component \tilde{G}_0 of \tilde{G} . Thus, each stabilizer is a Lie group with finitely many components and the result will follow from Proposition 2.11. \square

As mentioned above, the main idea for the proof of Proposition 2.11 is to use induction on the dimension $\dim(G)$ of the Lie group G . For this we were very much influenced by Pukánszky’s proof of the fact that locally algebraic groups (i.e., Lie groups having the same Lie algebra as some real algebraic group) have type I group C^* -algebras as presented in his recent book [49]. We split the induction argument into two main parts, which deal with the cases whether G is semi-simple or not. Note that even in the semi-simple case the result does not follow directly from the existent results, since all known results only work for the trivial coefficient and require that the groups have finite centers.

3. Baum-Connes for continuous fields of C^* -algebras

Let G be a separable locally compact group. Then G is called K -exact, if the functor $A \mapsto K_*(A \rtimes_r G)$ is half-exact, that is: whenever $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is

a short exact sequence of G -algebras, then the sequence

$$K_*(I \rtimes_r G) \rightarrow K_*(A \rtimes_r G) \rightarrow K_*(A/I \rtimes_r G)$$

is exact in the middle term. Clearly, every exact group is K -exact. Note that every almost connected group is exact by [35, Corollary 6.9]. Also, if k is a finite extension of the p -adic numbers \mathbf{Q}_p , then every linear algebraic group G over k is exact. To see this notice that by [10, Proposition 9.3] every such group has a solvable cocompact subgroup. Hence exactness of G follows from [35, §7]. Note that this gives in particular a correct proof for the statement of [12, Remark 4.4] that $GL(n, \mathbf{Q}_p)$ is exact (unfortunately, the argument given in [12] contains a mistake!).

Recall also that an element $\gamma \in KK_0^G(\mathbf{C}, \mathbf{C})$ is called a γ -element for G if there exists a locally compact proper G -space Y , a C^* -algebra D equipped with a nondegenerate and G -equivariant $*$ -homomorphism $\phi : C_0(Y) \rightarrow ZM(D)$, the center of the multiplier algebra $M(D)$ of D , and (Dirac and dual-Dirac) elements

$$\alpha \in KK_0^G(D, \mathbf{C}) \quad \beta \in KK_0^G(\mathbf{C}, D)$$

such that

$$\gamma = \beta \otimes_D \alpha \quad \text{and} \quad p_Z^*(\gamma) = 1 \in RKK_0^G(Z; \mathbf{C}, \mathbf{C})$$

for all locally compact proper G -spaces Z , where $p_Z : Z \rightarrow \{pt\}$. It is a basic result of Kasparov [32, Theorem 5.7] that every almost connected group has a γ -element and it follows also from the work of Kasparov (but see also [52, §5]) that a γ -element of G is unique and that it is an idempotent with the remarkable property that for every G -algebra B the image $\mu_B(K_*^{\text{top}}(G; B))$ of the assembly map is equal to the γ -part

$$\gamma \cdot K_*(B \rtimes_r G) := \{x \otimes_{B \rtimes_r G} j_G(\sigma_B(\gamma)) : x \in K_*(B \rtimes_r G)\}.$$

Here and below, we denote by $j_G : KK_*^G(A, B) \rightarrow KK_*(A \rtimes_r G, B \rtimes_r G)$ the (reduced) descent homomorphism of Kasparov and we denote by $\sigma_B : KK_*^G(A, D) \rightarrow KK_*^G(A \otimes B, D \otimes B)$ the external tensor product homomorphism (see [32, Definition 2.5]). Note that it follows from the above discussion that a group G with γ -element satisfies BC for a given G -algebra B if and only if γ (i.e., $j_G(\sigma_B(\gamma))$) acts as the identity on $K_*(B \rtimes_r G)$. We want to exploit these facts to prove the following basic result:

Proposition 3.1. — *Suppose that X is a separable locally compact space which satisfies one of the following conditions:*

- (a) X can be realized as the geometric realization of a (possibly infinite) finite dimensional simplicial complex.
- (b) X is totally disconnected.

Let A be the algebra of C_0 -sections of a continuous field of C^* -algebras $\{A_x : x \in X\}$, and let $\alpha : G \rightarrow \text{Aut}(A)$ be a $C_0(X)$ -linear action of G on A . Assume further that G is exact and has a γ -element $\gamma \in \text{KK}_0^G(\mathbf{C}, \mathbf{C})$. Then, if G satisfies BC for each fibre A_x , G satisfies BC for A .

For the general notion of continuous fields of C^* -algebras and their basic properties we refer to [23,22,6,34].

The idea of the proof is to show first that it holds for any closed interval $I \subseteq \mathbf{R}$. Then a short induction argument will show that it holds for any cube in \mathbf{R}^n . Then the result will follow from a Mayer-Vietoris argument. For the proof we first need the following lemma.

Lemma 3.2. — Assume that G is a \mathbf{K} -exact group with a γ -element $\gamma \in \text{KK}_0^G(\mathbf{C}, \mathbf{C})$. Let A be a G -algebra and let $I \subseteq A$ be a G -invariant closed ideal of A . Then there is a natural six-term exact sequence

$$\begin{array}{ccccc} (1-\gamma) \cdot \mathbf{K}_0(I \rtimes_r G) & \longrightarrow & (1-\gamma) \cdot \mathbf{K}_0(A \rtimes_r G) & \longrightarrow & (1-\gamma) \cdot \mathbf{K}_0(A/I \rtimes_r G) \\ & & \uparrow & & \downarrow \\ (1-\gamma) \cdot \mathbf{K}_1(A/I \rtimes_r G) & \longleftarrow & (1-\gamma) \cdot \mathbf{K}_1(A \rtimes_r G) & \longleftarrow & (1-\gamma) \cdot \mathbf{K}_1(I \rtimes_r G). \end{array}$$

Proof. — Since G is \mathbf{K} -exact, it follows that $A \mapsto \mathbf{K}_*(A \rtimes_r G)$ is a homotopy invariant and half-exact functor on the category of G - C^* -algebras which also satisfies Bott-periodicity (with respect to the trivial G -action on $C_0(\mathbf{R}^2)$). Then it follows from some general arguments (e.g., see [5, Chapter IX]) that there exists a six-term exact sequence

$$\begin{array}{ccccc} \mathbf{K}_0(I \rtimes_r G) & \longrightarrow & \mathbf{K}_0(A \rtimes_r G) & \longrightarrow & \mathbf{K}_0(A/I \rtimes_r G) \\ & & \uparrow & & \downarrow \\ \mathbf{K}_1(A/I \rtimes_r G) & \longleftarrow & \mathbf{K}_1(A \rtimes_r G) & \longleftarrow & \mathbf{K}_1(I \rtimes_r G). \end{array}$$

We want to show that all maps in the sequence commute with multiplication with the γ -element. By the construction of the connecting maps in the above sequence as given in [5, Chapters VIII and IX], it is enough to show that for any pair of G -algebras A and B and any $y \in \text{KK}_*^G(A, B)$

$$\mathbf{K}_*(A \rtimes_r G) \rightarrow \mathbf{K}_*(B \rtimes_r G); x \mapsto x \otimes_{A \rtimes_r G} j_G(y)$$

commutes with multiplication with γ . But for this it is enough to show that

$$j_G(y) \otimes_{B \rtimes_r G} j_G(\sigma_B(\gamma)) = j_G(\sigma_A(\gamma)) \otimes_{A \rtimes_r G} j_G(y).$$

This follows from the fact that the descent homomorphism j_G is compatible with Kasparov products and the fact that

$$y \otimes_B \sigma_B(\gamma) = y \otimes_{\mathbf{C}} \gamma = \gamma \otimes_{\mathbf{C}} y = \sigma_A(\gamma) \otimes_A y,$$

which follows from [32, Theorem 2.14].

It follows now that multiplication with $1 - \gamma$ also commutes with all maps in the above commutative diagram. Since $1 - \gamma$ is an idempotent, it is now easy to see that the full six-term exact sequence restricts to a six-term exact sequence on the $1 - \gamma$ -parts of the respective \mathbf{K} -theory groups of the crossed products. \square

Remark 3.3. — It is now a direct consequence of the above proposition that if G is a \mathbf{K} -exact group possessing a γ -element, and if $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is a short exact sequence of G -algebras, then G satisfying BC for two of the algebras in this sequence implies that G satisfies BC for all three algebras in the sequence. The same result holds without the assumption on the γ -element (see [12, Proposition 4.2] – which was actually deduced as an easy consequence of a result of Kasparov and Skandalis in [33]).

We also need the following easy lemma.

Lemma 3.4. — *Assume that X is a locally compact space and that A is the algebra of C_0 -sections of the continuous field $\{A_x : x \in X\}$ of C^* -algebras. Assume further that $z \in K_i(A)$, $i = 0, 1$, such that $q_{x,*}(z) = 0$ for some evaluation map $q_x : A \rightarrow A_x$. Then there exists a compact neighborhood C of x such that $q_{C,*}(z) = 0$ in $K_0(A|_C)$, where $A|_C$ denotes the restriction of A to C and $q_C : A \rightarrow A|_C$ denotes the quotient map.*

Proof. — We may assume without loss of generality that X is compact. Using suspension, it is enough to give a proof for the case $i = 0$. In what follows, if B is any C^* -algebra, we denote by B^1 the algebra obtained from B by adjoining a unit (even if B is already unital). Then $\{A_x^1 : x \in X\}$ is a continuous field of C^* -algebras in a canonical way. The algebra \tilde{A} of sections can be written as the set of pairs $\{(a, f) : a \in A, f \in C_0(X)\}$ with multiplication given pointwise by the multiplication rule of the fibres A_x^1 . Moreover, we have an obvious unital embedding $A^1 \rightarrow \tilde{A}$.

Assume now that $z \in K_0(A)$ and $x \in X$ are as in the lemma. We represent z as a formal difference $[p - p']$ for some projections $p, p' \in M_l(A^1)$. Since $q_{x,*}(z) = 0$ we may assume (after increasing dimension if necessary) that there exists a unitary $u_x \in M_l(A_x^1)$ such that $u_x p_x u_x^* = p'_x$. After passing to $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ if necessary, we may further assume that u_x lies in the connected component of the identity of $U(M_l(A_x^1))$. Thus, there exists a unitary $u \in M_l(A^1)$ such that $q_x(u) = u_x$. Since u is a continuous section in \tilde{A} , it follows that there exists a compact neighborhood C of x such that $\|u_y p_y u_y^* - p'_y\| < 1$ for all $y \in C$, which implies that $[p_C] = [u_C p_C u_C^*] = [p'_C] \in K_0(A|_C)$,

where $p_C, u_C,$ and p'_C denote the restrictions of p, u, p' to C , respectively. But this shows that $q_{C,*}(z) = [p_C - p'_C] = 0$ in $K_0(A|_C)$. \square

Proof of Proposition 3.1. — Since G is exact, it follows from [34, Theorem] that the crossed products $\{A_x \rtimes_r G : x \in X\}$ form a continuous bundle such that $A \rtimes_r G$ is the algebra of continuous sections of this bundle.

Assume first that we are in situation (a), i.e. that X is a geometric realization of a finite dimensional simplicial complex. Indeed, we first consider the special case where $X = [0, 1] \subseteq \mathbf{R}$.

Recall from the above discussions that G satisfies BC for a given G -algebra B if and only if $(1 - \gamma) \cdot K_*(B \rtimes_r G) = \{0\}$. In particular, it follows from our assumptions that $(1 - \gamma) \cdot K_*(A_x \rtimes_r G) = \{0\}$ for all $x \in [0, 1]$. Assume now that $z \in (1 - \gamma) \cdot K_i(A \rtimes_r G)$, $i = 0, 1$, and let $q_x : A \rtimes_r G \rightarrow A_x \rtimes_r G$ denote the evaluation maps for each $x \in X$. Then $q_{x,*}(z) \in (1 - \gamma) \cdot K_i(A_x \rtimes_r G) = \{0\}$ for all $x \in [0, 1]$. Thus, using Lemma 3.4, we see that there exists a partition $0 = x_0 < x_1 < \dots < x_l = 1$ such that $q_{[x_{j-1}, x_j],*}(z) = 0$ in $K_i(A|_{[x_{j-1}, x_j]} \rtimes_r G)$. Now let $O = [0, 1] \setminus \{x_0, \dots, x_l\}$ and let $A|_O = C_0(O) \cdot A \cong \bigoplus_{j=1}^l A|_{(x_{j-1}, x_j)}$. It follows from the exact sequence

$$(1 - \gamma) \cdot K_i(A_O \rtimes_r G) \rightarrow (1 - \gamma) \cdot K_i(A \rtimes_r G) \rightarrow \bigoplus_{j=0}^l (1 - \gamma) \cdot K_i(A_{x_j} \rtimes_r G) = \{0\}$$

that there exists a $z' \in (1 - \gamma) \cdot K_i(A_O \rtimes_r G)$ such that z is the image of z' under the inclusion. Since

$$(1 - \gamma) \cdot K_i(A_O \rtimes_r G) = \bigoplus_{j=1}^l (1 - \gamma) \cdot K_i(A|_{(x_{j-1}, x_j)} \rtimes_r G),$$

we may write $z' = \sum_{j=1}^l z'_j$ with $z'_j \in (1 - \gamma) \cdot K_i(A|_{(x_{j-1}, x_j)} \rtimes_r G)$ for each $1 \leq j \leq l$. Thus it is enough to show that $z'_j = 0$ for each $1 \leq j \leq l$. In what follows, we write $A_j = A|_{(x_{j-1}, x_j)}$ and $\bar{A}_j = A|_{[x_{j-1}, x_j]}$. Since $(1 - \gamma) \cdot K_i(A_{x_k} \rtimes_r G) = \{0\}$ for all $0 \leq k \leq l$ we obtain a six-term exact sequence

$$\begin{array}{ccccc} (1 - \gamma) \cdot K_0(A_j \rtimes_r G) & \longrightarrow & (1 - \gamma) \cdot K_0(\bar{A}_j \rtimes_r G) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & (1 - \gamma) \cdot K_1(\bar{A}_j \rtimes_r G) & \longleftarrow & (1 - \gamma) \cdot K_1(A_j \rtimes_r G) \end{array}$$

Since the image of z'_j in $K_i(\bar{A}_j \rtimes_r G)$ coincides with the image of z in $K_i(\bar{A}_j \rtimes_r G)$, we see that z'_j maps to 0 under the isomorphism $(1 - \gamma) \cdot K_0(A_j \rtimes_r G) \rightarrow (1 - \gamma) \cdot K_0(\bar{A}_j \rtimes_r G)$, so $z'_j = 0$.

We now show by induction on n that the result is true for $[0, 1]^n \subseteq \mathbf{R}^n$. For this assume that $\{A_x : x \in [0, 1]^n\}$ is a continuous field over the cube and A is the algebra of continuous sections of this field. We write $[0, 1]^n = \cup_{y \in [0, 1]} \{y\} \times [0, 1]^{n-1}$ and put $A_y = A|_{\{y\} \times [0, 1]^{n-1}}$. Then $\{A_y : y \in [0, 1]\}$ is a continuous field over $[0, 1]$ and A is also the section algebra of this bundle. If α is a $C([0, 1]^n)$ -linear action on A , it is also $C([0, 1])$ -linear with respect to the bundle structure of A over $[0, 1]$ coming from the above decomposition of the cube. Moreover, the actions on the fibres A_y are clearly $C([0, 1]^{n-1})$ -linear, so by the induction assumption we know that G satisfies BC with coefficient A_y for all $y \in [0, 1]$. We now apply the above result to the bundle $\{A_y : y \in [0, 1]\}$ to conclude that G satisfies BC with coefficient A .

In a next step we show that the result holds for the open cubes $(0, 1)^n \subseteq \mathbf{R}^n$. By similar arguments as given above it suffices to show that the result holds for open intervals. So assume that $\{A_x : x \in (0, 1)\}$ is a continuous field with section algebra A . Let $x_1 < x_2 \in (0, 1)$. Then it follows from the first part of the proof that G satisfies BC with coefficient $A_{[x_1, x_2]}$. Since, by assumption, G also satisfies BC for the fibres, a six-term sequence argument shows that it also satisfies BC with coefficient $A_{(x_1, x_2)}$. Writing $A = \lim_{n \rightarrow \infty} A|_{(\frac{1}{n}, 1 - \frac{1}{n})}$ and using continuity of the BC conjecture, it follows that G satisfies BC for A .

Since the result of the proposition is clearly invariant under replacing the space X by a homeomorphic space Y , we now see that the result holds for all open or closed simplices. We now prove the general result for simplicial complexes via induction on the dimension of the complex. By continuity of the conjecture, the result is clear for zero-dimensional complexes. If X has dimension n , let W_n denote the interiors of all n -dimensional simplices in X . Then W_n is homeomorphic to a disjoint union of open n -dimensional cubes, so the result holds for W_n . Since $X \setminus W_n$ is a simplicial complex of dimension $n - 1$, the result is true for $X \setminus W_n$ by the induction assumption. The result then follows from another easy application of the six-term sequence (see Remark 3.3).

We now come to the situation where X is totally disconnected. Let $z \in (1 - \gamma) \cdot \mathbf{K}_*(A \rtimes_r G)$ and let $q_x : A \rtimes_r G \rightarrow A_x \rtimes_r G$ denote evaluation at x . Then $q_{x,*}(z) \in (1 - \gamma) \cdot \mathbf{K}_*(A_x \rtimes_r G) = \{0\}$ for all $x \in X$. By Lemma 3.4, each $x \in X$ has a compact neighborhood C such that $q_{C,*}(z) = 0$. Since X is totally disconnected, we therefore find a disjoint covering $(C_i)_{i \in I}$ by compact open subsets of X such that $q_{C_i,*}(z) = 0$ for all $i \in I$. But then we have $A \rtimes_r G = \bigoplus_{i \in I} (A \rtimes_r G)|_{C_i}$ from which it follows that $\mathbf{K}_*(A \rtimes_r G) \cong \bigoplus_{i \in I} \mathbf{K}_*((A \rtimes_r G)|_{C_i})$. Since the projection $(q_{C_i,*}(z))_{i \in I}$ is zero for all $i \in I$, it follows that $z = 0$. \square

Remark 3.5. — Let G be a second countable locally compact group and let X be a second countable locally compact almost Hausdorff G -space (a topological space X is called *almost Hausdorff* if every closed subset $C \subseteq X$ contains a relatively open dense Hausdorff subset $O \subseteq C$). Following Glimm we say that the quotient space X/G is

countably separated, if all orbits $G(x)$ are locally closed, i.e. $G(x)$ is open in its closure $\overline{G(x)}$. Glimm showed in [24, Theorem] that X/G being countably separated is equivalent to each of the following conditions:

- (1) The canonical map $G/G_x \rightarrow G(x), gG_x \mapsto g \cdot x$ is a homeomorphism for each $x \in X$.
- (2) X/G is almost Hausdorff.
- (3) There exists a sequence of G -invariant open subsets $\{U_\nu\}_\nu$ of X , where ν runs through the ordinal numbers such that
 - (a) $U_\nu \subseteq U_{\nu+1}$ for each ν and $(U_{\nu+1} \setminus U_\nu)/G$ is Hausdorff.
 - (b) If ν is a limit ordinal, then $U_\nu = \cup_{\mu < \nu} U_\mu$.
 - (c) There exists an ordinal number ν_0 such that $X = U_{\nu_0}$.

A classical example of an almost Hausdorff space is the space \widehat{A} of all equivalence classes of irreducible representations of a type I C^* -algebra A equipped with the Jacobson topology (see [18]). Recall that A is type I if and only if $\pi(A)$ contains the compact operators $\mathcal{K}(H_\pi)$ whenever $\pi : A \rightarrow \mathcal{B}(H_\pi)$ is an irreducible representation of A . If A is a type I G -algebra, and if G_π is the stabilizer of $\pi \in \widehat{A}$ under the action of G , then the restriction of the G -action to G_π factorizes to a canonical action of G_π on $A_\pi := \mathcal{K}(H_\pi)$ and we get:

Theorem 3.6. — *Suppose that G is an exact group with γ -element $\gamma \in \text{KK}_0^G(\mathbf{C}, \mathbf{C})$ and let A be a type I G -algebra such that the action of G on \widehat{A} is countably separated. Suppose that there exists an ascending sequence $\{U_\nu\}_\nu$ of \widehat{A} as in item (3) of the above remark such that the difference sets $U_{\nu+1} \setminus U_\nu$ satisfy the following conditions:*

- (i) *There exists a locally compact Hausdorff space X_ν and a continuous and open surjection $q_\nu : U_{\nu+1} \setminus U_\nu \rightarrow X_\nu$ such that, for all $x \in X_\nu$, $q_\nu^{-1}(\{x\})$ is a finite union of G -orbits in $U_{\nu+1}$.*
- (ii) *The space X_ν of (i) is (the geometric realization of) a finite dimensional simplicial complex or X_ν is totally disconnected.*

Then G satisfies BC for A , if the stabilizer G_π satisfies BC for $A_\pi = \mathcal{K}(H_\pi)$ for all $\pi \in \widehat{A}$.

Proof. — For each ordinal ν let A_ν denote the ideal of A corresponding to the open subset U_ν of \widehat{A} (see [18, Chapter 3]). We show by transfinite induction that G satisfies BC with coefficient A_ν for each ν . Since $A = A_{\nu_0}$ for some ν_0 , the result will follow.

We start by showing that G satisfies BC with coefficient A_1 . Since the quotient map $q_1 : U_1 \rightarrow X_1$ is open, we can regard A_1 as a section algebra of a continuous bundle over X_1 with fibres isomorphic to $A|_{q_1^{-1}(x)}$ (see [41]). By condition (ii) it follows from Proposition 3.1 that it suffices to prove that G satisfies BC for $A|_{q_1^{-1}(x)}$ for all

$x \in X_1$. Fix $x \in X_1$ and put $Z := q_1^{-1}(x)$. Since Z is a finite union of G -orbits, we find a finite sequence

$$Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_l = \emptyset$$

of open invariant subsets of Z such that $(Z_{i-1} \setminus Z_i)/G$ is a discrete finite set. To see this let C_1 be the union of all closed G -orbits in Z (such orbits must exist by the finiteness of Z/G). Then C_1 is closed in Z and C_1/G is discrete. Put $Z_1 = Z \setminus C_1$ and then define the Z_i 's, $i > 1$, inductively by the same procedure. Using six-term sequences (e.g., see Remark 3.3), G satisfies BC for $A|_Z$ if G satisfies BC for all $A|_{Z_{i-1} \setminus Z_i}$, which in turn follows if G satisfies BC for $A|_{G(\pi)}$ for any G -orbit $G(\pi) \subseteq Z \subseteq \widehat{A}$ (where $A|_{G(\pi)}$ denotes the subquotient of A corresponding to the locally closed subset $G(\pi)$ of \widehat{A}). But this follows from the assumption that G_π satisfies BC for A_π and Corollary 2.10. This completes the proof for A_1 .

Assume now that ν is an ordinal number and that we have already shown that G satisfies BC for A_μ for all $\mu < \nu$. If $\nu = \mu + 1$ for some ordinal μ , it follows from the same reasoning as for the case $\nu = 1$ that G satisfies BC for A_ν/A_μ . Since G satisfies BC for A_μ by the induction assumption, it follows from Remark 3.3 that G satisfies BC for A_ν .

Assume now that ν is a limit ordinal and G satisfies BC for A_μ for each $\mu < \nu$. Then $U_\nu = \cup_{\mu < \nu} U_\mu$ which implies that $A_\nu = \lim_{\mu < \nu} A_\mu$ is the inductive limit of the A_μ . Thus it follows from Proposition 2.6 that G satisfies BC for A_ν . \square

4. The semi-simple case

In this section we want to show that Proposition 2.11 is true if G is semi-simple. For this we first have to obtain a slight extension of Lafforgue's results on the Baum-Connes conjecture for semi-simple groups with finite center.

Let us first recall the basic idea of Lafforgue's proof of the Baum-Connes conjecture for such groups. If G is a locally compact group we let $C_c(G)$ denote the convolution algebra of G consisting of continuous functions with compact supports. A norm $\|\cdot\|$ on $C_c(G)$ is called *unconditional* if convolution is continuous with respect to this norm and if $\|f\|$ only depends on the absolute value of f for all $f \in C_c(G)$ (i.e., $\|f\| = \|\|f\|\|$ for all $f \in C_c(G)$). An *unconditional completion* $\mathcal{A}(G)$ of $C_c(G)$ is a completion with respect to an unconditional norm on $C_c(G)$. Note that $L^1(G)$ is always an unconditional completion of $C_c(G)$, but $C^*(G)$ and $C_r^*(G)$ are in general not unconditional completions of $C_c(G)$.

If $\mathcal{A}(G)$ is an unconditional completion of $C_c(G)$, then Lafforgue constructed an assembly map

$$\mu_{\mathcal{A}(G)} : K_*^{\text{top}}(G; \mathbf{C}) \rightarrow K_*(\mathcal{A}(G)).$$

Moreover, if the identity on $C_c(G)$ extends to a continuous embedding $\iota : \mathcal{A}(G) \rightarrow C_r^*(G)$, he also shows that the assembly map $\mu : K_*^{\text{top}}(G; \mathbf{C}) \rightarrow K_*(C_r^*(G))$ factors through $K_*(\mathcal{A}(G))$, i.e.,

$$\mu = \iota_* \circ \mu_{\mathcal{A}(G)}$$

(see [38, Proposition 1.7.6]). Thus, if we know that $\mu_{\mathcal{A}(G)}$ is an isomorphism for all unconditional completions of $C_c(G)$, and if we further know that there exists an unconditional completion $\mathcal{A}(G) \subseteq C_r^*(G)$ such that the inclusion $\iota_* : K_*(\mathcal{A}(G)) \rightarrow K_*(C_r^*(G))$ is an isomorphism, it follows that $\mu : K_*^{\text{top}}(G; \mathbf{C}) \rightarrow K_*(C_r^*(G))$ is an isomorphism. Note that $\iota_* : K_*(\mathcal{A}(G)) \rightarrow K_*(C_r^*(G))$ is an isomorphism whenever $\mathcal{A}(G)$ is closed under holomorphic functional calculus in $C_r^*(G)$. Now Lafforgue was able to prove the following deep results:

Theorem 4.1 (cf [38, Théorème 1.7.12 and Chapitre 3]). — *Assume that G is a second countable locally compact group such that G satisfies one of the following conditions:*

- (a) *G acts isometrically and properly on a Riemannian manifold with nonpositive sectional curvature which is bounded below and has bounded covariant derivative (see [38, p. 61] for the details).*
- (b) *G acts properly and isometrically on a “bolic” space (X, d) in the sense of [38, Definition 2.2.1].*

Then $\mu_{\mathcal{A}(G)} : K_^{\text{top}}(G; \mathbf{C}) \rightarrow K_*(\mathcal{A}(G))$ is an isomorphism for every unconditional completion $\mathcal{A}(G)$ of $C_c(G)$.*

In fact, Lafforgue was even able to show that the above result holds with arbitrary C^* -algebra coefficients, but we do not need this more general result here. Note that if G is semi-simple with finite center, then the Riemannian manifold of the theorem can be chosen to be the symmetric space G/K , where K is the maximal compact subgroup of G .

In the second step for the proof of BC for semi-simple groups, Lafforgue constructed a *Schwartz algebra* $\mathcal{S}(G) \subseteq C_r^*(G)$ which is an unconditional completion of $C_c(G)$ which is closed under holomorphic functional calculus in $C_r^*(G)$. In fact, this construction followed a more general principle, which we are now going to describe in more detail.

Assume that G is a unimodular group and K is a compact subgroup of G . Then, following Lafforgue (see [38, Chapitre 4]), we say that the pair (G, K) satisfies property (HC) if the following conditions are satisfied

- (HC1) There exists a continuous function $d : G \rightarrow [0, \infty)$ such that $d(e) = 0$, $d(kgk') = d(g)$ for all $k, k' \in K, g \in G$ and $d(gg') \leq d(g) + d(g')$ for all $g, g' \in G$.

(HC2) There exists a continuous function $\phi : G \rightarrow (0, 1]$ such that $\phi(e) = 1$, $\phi(g^{-1}) = \phi(g)$, $\phi(kgk') = \phi(g)$ for all $g \in G$ and $k, k' \in K$, and

$$\int_K \phi(gkg') dk = \phi(g)\phi(g') \quad \text{for all } g, g' \in G,$$

with respect to the normalized Haar measure on K .

(HC3) There exists a $t_0 \in \mathbf{R}$ such that $(t \mapsto \phi(g)(1 + d(g))^{-t}) \in L^2(G)$ for all $t > t_0$.

If G is a connected semi-simple group with finite center, and if K is the maximal compact subgroup of G , then Lafforgue was able to show that (G, K) satisfies (HC). The following theorem then completes the proof of BC for connected semi-simple groups with finite center.

Theorem 4.2 (cf [38, Proposition 4.1.2]). — *Assume that G is a unimodular group and K is a compact subgroup of G such that the pair (G, K) satisfies (HC). Then there exists an unconditional completion $\mathcal{S}(G) \subseteq C_r^*(G)$ of $C_c(G)$ such that $\mathcal{S}(G)$ is closed under holomorphic functional calculus in $C_r^*(G)$.*

In order to extend Lafforgue's methods to extensions of semi-simple groups, we first show that property (HC) is closed under compact extensions.

Lemma 4.3. — *Assume that (G, K) satisfies property (HC). Assume further that*

$$1 \rightarrow C \rightarrow \tilde{G} \xrightarrow{q} G \rightarrow 1$$

is a group extension with C compact. Let $\tilde{K} = q^{-1}(K) \subseteq \tilde{G}$. Then (\tilde{G}, \tilde{K}) satisfies (HC).

Proof. — Since compact extensions of unimodular groups are unimodular, \tilde{G} is unimodular. Let (d, ϕ) be a pair of functions which satisfy HC1, HC2, and HC3 with respect to (G, K) . Define $\tilde{d}(g) = d(q(g))$ and $\tilde{\phi}(g) = \phi(q(g))$. Then a straightforward computation shows that $(\tilde{d}, \tilde{\phi})$ satisfies HC1, HC2, and HC3 with respect to (\tilde{G}, \tilde{K}) . \square

The second result is slightly more technical.

Lemma 4.4. — *Assume that G is a unimodular Lie group with finitely many components and let K be a maximal compact subgroup of G . Let G_0 denote the connected component of G and let K_0 be a maximal compact subgroup of G_0 such that (G_0, K_0) satisfies (HC). Then (G, K) satisfies (HC), too.*

Proof. — First note that we may assume that $K_0 = K \cap G_0$. To see this observe first that, since K_0 is a compact subgroup of G , we may assume without loss of generality that $K_0 \subseteq K \cap G_0$. But the maximality of K_0 then implies equality. It follows

from this that K_0 is a normal subgroup of K and since G/K is connected, it follows that the inclusion $K \rightarrow G$ induces a group isomorphism $K/K_0 \cong G/G_0$.

Let (d_0, ϕ_0) be a pair of functions satisfying conditions HC1–HC3 for (G_0, K_0) . It follows from the above remarks that we can write every element of G as a product kg with $k \in K, g \in G_0$. We then define

$$d(kg) = \int_K d_0(lgl^{-1}) dl \quad \text{and} \quad \phi(kg) = \int_K \phi_0(lgl^{-1}) dl.$$

To see that d and ϕ are well defined assume that we have two factorizations $kg = k'g'$ with $k, k' \in K, g, g' \in G_0$. Then $g = k^{-1}k'g'$ with $h := k^{-1}k' \in K_0$. Since K normalizes K_0 , it follows from the left and right K_0 -invariance of d_0 that

$$\begin{aligned} d(kg) &= \int_K d_0(lgl^{-1}) dl = \int_K d_0(lhg't^{-1}) dl = \int_K d_0((lhl^{-1})(lg't^{-1})) dl \\ &= \int_K d_0(lg't^{-1}) dl = d(k'g'). \end{aligned}$$

A similar computation shows that ϕ is well defined.

We are now going to check properties HC1–HC3 for (d, ϕ) . It follows directly from the definition of d and ϕ that they are left invariant under the action of K . To see right invariance, we compute for $h \in K$:

$$\begin{aligned} d(kgh) &= d(khh^{-1}gh) = d(h^{-1}gh) \\ &= \int_K d_0(lh^{-1}ghl^{-1}) dl \stackrel{l \mapsto lh}{=} \int_K d_0(lhl^{-1}) dl = d(kg). \end{aligned}$$

So d is also right invariant and a similar computation show that the same is true for ϕ . Since Haar measure on K is normalized, it follows that $d(e) = 0$ and $\phi(e) = 1$. Moreover, if $kg, hg' \in G$ with $k, h \in K, g, g' \in G_0$ we get

$$\begin{aligned} d(kghg') &= d(khh^{-1}ghg') = \int_K d_0(lh^{-1}ghg't^{-1}) dl = \int_K d_0(lh^{-1}ghl^{-1}lg't^{-1}) dl \\ &\leq \int_K d_0(lh^{-1}ghl^{-1}) dl + \int_K d_0(lg't^{-1}) dl = d(kg) + d(hg'). \end{aligned}$$

In order to prove the multiplication rule for ϕ we use Weil's formula

$$\int_K \varphi(l) dl = \int_{K/K_0} \left(\int_{K_0} \varphi(lm) dm \right) dl = \int_K \left(\int_{K_0} \varphi(lm) dm \right) dl,$$

with respect to normalized Haar measures on \mathbf{K} , \mathbf{K}_0 , and \mathbf{K}/\mathbf{K}_0 to compute

$$\begin{aligned}
\int_{\mathbf{K}} \phi(kglhg') \, dl &= \int_{\mathbf{K}} \phi(glhg'h^{-1}) \, dl = \int_{\mathbf{K}} \int_{\mathbf{K}_0} \phi(glmhg'h^{-1}) \, dm \, dl \\
&\stackrel{(*)}{=} \int_{\mathbf{K}} \int_{\mathbf{K}_0} \phi(l^{-1}glmhg'h^{-1}) \, dm \, dl \\
&= \int_{\mathbf{K}} \int_{\mathbf{K}_0} \int_{\mathbf{K}} \phi_0(nl^{-1}glmhg'h^{-1}n^{-1}) \, dn \, dm \, dl \\
&\stackrel{(**)}{=} \int_{\mathbf{K}} \int_{\mathbf{K}} \int_{\mathbf{K}_0} \phi_0(nl^{-1}gln^{-1}mnhg'h^{-1}n^{-1}) \, dm \, dl \, dn \\
&\stackrel{(***)}{=} \int_{\mathbf{K}} \int_{\mathbf{K}} \phi_0(nl^{-1}gln^{-1}) \phi_0(nhg'h^{-1}n^{-1}) \, dl \, dn \\
&\stackrel{h \mapsto l^{-1}n}{=} \left(\int_{\mathbf{K}} \phi_0(lgl^{-1}) \, dl \right) \left(\int_{\mathbf{K}} \phi_0(ng'n^{-1}) \, dn \right) \\
&= \phi(kg)\phi(hg').
\end{aligned}$$

Here the equation (*) follows from the \mathbf{K} -invariance of ϕ , (**) follows from Fubini together with the transformation $m \mapsto n^{-1}mn$, and (***) follows from property HC2 for ϕ_0 . This completes the proof of HC1 and HC2.

For the proof of HC3 we write $F_s(g) := \phi(g)(1+d(g))^{-s}$, $s > 0$. Since $\mathbf{G} = \mathbf{K}\mathbf{G}_0$, it follows from the \mathbf{K} -invariance of ϕ and d that the integrals of F_s^2 over the \mathbf{G}_0 -cosets coincide. Thus, since \mathbf{G}/\mathbf{G}_0 is finite, it is enough to show that $F_s|_{\mathbf{G}_0} \in L^2(\mathbf{G}_0)$ for some $s > 0$. For this we first choose a set of representatives $t_1, \dots, t_n \in \mathbf{K}$ for \mathbf{K}/\mathbf{K}_0 with $t_1 = e$. Then, for $g \in \mathbf{G}_0$, we obtain the inequality

$$1 + d(g) = 1 + \frac{1}{n} \sum_{i=1}^n d_0(t_i g t_i^{-1}) \geq 1 + \frac{1}{n} d_0(g),$$

from which it follows that $(1 + d(g))^n \geq 1 + d_0(g)$ for all $g \in \mathbf{G}_0$. Thus, if $t \in \mathbf{R}$ such that $(g \mapsto \phi_0(g)(1 + d_0(g))^{-t}) \in L^2(\mathbf{G}_0)$, then we also have $(g \mapsto \phi_0(g)(1 + d(g))^{-nt}) \in L^2(\mathbf{G}_0)$. So let $s = nt$ with t as above. Then we get

$$\begin{aligned}
F_s(g) &= \phi(g)(1 + d(g))^{-s} = \frac{1}{n} \sum_{i=1}^n \phi_0(t_i g t_i^{-1})(1 + d(g))^{-s} \\
&= \frac{1}{n} \sum_{i=1}^n \phi_0(t_i g t_i^{-1})(1 + d(t_i g t_i^{-1}))^{-s}.
\end{aligned}$$

Since \mathbf{G} is unimodular, it follows that each summand $(g \mapsto \phi_0(t_i g t_i^{-1})(1 + d(t_i g t_i^{-1}))^{-s}) \in L^2(\mathbf{G}_0)$, and hence $F_s|_{\mathbf{G}_0} \in L^2(\mathbf{G}_0)$. \square

We are now ready to combine the above results to get

Proposition 4.5. — *Let G be a locally compact group with finitely many components. Assume further that G has a compact normal subgroup $C \subseteq G_0$ such that G_0/C is a real semi-simple Lie group with finite center. Then G satisfies BC with trivial coefficient.*

Proof. — Let K_0 denote the maximal compact subgroup of G_0 . Then $G_0/K_0 \cong (G_0/C)/(K_0/C)$ is a symmetric space and therefore has nonpositive sectional curvature. Moreover, if K is a maximal compact subgroup of G such that $K \cap G_0 = K_0$, we see that $G/K \cong G_0/K_0$ as a Riemannian manifold. Since G acts isometrically and properly on G/K , G satisfies the assumptions of Theorem 4.1. By Lafforgue’s results we also know that $(G_0/C, K_0/C)$ satisfies (HC). Lemmas 4.3 and 4.4 then imply that (G, K) also satisfies property (HC). Thus, it follows from the combination of Theorem 4.1 with Theorem 4.2 that G satisfies BC with coefficient in \mathbf{C} . \square

Using the results on continuous fields of actions as presented in the previous section, we are now able to prove

Proposition 4.6. — *Assume that G is a real Lie group with finitely many components such that G_0 is reductive, i.e., the Lie algebra \mathfrak{g} of G is a direct sum of two ideals $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$ with \mathfrak{s} semi-simple and \mathfrak{z} abelian. Then G satisfies BC for \mathbf{C} .*

Proof. — Let $Z = Z(G_0)$ denote the center of G_0 . Using Theorem 2.1 it is enough to show that G/Z satisfies BC with coefficient $C^*(Z) \otimes \mathcal{K} \cong C_0(\widehat{Z}, \mathcal{K})$, where the action of G/Z on the dual space \widehat{Z} of Z is given via conjugation. Since Z is central in G_0 , it follows that this action factors through an action of the finite group G/G_0 . Moreover, since \widehat{Z} is a manifold (since Z is a compactly generated abelian group), it follows that the quotients of the orbit-types in \widehat{Z} are manifolds. From this we easily obtain a finite decomposition sequence

$$\emptyset = U_0 \subseteq U_1 \subseteq \dots \subseteq U_l = \widehat{Z}$$

of open G -invariant subsets of \widehat{Z} such that the quotients of the differences $U_j \setminus U_{j-1}$ are homeomorphic to geometric realizations of finite dimensional simplicial complexes. Moreover, since all stabilizers for the action of G/Z on \widehat{Z} contain G_0/Z , which is semi-simple with trivial center, it follows from a combination of Proposition 4.5 with Proposition 2.7 that all stabilizers satisfy BC for \mathcal{K} . The result then follows from Theorem 3.6. \square

Since any central extension of a semi-simple group is reductive, we now get the desired result for general semi-simple groups.

Corollary 4.7. — *Let G be a semi-simple Lie group with finitely many components and let*

$$1 \rightarrow \mathbf{T} \rightarrow \bar{G} \rightarrow G \rightarrow 1$$

be a central extension of G by \mathbf{T} . Then \bar{G} satisfies BC for \mathbf{C} . As a consequence (using Proposition 2.7), G satisfies BC for \mathcal{H} with respect to arbitrary actions of G on \mathcal{H} .

Remark 4.8. — When we first wrote down this article, we only had a copy of the original thesis of Lafforgue, where he shows that connected semi-simple Lie groups with finite center satisfy BC with trivial coefficient. In the meantime Lafforgue has extended this result (see [38]) to all reductive groups G with finitely many components such that the semi-simple part of the connected component G_0 of G has finite center (i.e. the semi-simple subgroup $S \subseteq G_0$ corresponding to the Levi summand $\mathfrak{s} \subseteq \mathfrak{g}$ has finite center). Using this more general result, we could have avoided Lemma 4.4, but we decided to stick to our original arguments. As was pointed out by Lafforgue, the results of [38] do not cover all reductive real groups. For instance, the universal covering group $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ of $\mathrm{SL}_2(\mathbf{R})$ is a semi-simple group with infinite center.

We close this section with a result on p -adic groups:

Proposition 4.9. — *Let k be a finite extension of some \mathbf{Q}_p and let G be the group of k -rational points of a reductive linear algebraic group over k . Then G satisfies BC for \mathcal{H} with respect to any action of G on \mathcal{H} .*

Proof. — Using Proposition 2.7, it suffices to show that every central extension \tilde{G} of G by \mathbf{T} satisfies BC with trivial coefficient. By the results of [33, 38] we know that G acts properly and isometrically on a “bolic” space (X, d) in the sense of [38, Definition 2.2.1] and that G has a compact subgroup K such that (G, K) satisfies property (HC) (see [38, Théorème 2.2.2, Proposition 4.1.2 and §4.3]). Lemma 4.3 then implies that (\tilde{G}, \tilde{K}) , where \tilde{K} denotes the inverse image of K in \tilde{G} , has the same properties. Hence the result follows from Theorems 4.1 and 4.2. \square

5. The general case of almost connected groups

We now prove Proposition 2.11. As outlined in §2, this implies Theorem 1.2 and therefore finishes the case of almost connected groups in Theorem 1.1. As indicated before, we are going to use an induction argument on the dimension $n = \dim(G)$. Since any one-dimensional Lie group with finitely many components is amenable, and since amenable groups satisfy BC for arbitrary coefficients, the case $n = 1$ is clear. Assume now that G is an arbitrary Lie group with finitely many components. Let G_0 denote the connected component of G . Let N denote the nilradical of G and let \mathfrak{n} and \mathfrak{g}

denote the Lie algebras of N and G , respectively. If $\mathfrak{n} = \{0\}$, then G is semi-simple and the result follows from the previous section. So we may assume that $\mathfrak{n} \neq \{0\}$.

It is shown in [49, Lemma 4 on p. 24] that the subgroup H of G_0 corresponding to the subalgebra $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{n}$ of \mathfrak{g} is closed in G_0 . Further, if \mathfrak{s} is a Levi section in \mathfrak{g} , i.e., \mathfrak{s} is a maximal semi-simple subalgebra of \mathfrak{g} , then $\mathfrak{h} = \mathfrak{s} + \mathfrak{n}$ (e.g., see the discussion in the proof of [49, Sublemma on p. 24]). In particular, H/N is semi-simple. Clearly, H is a normal subgroup of G and G/H is a finite extension of a connected abelian Lie group. Let M denote the inverse image of any compact subgroup of G/H in G . It follows then from Theorem 2.1 that G satisfies BC for \mathcal{K} if all such M satisfy BC for \mathcal{K} . From now on, let M denote the inverse image of a fixed compact subgroup of G/H . Note that the connected component of M/H is a compact connected abelian Lie group, hence a torus group.

Thus, replacing G by M , we may from now on assume that G has the following structure: There exist closed normal subgroups

$$(5.1) \quad N \subseteq H \subseteq G_0 \subseteq G$$

such that N is a non-trivial connected nilpotent Lie group, H/N is semi-simple, G_0/H is a torus group and G/G_0 is finite. Moreover, by induction we may assume that every almost connected Lie group with smaller dimension satisfies BC for \mathcal{K} with respect to arbitrary actions on \mathcal{K} , or, equivalently (by Proposition 2.7), every central extension by \mathbf{T} satisfies BC for \mathbf{C} . It is now useful to recall the following result of Chevalley (see [16, Proposition 5, p. 324]):

Proposition 5.1. — Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie-algebra of endomorphisms of the finite dimensional real vector space V . Then \mathfrak{g} is algebraic (i.e. it corresponds to a real algebraic subgroup $G \subseteq \mathbf{GL}(V)$) if and only if there exist subalgebras $\mathfrak{s}, \mathfrak{a}, \mathfrak{n}$ of \mathfrak{g} with $\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \mathfrak{n}$, \mathfrak{s} is semi-simple, \mathfrak{n} is the largest ideal of \mathfrak{g} consisting of nilpotent endomorphisms, and \mathfrak{a} is an algebraic abelian subalgebra of $\mathfrak{gl}(V)$ consisting of semi-simple endomorphisms such that $[\mathfrak{s}, \mathfrak{a}] \subseteq \mathfrak{a}$.

Using Ado’s theorem (see [16, Théorème 5 on p. 333]) and Proposition 5.1, it follows that the group H considered above is locally algebraic, i.e., the Lie algebra \mathfrak{h} has a faithful representation as an algebraic Lie subalgebra into some $\mathfrak{gl}(V)$.

Using this structure, the main idea is to apply the Mackey machine to a suitable abelian subgroup S of N which is normal in G . The fact that G is very close to an algebraic group implies that the action of G on the dual \widehat{S} of S has very good topological properties, which is precisely what we need to make everything work. As a first hint that this approach is feasible we prove:

Lemma 5.2. — Assume that G is a Lie group with finitely many components. Let $H \subseteq G_0$ be a connected closed normal subgroup of G such that H is locally algebraic, and such that G/H is compact. Let N denote the nilpotent radical of H , and let $S \subseteq N$ be a connected abelian normal

subgroup of G . Let \widehat{S} denote the character group of S and let G act on \widehat{S} via conjugation. Then the following assertions are true:

- (i) The orbit space \widehat{S}/G is countably separated, i.e., all G -orbits in \widehat{S} are locally closed.
- (ii) If G_χ is the stabilizer of some $\chi \in \widehat{S}$ for the action of G on \widehat{S} , then $G_\chi/(G_\chi)_0$ is amenable.

Proof. — We first show that it is sufficient to prove the result for the case $G = H$. Indeed, if we already know that \widehat{S}/H is countably separated, then we observe that \widehat{S}/H is a topological G/H -space such that $\widehat{S}/G \cong (\widehat{S}/H)/(G/H)$. But it is an easy exercise to prove that the quotient space of a countably separated space by a compact group action is countably separated.

Assume now that G_χ is the stabilizer of some $\chi \in \widehat{S}$ in G . Then $H_\chi = G_\chi \cap H$ is the stabilizer in H . It follows that H_χ is a normal subgroup of G_χ such that G_χ/H_χ is compact. If $H_\chi/(H_\chi)_0$ is amenable, it also follows that $G_\chi/(H_\chi)_0$, and hence also $G_\chi/(G_\chi)_0$ are amenable.

So, for the rest of the proof we assume that $G = H$. In the next step we reduce to the case where H is simply connected. For this let \widetilde{H} denote the universal covering group of H . Then \widetilde{H} has the same Lie algebra as H , and therefore it is locally algebraic. Let $q : \widetilde{H} \rightarrow H$ denote the quotient map and let $C = \ker q$. Then C is a discrete central subgroup of \widetilde{H} . Let $\mathfrak{t} \subseteq \mathfrak{h}$ denote the Lie algebra of S and let \widetilde{S} denote the connected closed normal subgroup of \widetilde{H} corresponding to \mathfrak{t} . Then \widetilde{S} is a vector subgroup of the nilpotent radical \widetilde{N} of \widetilde{H} and the quotient map $\widetilde{H} \rightarrow H$ maps \widetilde{S} surjectively onto S , i.e., we have $S \cong \widetilde{S}/(\widetilde{S} \cap C)$. In particular, it follows that we may view \widehat{S} as a closed \widetilde{H} -invariant subspace of $\widehat{\widetilde{S}}$, and we have $\widehat{S}/H = \widehat{\widetilde{S}}/\widetilde{H}$ (since the central subgroup C acts trivially on $\widehat{\widetilde{S}}$). Thus, if $\widehat{\widetilde{S}}/\widetilde{H}$ is countably separated, the same is true for \widehat{S}/H .

We now consider the stabilizers. It follows from the above considerations that if H_χ is the stabilizer of some $\chi \in \widehat{S}$, then $q^{-1}(H_\chi) \subseteq \widetilde{H}$ is the stabilizer of χ in \widetilde{H} . Thus it follows that $H_\chi = \widetilde{H}_\chi/(C \cap \widetilde{H}_\chi)$. Since the connected component of \widetilde{H}_χ is mapped onto the connected component of H_χ under the quotient map, it follows that $H_\chi/(H_\chi)_0$ is a quotient of $\widetilde{H}_\chi/(\widetilde{H}_\chi)_0$. Thus, if the latter is amenable, the same is true for $H_\chi/(H_\chi)_0$.

Thus, in what follows we may assume without loss of generality that H is simply connected. We are then in precisely the same situation as in the proof of Case (A) of the proof of the Theorem on page 2 of [49], and from now on we can follow the line of arguments as given on pages 2 and 3 of [49] to see that \widehat{S}/H is countably separated. Moreover, the arguments presented in steps c) and d) on page 3 of Pukánszky's book imply that for each stabilizer H_χ the quotient $H_\chi/(H_\chi)_0$ is a finite extension of an abelian group, and hence is amenable. \square

Remark 5.3. — Note that we want to use our induction assumption on the stabilizers G_x . This is not directly possible, since in general the groups G_x are not almost connected. However, if we know that every almost connected Lie group with dimension $\dim(G) < n$ satisfies BC for \mathcal{K} , then, by an easy application of Theorem 2.1, the same is true for all Lie groups G with $\dim(G) < n$ and G/G_0 amenable!

Unfortunately, the result on the orbit space \widehat{S}/G is not sufficient for a direct application of Theorem 3.6. So we need to obtain more information on the structure of \widehat{S}/G . To do this we have to do two steps:

- (i) Reduce to cases where the action of G on \widehat{S} factors through an algebraic action of some real algebraic group G' (or a subgroup of finite index in G').
- (ii) Show that the topological orbit-spaces of algebraic group actions on real affine varieties have nice stratifications as required by Theorem 3.6.

Note that Pukánszky does the first reduction for the cocompact subgroup H of G , which allowed us to draw the conclusions of the previous lemma. However, with a bit more work we obtain a similar conclusion for G . The following result is certainly well-known to the experts, but since we didn't find a direct reference we included the easy proof.

Lemma 5.4. — Assume that G is a Lie group with finitely many components such that G has a connected closed normal subgroup H with H semi-simple and G_0/H a torus group. Let V be a finite dimensional real vector space and let $\rho : G \rightarrow GL(V)$ be any continuous homomorphism. Then the Zariski closure G' of $\rho(G)$ is a (reductive) real algebraic group which contains $\rho(G)$ as a subgroup of finite index.

Proof. — Let $R : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ denote the differential of ρ and let \mathfrak{h} denote the ideal of \mathfrak{g} corresponding to H . Then $R(\mathfrak{h})$ is semi-simple (or trivial). Since $\rho(H)$ is a semi-simple subgroup of $GL(V)$ it is closed in $GL(V)$. This follows from the fact that every semi-simple subalgebra of $\mathfrak{gl}(V)$ is algebraic (by Proposition 5.1), which implies that $\rho(H)$ is the connected component of some algebraic linear subgroup of $GL(V)$. Since G/H is compact, it follows that $\rho(G)$ is a closed subgroup of $GL(V)$, too.

To simplify notation we assume from now on that G itself is a closed subgroup of $GL(V)$ and that ρ is the identity map. Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{z}$ be a Levi decomposition of \mathfrak{g} . Since G_0/H is abelian and H is semi-simple, it follows that $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = \{0\}$. Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ is reductive.

We now show that \mathfrak{g} is an algebraic subalgebra of $\mathfrak{gl}(V)$. By Proposition 5.1 it suffices to show that \mathfrak{z} is algebraic and consists of semi-simple elements. But this will follow if we can show that $Z = \exp(\mathfrak{z}) \subseteq GL(V)$ is compact, and hence a torus group. Since $Z \cap H$ is finite (since every linear semi-simple group has finite center), the restriction to Z of the quotient map $q : G \rightarrow G/H$ has finite kernel. Since $q(Z) = G_0/H$, $q(Z)$ is compact by assumption, and hence Z is compact, too.

It follows that the algebraic closure \tilde{G} of G_0 is a reductive algebraic group which contains G_0 as a subgroup of finite index. Since every element of G fixes the Lie algebra \mathfrak{g} via the adjoint action, it also normalizes \tilde{G} . Therefore, $G' = G\tilde{G}$ is a reductive algebraic group which contains G as a subgroup of finite index. \square

We now show that quotients of linear algebraic group actions on affine varieties have nice stratifications in the sense of Theorem 3.6. We are very grateful to Jörg Schürmann and Peter Slodowy for some valuable comments, which helped us to replace a previous version of the following result (which, as was pointed out to us by Jörg Schürmann, contained a gap) by

Proposition 5.5. — *Suppose that G is a closed subgroup of finite index of a Zariski closed subgroup G' of $\mathrm{GL}(n, \mathbf{R})$ and that $V \subseteq \mathbf{R}^n$ is a G' -invariant Zariski closed subset of \mathbf{R}^n . Then there exists a stratification*

$$\emptyset = V_0 \subseteq V_1 \subseteq \dots \subseteq V_l = V$$

of open G -invariant subsets V_i of V such that $(V_i \setminus V_{i-1})/G$ admits a continuous and open finite-to-one map onto a differentiable manifold.

Since every manifold has a triangulation, the above result really gives what we need to apply Theorem 3.6. For the proof we need the following lemma about certain decompositions of continuous semi-algebraic maps.

Lemma 5.6. — *Let X, Y be semi-algebraic sets and let $f : X \rightarrow Y$ be a continuous semi-algebraic map (see [7] for the notations). Then there exists a stratification*

$$\emptyset = Z_0 \subseteq Z_1 \cdots \subseteq Z_l = f(X),$$

with each Z_i open in $f(X)$, $Z_i \setminus Z_{i-1}$ is a differentiable manifold and

$$f : f^{-1}(Z_i \setminus Z_{i-1}) \rightarrow Z_i \setminus Z_{i-1}$$

is open (in the euclidean topology) for all $1 \leq i \leq l$.

Proof. — Since the image of a semi-algebraic set under a semi-algebraic map is semi-algebraic (see [7, Proposition 2.2.7]), we may assume without loss of generality that $Y = f(X)$. By [7, Corollary 9.3.3] there exists a closed semi-algebraic subset $Y_1 \subseteq Y$ with $\dim(Y_1) < \dim(Y)$, such that $Y \setminus Y_1$ is a finite disjoint union of connected components (combine with [7, Theorem 2.4.5]) and such that the restriction of f to the inverse image of each component is a projection, hence open. Thus $f : f^{-1}(Y \setminus Y_1) \rightarrow Y \setminus Y_1$ is open, too. Indeed, the construction (using [7, Proposition 9.18]) implies that $Y \setminus Y_1$ is homeomorphic to a submanifold of some \mathbf{R}^m . Put $Z_0 = Y \setminus Y_1$. Since $\dim(Y_1) < \dim(Y)$, the result follows by induction. \square

Remark 5.7. — Let $G \subseteq \mathrm{GL}(n, \mathbf{R})$ be a real linear algebraic group, and let $G_{\mathbf{C}} \subseteq \mathrm{GL}(n, \mathbf{C})$ be its complexification. Then it follows from [9, Proposition 2.3] that each $G_{\mathbf{C}}$ -orbit in \mathbf{C}^n contains at most finitely many G -orbits in $\mathbf{R}^n \subseteq \mathbf{C}^n$.

Proof of Proposition 5.5. — We first note that we may assume without loss of generality that $G = G'$. Indeed, since G has finite index in G' , every G' -orbit decomposes into finitely many G -orbits. Thus, if $\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_l = V$ is a stratification of V for the G' -action with the required properties, it is also a stratification for the G -action with the same properties. Thus we assume from now on that G is a Zariski closed subgroup of $\mathrm{GL}(V)$.

Let $V_{\mathbf{C}} \subseteq \mathbf{C}^n$ denote the complexification of V . Consider the diagram

$$\begin{array}{ccc} V & \longrightarrow & V_{\mathbf{C}} \\ \downarrow & & \downarrow \\ V/G & \longrightarrow & V_{\mathbf{C}}/G_{\mathbf{C}}. \end{array}$$

By the theorem of Rosenlicht ([51], but see also [36, Satz 2.2 on p. 23]), there exists a sequence

$$V_{\mathbf{C}} = W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots \supseteq W_r = \emptyset,$$

of Zariski-closed $G_{\mathbf{C}}$ -invariant subsets of strictly decreasing dimension such that $W_i \setminus W_{i+1}$ has closed $G_{\mathbf{C}}$ -orbits and the geometric quotient by $G_{\mathbf{C}}$ of $W_i \setminus W_{i+1}$ exists. This means that the quotient $(W_i \setminus W_{i+1})/G_{\mathbf{C}}$ can be realized as an algebraic set and the quotient map is also algebraic. Let \mathcal{O} be the first of the sets $W_i \setminus W_{i+1}$ which has nonempty intersection with V . Restricting the maps in the above diagram gives

$$\begin{array}{ccc} V \cap \mathcal{O} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ (V \cap \mathcal{O})/G & \longrightarrow & \mathcal{O}/G_{\mathbf{C}}. \end{array}$$

The resulting map f from $V \cap \mathcal{O}$ to $\mathcal{O}/G_{\mathbf{C}}$ is an algebraic map, and hence it is a continuous semi-algebraic map. Thus it follows from Lemma 5.6 that, if Y denotes the image of $X := V \cap \mathcal{O}$ in $\mathcal{O}/G_{\mathbf{C}}$, then Y has a stratification

$$\emptyset = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_s = Y$$

such that $f : f^{-1}(Z_i \setminus Z_{i-1}) \rightarrow Z_i \setminus Z_{i-1}$ is open for all $1 \leq i \leq s$, each Z_i is open in Y , and the difference sets $Z_i \setminus Z_{i-1}$ are submanifolds of some \mathbf{R}^m . Put $V_i = f^{-1}(Z_i)$ for $0 \leq i \leq s$. Then $V_s = V \cap \mathcal{O}$. By Remark 5.7, if we pass through the lower left corner of the diagram, the corresponding maps $(V_i \setminus V_{i-1})/G \rightarrow Z_i \setminus Z_{i-1}$ are open, finite-to-one, onto the manifolds $Z_i \setminus Z_{i-1}$.

Now replace V by the invariant Zariski-closed subset $V \setminus \mathcal{O}$. Repeating the above arguments finitely many times gives the desired stratification (the procedure stops after finitely many steps, since any increasing sequence of Zariski open sets eventually stabilizes). \square

Using the above results, we are now able to prove

Proposition 5.8. — *Suppose that G is a Lie group with finitely many components and with connected closed normal subgroups $N \subseteq H \subseteq G_0 \subseteq G$ as in (5.1), i.e., N is the nilradical of H , H/N is semi-simple and G_0/H is a torus group. Let $S \subseteq Z(N)$ be a connected closed subgroup which is normal in G , where $Z(N)$ denotes the center of N . Then \widehat{S} decomposes into a countable disjoint union of open G -invariant sets V_n such that each V_n has a stratification*

$$\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l = V_n$$

(where l may depend on n) of open G -invariant subsets of V_n , and continuous open surjections

$$q_i : U_i \setminus U_{i-1} \rightarrow Y_i, \quad 1 \leq i \leq l,$$

such that each Y_i is a differentiable manifold and inverse images of points in Y_i are finite unions of G -orbits in V_n for all $1 \leq i \leq l$.

Proof. — Let \mathfrak{s} denote the ideal of \mathfrak{g} corresponding to S . Then we may identify \widehat{S} with a closed G -stable subset of \mathfrak{s}^* of the form $R \times Z$ with R being a vector subgroup of \mathfrak{s}^* and Z a finitely generated free abelian group. Note that Z can be identified with the dual of the maximal compact subgroup in S , and therefore we can decompose Z into a disjoint union of G -orbits, which are all finite since G_0 acts trivially on Z . It then follows that \widehat{S} can be decomposed into a disjoint union of G -invariant sets of the form $R \times F$ with $F \subseteq Z$ finite.

The action of G on \widehat{S} is given via the coadjoint representation $\text{Ad}_{\mathfrak{s}}^* : G \rightarrow \text{GL}(\mathfrak{s}^*)$. Since $S \subseteq Z(N)$, it follows that this representation factors through a representation of G/N . Thus it follows from the general assumptions on G and Lemma 5.4 that the algebraic closure G' of $\text{Ad}_{\mathfrak{s}}^*(G)$ in $\text{GL}(\mathfrak{s}^*)$ is a reductive algebraic group which contains the image of $\text{Ad}_{\mathfrak{s}}^*(G)$ as a subgroup of finite index. Since the G -stable sets of the form $R \times F$ of the previous paragraph are closed algebraic subvarieties of \mathfrak{s}^* , it follows that these sets are also invariant under the action of the Zariski closure G' of $\text{Ad}_{\mathfrak{s}}^*(G)$. Thus it follows from Proposition 5.5 that for each such set we obtain a stratification

$$\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l = R \times F$$

with the required properties. \square

We are now ready for the final step:

Proof of Proposition 2.11. — By the discussion at the beginning of this section we may assume without loss of generality that G is as in (5.1), i.e., we have connected closed normal subgroups

$$N \subseteq H \subseteq G_0 \subseteq G$$

such that N is a nontrivial nilpotent group H/N is semi-simple and G_0/H is a torus group. For the induction step we have to show that every central extension

$$1 \rightarrow \mathbf{T} \rightarrow \bar{G} \rightarrow G \rightarrow 1$$

satisfies BC for \mathbf{C} . Let \bar{N} , \bar{H} and \bar{G}_0 denote the inverse images of N , H and G_0 in \bar{G} . Then the sequence of normal subgroups

$$\bar{N} \subseteq \bar{H} \subseteq \bar{G}_0 \subseteq \bar{G}$$

has the same general properties as the sequence $N \subseteq H \subseteq G_0 \subseteq G$, in particular, \bar{N} is the nilradical of \bar{H} and \bar{H} is locally algebraic. Let T denote the central copy of \mathbf{T} in \bar{G} coming from the given central extension. We now divide the proof into the following cases:

- C(1) The center $S = Z(\bar{N})$ of \bar{N} has dimension greater or equal to two.
- C(2) $Z(\bar{N}) = T$.

We start with Case C(1): By Theorem 2.1 (and the discussion following that theorem) it suffices to show that \bar{G}/S satisfies BC with coefficient $C_0(\widehat{S}, \mathcal{H})$ where the action of \bar{G}/S on \widehat{S} is given by conjugation. By Theorem 3.6 it suffices to show that all stabilizers $(\bar{G}/S)_\chi = \bar{G}_\chi/S$ satisfy BC for \mathcal{H} and that \widehat{S} has a nice stratification. While the latter follows from Proposition 5.8, the requirement on the stabilizers follows from Lemma 5.2, Remark 5.3, and the induction assumption since

$$\dim(\bar{G}_\chi/S) \leq \dim(\bar{G}) - 2 < \dim(G).$$

This finishes the proof in Case C(1).

For the proof of Case C(2) we have to do some more reduction steps in order to use the same line of arguments as in C(1). For this it is useful to consider the following two subcases:

- (2)a If $\bar{Z}(N)$ denotes the inverse image of the center $Z(N)$ of N in \bar{G} , then $Z(\bar{Z}(N)) = T$.
- (2)b $\dim(Z(\bar{Z}(N))) \geq 2$.

In Case (2)a we consider the normal subgroup $S = \bar{Z}(N)$ of G . Then S is a connected two-step nilpotent Lie group with one-dimensional center T , and therefore a Heisenberg group. It follows that $C_r^*(S) = C^*(S)$ can be written as the direct sum

$$C^*(S) = \bigoplus_{\chi \in \widehat{T}} A_\chi$$

with

$$A_\chi \cong \mathcal{K}, \quad \text{if } \chi \neq 1, \text{ and } A_1 = C_0(\widehat{S/T}).$$

Since \bar{G} acts trivially on \widehat{T} , it follows that the decomposition action of \bar{G}/S on $C^*(S) \otimes \mathcal{K}$ induces an action on each fibre A_χ , and, by Theorem 2.1 together with Proposition 2.6, it follows that \bar{G} satisfies BC with coefficient \mathbf{C} if \bar{G}/S satisfies BC with coefficient $A_\chi \otimes \mathcal{K}$ for each $\chi \in \widehat{T}$. If $\chi \neq 1$, we get $A_\chi \otimes \mathcal{K} \cong \mathcal{K}$, and the desired result follows from the induction assumption and the fact that $\dim(\bar{G}/S) < \dim(G)$.

So we only have to deal with the case $\chi = 1$, where we have to deal with the fibre $C_0(\widehat{S/T}, \mathcal{K}) = C_0(\widehat{Z(N)}, \mathcal{K})$. But here we are exactly in the same situation as in the proof of Case C(1), since the action of \bar{G} on $\widehat{Z(N)}$ factors through an action of G/N and all stabilizers of the characters of $Z(N)$ have dimension strictly smaller than $\dim(G)$.

We have to work a bit more for the Proof of Case (2)b. Here we put $S = Z(\bar{Z}(N))$. Then S is a connected abelian subgroup of \bar{N} and it follows from Lemma 5.2, Remark 5.3, the fact that $\dim(\bar{G}/S) < \dim(G)$ and the induction assumption that all stabilizers for the action of \bar{G}/S on \widehat{S} satisfy BC for \mathcal{K} .

Again we study the structure of the orbit space \widehat{S}/\bar{G} . For each $\chi \in \widehat{T}$ we define

$$\widehat{S}_\chi = \{\mu \in \widehat{S} : \mu|_T = \chi\}.$$

Since T is central in \bar{G} , it follows that \bar{G} acts trivially on \widehat{T} , and hence that \widehat{S}_χ is \bar{G} -invariant for all $\chi \in \widehat{T}$. Since \widehat{T} is discrete, we may write

$$C_0(\widehat{S}, \mathcal{K}) \cong \bigoplus_{\chi \in \widehat{T}} C_0(\widehat{S}_\chi, \mathcal{K})$$

with fiberwise action of \bar{G}/S . Thus by continuity of BC it suffices to deal with the single fibers. For $\chi = 1$ we are looking at the action of $\bar{G}/S \cong G/(S/T)$ on $\widehat{S}_1 \cong \widehat{S/T}$, and since S/T is a central subgroup of N we may again argue precisely as in the proof of Case C(1) to see that \bar{G}/S satisfies BC for $C_0(\widehat{S}_1, \mathcal{K})$.

In order to deal with the other fibers we are now going to show that \bar{G} acts transitively on \widehat{S}_χ for each nontrivial character $\chi \in \widehat{T}$. It follows then directly from Corollary 2.10 that \bar{G}/S satisfies BC for $C_0(\widehat{S}_\chi, \mathcal{K})$. In fact, Lemma 5.9 below shows that \bar{N} already acts transitively on \widehat{S}_χ for $\chi \neq 1$ and Proposition 2.11 will follow from that lemma. \square

The following lemma is certainly well known to the experts on the representation theory of nilpotent groups. For the readers convenience we give the elementary proof.

Lemma 5.9. — *Assume that N is a connected nilpotent Lie group with one-dimensional center $Z(N) = T$. Let S be a closed connected abelian normal subgroup of N such that $T \subseteq S$ and $S/T \subseteq Z(N/T)$. Let $1 \neq \chi \in \widehat{T}$ and let $\widehat{S}_\chi = \{\mu \in \widehat{S} : \mu|_T = \chi\}$. Then N acts transitively on \widehat{S}_χ by conjugation.*

Proof. — We may assume without loss of generality that N is simply connected. In fact, if this is not the case, we pass to the universal covering group \tilde{N} of N and the universal covering $\tilde{S} \subseteq \tilde{N}$ of S and observe that there exists a discrete subgroup $D \subseteq \tilde{T} = Z(\tilde{N})$ such that $N = \tilde{N}/D$, $S = \tilde{S}/D$, $T = \tilde{T}/D$ and \widehat{S}_χ can then be (equivariantly) identified with $\widehat{\tilde{S}}_\chi$ for all $\chi \in \widehat{T} \subseteq \widehat{\tilde{T}}$.

Let \mathfrak{n} , \mathfrak{s} and \mathfrak{t} denote the Lie algebras of N , S and T , respectively. Since N is simply connected, we can write

$$N = \{\exp(X) : X \in \mathfrak{n}\}$$

with multiplication given by the Campbell-Hausdorff formula. In particular, if $Y \in \mathfrak{s}$, then

$$\exp(X)\exp(Y) = \exp(X + Y + [X, Y])$$

for all $X \in \mathfrak{n}$, since it follows from the assumption that $S/T \subseteq Z(N/T)$ that $[X, Y] \in \mathfrak{t} = \mathfrak{z}(\mathfrak{n})$ and all commutators with $[X, Y]$ vanish. In particular, if we conjugate $\exp(Y)$ by $\exp(X)$ we get the formula

$$(5.2) \quad \exp(X)\exp(Y)\exp(-X) = \exp(Y + [X, Y])$$

for all $Y \in \mathfrak{s}$.

Assume now that $\dim(\mathfrak{s}) = n + 1$ and let $0 \neq Z \in \mathfrak{t}$. There exists a basis $\{Y_1, \dots, Y_n, Z\}$ of \mathfrak{s} and elements $X_1, \dots, X_n \in \mathfrak{n}$ such that

$$(5.3) \quad [X_i, Y_i] = Z \quad \text{and} \quad [X_i, Y_j] = 0$$

for all $1 \leq i, j \leq n, i \neq j$. Indeed this follows from an easy Schmidt-orthogonalization procedure applied to the bilinear form

$$(\cdot, \cdot) : \mathfrak{n} \times \mathfrak{s} \rightarrow \mathbf{R}; \quad (X, Y) = \lambda \Leftrightarrow [X, Y] = \lambda Z.$$

We identify S with \mathfrak{s} (via \exp) and \widehat{S} with \mathfrak{s}^* . The conjugation action of N on \widehat{S} is then transferred to the coadjoint action Ad^* . If $\{f_1, \dots, f_n, g\}$ is a dual basis for the basis $\{Y_1, \dots, Y_n, Z\}$ of \mathfrak{s} , the result will follow if we can show that

$$\text{Ad}^*(N)(\lambda g) = \text{span}\{f_1, \dots, f_n\} + \lambda g$$

for all $0 \neq \lambda \in \mathbf{R}$. By rescaling we may assume that $\lambda = 1$. But for $\lambda_1, \dots, \lambda_n \in \mathbf{R}$ we can compute

$$\begin{aligned} (\text{Ad}^*(\exp(\lambda_1 X_1 + \dots + \lambda_n X_n))(g))(Y_i) &= g(Y_i + \lambda_i [X_i, Y_i]) \\ &= \lambda_i g(Z) = \lambda_i. \end{aligned}$$

Since Z is central in \mathfrak{n} it follows that $(\text{Ad}^*(\exp(X))(g))(Z) = g(Z) = 1$ for all $X \in \mathfrak{n}$. Thus

$$\text{Ad}^*(\exp(\lambda_1 X_1 + \dots + \lambda_n X_n))(g) = \lambda_1 f_1 + \dots + \lambda_n f_n + g. \quad \square$$

6. The p -adic case

In this section, k denotes a finite extension of some field of p -adic numbers \mathbf{Q}_p . By a k -group G we shall always understand a Zariski closed subgroup of $\text{GL}(n, k)$. As pointed out in §3, all k -groups are exact. It is shown in [33] that all k -groups have a γ -element. Hence all k -groups G satisfy the general assumptions made in §3.

Since k has characteristic zero, it follows from general structure theory (e.g., see [27, VIII, Theorem 4.3]) that G is a semidirect product $N \rtimes R$ of the unipotent radical of G (which is a k -group) by some reductive k -subgroup R of G . We want to show by induction on the dimension of G (i.e., the dimension of the Lie-algebra \mathfrak{g} of G), that G satisfies the Baum-Connes conjecture with trivial coefficient. In fact, as in the real case, to perform the argument it is necessary to generalize the result a little bit, since we need to include certain actions of G on the algebra of compact operators $\mathcal{K}(H)$.

We first need some information on the unitary representation theory of unipotent k -groups. For this let \mathfrak{n} denote the Lie-algebra of N , and let $\exp : \mathfrak{n} \rightarrow N$ and $\log : N \rightarrow \mathfrak{n}$ denote the exponential map and its inverse. Let \mathfrak{n}^* denote the dual space of the underlying vector space of \mathfrak{n} and let $\text{Ad}^* : N \rightarrow \text{GL}(\mathfrak{n}^*)$ denote the coadjoint representation of N on \mathfrak{n}^* . By Kirillov's theory, established for p -adic unipotent groups by Moore in [43], there exists a bijection between the quotient space $\mathfrak{n}^*/\text{Ad}^*(N)$ and the unitary dual \widehat{N} of N as follows: Fix any character $\varepsilon \in \widehat{k}$ of order zero in the sense of [55, II, Definition 4]. For $f \in \mathfrak{n}^*$, let \mathfrak{m}_f be a maximal subalgebra of \mathfrak{n} such that $f([\mathfrak{m}_f, \mathfrak{m}_f]) = \{0\}$. Let $M_f = \exp(\mathfrak{m}_f) \subseteq N$. Then $m \mapsto \chi_f(m) := \varepsilon(f(\log(m)))$ is

a character of M_f and the induced representation $\pi_f := \text{ind}_{M_f}^N \chi_f$ is an irreducible representation of N , whose equivalence class does not depend on the choice of \mathfrak{m}_f . The resulting map

$$\mathfrak{n}^* \rightarrow \widehat{N}; f \rightarrow \pi_f$$

is constant on $\text{Ad}^*(N)$ -orbits and induces a bijection between $\mathfrak{n}^*/\text{Ad}^*(N)$ and \widehat{N} . By [28, Theorem II], this bijection is a homeomorphism. Since the $\text{Ad}^*(N)$ -orbits in \mathfrak{n}^* are closed, it follows that the points of $\widehat{N} = \widehat{C^*(N)}$ are closed, which implies that $C^*(N)$ is type I (cf [43, Theorem 4]).

We are now going to specify the actions of G on $\mathcal{K}(H)$ which we want to include into our picture: For this suppose that G and G' are k -groups such that G' is an algebraic semi-direct product $G' = M \rtimes G$ with M a normal unipotent k -subgroup of G' . Then $C_r^*(G') = C^*(M) \rtimes_r G$ and the unipotent radical N' of G' equals $M \rtimes N$, where N denotes the unipotent radical of G .

Let $\pi \in \widehat{M} \cong \widehat{C^*(M)}$ such that the unitary equivalence class of π is G -invariant. Since $C^*(M)$ is type I, the action of G on $C^*(M)$ factors through an action of G on $\mathcal{K}(H_\pi)$, and we may define

Definition 6.1. — Suppose that G is a k -group and that H is a Hilbert-space. An action of G on $\mathcal{K}(H)$ is called unipotent, if there exists an extension $G' = M \rtimes G$ as above, and a G -invariant element $\pi \in \widehat{M}$ such that $H = H_\pi$ and the action of G on $\mathcal{K}(H_\pi)$ is induced from the action of G on $C^*(M)$ as above.

Now, the p -adic case of Theorem 1.1 will follow from

Proposition 6.2. — Let k be a finite extension of \mathbf{Q}_p and let G be a k -group acting unipotently on $\mathcal{K}(H)$. Then G satisfies BC for $\mathcal{K}(H)$.

Proof. — Let G be any k -group. If $\dim G = 1$, then G is almost abelian, and the result is true by the general fact that all amenable groups satisfy BC for arbitrary coefficients ([26]). Assume now that $\dim(G) > 1$. If G is reductive, then the result is true by Proposition 4.9. If G is not reductive, then $G = N \rtimes R$, where N is the unipotent radical of G and R is a reductive k -subgroup of G with $\dim(R)$ strictly smaller than the dimension of G . Let $\alpha : G \rightarrow \mathcal{K}(H)$ be any unipotent action of G . Since R is totally disconnected and since N is an amenable closed subgroup of G , it follows from Theorem 2.1 that it suffices to show that R satisfies BC for $A := \mathcal{K}(H) \rtimes N$ with respect to the canonical action of R on this algebra (since the extension of R by N is split, there is no need to consider twisted actions at this point).

Let $G' = M \rtimes G$ be a semidirect product of G by a unipotent k -group M as in Definition 6.1, and let $\pi \in \widehat{C^*(M)}$ be G -invariant such that $H = H_\pi$ and such

that α is induced from the action of G on $C^*(M)$. Since π is also N -invariant, $A = \mathcal{H}(H) \rtimes N$ is the quotient of $C^*(M \rtimes N) \cong C^*(M) \rtimes N$ corresponding to the closed G -invariant subset $L := \{\rho \in \widehat{N \rtimes M} : \ker \rho|_M = \ker \pi\}$ (where the kernels are taken in $C^*(M)$). In particular, we have $L = \widehat{A}$. Moreover, since $C^*(M)$ is type I, the same is true for A .

Now write $N' = M \rtimes N$ and let \mathfrak{n}' denote the Lie algebra of N' . Let $\text{Ad}^* : G' \rightarrow \text{GL}((\mathfrak{n}')^*)$ denote the adjoint action of G' on $(\mathfrak{n}')^*$. Since this is an algebraic action, all $\text{Ad}^*(G')$ -orbits in $(\mathfrak{n}')^*$ are locally closed in the Zariski topology, and therefore also in the Hausdorff topology. For $\rho \in \widehat{N}'$ let $f \in (\mathfrak{n}')^*$ such that $\rho = \pi_f$. Then the R -orbit $R(\rho) \subseteq \widehat{N}'$ corresponds to the $\text{Ad}^*(G')$ -orbit of f under the Kirillov correspondence for \widehat{N}' (e.g., see [42, §4]). Since the Kirillov correspondence is a homeomorphism between $(\mathfrak{n}')^*/\text{Ad}^*(N')$ and \widehat{N}' , it follows that \widehat{N}'/R is homeomorphic to $(\mathfrak{n}')^*/\text{Ad}^*(G')$. In particular, it follows that every R -orbit in \widehat{N}' is locally closed. It follows that the action of R on \widehat{N}' and on $\widehat{A} \subseteq \widehat{N}'$ is countably separated (see Remark 3.5).

By Glimm's Theorem ([24, Theorem]) we obtain an ascending family $\{U_\nu\}_\nu$ of open subsets of \widehat{N}' satisfying the conditions as described in item (3) of Remark 3.5. Moreover, since all difference sets $U_{\nu+1} \setminus U_\nu$ are orbit spaces of locally compact subsets of the totally disconnected space $(\mathfrak{n}')^*$, it follows that all these difference sets are totally disconnected, and hence the Hausdorff quotients $(U_{\nu+1} \setminus U_\nu)/R$ are totally disconnected. Thus, taking the intersections $U_\nu^\Lambda := \widehat{A} \cap U_\nu$, we obtain an ascending family $\{U_\nu^\Lambda\}_\nu$ of \widehat{A} satisfying the conditions of Theorem 3.6.

Using Theorem 3.6 the proposition will follow as soon as we have checked that for all $\rho \in C^*(N')$ (and hence for all $\rho \in \widehat{A} \subseteq C^*(N')$), the stabilizer R_ρ of ρ in R satisfies BC for $\mathcal{H}(H_\rho)$. By [19, Lemme 12], if $\rho = \pi_f$ for some $f \in (\mathfrak{n}')^*$, then $R_\rho = G'_f \cap R$, where G'_f denotes the stabilizer of f in G' . Since the action of G' on $(\mathfrak{n}')^*$ is algebraic, G'_f and hence R_ρ is a k -group with dimension strictly smaller than $\dim(G)$. Since the action of R_ρ on $\mathcal{H}(H_\rho)$ is clearly unipotent, the result follows from our induction assumption. \square

7. Relations to the K-theory of the maximal compact subgroup

In this section we want to describe the relations between the K -theory of $C_r^*(G)$ and the K -theory of $C^*(L)$, where L denotes the maximal compact subgroup of the almost connected group G (we chose the letter L to avoid confusion). We should mention that all results presented here (except the conclusions drawn out of our main theorem) are well known, but since they have important impact on our results, we found it useful to give at least a brief report. The main references for these results are [17, 30, 31], and we refer especially to [17, §4] for a more geometric discussion of some of the results presented in this section.

If G and L are as above, it follows from work of Abels (see [1]) that G/L is a universal proper G -space. Thus we have

$$K_*^{\text{top}}(G; A) \cong KK_*^G(C_0(G/L), A) \cong^{\text{res}_L^G} KK_*^L(C_0(G/L), A),$$

where the second isomorphism follows from [32, Corollary to Theorem 5.7]. Also by the work of Abels [1], G/L is a Riemannian manifold which is L -equivariantly diffeomorphic to the tangent space $V := T_{eL}$, equipped with the adjoint action of L on V . It follows then from Kasparov’s work in [30] (see [11, Lemma 7.7] for a more extensive discussion) that tensoring with $C_0(V)$ gives a natural isomorphism

$$\sigma_{C_0(V)} : KK_*^L(C_0(V), A) \rightarrow KK_*^L(C_0(V) \otimes C_0(V), A \otimes C_0(V)),$$

and by Kasparov’s Bott-periodicity theorem (see [30, Theorem 7 of §5]) we know that $C_0(V) \otimes C_0(V)$, equipped with the diagonal action, is KK^L -equivalent to \mathbf{C} (but see also the discussion below). Thus we obtain the following chain of isomorphisms

$$\begin{aligned} KK_*^L(C_0(G/L), A) &\cong KK_*^L(C_0(V), A) \\ &\cong^{\sigma_{C_0(V)}} KK_*^L(C_0(V) \otimes C_0(V), A \otimes C_0(V)) \\ &\cong KK_*^L(\mathbf{C}, A \otimes C_0(V)) = K_*((A \otimes C_0(V)) \rtimes L), \end{aligned}$$

where the last isomorphism follows from the Green-Julg theorem. Hence, as a direct consequence of Theorem 1.2 we can deduce

Theorem 7.1. — *Assume that G is an almost connected (second countable) group with maximal compact subgroup L . Let $\mathcal{K} = \mathcal{K}(H)$ be the algebra of compact operators on the separable Hilbert space H equipped with any action of G . Then $K_*(\mathcal{K} \rtimes_r G)$ is naturally isomorphic to $K_*((\mathcal{K} \otimes C_0(V)) \rtimes L)$.*

By Kasparov’s Bott-periodicity theorem (see [30, Theorem 7 of §5]) it follows that $C_0(V)$ is KK^L -equivalent to the graded complex Clifford algebra $Cl(V)$ (with respect to a compatible inner product on V), equipped with the action of L induced by the given action on V . So we can replace $C_0(V)$ by the graded C^* -algebra $Cl(V)$, but then we have to use graded K -theory!

Let us look a bit closer to the implications of this Bott-periodicity theorem. Assume for the moment that V is even dimensional and that the action of L on V preserves a given orientation of V , i.e., the action factors through a homomorphism $\varphi : L \rightarrow SO(V)$. We have a central extension

$$0 \rightarrow \mathbf{T} \rightarrow \text{Spin}^c(V) \rightarrow SO(V) \rightarrow 0$$

of $\mathrm{SO}(V)$, where $\mathrm{Spin}^c(V) \subseteq \mathrm{Cl}(V)$ denotes the group of complex spinors (e.g. see [2]). The corresponding action of L on $\mathrm{Cl}(V)$ is given by the homomorphism

$$L \rightarrow \mathrm{SO}(V) \cong \mathrm{Spin}^c(V)/\mathbf{T} = \mathrm{Ad}(\mathrm{Spin}^c(V)).$$

Now choose a fixed orthonormal base $\{e_1, \dots, e_n\}$ of V . Then the grading of $\mathrm{Cl}(V)$ is given by conjugation with the symmetry $J = e_1 \cdot e_2 \cdots e_n \in \mathrm{Cl}(V)$. One can show that, up to a sign, J does not depend on the choice of this basis, and the sign only depends on the orientation of the basis. In particular, J is invariant under conjugation with elements in $\mathrm{Spin}^c(V)$. From this it follows that the graded L -algebra $\mathrm{Cl}(V)$ is L -equivariantly Morita equivalent to the trivially graded L -algebra $\mathrm{Cl}(V)$ – a Morita equivalence is given by the module $\mathrm{Cl}(V)$ with given L -action and grading automorphism given by left multiplication with J . Moreover, since $n = \dim(V)$ is even, $\mathrm{Cl}(V)$ is isomorphic to the simple matrix algebra $M_{2^n}(\mathbf{C})$.

Assume now that $\dim(G/L)$ is odd. Then, replacing G by $G \times \mathbf{R}$ (with trivial action of \mathbf{R} on \mathcal{K}) we get

$$K_*(\mathcal{K} \rtimes_r G) = K_{*+1}(\mathcal{K} \rtimes_r (G \times \mathbf{R})).$$

Moreover, if the action of L on $V = T_{eL}$ is orientation preserving, the same is true for the resulting action of L on $V \times \mathbf{R}$, which we identify with the tangent space at eL in the group $G \times \mathbf{R}$. Hence, modulo a dimension shift, we can use the above considerations also for this case. Thus, as a consequence of Theorem 7.1 we obtain

Theorem 7.2. — *Assume that G is an almost connected group with maximal compact subgroup L such that the adjoint action of L on $V = T_{eL}$ is orientation preserving. Then there are natural isomorphisms*

$$K_*(\mathcal{K} \rtimes_r G) \cong K_*((\mathcal{K} \otimes \mathrm{Cl}(V)) \rtimes L)$$

if $\dim(G/L)$ is even and

$$K_{*+1}(\mathcal{K} \rtimes_r G) \cong K_*((\mathcal{K} \otimes \mathrm{Cl}(V \times \mathbf{R})) \rtimes L)$$

if $\dim(G/L)$ is odd. Here all algebras are trivially graded!

Perhaps, the above result has its most satisfying formulation if translated into the language of twisted group algebras. For this let $\omega \in Z^2(G, \mathbf{T})$ denote a representative of the Mackey obstruction for the action of G on \mathcal{K} (see the discussion preceding Lemma 2.4). Then $\mathcal{K} \rtimes_r G$ is isomorphic to $C_r^*(G, \omega) \otimes \mathcal{K}$, where $C_r^*(G, \omega)$ denotes the reduced twisted group algebra $C_r^*(G, \omega)$ (e.g., see [25, Theorem 18]). Recall that $C_r^*(G, \omega)$ can be defined either as the reduced twisted crossed product $\mathbf{C} \rtimes_r (G_\omega, \mathbf{T})$ with respect to the twisted action (id, χ_1) (which, by Lemma 2.4, is Morita equivalent

to the given action on \mathcal{K} , or as the completion of $L^1(G) \subseteq B(L^2(G))$, where $L^1(G)$ acts on $L^2(G)$ by the twisted convolution

$$f * \xi(s) = \int_G f(t)\omega(t, t^{-1}s)\xi(t^{-1}s) dt, \quad f \in L^1(G), \xi \in L^2(G).$$

Up to isomorphism, $C_r^*(G, \omega)$ only depends on the class $[\omega] \in H^2(G, \mathbf{T})$. Conversely, given any cocycle, the representation $\lambda_\omega : G \rightarrow U(L^2(G))$ given by

$$(\lambda_\omega(t)\xi)(s) = \omega(t, t^{-1}s)\xi(t^{-1}s)$$

determines an action of G on $\mathcal{K}(L^2(G))$ with Mackey obstruction represented by ω .

Note that the Mackey obstruction for the action of L on \mathcal{K} is given by the restriction of ω to L and the obstruction for the action of L on $Cl(V) \cong M_{2^n}(\mathbf{C})$ (if $\dim(V)$ is even) is given by the pull-back, say μ_L , to L of a cocycle representing the central extension

$$1 \rightarrow \mathbf{T} \rightarrow \text{Spin}^c(V) \rightarrow \text{SO}(V) \rightarrow 1.$$

Since $\text{Spin}^c(V) \cong (\mathbf{T} \times \text{Spin}(V))/\mathbf{Z}_2$ (diagonal action), where

$$1 \rightarrow \mathbf{Z}_2 \rightarrow \text{Spin}(V) \rightarrow \text{SO}(V) \rightarrow 1$$

is the extension given by the real group of spinors, the cocycle μ_L can be chosen to take values in the subgroup $\mathbf{Z}_2 \subseteq \mathbf{T}$, and therefore $\mu_L^2 = 1$. Note that μ_L is trivial if and only if the homomorphism $\varphi : L \rightarrow \text{SO}(V)$ factorizes through $\text{Spin}^c(V)$ (i.e., if and only if G/L carries a G -invariant Spin^c -structure). If $\dim(V)$ is odd, we may define μ_L in the same way as above, noticing that this cocycle is equivalent to the pull back of (a cocycle representing) the extension

$$1 \rightarrow \mathbf{T} \rightarrow \text{Spin}^c(V \times \mathbf{R}) \rightarrow \text{SO}(V \times \mathbf{R}) \rightarrow 1,$$

which follows from the fact that L acts trivially on \mathbf{R} ! Since the Mackey obstruction of a tensor product of actions is the product of the Mackey obstructions of the factors, we obtain

Theorem 7.3. — Assume that G is an almost connected group with maximal compact subgroup L such that the adjoint action of L on $V = T_{eL}$ is orientation preserving. Let $n = \dim(G/L)$ and let $\omega \in Z^2(G, \mathbf{T})$ be any cocycle on G . Then

$$K_* (C_r^*(G, \omega)) \cong K_{*+n} (C^*(L, \omega \cdot \mu_L)).$$

In particular, in the special case where ω is trivial, we obtain an isomorphism

$$K_* (C_r^*(G)) \cong K_{*+n} (C^*(L, \mu_L)).$$

Again, μ_L is trivial if and only if G/L carries a G -invariant Spin^c -structure. In general, since $C^*(L, \omega \cdot \mu_L)$ is the quotient of the C^* -group algebra of the central extension L_{ω, μ_L} of L by \mathbf{T} corresponding to the character χ_1 of \mathbf{T} , it follows that $C^*(L, \omega \cdot \mu_L)$ is a direct sum of (possibly infinitely many) matrix algebras. Thus as a direct corollary of the above result we obtain:

Corollary 7.4. — *Assume that G , L and ω are as in Theorem 7.3. Then $K_{0+n}(C_r^*(G, \omega))$ is isomorphic to a free abelian group in at most countably many generators and $K_{1+n}(C_r^*(G, \omega)) = \{0\}$.*

This result has interesting consequences towards the question of existence of square integrable representations of connected unimodular Lie groups. In fact, combining the above corollary with [50, Theorem 4.6] gives:

Corollary 7.5 (cf [50, Corollary 4.7]). — *Let G be a connected unimodular Lie group. Then all square-integrable factor representations of G are type I. Moreover, G has no square-integrable factor representations if $\dim(G/L)$ is odd, where L denotes the maximal compact subgroup of G .*

We refer to [50] for more detailed discussions on this kind of applications of the positive solution of the Connes-Kasparov conjecture. Note that Theorem 7.3 and Corollary 7.4 do not hold in general without the assumption that the action of L on $V = T_{eL}$ is orientation preserving. In fact an easy six-term-sequence argument shows that it cannot hold for the group $G = \mathbf{R} \rtimes \mathbf{Z}_2$, where \mathbf{Z}_2 acts on \mathbf{R} by reflection through 0.

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