TAMENESS ON THE BOUNDARY AND AHLFORS' MEASURE CONJECTURE

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ABSTRACT

Let N be a complete hyperbolic 3-manifold that is an algebraic limit of geometrically finite hyperbolic 3-manifolds. We show N is homeomorphic to the interior of a compact 3-manifold, or *tame*, if one of the following conditions holds:

1. N has non-empty conformal boundary,
2. N is not homotopy equivalent to a compression body, or
3. N is a strong limit of geometrically finite manifolds.

The first case proves Ahlfors' measure conjecture for Kleinian groups in the closure of the geometrically finite locus: given any algebraic limit Γ of geometrically finite Kleinian groups, the limit set of Γ is either of Lebesgue measure zero or all of $\widehat{\mathbf{C}}$. Thus, Ahlfors' conjecture is reduced to the density conjecture of Bers, Sullivan, and Thurston.

1. Introduction

Let N be a complete hyperbolic 3-manifold. Then N is said to be *tame* if it is homeomorphic to the interior of a compact 3-manifold. A clear picture of the topology of hyperbolic 3-manifolds with finitely generated fundamental group rests on the following conjecture of A. Marden.

Conjecture **1.1** *(Marden's Tameness Conjecture). — Let* N *be a complete hyperbolic 3-manifold with finitely generated fundamental group. Then* N *is tame.*

In this paper, we employ new analytic techniques from the theory of hyperbolic cone-manifolds to fill in a step in W. Thurston's original program to prove Conjecture 1.1 [Th2].

Theorem **1.2.** *— Let* N *be an algebraic limit of geometrically finite hyperbolic 3-manifolds. If* N *has non-empty conformal boundary then* N *is tame.*

Each complete hyperbolic 3-manifold N is the quotient $\mathbf{H}^3/\Gamma$ of hyperbolic 3-space by a *Kleinian group*, namely, a discrete subgroup of $\mathrm{Isom}^+\mathbf{H}^3$, the orientation-preserving isometries of hyperbolic 3-space. The group Γ and its quotient $N = \mathbf{H}^3/\Gamma$ are called *geometrically finite* if a unit neighborhood of the convex

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core of N (the minimal convex subset whose inclusion is a homotopy equivalence) has finite volume, and N is an *algebraic limit* of the manifolds $N_i = \mathbf{H}^3/\Gamma_i$ if there are isomorphisms $\rho_i \colon \Gamma \to \Gamma_i$ so that ρ_i converges up to conjugacy to the identity as a sequence of maps to $\mathrm{Isom}^+\mathbf{H}^3$.

The extension to $\widehat{\mathbf{C}}$ of the action of Γ partitions the Riemann sphere into its *domain of discontinuity* $\Omega(\Gamma)$, where Γ acts properly discontinuously, and its *limit set* $\Lambda(\Gamma)$, where Γ acts chaotically. The quotient $\Omega(\Gamma)/\Gamma$, the *conformal boundary* of N, gives a bordification of N by finite area hyperbolic surfaces (see [Ah1]). In regard to the action of Γ on $\widehat{\mathbf{C}}$, L. Ahlfors made the following conjecture (see [Ah1, 1.4]).

Conjecture **1.3** *(Ahlfors' Measure Conjecture). — Let* Γ *be a finitely generated Kleinian group. Then either* $\Lambda(\Gamma)$ *is all of* $\widehat{\mathbf{C}}$ *or* $\Lambda(\Gamma)$ *has Lebesgue measure zero.*

Ahlfors established his conjecture for geometrically finite Γ in [Ah2]. Work of Thurston, Bonahon and Canary demonstrated the relevance of Conjecture 1.1 to Ahlfors' conjecture.

Theorem **1.4** *([Th1,Bon2,Can1]). — If* $N = \mathbf{H}^3/\Gamma$ *is tame, then Ahlfors' conjecture holds for* Γ.

Thus, Theorem 1.2 readily implies the following case of Ahlfors' conjecture.

Theorem **1.5.** *— Let* $N = \mathbf{H}^3/\Gamma$ *be an an algebraic limit of geometrically finite hyperbolic 3-manifolds. Then Ahlfors' conjecture holds for* Γ.

Theorem 1.5 reduces Ahlfors' conjecture to the following conjecture originally formulated by Bers and expanded upon by Sullivan and Thurston.

Conjecture **1.6** *(Bers-Sullivan-Thurston Density Conjecture). — If* N *is a complete hyperbolic 3-manifold with finitely generated fundamental group, then* N *is an algebraic limit of geometrically finite hyperbolic 3-manifolds.*

With the same methods, we obtain Conjecture 1.1 for limits of geometrically finite manifolds provided either $\pi_1(N)$ is not isomorphic to the fundamental group of a *compression body*, or N is a *strong* limit. We detail these consequences after providing some context for our results.

Theorem 1.2 is part of a history of tameness results for limits of either tame or geometrically finite manifolds. The first of these was proven by Thurston, who carried out his original suggested approach to Conjecture 1.1 (see [Th2]) by promoting *geometric* tameness, a geometric criterion on the ends of a hyperbolic 3-manifold, to algebraic limits N for which $\pi_1(N)$ is freely indecomposable (the condition is slightly different in the presence of cusps; see [Th1]). He also showed that his geometric tameness criterion was sufficient to guarantee the *topological* tameness condition of Conjecture 1.1 in this setting.

F. Bonahon later established that geometric tameness holds generally under such assumptions on $\pi_1(\mathrm{N})$ [Bon2], obviating any need for limiting arguments. Using Bonahon's work, Canary established the equivalence of geometric tameness and the topological condition of Conjecture 1.1 [Can1].

The inspiration for the present argument arises from the successful pursuit by R. Canary and Y. Minsky [CM] of Thurston's original limiting approach when $\pi_1(\mathrm{N})$ may decompose as a free product, and its recent extension by the third author [Ev2]. Each of these limiting arguments, however, makes strong working assumptions about the type of convergence and the role of parabolics in particular.

Our aim here is to employ the analytic theory of *cone-deformations* to force such assumptions to hold for *some* approximation of a given hyperbolic 3-manifold N. Before outlining our approach to Theorem 1.2, we record some other applications of our methods.

Algebraic and geometric limits. — One element of our proof of Theorem 1.2 relies on an in-depth study of the relationship between algebraic and geometric convergence carried out by Anderson and Canary [AC1,AC2] in their work on a conjecture of T. Jørgensen (see Conjecture 2.2). Their results are applicable in another setting, to which our techniques then also apply.

We will say a group G is a *compression body group* if it admits a non-trivial free product decomposition into orientable surface groups and infinite cyclic groups (then G is the fundamental group of a *compression body*, see [Bon1, App. B]).

Theorem **1.7.** — *Let* N *be an algebraic limit of geometrically finite hyperbolic 3-manifolds and assume* $\pi_1(\mathrm{N})$ *is not a compression body group. Then* N *is tame.*

When the algebraic limit N of N_i is also the *geometric limit*, or the *Gromov-Hausdorff limit* of N_i (with appropriately chosen basepoints), we say N_i converges *strongly* to N. As we will see, our study is closely related to this notion of strong convergence. Conjecture 1.1 also follows for this category of limits, with no assumptions on the limit itself.

Theorem **1.8.** — *Let* N *be a strong limit of geometrically finite* N_i. *Then* N *is tame.*

Drilling accidental parabolics. — The central new ingredient in our proof of Theorem 1.2 has its origins in the deformation theory of hyperbolic cone-manifolds as developed by S. Kerckhoff, C. Hodgson and the second author, and its utilization in the study of Conjecture 1.6 by the first and second authors (see [Brm4,BB2,BB1]). The key tool arising from these techniques is a *drilling theorem*, proven in [BB2], whose efficacy we briefly describe.

A sufficiently short closed geodesic η in a geometrically finite hyperbolic 3-manifold N can be "drilled out" to yield a new complete hyperbolic manifold N_0 homeomorphic to $\mathrm{N} \setminus \eta$. A "torus" or "rank-2" cusp remains in N_0 where η has receded to infinity. The Drilling Theorem (see Theorem 3.3) gives quantitative force to the idea one can drill out a short geodesic with small effect on the geometry of the ambient manifold away from a standard tubular neighborhood of the geodesic. In practice, the theorem allows one effectively to eliminate troublesome *accidental parabolics* in an algebraically convergent sequence $\mathrm{N}_i \to \mathrm{N}$, namely, parabolic elements of $\pi_1(\mathrm{N})$ whose corresponding elements in $\pi_1(\mathrm{N}_i)$ are not parabolic.

Drilling out of N_i the short geodesic representatives of the accidental parabolics in N changes the topology of N_i, but changes the geometry on a compact core carrying $\pi_1(\mathrm{N}_i)$ less and less. Passing to the cover corresponding to the core yields a manifold $\hat{\mathrm{N}}_i$ with the correct (marked) fundamental group, and the geometric convergence of the cores guarantees that this new sequence $\{\hat{\mathrm{N}}_i\}$ still converges to N. Moreover, the cusps of N are cusps in each $\hat{\mathrm{N}}_i$, so with respect to the approximation by $\hat{\mathrm{N}}_i$ the limit N has *no accidental parabolics*. The incipient cusps have been "drilled" to become cusps in the approximates.

When the Drilling Theorem is applied to an appropriate family of approximates for N, we obtain a convergent sequence $\hat{\mathrm{N}}_i \to \mathrm{N}$ that is *type-preserving*: cusps of N are in one-to-one correspondence with the cusps of $\hat{\mathrm{N}}_i$. In other words, we have the following theorem, which represents the central result of the paper.

Theorem **1.9** *(Limits are Type-Preserving Limits). — Each algebraic limit* N *of geometrically finite hyperbolic 3-manifolds is also a limit of a type-preserving sequence of geometrically finite hyperbolic 3-manifolds.*

(See Theorem 3.1 for a more precise statement).

Historically, accidental parabolics have represented the principal potential obstruction to strong convergence, as they often signal the presence of extra parabolic elements in the geometric limit (see, for example [BO], [Th4, Sect. 7], [Br], and Conjecture 2.2).

Theorem 1.9 represents the heart of the argument for Theorem 1.2. Indeed, applying the results of Anderson and Canary mentioned above, we are ready to give the proofs of Theorems 1.2, 1.5, and 1.7 assuming Theorem 1.9.

Proof of Theorems 1.2, 1.5, and 1.7. — Let N be an algebraic limit of geometrically finite hyperbolic 3-manifolds N_i, and assume that either

1. N has non-empty conformal boundary, or
2. $\pi_1(\mathrm{N})$ is not a compression body group.

Theorem 1.9 furnishes a type-preserving sequence $\hat{N}_i \to N$. Applying results of Anderson and Canary (see Theorem 2.3 or [AC2]), $\hat{N}_i$ converges *strongly* to N. By a theorem of the third author (see Theorem 2.1 or [Ev2]), any type-preserving strong limit of tame hyperbolic 3-manifolds is also tame. It follows that N is tame, proving Theorems 1.2 and 1.7.

Theorem 1.5 follows from observing that if $N = \mathbf{H}^3/\Gamma$, then either $\Lambda(\Gamma) = \widehat{\mathbf{C}}$ or $\Omega(\Gamma)$ is non-empty and N has non-empty conformal boundary. In the latter case, Theorem 1.2 implies that N is tame, and tameness of N guarantees that the Lebesgue measure of $\Lambda(\Gamma)$ is zero (see [Can1]). This proves Theorem 1.5. □

The strong topology. — Implicit in the proofs of Theorems 1.2 and 1.7 is the idea that a given algebraic limit can be realized as a *strong* limit. As an end in its own right, this step in the proof verifies a conjectural picture of the deformation space due to Thurston (see [Th3]) which we now briefly describe.

The space GF(M) of marked, geometrically finite hyperbolic 3-manifolds homotopy equivalent to M inherits its topology from its inclusion in the set

$$H(M) = \{\rho\colon \pi_1(M) \to \mathrm{Isom}^+\mathbf{H}^3 \mid \rho \text{ is discrete and faithful}\}/\text{conj}.$$

equipped with the quotient of the topology of convergence on generators (the *algebraic topology*). The set H(M) with this topology is denoted AH(M); in referring to a hyperbolic manifold N as an element of H(M), we assume an implicit representation $\rho\colon \pi_1(M) \to \mathrm{Isom}^+\mathbf{H}^3$ for which $N = \mathbf{H}^3/\rho(\pi_1(M))$.

Marden and Sullivan proved [Mar1,Sul2] that the interior of AH(M) is the subset MP(M) consisting of *minimally parabolic* geometrically finite structures, namely, those whose only cusps are rank-2 (and therefore are forced by the topology of M).

If one imposes the stronger condition that convergent representatives $\rho'_i \to \rho'$ from convergent conjugacy classes $[\rho_i] \to [\rho]$ have images $\left\{\rho'_i(\pi_1(M))\right\}$ that converge *geometrically* to $\rho'(\pi_1(M))$ (i.e. in the Hausdorff topology on closed subsets of $\mathrm{Isom}^+\mathbf{H}^3$) one obtains the *strong topology* on H(M), denoted GH(M) (the quotients converge strongly in the sense above). As a step in our proof of Theorem 1.9 we establish the following theorem, which generalizes results of W. Abikoff and Marden [Ab,Mar2] and seems to be well known.

Theorem **1.10.** — *Each* $N \in GF(M)$ *lies in the closure of* MP(M) *in* GH(M).

(See Theorem 3.4).

The identity map on H(M) determines a continuous mapping

$$\iota : GH(M) \to AH(M).$$

One can ask, however, whether ι sends the closure of the geometrically finite manifolds GF(M) taken in GH(M) onto its closure taken in AH(M). In other words,

> $(*)$ *is every algebraic limit of geometrically finite manifolds a strong limit of some sequence of geometrically finite manifolds?*

In particular, when $\pi_1(M)$ is not a compression body group, we have a positive answer to this question (see Corollary 4.2).

Corollary **1.11.** — *Let* M *be such that* $\pi_1(M)$ *is not a compression body group. Then for each* $N \in \overline{GF(M)} \subset AH(M)$, *there is a sequence* $\{N_i\} \subset GF(M)$ *converging strongly to* N.

(A similar result obtains for each algebraic limit N of geometrically finite manifolds such that N has non-empty conformal boundary; see Corollary 3.2).

In the language of Thurston's description of the case when M is acylindrical (see [Th3, Sect. 2]), Corollary 1.11 verifies the that "shell" adheres to the "hard-boiled egg" AH(M) after "thoroughly cracking the egg shell on a convenient hard surface" to produce GH(M).

Rigidity and ergodicity. — Historically, Ahlfors' conjecture fits within a framework of rigidity and ergodicity results for Kleinian groups and geodesic flows on their quotients due to Mostow, Sullivan and Thurston (see, e.g. [Mos], [Sul1], and [Th1]). In particular, Ahlfors' conjecture has come to be associated with the following complementary conjecture.

Conjecture **1.12** *(Ergodicity).* — *If the finitely generated Kleinian group* Γ *has limit set* $\Lambda = \widehat{\mathbf{C}}$, *then* Γ *acts ergodically on* $\widehat{\mathbf{C}} \times \widehat{\mathbf{C}}$.

Conjecture 1.12 guarantees the ergodicity of the geodesic flow on the unit tangent bundle $T_1(\mathbf{H}^3/\Gamma)$ as well as the non-existence of measurable Γ-invariant *line-fields* on $\widehat{\mathbf{C}}$ (Sullivan's *rigidity theorem* [Sul1]) which lies at the heart of the modern deformation theory of hyperbolic 3-manifolds (see [Mc1] or [Can2] for a nice discussion of these conjectures and their interrelations).

The results of Thurston, Bonahon, and Canary subsumed under Theorem 1.4 also establish Conjecture 1.12 as a consequence of the Tameness Conjecture (Conjecture 1.1). Thus, we have the following corollary of Theorems 1.7 and 1.8.

Corollary **1.13.** — *Let* $N = \mathbf{H}^3/\Gamma$ *be an algebraic limit of geometrically finite manifolds* N_i *and assume* $\Lambda(\Gamma) = \widehat{\mathbf{C}}$. *If* Γ *is not a compression body group, or if* N *is a strong limit of* N_i, *then* Γ *acts ergodically on* $\widehat{\mathbf{C}} \times \widehat{\mathbf{C}}$.

Plan of the paper. — In Section 2 we review background on hyperbolic 3-manifolds and their deformation spaces. Section 3 represents the heart of the paper, where we apply the Drilling Theorem to prove Theorem 1.9, assuming Theorem 1.10 (whose proof we defer to Section 5). In Section 4 we discuss strong convergence, proving Theorem 1.8 and Corollary 1.11.

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2. Preliminaries

Let $N = \mathbf{H}^3/\Gamma$ be the complete hyperbolic 3-manifold given as the quotient of $\mathbf{H}^3$ by a *Kleinian group* Γ, a discrete, torsion-free subgroup of $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{Isom}^+\mathbf{H}^3$. The action of Γ partitions $\widehat{\mathbf{C}}$ into its *limit set* $\Lambda(\Gamma) = \overline{\Gamma(0)} \cap \widehat{\mathbf{C}}$, the intersection of the closure of the orbit of a point $0 \in \mathbf{H}^3$ with the Riemann sphere, and its *domain of discontinuity* $\Omega(\Gamma) = \widehat{\mathbf{C}} \setminus \Lambda(\Gamma)$ where Γ acts properly discontinuously.

The hyperbolic manifold N extends to its *Kleinian manifold*

$$\overline{N} = (\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$$

by adjoining its conformal boundary $\partial N = \Omega(\Gamma)/\Gamma$ at infinity.

Algebraic and geometric convergence. — Let M be a compact, orientable *hyperbolizable* 3-manifold, namely, a compact, orientable 3-manifold whose interior admits some complete hyperbolic structure. We assume throughout for simplicity that all 3-manifolds in question are oriented and all homeomorphisms between them (local and otherwise) are orientation preserving.

Let $\mathscr{D}(M)$ denote the space of representations

$$\rho\colon \pi_1(M) \to \mathrm{Isom}^+\mathbf{H}^3$$

that are discrete and faithful; $\mathscr{D}(M)$ is topologized by convergence of the representations on generators as elements of $\mathrm{Isom}^+\mathbf{H}^3$. Convergence in $\mathscr{D}(M)$ is called *algebraic convergence*.

Each $\rho \in \mathscr{D}(M)$ determines an associated Kleinian *holonomy group* $\rho(\pi_1(M)) < \mathrm{Isom}^+\mathbf{H}^3$ and a complete quotient hyperbolic 3-manifold

$$\mathbf{H}^3/\rho(\pi_1(M)) = N_\rho,$$

but conjugate representations in $\mathscr{D}(\mathrm{M})$ determine isometric hyperbolic quotients. For a more geometric picture that eliminates this redundancy, we pass to the quotient of $\mathscr{D}(\mathrm{M})$ by conjugacy and denote this quotient with its quotient topology by AH(M). Since hyperbolic 3-manifolds are K(G, 1)s, elements of AH(M) are in bijection with equivalence classes of pairs (f, N) where N is a hyperbolic 3-manifold and

$$f: \mathrm{M} \to \mathrm{N}$$

is a homotopy equivalence (or *marking*), modulo the equivalence relation $(f, \mathrm{N}) \sim (f', \mathrm{N}')$ if there is an isometry $\phi: \mathrm{N} \to \mathrm{N}'$ so that $f \circ \phi$ is homotopic to f'. The marking f naturally determines a holonomy representation in $\mathscr{D}(\mathrm{M})$ up to conjugacy by the association

$$f \mapsto f_*.$$

It will be useful to view elements of AH(M) both as conjugacy classes of representations and as marked hyperbolic 3-manifolds at different points in our argument, and likewise we will from time to time view $\rho \in \mathscr{D}(\mathrm{M})$ as an isomorphism between $\pi_1(\mathrm{M})$ and $\pi_1(\mathrm{N}_\rho)$.

A related notion of convergence for hyperbolic 3-manifolds is that of *geometric convergence*. As a complete hyperbolic 3-manifold N determines a Kleinian group only up to conjugacy, we will pin down a unique representative of the conjugacy class by equipping N with the additional data of a *baseframe* ω, an orthonormal frame in $\mathrm{T}_p(\mathrm{N})$ at a basepoint p. Then there is a unique Kleinian group Γ so that if $\widetilde{\omega}$ denotes the standard baseframe at the origin in $\mathbf{H}^3$ then

$$(\mathbf{H}^3, \widetilde{\omega})/\Gamma = (\mathrm{N}, \omega),$$

in other words, the standard baseframe $\widetilde{\omega}$ covers the baseframe ω in the quotient under the locally isometric covering map.

A sequence of based hyperbolic 3-manifolds (N_i, ω_i) converges to a based hyperbolic 3-manifold $(\mathrm{N}_\mathrm{G}, \omega_\mathrm{G})$ *geometrically* if their associated Kleinian groups Γ_i converge *geometrically* to the Kleinian group Γ_G associated to $(\mathrm{N}_\mathrm{G}, \omega_\mathrm{G})$:

1. for each $\gamma \in \Gamma_\mathrm{G}$ there is a sequence of elements $\gamma_i \in \Gamma_i$ so that $\gamma_i \to \gamma$, and
2. for each convergent sequence of elements $\gamma_{i_j} \to \gamma$ in a subsequence Γ_{i_j} we have $\gamma \in \Gamma_\mathrm{G}$.

Fundamental compactness results (see, e.g. [CEG, Sect. 3]) guarantee that each algebraically convergent sequence $\rho_i \to \rho$ in $\mathscr{D}(\mathrm{M})$ has a subsequence for which the image Kleinian groups $\{\rho_i(\pi_1(\mathrm{M}))\}$ converge geometrically to a limit Γ_G. In such a setting, the algebraic limit $\rho(\pi_1(\mathrm{M}))$ is a subgroup of the geometric limit Γ_G by property (2) in the definition of geometric convergence.

Given an algebraically convergent sequence $(f_i, \mathrm{N}_i) \in \mathrm{AH}(\mathrm{M})$ converging to a limit (f, N), then, we may pass to a subsequence and choose baseframes $\omega_i \in \mathrm{N}_i$ so that (N_i, ω_i) converges geometrically to a geometric limit $(\mathrm{N}_{\mathrm{G}}, \omega_{\mathrm{G}})$ that is covered by N by a *local* isometry. Thus, any algebraic limit (f, N) has such an associated geometric limit N_{G}, although it may have many such geometric limits. In the case that N_{G} is unique and the covering $\mathrm{N} \to \mathrm{N}_{\mathrm{G}}$ is an isometry we say that the sequence (f_i, N_i) converges *strongly* to (f, N).

Here is a more internal formulation of geometric convergence. A diffeomorphism $g \colon \mathrm{M} \to \mathrm{N}$ is L-*bi-Lipschitz* if for each $p \in \mathrm{M}$ its derivative $\mathrm{D}g$ satisfies

$$\frac{1}{\mathrm{L}} \leq \frac{|\mathrm{D}g(v)|}{|v|} \leq \mathrm{L}$$

for each $v \in \mathrm{T}_p\mathrm{M}$. The least $\mathrm{L} \geq 1$ for which g is L-bi-Lipschitz is the *bi-Lipschitz constant* of g. Then the sequence (N_i, ω_i) converges to $(\mathrm{N}_{\mathrm{G}}, \omega_{\mathrm{G}})$ if for each compact submanifold $\mathrm{K} \subset \mathrm{N}_{\mathrm{G}}$ with $\omega_{\mathrm{G}} \in \mathrm{K}$, there are bi-Lipschitz embeddings

$$\phi_i \colon (\mathrm{K}, \omega_{\mathrm{G}}) \to (\mathrm{N}_i, \omega_i)$$

for all i sufficiently large, so that the 1-jet of ϕ_i sends ω_{G} to ω_i, and the bi-Lipschitz constant L_i for ϕ_i tends to 1 (cf. [BP, Thm. E.1.13] [Mc2, Sect. 2.2]).

Relative compact cores. — By a Theorem of Peter Scott (see [Scott]), each complete hyperbolic 3-manifold N with finitely generated fundamental group admits a *compact core* $\mathscr{M}$, namely, a compact submanifold whose inclusion is a homotopy equivalence. In the presence of cusps, one can relativize this compact core, aligning distinguished annuli and tori in $\partial\mathscr{M}$ with the cusps of N. We now describe this notion in detail.

By the Margulis lemma (see [BP, Thm. D.3.3]), there is a uniform constant $\mu > 0$, so that for any $\epsilon < \mu$ and any complete hyperbolic 3-manifold N, each component T of the ϵ-thin part $\mathrm{N}^{\leq\epsilon}$ of N where the injectivity radius is at most ϵ has a standard form: either

1. T is a *Margulis tube*: a solid torus neighborhood $\mathbf{T}^{\epsilon}(\gamma)$ of a short geodesic γ in N with $\ell_{\mathrm{N}}(\gamma) < 2\epsilon$ (T is the short geodesic itself if $\ell_{\mathrm{N}}(\gamma) = 2\epsilon$), or
2. T is a *cusp*: the quotient of a horoball $\mathrm{B} \subset \mathbf{H}^3$ by the action of a $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$ parabolic subgroup of $\mathrm{Isom}^+\mathbf{H}^3$ with fixed point at $\overline{\mathrm{B}} \cap \widehat{\mathbf{C}}$.

When $\mathrm{T} = \mathrm{B}/\mathbf{Z} \oplus \mathbf{Z}$, the component T is called a *rank-2 cusp*, and when $\mathrm{T} = \mathrm{B}/\mathbf{Z}$, T is called a *rank-1 cusp*. We will frequently denote rank-2 cusp components of $\mathrm{N}^{\leq\epsilon}$ by $\mathbf{P}^{\epsilon}$. The constant μ is called the *3-dimensional Margulis constant*.

Now let N be a complete hyperbolic 3-manifold with finitely generated fundamental group. For $\epsilon < \mu$, we denote by P^ϵ the *cuspidal ϵ-thin part of* N, namely, components of $N^{\leq \epsilon}$ corresponding to cusps of N.

By work of McCullough [McC] or Kulkarni and Shalen [KS] there is a compact submanifold $\mathscr{M}$ whose inclusion is a map of pairs

$$\iota \colon (\mathscr{M}, \mathscr{P}) \to (N \setminus \operatorname{int}(P^\epsilon), \partial P^\epsilon)$$

so that

1. $\mathscr{P} \subset \partial\mathscr{M}$ is a union of compact incompressible annuli and tori called the *parabolic locus*, and each component of $\partial\mathscr{M} \setminus \mathscr{P}$ has negative Euler characteristic,
2. ι is a homotopy equivalence, and
3. for each component $\hat{P}^\epsilon$ of P^ϵ there is a component $\hat{\mathscr{P}}$ of $\mathscr{P}$ so that $\iota(\hat{\mathscr{P}})$ lies in $\partial\hat{P}^\epsilon$.

Then we call the pair $\mathscr{K} = (\mathscr{M}, \mathscr{P})$ a *relative compact core* for N relative to its cuspidal ϵ-thin part P^ϵ.

A geometric criterion for algebraic convergence. — Given a sequence $\{(f_i, N_i)\}$ of marked hyperbolic 3-manifolds in AH(M), it is desirable to have geometric criteria on manifolds N_i to ensure algebraic convergence as in the case of geometric convergence.

Given $N_\rho \in AH(M)$, the holonomy representation $\rho \colon \pi_1(M) \to \operatorname{Isom}^+\mathbf{H}^3$ is determined by the restriction of the hyperbolic metric to a compact core for N. It follows that the sequence $\{(f_i, N_i)\} \subset AH(M)$ converges algebraically to its algebraic limit (f, N) if there is a compact core $\mathscr{K}$ for N and smooth homotopy equivalences $g_i \colon N \to N_i$ so that

1. $g_i \circ f$ is homotopic to f_i, and
2. g_i is an L_i-bi-Lipschitz *local* diffeomorphism on $\mathscr{K}$ with $L_i \to 1$.

The convergence of the bi-Lipschitz constant to 1 guarantees that the maps g_i are nearly local isometries for large i: lifts $\widetilde{g}_i$ of g_i (suitably normalized) are equicontinuous from $\widetilde{\mathscr{K}}$ to $\mathbf{H}^3$, and any limit on a compact subset of $\widetilde{\mathscr{K}}$ is a 1-bi-Lipschitz diffeomorphism, hence an isometry. Since $\mathscr{K}$ is a compact core for N, the convergence of $\widetilde{g}_i$ on a compact fundamental domain for the action of $\pi_1(N)$ on $\widetilde{\mathscr{K}}$ suffices to control the holonomy representations $(f_i)_*$ up to conjugation in $\operatorname{Isom}^+\mathbf{H}^3$ (cf. [CEG, Sect. 1.5, 3.2], [Mc2, Thm. B.24]).

Persistence of tameness. — The question of the persistence of tameness of hyperbolic 3-manifolds under algebraic convergence was first raised and answered by Thurston in the context of M with incompressible boundary with certain mild assumptions on cusps (see [Th1, Thm. 9.6.2a]). This result is now a consequence of Bonahon's tameness theorem [Bon2].

Work of Canary and Minsky [CM] (see also [Ohs]) removed the restrictions on M to establish that tameness persists under strong limits $\rho_i \to \rho$ in $\mathcal{D}(M)$ if the representations ρ_i and ρ are *purely hyperbolic*, namely, every element of $\pi_1(M)$ has image a hyperbolic element of $\mathrm{Isom}^+\mathbf{H}^3$. These results were generalized by the third author (see [Ev1,Ev2]) to the setting of *type-preserving limits*. An algebraically convergent sequence $\rho_i \to \rho$ is *type-preserving* if for each $g \in \pi_1(M)$, the element $\rho(g)$ is parabolic if and only if $\rho_i(g)$ is parabolic for all i. A convergent sequence $N_i \to N$ in AH(M) is type-preserving if $N_i = N_{\rho_i}$ and $N = N_\rho$ for some type-preserving sequence $\rho_i \to \rho$.

Theorem **2.1** *(Evans). — Let* $N_i \to N$ *be a type-preserving sequence of representations in* AH(M) *converging strongly. Then if each* N_i *is tame, the limit* N *is tame.*

Strong convergence and Jørgensen's conjecture. — In light of Theorem 2.1 a conjecture of Jørgensen is an undercurrent to the paper.

Conjecture **2.2** *(Jørgensen). — Let* $\rho_i \to \rho$ *be a type-preserving sequence in* $\mathcal{D}(M)$ *with limit* ρ. *Then* ρ_i *converges strongly to* ρ.

Anderson and Canary have resolved Jørgensen's conjecture in many cases [AC2, Thm. 3.1] (see also [Ohs]).

Theorem **2.3** *(Anderson-Canary). — Let* $\rho_i \to \rho$ *be a type-preserving sequence in* $\mathcal{D}(M)$ *with limit* ρ. *If either*

1. $\{\rho(\pi_1(M))\}$ *has non-empty domain of discontinuity, or*
2. $\pi_1(M)$ *is not a compression body group,*

then ρ_i *converges strongly to* ρ.

For the purposes of addressing Ahlfors' conjecture, it is case (1) that will be of interest to us, but in each case our techniques produce new strong approximation theorems (see Sect. 4).

3. Cone-manifolds, drilling, and strong convergence

The aim of this section is to promote algebraic approximation of a hyperbolic 3-manifold N by geometrically finite manifolds to *type-preserving* approximation by geometrically finite manifolds. As seen in the last section, the type-preserving condition is sufficient to ensure strong convergence with certain assumptions on N.

Given a compact hyperbolizable 3-manifold M, we will focus on the closure $\overline{GF(M)} \subset AH(M)$ of the geometrically finite locus (Conjecture 1.6 predicts

$\overline{\mathrm{GF(M)}} = \mathrm{AH(M)}$). We will assume here and in the sequel that $\pi_1(\mathrm{M})$ is non-abelian to avoid the trivial case of elementary Kleinian groups.

Our goal in this section will be to prove the following theorem.

Theorem **3.1** *(Limits are Type-Preserving Limits). — Let* $\mathrm{N} \in \overline{\mathrm{GF(M)}}$ *be the algebraic limit of the manifolds* $\mathrm{N}_i \in \mathrm{GF(M)}$. *Then there is a type-preserving sequence* $\hat{\mathrm{N}}_i \to \mathrm{N}$ *for which each* $\hat{\mathrm{N}}_i$ *lies in* $\mathrm{GF(M)}$.

Then applying Theorem 2.3 of Anderson and Canary [AC2, Thm. 3.1], we have the following corollary.

Corollary **3.2.** — *Let* $\mathrm{N} \in \overline{\mathrm{GF(M)}}$ *have non-empty conformal boundary* $\partial\mathrm{N}$. *Then there is a type-preserving sequence* $\hat{\mathrm{N}}_i \to \mathrm{N}$ *for which each* $\hat{\mathrm{N}}_i$ *lies in* $\mathrm{GF(M)}$ *and the convergence* $\hat{\mathrm{N}}_i \to \mathrm{N}$ *is strong.*

The following theorem of the first and second authors will play a central role in all that follows.

Theorem **3.3** *(Brock-Bromberg) (The Drilling Theorem). — Given* $\mathrm{L} > 1$ *and* $\epsilon_0 < \mu$, *there is an* $\epsilon > 0$ *so that if* N *is a geometrically finite hyperbolic 3-manifold with no rank-1 cusps and* η *is a closed geodesic in* N *with length at most* ϵ, *then there is an* L*-bi-Lipschitz diffeomorphism of pairs*

$$h\colon (\mathrm{N} \setminus \mathbf{T}^{\epsilon_0}(\eta), \partial\mathbf{T}^{\epsilon_0}(\eta)) \to (\mathrm{N}_0 \setminus \mathbf{P}^{\epsilon_0}(\eta), \partial\mathbf{P}^{\epsilon_0}(\eta))$$

where N_0 *is the complete hyperbolic structure on* $\mathrm{N} \setminus \eta$ *with the same conformal boundary, and* $\mathbf{P}^{\epsilon_0}(\eta)$ *is the rank-2 cusp component of the thin part* $(\mathrm{N}_0)^{\leq\epsilon_0}$ *corresponding to* η.

A similar statement holds for drilling multiple short geodesics in a collection $\mathscr{C}$ (see [BB2, Thm. 6.2], [Brm3]).

The theorem relies on fundamental work of C. Hodgson and S. Kerckhoff on the deformation theory of 3-dimensional hyperbolic cone-manifolds. The key estimate gives control on the L^2 norm outside of $\mathbf{T}^{\epsilon_0}(\eta)$ of a harmonic cone-deformation that sends the cone angle at η from 2π to *zero*: cone-angle zero corresponds to a torus cusp at η. As the length of η tends to zero, the L^2 norm also tends to zero. Mean value estimates then give pointwise C^2 control over the metric distortion in the thick part. One then uses this control to extend the deformation over the thin parts other than $\mathbf{T}^{\epsilon_0}(\eta)$.

Remark. — While the use of the Drilling Theorem in [BB2] requires cone-deformations involving cone angles greater than 2π, and thence an application of [HK3], the cone-deformations implicit in the version of the Drilling Theorem stated above will only involve cone angles in the interval $[0, 2\pi]$. These cases are addressed in [HK2], [HK1], [Brm2] and [Brm3].

An important approximation theorem we will use is the following result, whose proof appears in Section 5. While this result seems reasonably well-known, and cases have appeared in work of W. Abikoff [Ab] and Marden [Mar2] (cf. [EM] [KT, Sect. 3]), we have been unable to find a proof in the published literature. For completeness we devote Section 5 to a proof using now standard techniques of Marden, Maskit, Kerckhoff and Thurston.

Theorem **3.4.** — *Each* $N \in GF(M)$ *is a strong limit of manifolds in* $MP(M)$.

Recall from Section 1 that $MP(M) \subset GF(M)$ denotes the minimally parabolic structures in $GF(M)$, which comprise the interior of $AH(M)$ [Mar1,Sul2]. Hyperbolic 3-manifolds $N \in MP(M)$ are characterized by the property that each cusp of N is rank-2 and therefore corresponds to a torus boundary component of M. Assuming Theorem 3.4, we proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. — We seek geometrically finite manifolds $\hat{N}_i \in GF(M)$ converging in a type-preserving manner to N. For reference, let $\rho_i \to \rho$ in $\mathscr{D}(M)$ be an algebraically convergent sequence for which $N_i = N_{\rho_i}$ is geometrically finite and $N = N_\rho$. Applying Theorem 3.4, and a diagonal argument, we may assume that N_{ρ_i} lies in $MP(M)$ for each i. Let $f : M \to N$ and $f_i : M \to N_i$ be markings for N and N_i that are compatible with ρ and ρ_i.

The idea of the proof is as follows: let $a \in \pi_1(M)$ be a primitive element so that $\rho(a)$ is parabolic but $\rho_i(a)$ is not parabolic for all i. For each $\epsilon > 0$ there is an I so that for all $i > I$, the translation length of $\rho_i(a)$ is less than ϵ. We may apply Theorem 3.3 to N_i once the geodesic η_i^* corresponding to $\rho_i(a)$ is sufficiently short: we may drill out the geodesic η_i^* leaving the conformal boundary of N_i fixed. Since the length $\ell_{N_i}(\eta_i^*)$ of η_i^* in N_i is tending to zero, the bi-Lipschitz constants for the drilling diffeomorphisms h_i are tending to 1 as i tends to infinity. Thus, the drillings force parabolicity of the incipient parabolic in each approximate by a geometric perturbation that becomes smaller and smaller as the length of η_i^* tends to zero.

The drilling diffeomorphisms transport a compact core to the drilled manifold, so the algebraic effect of the drilling is small as well: passing to the cover corresponding to the image of the core, we obtain representations $\hat{\rho}_i \to \rho$, for which $\hat{\rho}_i(a)$ is parabolic for each i and $\hat{\rho}_i$ converges to ρ. Performing this process simultaneously for all such a produces the desired type-preserving sequence.

Now we fill in the details. By a theorem of Brooks and Matelski [BM], given $d > 0$ there is a constant $\epsilon_{\text{collar}}(d) > 0$ so that the distance from the boundary of the $\epsilon_{\text{collar}}(d)$-thin part to the μ-thick part of a hyperbolic 3-manifold is at least d (recall μ is the 3-dimensional Margulis constant). Moreover, given any $\delta > 0$, there is a constant $\epsilon_{\text{short}}(\delta) > 0$ so that the arclength of a shortest essential closed curve on the boundary of any component of the $\epsilon_{\text{short}}(\delta)$-thin part is at most δ. We choose ϵ' so that

$$\epsilon' < \min\{\epsilon_{\text{collar}}(2), \epsilon_{\text{short}}(1), \mu/2\}.$$

Let $\mathscr{K} = (\mathscr{M}, \mathscr{P})$ be a relative compact core for N relative to the ϵ'-cuspidal thin part P of N. Since ρ_i converges algebraically to ρ, there are smooth homotopy equivalences

$$g_i \colon \mathrm{N} \to \mathrm{N}_i$$

with $g_i \circ f$ homotopic to f_i, so that g_i is a local diffeomorphism on $\mathscr{K}$ for i sufficiently large, and the bi-Lipschitz constant for g_i on $\mathscr{K}$ goes to 1.

The core $\mathscr{K}$ and its images $g_i(\mathscr{K})$ have diameters bounded by a constant D. Since $\pi_1(\mathscr{K}) \cong \pi_1(\mathrm{N})$ contains a pair of non-commuting elements, the Margulis lemma implies that $\mathscr{K}$ and its images $g_i(\mathscr{K})$ cannot lie entirely in the μ-thin part. Thus, we may apply [BM] and take

$$\epsilon_0 < \epsilon_{\text{collar}}(\mathrm{D})/2$$

to ensure $\mathscr{K}$ and $g_i(\mathscr{K})$ avoid the $2\epsilon_0$-thin parts of N and of N_i respectively.

Since each manifold N_i lies in MP(M), each N_i is geometrically finite without rank-1 cusps, so we may apply Theorem 3.3 to "drill" any sufficiently short geodesic in N_i. Choose real numbers $\mathrm{L}_n \to 1^+$, and let $\epsilon_n \to 0^+$ be corresponding real numbers so that the conclusions of Theorem 3.3 obtain.

There is an integer I_n so that for all $i > \mathrm{I}_n$ we have

$$\ell_{\mathrm{N}_i}(\eta^*) < \epsilon_n.$$

Applying Theorem 3.3, there are diffeomorphisms of pairs

$$h_i \colon (\mathrm{N}_i \setminus \mathbf{T}^{\epsilon_0}(\eta), \partial \mathbf{T}^{\epsilon_0}(\eta)) \to ((\mathrm{N}_i)_0 \setminus \mathbf{P}^{\epsilon_0}(\eta), \partial \mathbf{P}^{\epsilon_0}(\eta))$$

from the complement of the ϵ_0-Margulis tube $\mathbf{T}^{\epsilon_0}(\eta)$ about η^* in N_i to the complement of the ϵ_0-torus cusp $\mathbf{P}^{\epsilon_0}(\eta)$ corresponding to η in the drilled manifold $(\mathrm{N}_i)_0$, so that h_i is L_n-bi-Lipschitz. Assume we have re-indexed so that all i are greater than I_0.

Let $(\Gamma_i)_0$ be the holonomy group of $(\mathrm{N}_i)_0$. We claim there are natural injective homomorphisms

$$\hat{\rho}_i \colon \pi_1(\mathrm{M}) \to (\Gamma_i)_0$$

that converge algebraically to ρ as representations from $\pi_1(\mathrm{M})$ to $\mathrm{Isom}^+(\mathbf{H}^3)$, and so that $\hat{\rho}_i(a)$ is parabolic for all i.

Letting $(\mathbf{T}^{\epsilon_0}(\eta))_i$ be the ϵ_0-Margulis tube about the geodesic η^* in N_i, recall we have chosen ϵ_0 so that

$$g_i(\mathscr{K}) \cap (\mathbf{T}^{\epsilon_0}(\eta))_i = \varnothing$$

for each i. Then the mappings

$$h_i \circ g_i|_{\mathscr{K}} \colon \mathscr{K} \to (N_i)_0,$$

which we denote by φ_i, are bi-Lipschitz local diffeomorphisms with bi-Lipschitz constant $L_i' \to 1^+$.

Since $\mathscr{K}$ is a compact core for N, the mappings φ_i are π_1-injective so we may consider the locally isometric covers $\hat{N}_i$ of $(N_i)_0$ corresponding to the subgroups

$$(\varphi_i)_*(\pi_1(\mathscr{K}))$$

of $\pi_1((N_i)_0)$. Let $\widetilde{\varphi}_i$ denote the lift of φ_i to $\hat{N}_i$. Then we have

$$\hat{N}_i = \mathbf{H}^3/\hat{\rho}_i(\pi_1(M))$$

where $\hat{\rho}_i$ is induced by the isomorphism $(\widetilde{\varphi}_i \circ \iota^{-1} \circ f)_*$ and ι^{-1} denotes a homotopy inverse for the inclusion $\iota \colon \mathscr{K} \to N$. Since the bi-Lipschitz constants L_i' for φ_i, and hence for $\widetilde{\varphi}_i$ converge to 1, we may conclude that (after possibly conjugating each $\hat{\rho}_i$ in $\mathrm{Isom}^+\mathbf{H}^3$) we have $\hat{\rho}_i \to \rho$ in $\mathscr{D}(M)$.

We now claim that $\hat{\rho}_i(a)$ is parabolic for all i. The parabolic locus $\mathscr{P}$ sits in the boundary of the cuspidal ϵ'-thin part $P^{\epsilon'}$. We may assume, after modifying our choice of $\mathscr{K}$ by an isotopy, that each annular component of $\mathscr{P}$ of the parabolic locus of $\mathscr{K}$ contains an essential closed curve of shortest length on the boundary of the component of $P^{\epsilon'}$ in which it lies.

Let a' be such a shortest curve in the annular component $\mathscr{A}$ of $P^{\epsilon'}$ representing the free homotopy class of $\rho(a)$ in $\pi_1(N)$. Since the bi-Lipschitz constants for g_i are converging to 1 on $\mathscr{K}$, the arc length $\ell_{N_i}(g_i(a'))$ of the loop $g_i(a')$ in N_i is less than $2\ell_N(a')$ for all i sufficiently large. It follows from our choice of ϵ' that that the image $g_i(a')$ lies entirely within the Margulis tube $(\mathbf{T}^{\mu}(\eta))_i$ in N_i for all i sufficiently large. Moreover, since we chose ϵ_0 so that

$$g_i(\mathscr{K}) \cap (N_i)^{\leq \epsilon_0} = \varnothing,$$

we may conclude that $g_i(a')$ does not intersect the Margulis tube $(\mathbf{T}^{\epsilon_0}(\eta))_i$.

Thus, if n is taken sufficiently large so that $\epsilon_n < \epsilon_0$, the curve $g_i(a')$ is homotopic within the Margulis tube $(\mathbf{T}^{\mu}(\eta))_i$ in the complement of the Margulis tube $(\mathbf{T}^{\epsilon_0}(\eta))_i$ to a curve a'' on $\partial(\mathbf{T}^{\epsilon_0}(\eta))_i$ for all $i > I_n$. Let $H_t \colon S^1 \to N_i \setminus (\mathbf{T}^{\epsilon_0}(\eta))_i$ denote this homotopy (one can use radial lines from the core geodesic η^* through $g_i(a')$ to construct H_t).

Since the diffeomorphisms h_i are maps of pairs, the restriction $h_i|_{\partial(\mathbf{T}^{\epsilon_0}(\eta))_i}$ is a diffeomorphism of $\partial(\mathbf{T}^{\epsilon_0}(\eta))_i$ to $\partial(\mathbf{P}^{\epsilon_0}(\eta))_i$. Thus, the homotopy H_t gives a homotopy

$$h_i \circ \mathrm{H}_t \colon \mathrm{S}^1 \to (\mathrm{N}_i)_0 \setminus (\mathbf{P}^{\epsilon_0}(\eta))_i$$

from $\varphi_i(a')$ to $\varphi_i(a'')$, and $\varphi_i(a'')$ has image in $\partial(\mathbf{P}^{\epsilon_0}(\eta))_i$. It follows that the curve $a' \subset \mathscr{A}$ has image under φ_i homotopic into the component $(\mathbf{P}^{\mu}(\eta))_i$ of the cuspidal μ-thin part of $(\mathrm{N}_i)_0$, and therefore that $\left(\widetilde{\varphi}_i \circ \iota^{-1} \circ f\right)_*$ sends a to a parabolic element in $\pi_1(\hat{\mathrm{N}}_i)$. We conclude that $\hat{\rho}_i(a)$ is parabolic for all i.

When $\mathscr{P}$ has many annular components $\mathscr{A}_1, \ldots, \mathscr{A}_m$, the argument proceeds similarly. Letting a_j be the core curve of $\mathscr{A}_j$, we first simultaneously drill short geodesics in the collection $\mathscr{C}_i$ of geodesic representatives in N_i of the curves $g_i(a_j)$, $j = 1, \ldots, m$. Taking covers corresponding to the image of the core under drilling again yields representations $\hat{\rho}_i \in \mathscr{D}(\mathrm{M})$ and quotient manifolds $\hat{\mathrm{N}}_i = \mathbf{H}^3/\hat{\rho}_i(\pi_1(\mathrm{M}))$ that converge algebraically to N. Repeating the above arguments cusp by cusp demonstrates that $\hat{\rho}_i(a_j)$ is parabolic for each i and each $j = 1, \ldots, m$, so the convergence $\hat{\mathrm{N}}_i \to \mathrm{N}$ is type-preserving. □

Corollary 3.2 is a simple application of Theorem 2.3.

Proof of Corollary 3.2. — When N_ρ has non-empty conformal boundary, the holonomy group $\rho(\pi_1(\mathrm{M}))$ has non-empty domain of discontinuity. Since $\{\hat{\rho}_i\}$ is a type-preserving sequence with limit ρ, we may apply Theorem 2.3 to conclude that ρ is a strong limit of $\hat{\rho}_i$. This proves the Corollary. □

4. The strong topology

The application of the Drilling Theorem to Ahlfors' conjecture exploits the solution of Anderson and Canary to Jørgensen's conjecture for type-preserving limits with non-empty domain of discontinuity [AC2].

For this section, we focus on the second conclusion of Theorem 2.3.

Theorem **4.1** *(Anderson-Canary).* — *If* $\pi_1(\mathrm{M})$ *is not a compression body group, then any type-preserving sequence* $\mathrm{N}_i \to \mathrm{N}$ *in* $\mathrm{AH}(\mathrm{M})$ *converges strongly.*

As remarked in [AC1], their result holds under the weaker assumption that a relative compact core $(\mathscr{M}, \mathscr{P})$ for N relative to its cusps is not a *relative compression body* (see [AC1, Sect. 11] for a details).

Applying Theorem 3.1 and Theorem 2.3, then, we have the following corollary of the proof of Theorem 1.7.

Corollary **4.2.** — *If* $\pi_1(\mathrm{M})$ *is not a compression body group, then each* $\mathrm{N} \in \overline{\mathrm{GF(M)}}$ *is a strong limit of a sequence* $\hat{\mathrm{N}}_i$ *of manifolds in* GF(M).

Finally, we conclude with an application of Theorem 3.1 to all strong limits of geometrically finite hyperbolic 3-manifolds.

Theorem **4.3.** — *If* N *is a strong limit of geometrically finite hyperbolic 3-manifolds, then* N *is tame.*

Proof. — If N is a strong limit of geometrically finite hyperbolic 3-manifolds, then we may once again assume that N is a strong limit of manifolds N_i lying in MP(M), by a diagonal argument applying Theorem 3.4. We show that the type-preserving sequence $\hat{\mathrm{N}}_i$ furnished by Theorem 3.1 can be chosen to converge strongly; the theorem then follows from Theorem 2.1.

To this end, let $\omega \in \mathrm{N}$ be a baseframe in the convex core of N. By strong convergence, we may choose $\omega_i \in \mathrm{N}_i$ so that (N_i, ω_i) converges geometrically to (N, ω). Given any compact submanifold $\mathrm{K} \subset \mathrm{N}$ with $\omega \in \mathrm{K}$, geometric convergence provides bi-Lipschitz embeddings

$$\phi_i \colon \mathrm{K} \to \mathrm{N}_i$$

so that ϕ_i sends ω to ω_i and so that the bi-Lipschitz constant of ϕ_i tends to 1.

We take $\epsilon_0 > 0$ so that

$$2\epsilon_0 < \inf_{x \in \mathrm{K}} \operatorname{inj}(x),$$

where $\operatorname{inj} \colon \mathrm{N} \to \mathbf{R}^+$ is the injectivity radius on N. There is, then, an $\mathrm{I} \in \mathbf{N}$ so that for $i > \mathrm{I}$, $\phi_i(\mathrm{K})$ misses the ϵ_0-thin part $(\mathrm{N}_i)^{\leq \epsilon_0}$.

At the drilling stage in the proof of Theorem 3.1, we may take ϵ_0 as input for Theorem 3.3, to obtain drilled manifolds $(\mathrm{N}_i)_0$ together with drilling diffemorphisms h_i so that the compositions

$$h_i \circ \phi_i \colon \mathrm{K} \to (\mathrm{N}_i)_0,$$

which we denote by Φ_i, are embeddings with bi-Lipschitz constant $\mathrm{L}_i \to 1^+$.

As in the proof of Theorem 3.1, there are resulting locally isometric covers $\hat{\mathrm{N}}_i(\mathrm{K})$ of these drillings that converge to N in a type-preserving manner. In the case at hand, the approximates $\hat{\mathrm{N}}_i(\mathrm{K})$ have the additional property that the embeddings Φ_i lift to embeddings

$$\widetilde{\Phi}_i \colon \mathrm{K} \to \hat{\mathrm{N}}_i(\mathrm{K})$$

of K into $\hat{\mathrm{N}}_i(\mathrm{K})$ with bi-Lipschitz constant L_i. Letting K_n be an exhaustion of N by compact subsets containing K and letting $\hat{\mathrm{N}}_i(\mathrm{K}_n)$ be the type-preserving approximates converging to N resulting from the above procedure, we may diagonalize to obtain a type-preserving sequence converging strongly to N. An application of Theorem 2.1 completes the proof. □

5. Strong approximation of geometrically finite manifolds

The aim of this section is to give a proof of Theorem 3.4. Our method of proof follows the ideas of [EM] and [KT] to promote rank-1 cusps to rank-2 cusps and then fill them in using Thurston's hyperbolic Dehn surgery theorem. By choosing the appropriate promotions and fillings for rank-1 cusps in a sequence of approximates, one easily obtains a sequence of strongly convergent minimally parabolic approximates.

We first establish the following lemma, a simple application of the Klein-Maskit *combination theorems* (see [Msk2]).

Lemma **5.1.** — *Let* N *lie in* GF(M) *and let* $(\mathscr{M}, \mathscr{P})$ *be a relative compact core for* N. *Let* $\mathscr{A}_1, \ldots, \mathscr{A}_m$ *be annular components of the parabolic locus* $\mathscr{P}$. *Then there is a geometrically finite hyperbolic 3-manifold* $\check{N}$ *with no rank-1 cusps so that*

1. $\check{N}$ *is homeomorphic to* $N \setminus \mathscr{A}_1 \sqcup \ldots \sqcup \mathscr{A}_m$, *and*
2. *there is a locally isometric covering map* $\Pi : N \to \check{N}$ *that restricts to an embedding on* $(\mathscr{M}, \mathscr{P})$.

Moreover, given a choice of baseframe $\omega \in N$ *and any neighborhood* U *of* (N, ω) *in the geometric topology, there exists such a manifold* $\check{N}$ *and a baseframe* $\check{\omega} \in \check{N}$ *so that* $(\check{N}, \check{\omega})$ *lies in* U.

We call the manifold $\check{N}$ a *promotion* of the rank-1 cusps of N. The topological structure of $\check{N}$ is that of the original manifold with the core of each $\mathscr{A}_i$ removed (see, e.g. [KT,EM]).

Proof. — Let $N = N_\rho$ for $\rho \in \mathscr{D}(M)$. Consider a primitive element $g \in \pi_1(M)$ so that g is homotopic into an annular component of the parabolic locus $\mathscr{P}$. Let $\mathscr{A}_g$ denote the annular component of the parabolic locus $\mathscr{P}$ corresponding to g, so that $\pi_1(\mathscr{A}_g)$ is conjugate to the cyclic subgroup $\langle g \rangle$ in $\pi_1(\mathscr{M}) = \pi_1(M)$ under inclusion, and let $\gamma = \rho(g)$. Since $\rho(\pi_1(M))$ is geometrically finite, the parabolic subgroup $\langle \gamma \rangle$ is *doubly cusped*: there are two disjoint components Ω and Ω' in the domain of discontinuity $\Omega(\rho)$ so that $\langle \gamma \rangle$ is a subgroup of the stabilizers $\mathrm{Stab}_\rho(\Omega)$ and $\mathrm{Stab}_\rho(\Omega')$ of Ω and Ω' in $\rho(\pi_1(M))$.

There are disks $B \subset \Omega$ and $B' \subset \Omega'$ so that ∂B and $\partial B'$ are round circles in $\widehat{\mathbf{C}}$ that are tangent at the fixed point p of γ (with B and B′ each invariant by γ, see [Msk2, Prop. A.10]), and a parabolic element $\delta \in \mathrm{PSL}_2(\mathbf{C})$ with fixed point p so that the interior of B is taken to the exterior of B′ by δ. The triple $(\mathrm{B}, \mathrm{B}', \delta)$ satisfies the hypotheses of the Klein-Maskit combination theorem (see [Msk1, Sect. 9, Combination II]) for the cyclic subgroups $\mathrm{H} = \langle\gamma\rangle = \mathrm{H}'$ of $\rho(\pi_1(\mathrm{M}))$, so the group

$$\check{\Gamma} = \langle \rho(\pi_1(\mathrm{M})), \delta \rangle$$

generated by $\rho(\pi_1(\mathrm{M}))$ and δ is again a Kleinian group; the subgroup generated by δ and γ is a rank-2 parabolic subgroup with fixed point p that corresponds to a torus-cusp of the quotient

$$\check{\mathrm{N}} = \mathbf{H}^3/\check{\Gamma}.$$

The manifold $\check{\mathrm{N}}$ is easily seen to be homeomorphic to $\mathrm{N} \setminus \mathscr{A}_g$. Letting P_g be the component of the cuspidal thin part $\mathrm{P} = \mathrm{P}^{\mu}$ of N whose boundary contains $\mathscr{A}_g$, we call $\check{\mathrm{N}}$ a *promotion* of the rank-1 cusp P_g corresponding to $\mathscr{A}_g$ to rank-2.

If $\{\mathscr{A}_1, \dots, \mathscr{A}_m\}$ is an enumeration of the annular components of parabolic locus $\mathscr{P}$ for $\mathscr{M}$, we can promote each rank-1 cusp $\{\mathrm{P}_1, \dots, \mathrm{P}_m\}$ in P to rank two cusps to obtain a hyperbolic 3-manifold

$$\check{\mathrm{N}}(\mathrm{P}_1, \dots, \mathrm{P}_m).$$

The manifold $\check{\mathrm{N}}(\mathrm{P}_1, \dots, \mathrm{P}_m)$ is homeomorphic to $\mathrm{N} \setminus (\mathscr{A}_1 \sqcup \dots \sqcup \mathscr{A}_m)$, and since the corresponding Kleinian group $\check{\Gamma}$ is given as

$$\check{\Gamma} = \langle \rho(\pi_1(\mathrm{M})), \delta_1, \dots, \delta_m \rangle,$$

the group generated by $\rho(\pi_1(\mathrm{M}))$ and parabolic elements $\delta_1, \dots, \delta_m$, there is a natural locally isometric covering map

$$\Pi : \mathrm{N} \to \check{\mathrm{N}}(\mathrm{P}_1, \dots, \mathrm{P}_m).$$

Choosing δ_j appropriately, we can ensure that the relative compact core $(\mathscr{M}, \mathscr{P})$ is contained in the complement

$$\mathrm{N} \setminus \big((\mathscr{H}_1 \sqcup \mathscr{H}_1') \sqcup \dots \sqcup (\mathscr{H}_m \sqcup \mathscr{H}_m') \big)$$

where $\mathscr{H}_j$ and $\mathscr{H}_j'$ are the quotients of half spaces bounded by the invariant circles $\partial\mathrm{B}_j$ and $\partial\mathrm{B}_j'$ for each Klein-Maskit combination. It follows that Π is an isometric embedding restricted to $(\mathscr{M}, \mathscr{P})$. Letting $\check{\mathrm{N}} = \check{\mathrm{N}}(\mathrm{P}_1, \dots, \mathrm{P}_m)$ proves parts (1) and (2) of the lemma.

We now verify the final conclusion, which asserts the existence of promotions $\check{\mathrm{N}}_n = \check{\mathrm{N}}_n(\mathrm{P}_1, \dots, \mathrm{P}_m)$ with baseframes ω_n, so that $(\check{\mathrm{N}}_n, \omega_n)$ converges geometrically to (N, ω), where ω is a baseframe in N. Indeed, for each compact subset K of N with $\omega \in \mathrm{K}$, we may choose B_j and B_j' so that the quotient half-spaces $\mathscr{H}_j$ and $\mathscr{H}_j'$ avoid K. Thus, given an exhaustion of N by compact submanifolds K_n

containing ω, we may choose $\check{N}_n$ so that K_n embeds isometrically into $\check{N}_n$ by the covering projection $\Pi_n\colon (N, \omega) \to (\check{N}_n, \omega_n)$. It follows that $(\check{N}_n, \omega_n)$ converges geometrically to (N, ω). □

Proof of Theorem 3.4. — Let $N = N_\rho$ lie in GF(M), and let $(\mathscr{M}, \mathscr{P})$ denote a relative compact core for N. We assume $(\mathscr{M}, \mathscr{P})$ has the structure of the relative compact core in Lemma 5.1. In particular, let $\mathscr{A}_1, \ldots, \mathscr{A}_m$ denote the annular components of the parabolic locus $\mathscr{P}$, and let $g_1, \ldots, g_m$ denote primitive elements of $\pi_1(\mathscr{M})$, so that g_j is homotopic into $\mathscr{A}_j$, for $j = 1, \ldots, m$.

Applying Lemma 5.1, we let $\check{N}$ be a promotion of all rank-1 cusps of N so that the locally isometric covering map $\Pi\colon N \to \check{N}$ restricts to an embedding on $\mathscr{M}$. Let $T_1, \ldots, T_m$ denote the torus cusps of $\check{N}$ so that $\Pi_*(g_j)$ lies in $\pi_1(T_j)$ up to conjugacy in $\pi_1(\check{N})$.

Performing $(1, n)$ hyperbolic Dehn-fillings on each torus-cusp $T_1, \ldots, T_m$ (see [Brm1, Thm. 7.3] or [BO]) we obtain a hyperbolic 3-manifold N_n that is homeomorphic to N, and so that there are baseframes ω_n in N_n and $\check{\omega}$ in $\check{N}$ with (N_n, ω_n) converging geometrically to $(\check{N}, \check{\omega})$ as n tends to ∞. Since such promotions $(\check{N}, \check{\omega})$ lie in every neighborhood of (N, ω) in the geometric topology by Lemma 5.1, we may assume $\{(N_n, \omega_n)\}$ converges geometrically to (N, ω) by a diagonal argument.

The natural embeddings $\phi_n\colon \mathscr{M} \to N_n$ determined by geometric convergence (for n sufficiently large) are homotopy equivalences whose bi-Lipschitz constant L_n tends to 1. Thus the manifolds N_n determine a sequence in MP(M) that converges algebraically, and thus strongly, to N. □

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