THE SIX OPERATIONS FOR SHEAVES ON ARTIN STACKS I: FINITE COEFFICIENTS

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ABSTRACT

In this paper we develop a theory of Grothendieck's six operations of lisse-étale constructible sheaves on Artin stacks locally of finite type over certain excellent schemes of finite Krull dimension. We also give generalizations of the classical base change theorems and Kunneth formula to stacks, and prove new results about cohomological descent for unbounded complexes.

1. Introduction

We denote by Λ a Gorenstein local ring of dimension 0 and characteristic l. Let S be an affine excellent finite-dimensional scheme and assume l is invertible on S. We assume that all S-schemes of finite type X satisfy $\mathrm{cd}_l(\mathrm{X}) < \infty$ (see 1.0.1 for more discussion of this). For an algebraic stack $\mathscr{X}$ locally of finite type over S and $* \in \{+, -, b, \varnothing, [a, b]\}$ we write $\mathrm{D}_c^*(\mathscr{X})$ for the full subcategory of the derived category $\mathrm{D}^*(\mathscr{X})$ of complexes of Λ-modules on the lisse-étale site of $\mathscr{X}$ with constructible cohomology sheaves. We will also consider the variant subcategories $\mathrm{D}_c^{(*)}(\mathscr{X}) \subset \mathrm{D}_c(\mathscr{X})$ consisting of complexes K such that for any quasi-compact open $\mathscr{U} \subset \mathscr{X}$ the restriction $\mathrm{K}|_{\mathscr{U}}$ is in $\mathrm{D}_c^*(\mathscr{U})$.

In this paper we develop a theory of Grothendieck's six operations of lisse-étale constructible sheaves on Artin stacks locally of finite type over S[1]. In forthcoming papers, we will also develop a theory of adic sheaves and perverse sheaves for Artin stacks. In addition to being of basic foundational interest, we hope that the development of these six operations for stacks will have a number of applications. Already the work done in this paper (and the forthcoming ones) provides the necessary tools needed in several papers on the geometric Langland's program (e.g. [21], [19], [7]). We hope that it will also shed further light on the Lefschetz trace formula for stacks proven by Behrend [1], and also to versions of such a formula for stacks not necessarily of finite type. We should also remark that recent work of Toen should provide another approach to defining the six operations for stacks, and in fact should generalize to a theory for n-stacks.

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[1] In fact our method could apply to other situations like analytic stacks or non separated analytic varieties.

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Let us describe more precisely the contents of this papers. For a finite type morphism $f : \mathscr{X} \to \mathscr{Y}$ of stacks locally of finite type over S we define functors

$$\begin{aligned} &\mathrm{R}f_* : \mathrm{D}_c^{(+)}(\mathscr{X}) \to \mathrm{D}_c^{(+)}(\mathscr{Y}), \quad \mathrm{R}f_! : \mathrm{D}_c^{(-)}(\mathscr{X}) \to \mathrm{D}_c^{(-)}(\mathscr{Y}),\\ &\mathrm{L}f^* : \mathrm{D}_c(\mathscr{Y}) \to \mathrm{D}_c(\mathscr{X}), \quad \mathrm{R}f^! : \mathrm{D}_c(\mathscr{Y}) \to \mathrm{D}_c(\mathscr{X}),\\ &\mathscr{R}hom : \mathrm{D}_c^{(-)}(\mathscr{X})^{\mathrm{op}} \times \mathrm{D}_c^{(+)}(\mathscr{X}) \to \mathrm{D}_c^{(+)}(\mathscr{X}), \end{aligned}$$

and

$$(-)\overset{\mathbf{L}}{\otimes}(-) : \mathrm{D}_c^{(-)}(\mathscr{X}) \times \mathrm{D}_c^{(-)}(\mathscr{X}) \to \mathrm{D}_c^{(-)}(\mathscr{X})$$

satisfying all the usual adjointness properties that one has in the theory for schemes[2].

The main tool is to define $f_!, f^!$, even for unbounded constructible complexes, by duality. One of the key points is that, as observed by Laumon, the dualizing complex is a local object of the derived category and hence has to exist for stacks by glueing (see 2.3.3). Notice that this formalism applies to non-separated schemes, giving a theory of cohomology with compact supports in this case. Previously, Laumon and Moret-Bailly constructed the truncations of dualizing complexes for Bernstein–Lunts stacks (see [20]). Our constructions reduces to theirs in this case. Another approach using a dual version of cohomological descent has been suggested by Gabber but seems to be technically much more complicated.

1.0.1. *Remark.* — The cohomological dimension hypothesis on schemes of finite type over S is achieved for instance if S is the spectrum of a finite field or of a separably closed field. In dimension 1, it will be achieved for instance for the spectrum of a complete discrete valuation field with residue field either finite or separably closed, or if S is a smooth curve over $\mathbf{C}, \mathbf{F}_q$ (cf. [13, exp. X] and [26]). In these situations, $\mathrm{cd}_l(\mathrm{X})$ is bounded by a function of the dimension $\dim(\mathrm{X})$.

1.1. *Conventions.* — In order to develop the theory over an excellent base S as above, we use the recent finiteness results of Gabber [9] and [10]. A complete account of these results will soon appear in a writeup of the seminar on Gabber's work [15]. However, the reader uncomfortable with this theory may assume that S is an affine regular, noetherian scheme of dimension ≤ 1.

Our conventions about stacks are those of [20].

Let $\mathscr{X}$ be an algebraic stack locally of finite type over S and let $\mathrm{K} \in \mathrm{D}_c^+(\mathscr{X})$ be a complex of Λ-modules. Following [14, I.1.1], we say that K has *finite quasi-injective dimension* if there exists an integer n such that for any constructible sheaf of Λ-modules F we have

$$\mathscr{E}xt^i(\mathrm{F}, \mathrm{K}) = 0, \qquad \text{for } i > n.$$

[2] We will often write $f^*, f^!, f_*, f_!$ for $\mathrm{L}f^*$, $\mathrm{R}f^!$, $\mathrm{R}f_*$, $\mathrm{R}f_!$.

For $K \in D_c^{(+)}(\mathscr{X})$ we say that K has *locally finite quasi-injective dimension* if for every quasi-compact open substack $\mathscr{U} \hookrightarrow \mathscr{X}$ the restriction $K|_{\mathscr{U}} \in D_c^+(\mathscr{U})$ has finite quasi-injective dimension.

In this paper we work systematically with unbounded complexes. The theory of derived functors for unbounded complexes on topological spaces is due to Spaltenstein [27], and for Grothendieck categories Serpé [25]. An excellent reference for unbounded homological algebra is the book of Kashiwara and Schapira [16].

Recall that for a ringed topos $(T, \mathscr{O}_T)$ one has functors [16, Theorem 18.6.4],

$$(-)\overset{\mathbf{L}}{\otimes}(-) : D(\mathscr{O}_T) \times D(\mathscr{O}_T) \to D(\mathscr{O}_T)$$

and

$$\mathscr{R}hom(-,-) : D(\mathscr{O}_T) \times D(\mathscr{O}_T)^{op} \to D(\mathscr{O}_T),$$

and for a morphism $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ of ringed topos one has functors [16, Theorem 18.6.9] and the line preceding this theorem,

$$Rf_* : D(\mathscr{O}_T) \to D(\mathscr{O}_T)$$

and

$$Lf^* : D(\mathscr{O}_S) \to D(\mathscr{O}_T).$$

Moreover these satisfy the usual adjunction properties that one would expect from the theory for the bounded derived category.

All the stacks we will consider will be locally of finite type over S. As in [20, lemme 12.1.2], the lisse-étale topos $\mathscr{X}_{\text{lis-ét}}$ can be defined using the site Lisse-Et($\mathscr{X}$) whose objects are S-morphisms $u : U \to \mathscr{X}$ where U is an algebraic space which is *separated and of finite type over* S. The topology is generated by the pretopology such that the covering families are finite families $(U_i, u_i) \to (U, u)$ such that $\coprod U_i \to U$ is surjective and étale (use the comparison theorem [13, III.4.1] remembering $\mathscr{X}$ is locally of finite type over S). Notice that products over $\mathscr{X}$ are representable in Lisse-Et($\mathscr{X}$), simply because the diagonal morphism $\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable and separated by definition [20].

If C is a complex of sheaves and d a locally constant valued function $C(d)$ is the Tate twist and $C[d]$ the shifted complex. We denote $C(d)[2d]$ by $C\langle d\rangle$. We fix once and for all a dualizing complex Ω_S on S. In the case when S is regular of dimension 0 or 1 we take $\Omega_S = \Lambda\langle \dim(S)\rangle$ [4, "Dualité"].

2. Homological algebra

2.1. *Existence of* K*-injectives.* — Let $(\mathscr{S}, \mathscr{O})$ denote a ringed site, and let $\mathscr{C}$ denote a full subcategory of the category of $\mathscr{O}$-modules on $\mathscr{S}$. Let M be a complex of $\mathscr{O}$-modules on $\mathscr{S}$. By [27, 3.7], there exists a morphism of complexes $f : \mathrm{M} \to \mathrm{I}$ with the following properties:

(i) $\mathrm{I} = \varprojlim \mathrm{I}_n$ where each I_n is a bounded below complex of flasque $\mathscr{O}$-modules.
(ii) The morphism f is induced by a compatible collection of quasi-isomorphisms $f_n : \tau_{\geq -n}\mathrm{M} \to \mathrm{I}_n$.
(iii) For every n the map $\mathrm{I}_n \to \mathrm{I}_{n-1}$ is surjective with kernel K_n a bounded below complex of flasque $\mathscr{O}$-modules.
(iv) For any pair of integers n and i the sequence

$$0 \to \mathrm{K}_n^i \to \mathrm{I}_n^i \to \mathrm{I}_{n-1}^i \to 0 \qquad (\mathbf{2.1.i})$$

is split.

2.1.1. *Remark.* — In fact [27, 3.7], shows that we can choose I_n and K_n to be complexes of injective $\mathscr{O}$-modules (in which case (iv) follows from (iii)). However, for technical reasons it is sometimes useful to know that one can work just with flasque sheaves.

We make the following finiteness assumption, which is the analog of [27, 3.12 (1)].

2.1.2. *Assumption.* — *For any object* $\mathrm{U} \in \mathscr{S}$ *there exists a covering* $\{\mathrm{U}_i \to \mathrm{U}\}_{i\in\mathrm{I}}$ *and an integer* n_0 *such that for any sheaf of* $\mathscr{O}$*-modules* $\mathrm{F} \in \mathscr{C}$ *we have* $\mathrm{H}^n(\mathrm{U}_i, \mathrm{F}) = 0$ *for all* $n \geq n_0$.

2.1.3. *Example.* — Let $\mathscr{S} = \text{Lisse-Et}(\mathscr{X})$ be the lisse-étale site of an algebraic S-stack locally of finite type $\mathscr{X}$ and $\mathscr{O}$ a constant local Gorenstein ring of dimension 0 and characteristic invertible on S. Then the class $\mathscr{C}$ of all $\mathscr{O}$-sheaves, cartesian or not, satisfies the assumption. Indeed, if $\mathrm{U} \in \mathscr{S}$ is of finite type over S and $\mathrm{F} \in \mathscr{S}$, one has $\mathrm{H}^n(\mathrm{U}, \mathrm{F}) = \mathrm{H}^n(\mathrm{U}_{\text{ét}}, \mathrm{F}_\mathrm{U})$[3] which is zero for n bigger than a constant depending only on U (and not on F). Therefore, one can take the trivial covering in this case. We could also take $\mathscr{O} = \mathscr{O}_\mathscr{X}$ and $\mathscr{C}$ to be the class of quasi-coherent sheaves.

With Hypothesis 2.1.2, one has the following criterion for f being a quasi-isomorphism (cf. [27, 3.13]).

2.1.4. *Proposition.* — *Assume that* $\mathscr{H}^j(\mathrm{M}) \in \mathscr{C}$ *for all* j. *Then the map* f *is a quasi-isomorphism. In particular, if each* I_n *is a complex of injective* $\mathscr{O}$*-modules then by* [27, 2.5], $f : \mathrm{M} \to \mathrm{I}$ *is a* K*-injective resolution of* M.

[3] Cf. 3.3.1 below

Proof. — For a fixed integer j, the map $\mathscr{H}^j(\mathrm{M}) \to \mathscr{H}^j(\mathrm{I}_n)$ is an isomorphism for n sufficiently big. Since this isomorphism factors as

$$\mathscr{H}^j(\mathrm{M}) \to \mathscr{H}^j(\mathrm{I}) \to \mathscr{H}^j(\mathrm{I}_n) \tag{2.1.ii}$$

it follows that the map $\mathscr{H}^j(\mathrm{M}) \to \mathscr{H}^j(\mathrm{I})$ is injective.

To see that $\mathscr{H}^j(\mathrm{M}) \to \mathscr{H}^j(\mathrm{I})$ is surjective, let $\mathrm{U} \in \mathscr{S}$ be an object and $\gamma \in \Gamma(\mathrm{U}, \mathrm{I}^j)$ an element with $d\gamma = 0$ defining a class in $\mathscr{H}^j(\mathrm{I})(\mathrm{U})$. Since $\mathrm{I} = \varprojlim \mathrm{I}_n$ the class γ is given by a compatible collection of sections $\gamma_n \in \Gamma(\mathrm{U}, \mathrm{I}_n^j)$ with $d\gamma_n = 0$.

Let $(\mathscr{U} = \{\mathrm{U}_i \to \mathrm{U}\}, n_0)$ be the data provided by 2.1.2. Let N be an integer greater than $n_0 - j$. For $m \geq \mathrm{N}$ and $\mathrm{U}_i \in \mathscr{U}$ the sequence

$$\Gamma\big(\mathrm{U}_i, \mathrm{K}_m^{j-1}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{K}_m^{j}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{K}_m^{j+1}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{K}_m^{j+2}\big) \tag{2.1.iii}$$

is exact. Indeed K_m is a bounded below complex of flasque sheaves quasi-isomorphic to $\mathscr{H}^{-m}(\mathrm{M})[m]$, and therefore the exactness is equivalent to the statement that the groups

$$\mathrm{H}^j(\mathrm{U}_i, \mathscr{H}^{-m}(\mathrm{M})[m]) = \mathrm{H}^{j+m}(\mathrm{U}_i, \mathscr{H}^{-m}(\mathrm{M}))$$

and

$$\mathrm{H}^{j+1}(\mathrm{U}_i, \mathscr{H}^{-m}(\mathrm{M})[m]) = \mathrm{H}^{j+m+1}(\mathrm{U}_i, \mathscr{H}^{-m}(\mathrm{M}))$$

are zero. This follows from the assumptions and the observation that

$$j + m \geq j + \mathrm{N} > j + n_0 - j = n_0,$$
$$j + 1 + m \geq j + 1 + \mathrm{N} \geq j + n_0 - j = n_0.$$

Since the maps $\Gamma(\mathrm{U}_i, \mathrm{I}_m^r) \to \Gamma(\mathrm{U}_i, \mathrm{I}_{m-1}^r)$ are also surjective for all m and r, it follows from [27, 0.11], applied to the system

$$\Gamma\big(\mathrm{U}_i, \mathrm{I}_m^{j-1}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{I}_m^{i}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{I}_m^{j+1}\big) \to \Gamma\big(\mathrm{U}_i, \mathrm{I}_m^{j+2}\big) \tag{2.1.iv}$$

that the map

$$\mathrm{H}^j(\Gamma(\mathrm{U}_i, \mathrm{I})) \to \mathrm{H}^j(\Gamma(\mathrm{U}_i, \mathrm{I}_m)) \tag{2.1.v}$$

is an isomorphism.

Then since the map $\mathscr{H}^j(\mathrm{M}) \to \mathscr{H}^j(\mathrm{I}_m)$ is an isomorphism it follows that for every i the restriction of γ to U_i is in the image of $\mathscr{H}^j(\mathrm{M})(\mathrm{U}_i)$. □

Next consider a fibred topos $\mathscr{T} \to \mathrm{D}$ with corresponding total topos $\mathscr{T}_\bullet$ [13, VI.7]. We call $\mathscr{T}_\bullet$ a D-*simplicial topos*. Concretely, this means that for each $i \in \mathrm{D}$ the fiber $\mathscr{T}_i$ is a topos and that any $\delta \in \mathrm{Hom}_\mathrm{D}(i,j)$ comes together with a morphism of topos $\delta : \mathscr{T}_i \to \mathscr{T}_j$ such that δ^{-1} is the inverse image functor of the fibred structure. The objects of the total topos are simply collections $(\mathrm{F}_i \in \mathscr{T}_i)_{i\in\mathrm{D}}$ together with functorial transition morphisms $\delta^{-1}\mathrm{F}_j \to \mathrm{F}_i$ for any $\delta \in \mathrm{Hom}_\mathrm{D}(i,j)$. We assume furthermore that $\mathscr{T}_\bullet$ is ringed by a $\mathscr{O}_\bullet$ and that for any $\delta \in \mathrm{Hom}_\mathrm{D}(i,j)$, the morphism $\delta : (\mathscr{T}_i, \mathscr{O}_i) \to (\mathscr{T}_j, \mathscr{O}_j)$ is flat.

2.1.5. *Example.* — Let Δ^+ be the category whose objects are the ordered sets $[n] = \{0, \dots, n\}$ ($n \in \mathbf{N}$) and whose morphisms are injective order-preserving maps. Let D be the opposite category of Δ^+. In this case $\mathscr{T}_.$ is called a *strict simplicial topos*. For instance, if $\mathrm{U} \to \mathscr{X}$ is a presentation, the simplicial algebraic space $\mathrm{U}_\bullet = \mathrm{cosq}_0(\mathrm{U}/\mathscr{X})$ defines a strict simplicial topos $\mathrm{U}_{\bullet\text{lis-ét}}$ whose fiber over $[n]$ is $\mathrm{U}_{n\text{lis-ét}}$. For a morphism $\delta : [n] \to [m]$ in Δ^+ the morphism $\delta : \mathscr{T}_m \to \mathscr{T}_n$ is induced by the (smooth) projection $\mathrm{U}_m \to \mathrm{U}_n$ defined by $\delta \in \mathrm{Hom}_{\Delta^{+\mathrm{opp}}}([m], [n])$.

2.1.6. *Example.* — Let $\mathbf{N}$ be the natural numbers viewed as a category in which $\mathrm{Hom}(n, m)$ is empty unless $m \geq n$ in which case it consists of a unique element. For a topos T we can then define an $\mathbf{N}$-simplicial topos $\mathrm{T}^\mathbf{N}$. The fiber over n of $\mathrm{T}^\mathbf{N}$ is T and the transition morphisms by the identity of T. The topos $\mathrm{T}^\mathbf{N}$ is the category of projective systems in T. If $\mathscr{O}_\bullet$ is a constant projective system of rings then the flatness assumption is also satisfied, or more generally if $\delta^{-1}\mathscr{O}_n \to \mathscr{O}_m$ is an isomorphism for any morphism $\delta : m \to n$ in $\mathbf{N}$ then the flatness assumption holds.

Let $\mathscr{C}_\bullet$ be a full subcategory of the category of $\mathscr{O}_\bullet$-modules on a ringed D-simplicial topos $(\mathscr{T}_\bullet, \mathscr{O}_\bullet)$. For $i \in \mathrm{D}$, let $e_i : \mathscr{T}_n \to \mathscr{T}_\bullet$ the morphism of topos defined by $e_i^{-1}\mathrm{F}_\bullet = \mathrm{F}_n$ (cf. [13, $\mathrm{V}^{\mathrm{bis}}$.1.2.11]). Recall that the family $e_i^{-1}, i \in \mathrm{D}$ is conservative. Let $\mathscr{C}_i$ denote the essential image of $\mathscr{C}_\bullet$ under e_i^{-1} (which coincides with e_i^* on $\mathrm{Mod}(\mathscr{T}_\bullet, \mathscr{O}_\bullet)$ because $e_i^{-1}\mathscr{O}_\bullet = \mathscr{O}_i$).

2.1.7. *Assumption.* — *For every $i \in \mathrm{D}$ the ringed topos $(\mathscr{T}_i, \mathscr{O}_i)$ is isomorphic to the topos of a ringed site satisfying 2.1.2 with respect to $\mathscr{C}_i$.*

2.1.8. *Example.* — Let $\mathscr{T}_\bullet$ be the topos $(\mathscr{X}_{\text{lis-ét}})^\mathbf{N}$ of a S-stack locally of finite type. Then, the full subcategory $\mathscr{C}_\bullet$ of $\mathrm{Mod}(\mathscr{T}_\bullet, \mathscr{O}_\bullet)$ whose objects are families F_i of *cartesian* modules satisfies the hypothesis.

Let M be a complex of $\mathscr{O}_\bullet$-modules on $\mathscr{T}_\bullet$. Again by [27, 3.7], there exists a morphism of complexes $f : \mathrm{M} \to \mathrm{I}$ with the following properties:

(S i) $\mathrm{I} = \varprojlim \mathrm{I}_n$ where each I_n is a bounded below complex of injective modules.

(S ii) The morphism f is induced by a compatible collection of quasi-isomorphisms $f_n : \tau_{\geq -n}\mathrm{M} \to \mathrm{I}_n$.

(S iii) For every n the map $\mathrm{I}_n \to \mathrm{I}_{n-1}$ is surjective with kernel K_n a bounded below complex of injective $\mathscr{O}$-modules.

(S iv) For any pair of integers n and i the sequence

$$0 \to \mathrm{K}_n^i \to \mathrm{I}_n^i \to \mathrm{I}_{n-1}^i \to 0 \qquad (\mathbf{2.1.vi})$$

is split.

2.1.9. *Proposition.* — *Assume that $\mathscr{H}^j(\mathrm{M}) \in \mathscr{C}_\bullet$ for all j. Then the morphism f is a quasi-isomorphism and $f : \mathrm{M} \to \mathrm{I}$ is a* K-*injective resolution of* M.

Proof. — By [27, 2.5], it suffices to show that f is a quasi-isomorphism. For this in turn it suffices to show that for every $i \in \mathrm{D}$ the restriction $e_i^* f : e_i^*\mathrm{M} \to e_i^*\mathrm{I}$ is a quasi-isomorphism of complexes of $\mathscr{O}_i$-modules since the family $e_i^* = e_i^{-1}$ is conservative. But $e_i^* : \mathrm{Mod}(\mathscr{T}_\bullet, \mathscr{O}_\bullet) \to \mathrm{Mod}(\mathscr{T}_i, \mathscr{O}_i)$ has a left adjoint $e_{i!}$ defined by

$$[e_{i!}(\mathrm{F})]_j = \bigoplus_{\delta \in \mathrm{Hom}_\mathrm{D}(j,i)} \delta^*\mathrm{F}$$

with the obvious transition morphisms. It is exact by the flatness of the morphisms δ. It follows that e_i^* takes injectives to injectives and commutes with direct limits. We can therefore apply 2.1.4 to $e_i^*\mathrm{M} \to e_i^*\mathrm{I}$ to deduce that this map is a quasi-isomorphism. □

In what follows we call a K-injective resolution $f : \mathrm{M} \to \mathrm{I}$ obtained from data (i)–(iv) as above a *Spaltenstein resolution.*

The main technical lemma is the following.

2.1.10. *Lemma.* — *Let $\epsilon : (\mathscr{T}_\bullet, \mathscr{O}_\bullet) \to (\mathrm{S}, \Psi)$ be a morphism of ringed topos, and let* C *be a complex of $\mathscr{O}_\bullet$-modules. Assume that*

1. $\mathscr{H}^n(\mathrm{C}) \in \mathscr{C}_\bullet$ for all n.

2. There exists i_0 such that $\mathrm{R}^i\epsilon_*\mathscr{H}^n(\mathrm{C}) = 0$ for any n and any $i > i_0$.

Then, if $j \geq -n + i_0$, we have $\mathrm{R}^j\epsilon_*\mathrm{C} = \mathrm{R}^j\epsilon_*\tau_{\geq -n}\mathrm{C}$.

Proof. — By 2.1.9 and assumption (1), there exists a Spaltenstein resolution $f : \mathrm{C} \to \mathrm{I}$ of C. Let $\mathrm{J}_n := \epsilon_*\mathrm{I}_n$ and $\mathrm{F}_n := \epsilon_*\mathrm{K}_n$. Since the sequences (2.1.vi) are split, the sequences

(**2.1.vii**) $$0 \to F_n \to J_n \to J_{n-1} \to 0$$

are exact.

The exact sequence (2.1.vi) and property (S ii) defines a distinguished triangle

$$K_n \to \tau_{\geq -n}C \to \tau_{\geq -n+1}C$$

showing that K_n is quasi-isomorphic to $\mathscr{H}^{-n}(C)[n]$. Because K_n is a bounded below complex of injectives, one gets

$$R\epsilon_*\mathscr{H}^{-n}(C)[n] = \epsilon_*K_n$$

and accordingly

$$R^{j+n}\epsilon_*\mathscr{H}^{-n}(C) = \mathscr{H}^j(\epsilon_*K_n) = \mathscr{H}^j(F_n).$$

By assumption (2), we have therefore

$$\mathscr{H}^j(F_n) = 0 \text{ for } j > -n + i_0.$$

By [27, 0.11], this implies that

$$\mathscr{H}^j(\varprojlim J_n) \to \mathscr{H}^j(J_n)$$

is an isomorphism for $j \geq -n + i_0$. But, by adjunction, ϵ_* commutes with projective limit. In particular, one has

$$\varprojlim J_n = \epsilon_*I,$$

and by (S i) and (S ii)

$$R\epsilon_*C = \epsilon_*I \text{ and } R\epsilon_*\tau_{\geq -n}C = \epsilon_*J_n.$$

Thus for any n such that $j \geq -n + i_0$ one has

(**2.1.viii**) $$R^j\epsilon_*C = \mathscr{H}^j(\epsilon_*I) = \mathscr{H}^j(J_n) = R^j\epsilon_*\tau_{\geq -n}C.$$

□

2.1.11. *Remark.* — An important special case of Lemma 2.1.10 is the following. Take D to be the category with one element and one morphism so that $\epsilon : (\mathscr{T}, \mathscr{O}) \to (S, \Psi)$ is just a morphism of ringed topos. Let $\mathscr{C}$ be a full subcategory of the category of $\mathscr{O}$-modules such that $(\mathscr{T}, \mathscr{O})$ is isomorphic to the ringed topos associated to a ringed site satisfying Assumption 2.1.2 with respect to $\mathscr{C}$. Let C be a complex of $\mathscr{O}$-modules such that

1. $\mathscr{H}^n(C) \in \mathscr{C}$ for all n.
2. There exists an integer i_0 such that $R^i\epsilon_*\mathscr{H}^n(C) = 0$ for any n and $i > i_0$.

Then by Lemma 2.1.10 the natural map $R^j\epsilon_*C \to R^j\epsilon_*\tau_{\geq -n}C$ is an isomorphism for $j \geq -n + i_0$.

2.2. *The descent theorem.* — Let $(\mathscr{T}_\bullet, \mathscr{O}_\bullet)$ be a simplicial or strictly simplicial[4] ringed topos ($D = \Delta^{\rm opp}$ or $D = \Delta^{+{\rm opp}}$), let (S, Ψ) be another ringed topos, and let $\epsilon : (\mathscr{T}_\bullet, \mathscr{O}_\bullet) \to (S, \Psi)$ be an augmentation. Assume that ϵ is a flat morphism (i.e. for every $i \in D$, the morphism of ringed topos $(\mathscr{T}_i, \mathscr{O}_i) \to (S, \Psi)$ is a flat morphism), and furthermore that for every morphism $\delta : i \to j$ in D the corresponding morphism of ringed topos $(\mathscr{T}_i, \mathscr{O}_i) \to (\mathscr{T}_j, \mathscr{O}_j)$ is flat.

Let $\mathscr{C}$ be a full subcategory of the category of Ψ-modules, and assume that $\mathscr{C}$ is closed under kernels, cokernels and extensions (one says that $\mathscr{C}$ is a *Serre subcategory*). Let $D(S)$ denote the derived category of Ψ-modules, and let $D_{\mathscr{C}}(S) \subset D(S)$ be the full subcategory consisting of complexes whose cohomology sheaves are in $\mathscr{C}$. Let $\mathscr{C}_\bullet$ denote the essential image of $\mathscr{C}$ under the functor $\epsilon^* : \mathrm{Mod}(\Psi) \to \mathrm{Mod}(\mathscr{O}_\bullet)$.

We assume the following condition holds:

2.2.1. *Assumption.* — *Assumption 2.1.7 holds (with respect to $\mathscr{C}_\bullet$), and $\epsilon^* : \mathscr{C} \to \mathscr{C}_\bullet$ is an equivalence of categories with quasi-inverse* $R\epsilon_*$.

2.2.2. *Lemma.* — *The full subcategory $\mathscr{C}_\bullet \subset \mathrm{Mod}(\mathscr{O}_\bullet)$ is closed under extensions, kernels and cokernels.*

Proof. — Consider an extension of sheaves of $\mathscr{O}_\bullet$-modules

$$0 \longrightarrow \epsilon^*F_1 \longrightarrow E \longrightarrow \epsilon^*F_2 \longrightarrow 0, \tag{2.2.i}$$

where $F_1, F_2 \in \mathscr{C}$. Since $R^1\epsilon_*\epsilon^*F_1 = 0$ and the maps $F_i \to R^0\epsilon_*\epsilon^*F_i$ are isomorphisms, we obtain by applying $\epsilon^*\epsilon_*$ a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \epsilon^*F_1 & \longrightarrow & \epsilon^*\epsilon_*E & \longrightarrow & \epsilon^*F_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \epsilon^*F_1 & \longrightarrow & E & \longrightarrow & \epsilon^*F_2 & \longrightarrow & 0. \end{array} \tag{2.2.ii}$$

It follows that α is an isomorphism. Furthermore, since $\mathscr{C}$ is closed under extensions we have $\epsilon_*E \in \mathscr{C}$. Let $f \in \mathrm{Hom}(\epsilon^*F_1, \epsilon^*F_2)$. There exists a unique $\varphi \in \mathrm{Hom}(F_1, F_2)$ such that $f = \epsilon^*\varphi$. Because ϵ^* is exact, it maps the kernel and cokernel of φ, which are objects of $\mathscr{C}$, to the kernel and cokernel of f respectively. Therefore, the latter are objects of $\mathscr{C}_\bullet$. □

Let $D(\mathscr{T}_\bullet)$ denote the derived category of $\mathscr{O}_\bullet$-modules, and let $D_{\mathscr{C}_\bullet}(\mathscr{T}_\bullet) \subset D(\mathscr{T}_\bullet)$ denote the full subcategory of complexes whose cohomology sheaves are in $\mathscr{C}_\bullet$.

---

[4] One could replace simplicial by multisimplicial

Since ϵ is a flat morphism, we obtain a morphism of triangulated categories (the fact that these categories are triangulated comes precisely from the fact that both $\mathscr{C}$ and $\mathscr{C}_\bullet$ are Serre categories [11]).

$$\epsilon^* : D_{\mathscr{C}}(S) \to D_{\mathscr{C}_\bullet}(\mathscr{T}_\bullet). \tag{2.2.iii}$$

2.2.3. *Theorem. — The functor* ϵ^* *of* (2.2.iii) *is an equivalence of triangulated categories with quasi-inverse given by* $R\epsilon_*$.

Proof. — Note first that if $M_\bullet \in D_{\mathscr{C}_\bullet}(\mathscr{T}_\bullet)$, then by Lemma 2.1.10, for any integer j there exists n_0 such that $R^j\epsilon_*M_\bullet = R^j\epsilon_*\tau_{\geq n_0}M_\bullet$. In particular, we get by induction $R^j\epsilon_*M_\bullet \in \mathscr{C}$. Thus $R\epsilon_*$ defines a functor

$$R\epsilon_* : D_{\mathscr{C}_\bullet}(\mathscr{T}_\bullet) \to D_{\mathscr{C}}(S). \tag{2.2.iv}$$

To prove 2.2.3 it suffices to show that for $M_\bullet \in D_{\mathscr{C}_\bullet}(\mathscr{T}_\bullet)$ and $F \in D_{\mathscr{C}}(S)$ the adjunction maps

$$\epsilon^*R\epsilon_*M_\bullet \to M_\bullet, \qquad F \to R\epsilon_*\epsilon^*F \tag{2.2.v}$$

are isomorphisms. For this note that for any integers j and n there are commutative diagrams

$$\begin{array}{ccc} \epsilon^*R^j\epsilon_*M_\bullet & \longrightarrow & \mathscr{H}^j(M_\bullet) \\ \downarrow & & \downarrow \\ \epsilon^*R^j\epsilon_*\tau_{\geq n}M_\bullet & \longrightarrow & \mathscr{H}^j(\tau_{\geq n}M_\bullet), \end{array} \tag{2.2.vi}$$

and

$$\begin{array}{ccc} \mathscr{H}^j(F) & \longrightarrow & R^j\epsilon_*\epsilon^*F \\ \downarrow & & \downarrow \\ \mathscr{H}^j(\tau_{\geq n}F) & \longrightarrow & R^j\epsilon_*\epsilon^*\tau_{\geq n}F. \end{array} \tag{2.2.vii}$$

By the observation at the begining of the proof, there exists an integer n so that the vertical arrows in the above diagrams are isomorphisms. This reduces the proof 2.2.3 to the case of a bounded below complex. In this case one reduces by devissage to the case when $M_\bullet \in \mathscr{C}_\bullet$ and $F \in \mathscr{C}$ in which case the result holds by assumption. □

The theorem applies in particular to the following examples.

2.2.4. *Example.* — Let S be an algebraic space and $X_\bullet \to S$ a flat hypercover by algebraic spaces. Let $X_\bullet^+ \to S$ denote the associated strictly simplicial space with S-augmentation. We then obtain an augmented strictly simplicial topos $\epsilon : (X^+_{\bullet,\text{ét}}, \mathscr{O}_{X^+_{\bullet,\text{ét}}}) \to (S_{\text{ét}}, \mathscr{O}_{\text{ét}})$. Note that this augmentation is flat. Let $\mathscr{C}$ denote the category of quasi-coherent sheaves on $S_{\text{ét}}$. Then the category $\mathscr{C}_\bullet$ is the category of cartesian sheaves of $\mathscr{O}_{X^+_{\bullet,\text{ét}}}$-modules whose restriction to each X_n is quasi-coherent. Let $D_{\text{qcoh}}(X^+_\bullet)$ denote the full subcategory of the derived category of $\mathscr{O}_{X^+_\bullet,\text{ét}}$-modules whose cohomology sheaves are quasi-coherent, and let $D_{\text{qcoh}}(S)$ denote the full subcategory of the derived category of $\mathscr{O}_{S_{\text{ét}}}$-modules whose cohomology sheaves are quasi-coherent. Theorem 2.2.3 then shows that the pullback functor

$$\epsilon^* : D_{\text{qcoh}}(S) \to D_{\text{qcoh}}(X^+_\bullet) \tag{2.2.viii}$$

is an equivalence of triangulated categories with quasi-inverse $R\epsilon_*$.

2.2.5. *Example.* — Let $\mathscr{X}$ be an algebraic stack and let $U_\bullet \to \mathscr{X}$ be a smooth hypercover by algebraic spaces. Let $D(\mathscr{X})$ denote the derived category of sheaves of $\mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}$-modules in the topos $\mathscr{X}_{\text{lis-ét}}$, and let $D_{\text{qcoh}}(\mathscr{X}) \subset D(\mathscr{X})$ be the full subcategory of complexes with quasi-coherent cohomology sheaves.

Let $U^+_\bullet$ denote the strictly simplicial algebraic space obtained from $U_\bullet$ by forgetting the degeneracies. Since the Lisse-Étale topos is functorial with respect to smooth morphisms, we therefore obtain a strictly simplicial topos $U^+_{\bullet\text{lis-ét}}$ and a flat morphism of ringed topos

$$\epsilon : \left(U^+_{\bullet\text{lis-ét}}, \mathscr{O}_{U^+_{\bullet\text{lis-ét}}}\right) \to (\mathscr{X}_{\text{lis-ét}}, \mathscr{O}_{\mathscr{X}_{\text{lis-ét}}}).$$

Then 2.2.1 holds with $\mathscr{C}$ equal to the category of quasi-coherent sheaves on $\mathscr{X}$. The category $\mathscr{C}_\bullet$ in this case is the category of cartesian $\mathscr{O}_{U^+_{\bullet\text{lis-ét}}}$-modules $M_\bullet$ such that the restriction M_n is a quasi-coherent sheaf on U_n for all n. By 2.2.3 we then obtain an equivalence of triangulated categories

$$D_{\text{qcoh}}(\mathscr{X}) \to D_{\text{qcoh}}\left(U^+_{\bullet,\text{lis-ét}}\right), \tag{2.2.ix}$$

where the right side denotes the full subcategory of the derived category of $\mathscr{O}_{U^+_{\bullet\text{lis-ét}}}$-modules with cohomology sheaves in $\mathscr{C}_\bullet$.

On the other hand, there is also a natural morphism of ringed topos

$$\pi : \left(U^+_{\bullet\text{lis-ét}}, \mathscr{O}_{U^+_{\bullet\text{lis-ét}}}\right) \to \left(U^+_{\bullet\text{ét}}, \mathscr{O}_{U^+_{\bullet\text{ét}}}\right)$$

with π_* and π^* both exact functors. Let $D_{\text{qcoh}}(U^+_{\bullet\text{ét}})$ denote the full subcategory of the derived category of $\mathscr{O}_{U^+_{\bullet\text{ét}}}$-modules consisting of complexes whose cohomology sheaves are quasi-coherent (i.e. cartesian and restrict to a quasi-coherent sheaf on each $U_{n\text{ét}}$). Then π induces an equivalence of triangulated categories $D_{\text{qcoh}}(U^+_{\bullet\text{ét}}) \simeq D_{\text{qcoh}}(U^+_{\bullet\text{lis-ét}})$. Putting it all together we obtain an equivalence of triangulated categories $D_{\text{qcoh}}(\mathscr{X}_{\text{lis-ét}}) \simeq D_{\text{qcoh}}(U^+_{\bullet\text{ét}})$.

2.2.6. *Example.* — Let $\mathscr{X}$ be an algebraic stack locally of finite type over S and $\mathscr{O}$ be a constant local Gorenstein ring of dimension 0 and of characteristic invertible on S. Let $U_\bullet \to \mathscr{X}$ be a smooth hypercover by algebraic spaces, and $\mathscr{T}_\bullet$ the localized topos $\mathscr{X}_{\text{lis-ét}}|_{U_\bullet}$. Take $\mathscr{C}$ to be the category of constructible sheaves of $\mathscr{O}$-modules. Then 2.2.3 gives an equivalence $D_c(\mathscr{X}_{\text{lis-ét}}) \simeq D_c(\mathscr{T}_\bullet, \Lambda)$. On the other hand, there is a natural morphism of topos $\lambda : \mathscr{T}_\bullet \to U_{\bullet,\text{ét}}$ and one sees immediately that λ_* and λ^* induce an equivalence of derived categories $D_c(\mathscr{T}_\bullet, \Lambda) \simeq D_c(U_{\bullet,\text{ét}}, \Lambda)$. It follows that $D_c(\mathscr{X}_{\text{lis-ét}}) \simeq D_c(U_{\bullet,\text{ét}})$.

2.3. *The BBD gluing lemma.* — The purpose of this section is to explain how to modify the proof of the gluing lemma [2, 3.2.4], for unbounded complexes.

Let $\widetilde{\Delta}$ denote the strictly simplicial category of finite ordered sets with injective order preserving maps, and let $\Delta^+ \subset \widetilde{\Delta}$ denote the full subcategory of nonempty finite ordered sets. For a morphism α in $\widetilde{\Delta}$ we write $s(\alpha)$ (resp. $b(\alpha)$) for its source (resp. target).

Let T be a topos and $U. \to e$ a strictly simplicial hypercovering of the initial object $e \in T$. For $[n] \in \widetilde{\Delta}$ write U_n for the localized topos $T|_{U_n}$ where by definition we set $U_\varnothing = T$. Then we obtain a strictly simplicial topos U. with an augmentation $\pi : U. \to T$.

Let Λ be a sheaf of rings in T and write also Λ for the induced sheaf of rings in U. so that π is a morphism of ringed topos.

Let $\mathscr{C}.$ denote a full substack of the fibered and cofibered category over $\widetilde{\Delta}$

$$[n] \mapsto (\text{category of sheaves of } \Lambda\text{-modules in } U_n)$$

such that each $\mathscr{C}_n$ is a Serre subcategory of the category of Λ-modules in U_n. For any $[n]$ we can then form the derived category $D_{\mathscr{C}}(U_n, \Lambda)$ of complexes of Λ-modules whose cohomology sheaves are in $\mathscr{C}_n$. The categories $D_{\mathscr{C}}(U_n, \Lambda)$ form a fibered and cofibered category over $\widetilde{\Delta}$.

We make the following assumptions on $\mathscr{C}$:

2.3.1. *Assumption.* — *(i) For any $[n]$ the topos U_n is equivalent to the topos associated to a site $\mathscr{S}_n$ such that for any object $V \in \mathscr{S}_n$ there exists an integer n_0 and a covering $\{V_j \to V\}$ in $\mathscr{S}_n$ such that for any $F \in \mathscr{C}_n$ we have $H^n(V_j, F) = 0$ for all $n \geq n_0$.*

(ii) The natural functor

$$\mathscr{C}_\varnothing \to (\textit{cartesian sections of } \mathscr{C}|_{\Delta^+} \textit{ over } \Delta^+)$$

is an equivalence of categories.

(iii) The category $D(T, \Lambda)$ is compactly generated.

2.3.2. *Remark.* — The case we have in mind is when T is the lisse-étale topos of an algebraic stack $\mathscr{X}$ locally of finite type over an affine regular, noetherian scheme of dimension ≤ 1, $U.$ is given by a hypercovering of $\mathscr{X}$ by schemes, Λ is a Gorenstein local ring of dimension 0 and characteristic l invertible on $\mathscr{X}$, and $\mathscr{C}$ is the category of constructible Λ-modules. In this case the category $D_c(\mathscr{X}_{\text{lis-et}}, \Lambda)$ is compactly generated. Indeed a set of generators is given by sheaves $j_!\Lambda[i]$ for $i \in \mathbf{Z}$ and $j : U \to \mathscr{X}$ an object of the lisse-étale site of $\mathscr{X}$.

There is also a natural functor

$$D_{\mathscr{C}}(T, \Lambda) \to (\text{cartesian sections of } [n] \mapsto D_{\mathscr{C}}(U_n, \Lambda) \text{ over } \Delta^+). \tag{2.3.i}$$

2.3.3. *Theorem.* — *Let* $[n] \mapsto K_n \in D_{\mathscr{C}}(U_n, \Lambda)$ *be a cartesian section of* $[n] \mapsto D_{\mathscr{C}}(U_n, \Lambda)$ *over* Δ^+ *such that* $\mathscr{E}xt^i(K_0, K_0) = 0$ *for all* $i < 0$. *Then* (K_n) *is induced by a unique object* $K \in D_{\mathscr{C}}(T, \Lambda)$ *via the functor* (2.3.i).

The uniqueness is the easy part:

2.3.4. *Lemma.* — *Let* $K, L \in D(T, \Lambda)$ *and assume that* $\mathscr{E}xt^i(K, L) = 0$ *for* $i < 0$. *Then* $U \mapsto \mathrm{Hom}_{D(U,\Lambda)}(K|_U, L|_U)$ *is a sheaf.*

Proof. — Let $\mathscr{H}$ denote the complex $\mathscr{R}hom(K, L)$. By assumption the natural map $\mathscr{H} \to \tau_{\geq 0}\mathscr{H}$ is an isomorphism. It follows that $\mathrm{Hom}_{D(U,\Lambda)}(K|_U, L|_U)$ is equal to the value of $\mathscr{H}^0(\mathscr{H})$ on U which implies the lemma. □

The existence part is more delicate. Let $\mathscr{A}$ denote the fibered and cofibered category over $\widetilde{\Delta}$ whose fiber over $[n] \in \widetilde{\Delta}$ is the category of Λ-modules in U_n. For a morphism $\alpha : [n] \to [m]$, $F \in \mathscr{A}(n)$ and $G \in \mathscr{A}(m)$ we have

$$\mathrm{Hom}_\alpha(F, G) = \mathrm{Hom}_{\mathscr{A}(m)}(\alpha^*F, G) = \mathrm{Hom}_{\mathscr{A}(n)}(F, \alpha_*G).$$

We write $\mathscr{A}^+$ for the restriction of $\mathscr{A}$ to Δ^+.

Define a new category $\mathrm{tot}(\mathscr{A}^+)$ as follows:

- The objects of $\mathrm{tot}(\mathscr{A}^+)$ are collections of objects $(A^n)_{n\geq 0}$ with $A^n \in \mathscr{A}(n)$.
- For two objects (A^n) and (B^n) we define

  $$\mathrm{Hom}_{\mathrm{tot}(\mathscr{A}^+)}((A^n), (B^n)) := \prod_\alpha \mathrm{Hom}_\alpha(A^{s(\alpha)}, B^{b(\alpha)}),$$

  where the product is taken over all morphisms in Δ^+.
- If $f = (f_\alpha) \in \mathrm{Hom}((A^n), (B^n))$ and $g = (g_\alpha) \in \mathrm{Hom}((B^n), (C^n))$ are two morphisms then the composite is defined to be the collection of morphisms whose α component is defined to be

  $$(g \circ f)_\alpha := \sum_{\alpha=\beta\gamma} g_\beta f_\gamma$$

  where the sum is taken over all factorizations of α (note that this sum is finite).

The category $\text{tot}(\mathscr{A}^+)$ is an additive category.

Let (K, d) be a complex in $\text{tot}(\mathscr{A}^+)$ so for every degree n we are given a family of objects $(K^n)^m \in \mathscr{A}(m)$. Set

$$K^{n,m} := (K^{n+m})^n.$$

For $\alpha : [n] \to [m]$ in Δ^+ let $d(\alpha)$ denote the α-component of d so

$$d(\alpha) \in \text{Hom}_\alpha((K^p)^n, (K^{p+1})^m) = \text{Hom}_\alpha(K^{n,p-n}, K^{m,p+1-m})$$

or equivalently $d(\alpha)$ is a map $K^{n,p} \to K^{m,p+n-m+1}$. In particular, $d(\text{id}_{[n]})$ defines a map $K^{n,m} \to K^{n,m+1}$ and as explained in [2, 3.2.8], this map makes $K^{n,*}$ a complex. Furthermore for any α the map $d(\alpha)$ defines an α-map of complexes $K^{n,*} \to K^{m,*}$ of degree $n - m + 1$. The collection of complexes $K^{n,*}$ can also be defined as follows. For an integer p let L^pK denote the subcomplex with $(L^pK)^{n,m}$ equal to 0 if $n < p$ and $K^{n,m}$ otherwise. Note that for any $\alpha : [n] \to [m]$ which is not the identity map $[n] \to [n]$ the image of $d(\alpha)$ is contained in $L^{p+1}K$. Taking the associated graded of L we see that

$$\text{gr}^n_L K[n] = (K^{n,*}, d'')$$

where d'' denote the differential $(-1)^n d(\text{id}_{[n]})$. Note that the functor $(K, d) \mapsto K^{n,*}$ commutes with the formation of cones and with shifting of degrees.

As explained in [2, 3.2.8], a complex in $\text{tot}(\mathscr{A}^+)$ is completely characterized by the data of a complex $K^{n,*} \in C(\mathscr{A}^+)$ for every $[n] \in \Delta^+$ and for every morphism $\alpha : [n] \to [m]$ an α-morphism $d(\alpha) : K^{n,*} \to K^{m,*}$ of degree $n - m + 1$, such that $d(\text{id}_{[n]})$ is equal to $(-1)^n$ times the differential of $K^{n,*}$ and such that for every α we have

$$\sum_{\alpha=\beta\gamma} d(\beta)d(\gamma) = 0.$$

Via this dictionary, a morphism $f : K \to L$ in $C(\text{tot}(\mathscr{A}^+))$ is given by an α-map $f(\alpha) : K^{n,*} \to K^{m,*}$ of degree $n - m$ for every morphism $\alpha : [n] \to [m]$ in Δ^+ such that for any morphism α we have

$$\sum_{\alpha=\beta\gamma} d(\beta)f(\gamma) = \sum_{\alpha=\beta\gamma} f(\beta)d(\gamma).$$

Let $K(\text{tot}(\mathscr{A}^+))$ denote the category whose objects are complexes in $\text{tot}(\mathscr{A}^+)$ and whose morphisms are homotopy classes of morphisms of complexes. The category $K(\text{tot}(\mathscr{A}^+))$ is a triangulated category. Let $L \subset K(\text{tot}(\mathscr{A}^+))$ denote the full subcategory of objects K for which each $K^{n,*}$ is acyclic for all n. The category L is a localizing subcategory of $K(\mathrm{tot}(\mathscr{A}^+))$ in the sense of [3, 1.3], and hence the localized category $D(\mathrm{tot}(\mathscr{A}^+))$ exists. The category $D(\mathrm{tot}(\mathscr{A}^+))$ is obtained from $K(\mathrm{tot}(\mathscr{A}^+))$ by inverting quasi-isomorphisms. Recall that an object $K \in K(\mathrm{tot}(\mathscr{A}^+))$ is called L-*local* if for any object $X \in L$ we have $\mathrm{Hom}_{K(\mathrm{tot}(\mathscr{A}^+))}(X, K) = 0$. Note that the functor $K \mapsto K^{n,*}$ descends to a functor

$$D(\mathrm{tot}(\mathscr{A}^+)) \to D(U_n, \Lambda).$$

We define $D^+(\mathrm{tot}(\mathscr{A}^+)) \subset D(\mathrm{tot}(\mathscr{A}^+))$ to be the full subcategory of objects K for which there exists an integer N such that $\mathscr{H}^j(K^{n,*}) = 0$ for all n and all $j \leq N$.

Recall [3, 4.3], that a *localization* for an object $K \in K(\mathrm{tot}(\mathscr{A}^+))$ is a morphism $K \to I$ with I an L-local object such that for any L-local object Z the natural map

$$\text{(2.3.ii)} \qquad \mathrm{Hom}_{K(\mathrm{tot}(\mathscr{A}^+))}(I, Z) \to \mathrm{Hom}_{K(\mathrm{tot}(\mathscr{A}^+))}(K, Z)$$

is an isomorphism.

2.3.5. *Lemma. — A morphism* $K \to I$ *is a localization if* I *is* L-*local and for every* n *the map* $K^{n,*} \to I^{n,*}$ *is a quasi-isomorphism.*

Proof. — By [3, 2.9], the morphism (2.3.ii) can be identified with the natural map

$$\text{(2.3.iii)} \qquad \mathrm{Hom}_{D(\mathrm{tot}(\mathscr{A}^+))}(I, Z) \to \mathrm{Hom}_{D(\mathrm{tot}(\mathscr{A}^+))}(K, Z),$$

which is an isomorphism if $K \to I$ induces an isomorphism in $D(\mathrm{tot}(\mathscr{A}^+))$. □

2.3.6. *Proposition. — Let* $K \in C(\mathrm{tot}(\mathscr{A}^+))$ *be an object with each* $K^{n,*}$ *homotopically injective. Then* K *is* L-*local.*

Proof. — Let $X \in L$ be an object. We have to show that any morphism $f : X \to K$ in $C(\mathrm{tot}(\mathscr{A}^+))$ is homotopic to zero. Such a homotopy h is given by a collection of maps $h(\alpha)$ such that

$$f(\alpha) = -\sum_{\alpha=\beta\gamma} d(\beta)h(\gamma) + h(\beta)d(\gamma).$$

We usually write just h for $h(\mathrm{id}_{[n]})$.

We construct these maps $h(\alpha)$ by induction on $b(\alpha) - s(\alpha)$. For $s(\alpha) = b(\alpha)$ we choose the $h(\alpha)$ to be any homotopies between the maps $f(\mathrm{id}_{[n]})$ and the zero maps.

For the inductive step, it suffices to show that

$$\Psi(\alpha) = f(\alpha) + d(\alpha)h + hd(\alpha) + \sideset{}{'}\sum_{\alpha=\beta\gamma} d(\beta)h(\gamma) + h(\beta)d(\gamma)$$

commutes with the differentials d, where $\sum'_{\alpha=\beta\gamma}$ denotes the sum over all possible factorizations with β and γ not equal to the identity maps. For then $\Psi(\alpha)$ is homotopic to zero and we can take $h(\alpha)$ to be a homotopy between $\Psi(\alpha)$ and 0.

Define

$$A(\alpha) = \sideset{}{'}\sum_{\alpha=\beta\gamma} d(\beta)h(\gamma) + h(\gamma)d(\beta)$$

and

$$B(\alpha) = d(\alpha)h + hd(\alpha) + A(\alpha).$$

2.3.7. *Lemma.* — *One has the identity*

$$\sideset{}{'}\sum_{\alpha=\beta\gamma} A(\beta)d(\gamma) - d(\beta)A(\gamma) = \sideset{}{'}\sum_{\alpha=\beta\gamma} h(\beta)S(\gamma) - S(\beta)h(\gamma),$$

where $S(\alpha)$ *denotes* $\sum'_{\alpha=\beta\gamma} d(\beta)d(\gamma)$.

Proof

$$\begin{aligned}
&\sideset{}{'}\sum_{\alpha=\beta\gamma} A(\beta)d(\gamma) - d(\beta)A(\gamma)\\
&= \sideset{}{'}\sum_{\alpha=\epsilon\rho\gamma} d(\epsilon)h(\rho)d(\gamma) + h(\epsilon)d(\rho)d(\gamma) - d(\epsilon)h(\rho)d(\gamma) - d(\epsilon)d(\rho)h(\gamma)\\
&= \sideset{}{'}\sum_{\alpha=\beta\gamma} h(\beta)S(\gamma) - S(\beta)h(\gamma),
\end{aligned}$$

where $\sum'_{\alpha=\epsilon\rho\gamma}$ denotes the sum over all possible factorizations with ϵ, ρ, and γ not equal to the identity maps. □

2.3.8. *Lemma.* — *One has the identity*

$$\begin{aligned}
&\sideset{}{'}\sum_{\alpha=\beta\gamma} B(\beta)d(\gamma) - d(\beta)B(\gamma)\\
&= -h(d(\alpha)d + dd(\alpha)) + (d(\alpha)d + dd(\alpha))h + \sideset{}{'}\sum_{\alpha=\beta\gamma} h(\beta)S(\gamma) - S(\beta)h(\gamma).
\end{aligned}$$

Proof

$$
\begin{aligned}
&\sum_{\alpha=\beta\gamma}{}' \mathrm{B}(\beta)d(\gamma) - d(\beta)\mathrm{B}(\gamma) \\
&= \sum_{\alpha=\beta\gamma}{}' d(\beta)hd(\gamma) + hd(\beta)d(\gamma) + \mathrm{A}(\beta)d(\gamma) - d(\beta)d(\gamma)h \\
&\quad - d(\beta)hd(\gamma) - d(\beta)\mathrm{A}(\gamma) \\
&= -h(d(\alpha)d + dd(\alpha)) + (d(\alpha)d + dd(\alpha))h + \sum_{\alpha=\beta\gamma}{}' h(\beta)\mathrm{S}(\gamma) - \mathrm{S}(\beta)h(\gamma).
\end{aligned}
$$

□

We can now prove 2.3.6. We compute

$$
\begin{aligned}
&d\mathrm{A}(\alpha) - \mathrm{A}(\alpha)d \\
&= \sum_{\alpha=\beta\gamma}{}' dd(\beta)h(\gamma) + dh(\beta)d(\gamma) - d(\beta)h(\gamma)d - h(\beta)d(\gamma)d \\
&= \sum_{\alpha=\beta\gamma}{}' dd(\beta)h(\gamma) + (-f(\beta) - \mathrm{B}(\beta) - h(\beta)d)d(\gamma) \\
&\quad - d(\beta)(-f(\gamma) - \mathrm{B}(\gamma) - dh(\gamma)) - h(\beta)d(\gamma)d \\
&= \sum_{\alpha=\beta\gamma}{}' dd(\beta)h(\gamma) - f(\beta)d(\gamma) - \mathrm{B}(\beta)d(\gamma) \\
&\quad - h(\beta)dd(\gamma) + d(\beta)f(\gamma) + d(\beta)\mathrm{B}(\gamma) + d(\beta)dh(\gamma) - h(\beta)d(\gamma)d \\
&= \Big[\sum_{\alpha=\beta\gamma}{}' (-\mathrm{S}(\beta)h(\gamma)) + h(\beta)\mathrm{S}(\gamma) - f(\beta)d(\gamma) + d(\beta)f(\gamma)\Big] \\
&\quad + h(d(\alpha)d + dd(\alpha)) - (d(\alpha)d + dd(\alpha))h - \sum_{\alpha=\beta\gamma}{}' h(\beta)\mathrm{S}(\gamma) - \mathrm{S}(\beta)h(\gamma) \\
&= f(\alpha)d - df(\alpha) + fd(\alpha) - d(\alpha)f + h(d(\alpha)d + dd(\alpha)) \\
&\quad - (d(\alpha)d + dd(\alpha))h.
\end{aligned}
$$

So finally

$$
\begin{aligned}
&d\Psi(\alpha) - \Psi(\alpha)d \\
&= df(\alpha) + dd(\alpha)h + dhd(\alpha) + d\mathrm{A}(\alpha) - f(\alpha)d - d(\alpha)hd - hd(\alpha)d \\
&\quad - \mathrm{A}(\alpha)d \\
&= df(\alpha) + dd(\alpha)h + dhd(\alpha) - f(\alpha)d - d(\alpha)hd - hd(\alpha)d + f(\alpha)d \\
&\quad - df(\alpha) + fd(\alpha) - d(\alpha)f + hd(\alpha)d + hdd(\alpha) - d(\alpha)dh - dd(\alpha)h \\
&= 0.
\end{aligned}
$$

This completes the proof of 2.3.6. □

Let

$$\epsilon^* : \mathrm{C}(\mathscr{A}(\varnothing)) \to \mathrm{C}(\mathrm{tot}(\mathscr{A}^+))$$

be the functor sending a complex K to the object of $\mathrm{C}(\mathrm{tot}(\mathscr{A}^+))$ with $\epsilon^*\mathrm{K}^{n,*} = \mathrm{K}|_{\mathrm{U}_n}$ with maps $d(\mathrm{id}_{[n]})$ equal to $(-1)^n$ times the differential, for $\partial_i : [n] \to [n+1]$ the map $d(\partial_i)$ is the canonical map of complexes, and all other $d(\alpha)$'s are zero. The functor ϵ^* takes quasi-isomorphisms to quasi-isomorphisms and hence induces a functor

$$\epsilon^* : \mathrm{D}(\mathscr{A}(\varnothing)) \to \mathrm{D}(\mathrm{tot}(\mathscr{A}^+)).$$

2.3.9. *Lemma.* — *The functor* ϵ^* *has a right adjoint* $\mathrm{R}\epsilon_* : \mathrm{D}(\mathrm{tot}(\mathscr{A}^+)) \to \mathrm{D}(\mathscr{A}(\varnothing))$ *and* $\mathrm{R}\epsilon_*$ *is a triangulated functor.*

Proof. — We apply the adjoint functor theorem [22, 4.1]. By our assumptions the category $\mathrm{D}(\mathscr{A}(\varnothing))$ is compactly generated. Therefore it suffices to show that ϵ^* commutes with coproducts (direct sums) which is immediate. □

More concretely, the functor $\mathrm{R}\epsilon_*$ can be computed as follows. If K is L-local and there exists an integer N such that for every n we have $\mathrm{K}^{n,m} = 0$ for $m < \mathrm{N}$, then $\mathrm{R}\epsilon_*\mathrm{K}$ is represented by the complex with

$$(\epsilon_*\mathrm{K})^p = \bigoplus_{n+m=p} \epsilon_{n*}\mathrm{K}^{n,m}$$

with differential given by $\sum d(\alpha)$. This follows from Yoneda's lemma and the observation that for any $\mathrm{F} \in \mathrm{D}(\mathscr{A}(\varnothing))$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\mathscr{A}(\varnothing))}(\mathrm{F}, \mathrm{R}\epsilon_*\mathrm{K}) &= \mathrm{Hom}_{\mathrm{D}(\mathrm{tot}(\mathscr{A}^+))}(\epsilon^*\mathrm{F}, \mathrm{K}) \\
&= \mathrm{Hom}_{\mathrm{K}(\mathrm{tot}(\mathscr{A}^+))}(\epsilon^*\mathrm{F}, \mathrm{K}) \qquad \text{since K is L-local} \\
&= \mathrm{Hom}_{\mathrm{K}(\mathscr{A}(\varnothing))}(\mathrm{F}, \epsilon_*\mathrm{K}) \qquad \text{by [2, 3.2.12].}
\end{aligned}$$

2.3.10. *Lemma.* — *For any* $\mathrm{F} \in \mathrm{D}^+(\mathscr{A}(\varnothing))$ *the natural map* $\mathrm{F} \to \mathrm{R}\epsilon_*\epsilon^*\mathrm{F}$ *is an isomorphism.*

Proof. — Represent F by a complex of injectives. Then $\epsilon^*\mathrm{F}$ is L-local by 2.3.6. The result then follows from cohomological descent. □

Let $\mathrm{D}^+_{\mathrm{cart}}(\mathrm{tot}(\mathscr{A}^+) \subset \mathrm{D}^+(\mathrm{tot}(\mathscr{A}^+))$ denote the full subcategory of objects K such that for every n and inclusion $\partial_i : [n] \hookrightarrow [n+1]$ the map of complexes

$$\mathrm{K}^{n,*}|_{\mathrm{U}_{n+1}} \to \mathrm{K}^{n+1,*}$$

is a quasi-isomorphism.

2.3.11. *Proposition. — Let* $\mathrm{K} \in \mathrm{D}^+_{\mathrm{cart}}(\mathrm{tot}(\mathscr{A}^+))$ *be an object. Then* $\epsilon^*\mathrm{R}\epsilon_*\mathrm{K} \to \mathrm{K}$ *is an isomorphism. In particular,* $\mathrm{R}\epsilon_*$ *and* ϵ^* *induce an equivalence of categories between* $\mathrm{D}^+_{\mathrm{cart}}(\mathrm{tot}(\mathscr{A}^+))$ *and* $\mathrm{D}^+(\mathscr{A}(\varnothing))$.

Proof. — For any integer s and system $(\mathrm{K}^{n,*}, d(\alpha))$ defining an object of $\mathrm{C}(\mathrm{tot}(\mathscr{A}^+))$ we obtain a new object by $(\tau_{\leq s}\mathrm{K}^{n,*}, d(\alpha))$ since for any α which is not the identity morphism the map $d(\alpha)$ has degree ≤ 0. We therefore obtain a functor $\tau_{\leq s} : \mathrm{C}(\mathrm{tot}(\mathscr{A}^+)) \to \mathrm{C}(\mathrm{tot}(\mathscr{A}^+))$ which takes quasi-isomorphisms to quasi-isomorphisms and hence descends to a functor

$$\tau_{\leq s} : \mathrm{D}(\mathrm{tot}(\mathscr{A}^+)) \to \mathrm{D}(\mathrm{tot}(\mathscr{A}^+)).$$

Furthermore, there is a natural morphism of functors $\tau_{\leq s} \to \tau_{\leq s+1}$ and we have

$$\mathrm{K} \simeq \mathrm{hocolim}\, \tau_{\leq s}\mathrm{K}.$$

Note that the functor ϵ^* commutes with homotopy colimits since it commutes with direct sums. If we show the proposition for the $\tau_{\leq s}\mathrm{K}$ then we see that the natural map

$$\epsilon^*(\mathrm{hocolim}\, \mathrm{R}\epsilon_*\tau_{\leq s}\mathrm{K}) \simeq \mathrm{hocolim}\, \epsilon^*\mathrm{R}\epsilon_*\tau_{\leq s}\mathrm{K} \to \mathrm{hocolim}\, \tau_{\leq s}\mathrm{K} \simeq \mathrm{K}$$

is an isomorphism. In particular K is in the essential image of ϵ^*. Write $\mathrm{K} = \epsilon^*\mathrm{F}$. Then by 2.3.10 $\mathrm{R}\epsilon_*\mathrm{K} \simeq \mathrm{F}$ whence $\epsilon^*\mathrm{R}\epsilon_*\mathrm{K} \to \mathrm{K}$ is an isomorphism.

It therefore suffices to prove the proposition for K bounded above. Considering the distinguished triangles associated to the truncations $\tau_{\leq s}\mathrm{K}$ we further reduce to the case when K is concentrated in just a single degree. In this case, K is obtained by pullback from an object of $\mathscr{A}(\varnothing)$ and the proposition again follows from 2.3.10. □

For an object $\mathrm{K} \in \mathrm{K}(\mathrm{tot}(\mathscr{A}^+))$, we define $\tau_{\geq s}\mathrm{K}$ to be the cone of the natural map $\tau_{\leq s-1}\mathrm{K} \to \mathrm{K}$.

Observe that the category $\mathrm{K}(\mathrm{tot}(\mathscr{A}^+))$ has products and therefore also homotopy limits. Let $\mathrm{K} \in \mathrm{K}_{\mathscr{C}}(\mathrm{tot}(\mathscr{A}^+))$ be an object. By 2.3.11, for each s we can find a bounded below complex of injectives $\mathrm{I}_s \in \mathrm{C}(\mathscr{A}(\varnothing))$ and a quasi-isomorphism $\sigma_s : \tau_{\geq s}\mathrm{K} \to \epsilon^*\mathrm{I}_s$. Since $\epsilon^*\mathrm{I}_s$ is L-local and $\epsilon^* : \mathrm{D}^+(\mathscr{A}(\varnothing)) \to \mathrm{D}(\mathrm{tot}(\mathscr{A}^+))$ is fully faithful by 2.3.11, the maps $\tau_{\geq s-1}\mathrm{K} \to \tau_{\geq s}\mathrm{K}$ induce a unique morphism $t_s : \mathrm{I}_{s-1} \to \mathrm{I}_s$ in $\mathrm{K}(\mathscr{A}(\varnothing))$ such that the diagrams

$$\begin{array}{ccc} \tau_{\geq s-1}\mathrm{K} & \longrightarrow & \tau_{\geq s}\mathrm{K} \\ \downarrow{\scriptstyle \sigma_{s-1}} & & \downarrow{\scriptstyle \sigma_s} \\ \epsilon^*\mathrm{I}_{s-1} & \xrightarrow{t_s} & \epsilon^*\mathrm{I}_s \end{array}$$

commutes in $\mathrm{K}(\mathrm{tot}(\mathscr{A}^+))$.

2.3.12. *Proposition. — The natural map* $\mathrm{K} \to \mathrm{holim}\, \epsilon^*\mathrm{I}_s$ *is a quasi-isomorphism.*

Proof. — It suffices to show that for all n the map $K^{n,*} \to \operatorname{holim} \epsilon_n^* I_s$ is a quasi-isomorphism, where $\epsilon_n : U_n \to T$ is the projection. Let $\mathscr{S}_n$ be a site inducing U_n as in 2.3.1. We show that for any integer i the map of presheaves on the subcategory of $\mathscr{S}_n$ satisfying the finiteness assumption in 2.3.1(i)

$$(V \to U_n) \mapsto H^i(V, K^{n,*}) \to H^i(V, \operatorname{holim} \epsilon_n^* I_s)$$

is an isomorphism. For this note that for every s there is a distinguished triangle

$$\mathscr{H}^s(K^{n,*})[-s] \to \epsilon_n^* I_s \to \epsilon_n^* I_{s-1}$$

and hence by the Assumption 2.3.1(i) the map

$$H^i(V, \epsilon_n^* I_s) \to H^i(V, \epsilon_n^* I_{s-1}) \qquad (\mathbf{2.3.iv})$$

is an isomorphism for $s < i - n_0$. Since each $\epsilon_n^* I_s$ is a complex of injectives, the complex $\prod_s \epsilon_n^* I_s$ is also a complex of injectives. Therefore

$$H^i(V, \prod_s \epsilon_n^* I_s) = H^i(\prod_s \epsilon_n^* I_s(V)) = \prod_s H^i(\epsilon_n^* I_s(V)).$$

It follows that there is a canonical long exact sequence

$$\longrightarrow \prod_s H^i(\epsilon_n^* I_s(V)) \xrightarrow{\text{1-shift}} \prod_s H^i(\epsilon_n^* I_s(V)) \longrightarrow H^i(V, \operatorname{holim} \epsilon_n^* I_s) \longrightarrow.$$

From this and the fact that the maps (2.3.iv) are isomorphisms for s sufficiently negative it follows that the cohomology group $H^i(V, \operatorname{holim} \epsilon_n^* I_s)$ is isomorphic to $H^i(V, K^{n,*})$ via the canonical map. Passing to the associated sheaves we obtain the proposition. □

2.3.13. *Corollary.* — *Every object* $K \in D_{\mathscr{C}}(\operatorname{tot}(\mathscr{A}^+))$ *is in the essential image of the functor*

$$\epsilon^* : D_{\mathscr{C}}(\mathscr{A}(\varnothing)) \to D_{\mathscr{C}}(\operatorname{tot}(\mathscr{A}^+)).$$

Proof. — Since ϵ^* also commutes with products and hence also homotopy limits we find that $K \simeq \epsilon^*(\operatorname{holim} I_s)$ in $D_{\mathscr{C}}(\operatorname{tot}(\mathscr{A}^+))$ (note that $\mathscr{H}^i(\operatorname{holim} I_s)$ is in $\mathscr{C}$ since this can be checked after applying ϵ^*). □

2.3.14. *Lemma.* — *Let* $[n] \mapsto K^n$ *be a cartesian section of* $[n] \mapsto D(U_n, \Lambda)$ *such that* $\mathscr{E}xt^i(K^n, K^n) = 0$ *for all* $i < 0$. *Then* (K^n) *is induced by an object of* $D(\operatorname{tot}(\mathscr{A}^+))$.

Proof. — Represent each K^n by a homotopically injective complex (denoted by the same letter) in $C(U_n, \Lambda)$ for every n. For each morphism $\partial_i : [n] \to [n+1]$ (the unique morphism whose image does not contain i) choose a ∂_i-map of complexes $\partial_i^* : K^n \to K^{n+1}$ inducing the given map in $D(U_{n+1}, \Lambda)$ by the strictly simplicial structure. The proof then proceeds by the same argument used to prove [2, 3.2.9]. □

Combining this with 2.3.13 we obtain 2.3.3. □

3. Dualizing complex

3.1. *Dualizing complexes on algebraic spaces.* — Let $w : W \to S$ be a separated[5] morphism of finite type with W an algebraic space. We'll define Ω_w by glueing as follows. By the comparison lemma [13, III.4.1], the étale topos $W_{\text{ét}}$ can be defined using the site Étale(W) whose objects are étale morphisms $A : U \to W$ where $a : U \to S$ is affine of finite type. The localized topos $W_{\text{ét}|U}$ coincides with $U_{\text{ét}}$. Notice that this is not true for the corresponding lisse-étale topos. This fact will cause some difficulties below.

Unless otherwise explicitly stated, we will ring the various étale or lisse-étale topos which will be appear by the constant Gorenstein ring Λ of dimension 0 of the introduction.

Let Ω denote the dualizing complex of S, and let $\alpha : U \to S$ denote the structural morphism. We define

(**3.1.i**) $$\Omega_A = \alpha^! \Omega \in D(U_{\text{ét}}, \Lambda) = D(W_{\text{ét}|U})$$

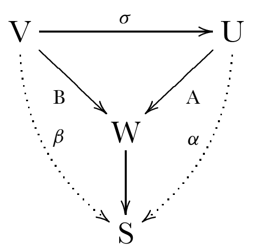
which is the (relative) dualizing complex of U, and therefore one gets by biduality [4, «Th. finitude» 4.3],

(**3.1.ii**) $$\mathscr{R}hom(\Omega_A, \Omega_A) = \Lambda$$

implying at once

(**3.1.iii**) $$\mathscr{E}xt^i_{W_{\text{ét}|U}}(\Omega_A, \Omega_A) = 0 \text{ if } i < 0.$$

We want to apply the glueing Theorem 2.3.3. Let us therefore consider a diagram



with a commutative triangle and $A, B \in$ Étale(W). Since A and B are étale, the morphism σ is also étale so there is a functorial isomorphism

$$\sigma^* \Omega_A = \Omega_B.$$

Therefore $(\Omega_A)_{A \in \text{Étale}(W)}$ defines locally an object Ω_w of D(W) with vanishing negative $\mathscr{E}xt$'s (recall that $w : W \to S$ is the structural morphism). By 2.3.3, we get

3.1.1. *Proposition.* — *There exists a unique* $\Omega_w \in D(W_{\text{ét}})$ *such that* $\Omega_{w|U} = \Omega_A$.

[5] Probably one can assume only that w quasi-separated, cf. [13, XVII.7]; but we do not need this more general version.

We need functoriality for smooth morphisms.

3.1.2. *Lemma. — If* $f : W_1 \to W_2$ *is a smooth* S*-morphism of relative dimension* d *between algebraic space separated and of finite type over* S *with dualizing complexes* Ω_1, Ω_2, *then*

$$f^*\Omega_2 = \Omega_1\langle -d\rangle.$$

Proof. — Start with $U_2 \to W_2$ étale and surjective with U_2 affine say. Then, $\tilde{W}_1 = W_1 \times_{U_2} W_2$ is an algebraic space separated and of finite type over S. Let $U_1 \to \tilde{W}_1$ be a surjective étale morphism with U_1 affine and let $g : U_1 \to U_2$ be the composition $U_1 \to \tilde{W}_1 \to U_2$. It is a smooth morphism of relative dimension d between affine schemes of finite type from which follows the formula $g^!(-) = g^*(-)\langle d\rangle$. Therefore, the pull-backs of $L_1 = \Omega_1\langle -d\rangle$ and $f^*\Omega_2$ on U_1 are the same, namely Ω_{U_1}. One deduces that these complexes coincide on the covering sieve $W_{1\text{ét}|U_1}$ and therefore coincide by 2.3.4 (because the relevant negative $\mathscr{E}xt^i$'s vanish). □

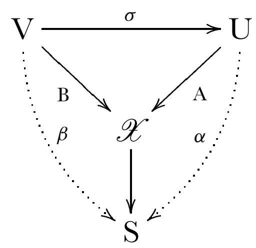
3.2. *Étale dualizing data.* — Let $\mathscr{X} \to S$ be an algebraic S-stack locally of finite type. Let $A : U \to \mathscr{X}$ in Lisse-Et($\mathscr{X}$) and $\alpha : U \to S$ the composition $U \to \mathscr{X} \to S$. We define

(**3.2.i**) $$K_A = \Omega_\alpha\langle -d_A\rangle \in D_c(U_{\text{ét}}, \Lambda)$$

where d_A is the relative dimension of A (which is locally constant). Up to shift and Tate torsion, K_A is the (relative) dualizing complex of U and therefore one gets by biduality

(**3.2.ii**) $$\mathscr{R}hom(K_A, K_A) = \Lambda \text{ and } \mathscr{E}xt^i_{U_{\text{ét}}}(K_A, K_A) = 0 \text{ if } i < 0.$$

We need again a functoriality property of K_A. Let us consider a diagram



with a 2-commutative triangle and $A, B \in$ Lisse-Et($\mathscr{X}$).

3.2.1. *Lemma. — There is a functorial identification*

$$\sigma^*K_A = K_B.$$

Proof. — Let $W = U \times_{\mathscr{X}} V$ which is an algebraic space. One has a commutative diagram with cartesian square

$$\begin{array}{ccc} W & \xrightarrow{b} & U \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle a} & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle A} \\ V & \xrightarrow[B]{} & \mathscr{X}. \end{array}$$

In particular, a, b are smooth and separated like A, B. One deduces a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{b} & U \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle a} & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle \alpha} \\ V & \xrightarrow[\beta]{} & S. \end{array}$$

I claim that

$$(\mathbf{3.2.iii}) \qquad b^*K_A = a^*K_B = K_w$$

where w denotes the morphism $W \to \mathscr{X}$.

Indeed, a, b being smooth of relative dimensions d_A, d_B, one has 3.1.2

$$b^*K_A = b^*\Omega_\alpha\langle -d_A\rangle = \Omega_\gamma\langle -d_A - d_B\rangle$$

and analogously

$$a^*K_B = a^*\Omega_\beta\langle -d_B\rangle = \Omega_\gamma\langle -d_B - d_A\rangle,$$

where $\gamma : W \to S$ is the structure morphism. Pulling back by s gives the result. □

3.2.2. *Remark.* — Because all S-schemes of finite type satisfy $\mathrm{cd}_\Lambda(X) < \infty$, we know that K_X is not only of finite quasi-injective dimension but of finite injective dimension [14, I.1.5]. By construction this implies that K_A is of finite injective dimension for A as above.

3.3. *Lisse-étale dualizing data.* — In order to define $\Omega_{\mathscr{X}} \in D(\mathscr{X}_{\text{lis-ét}})$ by glueing, we need glueing data $\kappa_A \in D(\mathscr{X}_{\text{lis-ét}|U})$ for objects $A : U \to \mathscr{X}$ of Lisse-Et($\mathscr{X}$). The inclusion

$$\text{Étale}(U) \hookrightarrow \text{Lisse-Et}(\mathscr{X})_{|U}$$

induces a continuous morphism of sites. Since finite inverse limits exist in Étale(U) and this morphism of sites preserves such limits, it defines by [13, IV.4.9.2], a morphism of topos (we abuse notation slightly and omit the dependence on A from the notation)

$$\epsilon : \mathscr{X}_{\text{lis-ét}|U} \to U_{\text{ét}}.$$

3.3.1. Let us describe more explicitely the morphism ϵ. Let Lisse-Et$(\mathscr{X})_{|U}$ denote the category of morphisms $V \to U$ in Lisse-Et$(\mathscr{X})$. The category Lisse-Et$(\mathscr{X})_{|U}$ has a Grothendieck topology induced by the topology on the site Lisse-Et$(\mathscr{X})$, and the resulting topos is canonicallly isomorphic to the localized topos $\mathscr{X}_{\text{lis-ét}|U}$. Note that there is a natural inclusion Lisse-Et(U) $\hookrightarrow$ Lisse-Et$(\mathscr{X})_{|U}$ but this is not an equivalence of categories since for an object $(V \to U) \in$ Lisse-Et$(\mathscr{X})_{|U}$ the morphism $V \to U$ need not be smooth. It follows that an element of $\mathscr{X}_{\text{lis-ét}|U}$ is equivalent to giving for every U-scheme of finite type $V \to U$, such that the composite $V \to U \to \mathscr{X}$ is smooth, a sheaf $\mathscr{F}_V \in V_{\text{ét}}$ together with morphisms $f^{-1}\mathscr{F}_V \to \mathscr{F}_{V'}$ for U-morphisms $f : V' \to V$. Furthermore, these morphisms satisfy the usual compatibility with compositions. Viewing $\mathscr{X}_{\text{lis-ét}|U}$ in this way, the functor ϵ^{-1} maps $\mathscr{F}$ on $U_{\text{ét}}$ to $\mathscr{F}_V = \pi^{-1}\mathscr{F} \in V_{\text{ét}}$ where $\pi : V \to U \in$ Lisse-Et$(\mathscr{X})_{|U}$. For a sheaf $F \in \mathscr{X}_{\text{lis-ét}|U}$ corresponding to a collection of sheaves $\mathscr{F}_V$, the sheaf ϵ_*F is simply the sheaf $\mathscr{F}_U$.

In particular, the functor ϵ_* is exact and therefore $H^*(U, F) = H^*(U_{\text{ét}}, F_U)$ for any sheaf of Λ-modules of $\mathscr{X}$.

3.3.2. A morphism $f : U \to V$ of Lisse-Et$(\mathscr{X})$ induces a diagram

$$\begin{array}{ccc} \mathscr{X}_{\text{lis-ét}}|_U & \xrightarrow{\epsilon} & U_{\text{ét}} \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathscr{X}_{\text{lis-ét}}|_V & \xrightarrow{\epsilon} & V_{\text{ét}} \end{array} \tag{3.3.i}$$

where $\mathscr{X}_{\text{lis-ét}|U} \to \mathscr{X}_{\text{lis-ét}|V}$ is the localization morphism [13, IV.5.5.2], which we still denote by f slightly abusively. For a sheaf $\mathscr{F} \in V_{\text{ét}}$, the pullback $f^{-1}\epsilon^{-1}\mathscr{F}$ is the sheaf corresponding to the system which to any $p : U' \to U$ associates $p^{-1}f^{-1}\mathscr{F}$. In particular, $f^{-1} \circ \epsilon^{-1} = \epsilon^{-1} \circ f^{-1}$ which implies that (3.3.i) is a commutative diagram of topos. We define

$$\kappa_A = \epsilon^* K_A \in D(\mathscr{X}_{\text{lis-ét}|U}). \tag{3.3.ii}$$

By the preceding discussion, if

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle A}\searrow \quad \swarrow{\scriptstyle B} & \\ & \mathscr{X} & \end{array}$$

is a morphism in Lisse-Et$(\mathscr{X})$, we get

$$f^*\kappa_B = \kappa_A$$

showing that the family (κ_A) defines locally an object of $D(\mathscr{X}_{\text{lis-ét}})$.

3.4. *Glueing the local dualizing data.* — Let $A \in \text{Lisse-Et}(\mathcal{X})$ and $\epsilon : \mathcal{X}_{\text{lis-ét}|U} \to U_{\text{ét}}$ be as above. We need first the vanishing of $\mathcal{E}xt^i(\kappa_A, \kappa_A)$, $i < 0$.

3.4.1. *Lemma.* — *Let* $\mathcal{F}, \mathcal{G} \in D(U_{\text{ét}})$. *One has*

(i) $\text{Ext}^i(\epsilon^*\mathcal{F}, \epsilon^*\mathcal{G}) = \text{Ext}^i(\mathcal{F}, \mathcal{G})$.

(ii) The étale sheaf $\mathcal{E}xt^i(\epsilon^*\mathcal{F}, \epsilon^*\mathcal{G})_U$ *on* $U_{\text{ét}}$ *is* $\mathcal{E}xt^i_{U_{\text{ét}}}(\mathcal{F}, \mathcal{G})$.

Proof. — Since ϵ_* is exact and for any sheaf $F \in U_{\text{ét}}$ one has $F = \epsilon_*\epsilon^*F$, the adjunction map $F \to R\epsilon_*\epsilon^*F$ is an isomorphism for any $F \in D(U_{\text{ét}})$. By trivial duality, one gets

$$\epsilon_* \mathcal{R}hom(\epsilon^*\mathcal{F}, \epsilon^*\mathcal{G}) = \mathcal{R}hom(\mathcal{F}, \epsilon_*\epsilon^*\mathcal{G}) = \mathcal{R}hom(\mathcal{F}, \mathcal{G}).$$

Taking $\mathcal{H}^i R\Gamma$ gives (i).

By construction, $\mathcal{E}xt^i(\epsilon^*\mathcal{F}, \epsilon^*\mathcal{F})_U$ is the sheaf associated to the presheaf on $U_{\text{ét}}$ which to any étale morphism $\pi : V \to U$ associates $\text{Ext}^i(\pi^*\epsilon^*\mathcal{F}, \pi^*\epsilon^*\mathcal{G})$ where π^* is the the pull-back functor associated to the localization morphism

$$(\mathcal{X}_{\text{lis-ét}}|_U)_{|V} = \mathcal{X}_{\text{lis-ét}|V} \to \mathcal{X}_{\text{lis-ét}|U}$$

[13, V.6.1]. By the commutativity of the Diagram (3.3.i), one has $\pi^*\epsilon^* = \epsilon^*\pi^*$. Therefore

$$\text{Ext}^i(\pi^*\epsilon^*\mathcal{F}, \pi^*\epsilon^*\mathcal{G}) = \text{Ext}^i(\epsilon^*\pi^*\mathcal{F}, \epsilon^*\pi^*\mathcal{G}) = \text{Ext}^i_{V_{\text{ét}}}(\pi^*\mathcal{F}, \pi^*\mathcal{G}),$$

the last equality is by (i). Since $\mathcal{E}xt^i_{U_{\text{ét}}}(\mathcal{F}, \mathcal{G})$ is also the sheaf associated to this presheaf we obtain (ii). □

Using (3.2.ii), one obtains

3.4.2. *Corollary.* — *One has* $\mathcal{R}hom(\kappa_A, \kappa_A) = \Lambda$ *and therefore* $\mathcal{E}xt^i(\kappa_A, \kappa_A) = 0$ *if* $i < 0$.

Now choose a presentation $p : X \to \mathcal{X}$ with X a scheme. The object $\kappa_p \in D(\mathcal{X}_{\text{lis-ét}}|_X)$ then comes with descent data to $\mathcal{X}_{\text{lis-ét}}$ which by 2.3.3 and the above discusison is effective. We therefore obtain:

3.4.3. *Proposition.* — *There exists* $\Omega_{\mathcal{X}}(p) \in D^{(b)}(\mathcal{X}_{\text{lis-ét}})$ *inducing* κ_A *for all* $A \in \text{Lisse-Et}(\mathcal{X})_{|X}$. *It is well defined up to unique isomorphism.*

The independence of the presentation is straightforward and is left to the reader:

3.4.4. *Lemma.* — *Let* $p_i : X_i \to \mathscr{X}, i = 1, 2$ *two presentations as above. There exists a canonical, functorial isomorphism* $\Omega_{\mathscr{X}}(p_1) \xrightarrow{\sim} \Omega_{\mathscr{X}}(p_2)$.

3.4.5. *Definition.* — *The* dualizing complex *of* $\mathscr{X}$ *is the "essential" value* $\Omega_{\mathscr{X}} \in D^{(b)}(\mathscr{X}_{\text{lis-ét}})$ *of* $\Omega_{\mathscr{X}}(p)$, *where* p *runs over presentations of* $\mathscr{X}$. *It is well defined up to canonical functorial isomorphism and is characterized by* $\Omega_{\mathscr{X}|U} = \epsilon^* K_A$ *for any* $A : U \to \mathscr{X}$ *in* Lisse-Et($\mathscr{X}$).

3.5. *Biduality.* — For A, B any abelian complexes of some topos, there is a biduality *morphism*

$$A \to \mathscr{R}hom(\mathscr{R}hom(A, B), B) \tag{3.5.i}$$

(replace B by some homotopically injective complex isomorphic to it in the derived catgory).

In general, it is certainly not an isomorphism.

3.5.1. *Lemma.* — *Let* $u : U \to S$ *be a separated* S-*scheme (or algebraic space) of finite type and* $A \in D_c(U_{\text{ét}}, \Lambda)$. *Then the biduality morphism*

$$A \to \mathscr{R}hom(\mathscr{R}hom(A, K_U), K_U)$$

is an isomorphism (where K_U *is – up to shift and twist – the dualizing complex of* $U_{\text{ét}}$*).*

Proof. — If A is moreover bounded, it is the usual theorem of [4]. Let us denote by τ_n the two-sides truncation functor

$$\tau_{\geq -n}\tau_{\leq n}.$$

We know that K_U is a dualizing complex [14, exp. I], and is of *finite injective dimension* (3.2.2); the homology in degree n of the biduality morphism $A \to DD(A)$ is therefore the same as the homology in degree n of the biduality morphism $\tau_m A \to DD(\tau_m A)$ for m large enough and the lemma follows. □

We will be interested in a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & U \\ & {\scriptstyle B}\searrow \qquad \swarrow{\scriptstyle A} & \\ & \mathscr{X} & \end{array}$$

as above.

3.5.2. *Lemma. — Let* $\mathscr{F} \in D_c(\mathscr{X}_{\text{lis-ét}})$ *and let* $\mathscr{F}_U \in D_c(U_{\text{ét}})$ *be the object obtained by restriction.*

(i) One has $f^* \mathscr{R}hom(\mathscr{F}_U, K_A) = \mathscr{R}hom(f^*\mathscr{F}_U, f^*K_A) = \mathscr{R}hom(f^*\mathscr{F}_U, K_B)$.

(ii) There exist integers $a < b$ *(independent of* $\mathscr{F}$*) such that for every* $i \in \mathbf{Z}$

$$\mathscr{E}xt^i(\mathscr{F}_U, K_A) = \mathscr{E}xt^i(\tau_{\geq i-b}\tau_{\leq i-a}\mathscr{F}_U, K_A).$$

Moreover, $\mathscr{R}hom(\mathscr{F}_U, K_A)$ *is constructible.*

Proof. — Let's prove (i). By 3.2.1, one has $f^*K_A = K_B$, therefore one has a morphism

$$f^* \mathscr{R}hom(\mathscr{F}_U, K_A) \to \mathscr{R}hom(f^*\mathscr{F}_U, K_B).$$

To prove that it is an isomorphism, consider first the case when f is smooth. Because both K_A and K_B are of finite injective dimension (3.2.2), one can assume that $\mathscr{F}$ is bounded where it is obviously true by reduction to $\mathscr{F}$ the constant sheaf (or use [14, I.7.2]). Therefore the result holds when f is smooth.

From the case of a smooth morphism, one reduces the proof in general to the case when $\mathscr{X}$ is a scheme. Let $\mathscr{F}_{\mathscr{X}} \in D_c(\mathscr{X}_{\text{ét}})$ denote the complex obtained by restricting $\mathscr{F}$. By the smooth case already considered, we have

$$\begin{aligned} f^* \mathscr{R}hom(\mathscr{F}_U, K_A) &\simeq f^*A^* \mathscr{R}hom(\mathscr{F}_{\mathscr{X}}, K_{\mathscr{X}}) \\ &= B^* \mathscr{R}hom(\mathscr{F}_{\mathscr{X}}, K_{\mathscr{X}}) \\ &\simeq \mathscr{R}hom(B^*\mathscr{F}_{\mathscr{X}}, B^*K_{\mathscr{X}}) \\ &\simeq \mathscr{R}hom(f^*\mathscr{F}_U, f^*K_A). \end{aligned}$$

The existence of $a < b$ as in (ii) follows immediately from the fact that K_A has finite injective dimension 3.2.2. To verify that $\mathscr{R}hom(\mathscr{F}_U, K_A)$ is constructible it suffices by the first statement in (ii) to consider the case when $\mathscr{F}$ is bounded in which case the result follows from [14, I.7.1]. □

3.5.3. *Lemma. — Let* $\mathscr{F} \in D_c(\mathscr{X}_{\text{lis-ét}})$. *Then,*

$$\epsilon^* \mathscr{R}hom_{U_{\text{ét}}}(\mathscr{F}_U, K_A) = \mathscr{R}hom(\mathscr{F}, \Omega_{\mathscr{X}})_{|U}$$

where $\mathscr{F}_U = \epsilon_*\mathscr{F}_{|U}$ *is the restriction of* $\mathscr{F}$ *to* Étale(U).

Proof. — By definition of constructibility, $\mathscr{H}^i(\mathscr{F})$ are cartesian sheaves. In other words, ϵ_* being exact, the adjunction morphism

$$\epsilon^*\mathscr{F}_U = \epsilon^*\epsilon_*\mathscr{F}_{|U} \to \mathscr{F}_{|U}$$

is an isomorphism. We therefore have

$$\begin{aligned}\mathscr{R}hom(\mathscr{F}, \Omega)_{|U} &= \mathscr{R}hom(\mathscr{F}_{|U}, \Omega_{|U}) \\ &= \mathscr{R}hom(\epsilon^* \mathscr{F}_U, \epsilon^* K_A).\end{aligned}$$

Therefore, we get a morphism

$$\epsilon^* \mathscr{R}hom_{U_{ét}}(\mathscr{F}_U, K_A) \to \mathscr{R}hom(\epsilon^* \mathscr{F}_U, \epsilon^* K_A) = \mathscr{R}hom(\mathscr{F}, \Omega_{\mathscr{X}})_{|U}.$$

By 3.4.1, one has for any object $f : V \to U$ in Lisse-Et$(\mathscr{X})|_U$

$$\mathscr{E}xt^i(\epsilon^* \mathscr{F}_U, \epsilon^* K_A)_V = \mathscr{E}xt^i_{V_{ét}}(f^* \mathscr{F}_U, f^* K_A).$$

But, one has

$$\mathscr{H}^i(\epsilon^* \mathscr{R}hom_{U_{ét}}(\mathscr{F}_U, K_A))_V = f^* \mathscr{E}xt^i_{U_{ét}}(\mathscr{F}_U, K_A)$$

and the lemma follows from 3.5.2. □

One gets immediately (cf. [14, I.1.4])

3.5.4. *Corollary.* — $\Omega_{\mathscr{X}}$ *is of locally finite quasi-injective dimension.*

3.5.5. *Remark.* — It seems over-optimistic to think that $\Omega_{\mathscr{X}}$ would be of finite injective dimension even if $\mathscr{X}$ is a scheme.

3.5.6. *Lemma.* — *If* $A \in D_c(\mathscr{X})$, *then* $\mathscr{R}hom(A, \Omega_{\mathscr{X}}) \in D_c(\mathscr{X})$.

Proof. — Immediate consequence of 3.5.2 and 3.5.3. □

3.5.7. *Corollary.* — *The (contravariant) functor*

$$D_{\mathscr{X}} : \begin{cases} D_c(\mathscr{X}) \to & D_c(\mathscr{X}) \\ \quad \mathscr{F} & \mapsto \mathscr{R}hom(\mathscr{F}, \Omega_{\mathscr{X}}) \end{cases}$$

is an involution. More precisely, the morphism

$$\iota : \mathrm{Id} \to D_{\mathscr{X}} \circ D_{\mathscr{X}}$$

induced by (3.5.i) *is an isomorphism.*

Proof. — We have to prove that the cone C of the biduality morphism is zero in the derived category, that is to say

$$C_U = \epsilon_* C_{|U} = 0 \text{ in } D_c(U_{ét}).$$

But we have

$$\begin{aligned}\epsilon_*(\mathscr{R}hom(\mathscr{R}hom(\mathscr{F},\Omega_{\mathscr{X}}),\Omega_{\mathscr{X}}))_{|\mathrm{U}} &= \epsilon_*\,\mathscr{R}hom(\mathscr{R}hom(\mathscr{F},\Omega_{\mathscr{X}})_{|\mathrm{U}},\Omega_{\mathscr{X}|\mathrm{U}})\\ &\overset{3.5.3}{=} \epsilon_*\,\mathscr{R}hom(\epsilon^*\,\mathscr{R}hom(\mathscr{F}_{\mathrm{U}},\mathrm{K}_{\mathrm{A}}),\Omega_{\mathscr{X}|\mathrm{U}})\\ &= \mathscr{R}hom(\mathscr{R}hom(\mathscr{F}_{\mathrm{U}},\mathrm{K}_{\mathrm{A}}),\epsilon_*\epsilon^*\mathrm{K}_{\mathrm{A}})\\ &= \mathscr{R}hom(\mathscr{R}hom(\mathscr{F}_{\mathrm{U}},\mathrm{K}_{\mathrm{A}}),\mathrm{K}_{\mathrm{A}})\\ &\overset{3.5.1}{=} \mathscr{F}_{\mathrm{U}},\end{aligned}$$

where the third equality is by trivial duality. □

3.5.8. *Remark.* — Verdier duality $\mathrm{D}_{\mathscr{X}}$ identifies $\mathrm{D}_c^{(a)}$ and $\mathrm{D}_c^{(-a)}$ with $a = \varnothing, \pm, b$ and the usual conventions $-\varnothing = \varnothing$ and $-b = b$.

3.5.9. *Proposition.* — *One has a canonical (bifunctorial) morphism*

$$\mathscr{R}hom(\mathrm{A},\mathrm{B}) = \mathscr{R}hom(\mathrm{D}(\mathrm{B}),\mathrm{D}(\mathrm{A}))$$

for all $\mathrm{A},\mathrm{B} \in \mathrm{D}_c(\mathscr{X})$.

Proof. — By [16, 18.6.9], we have canonical identifications for any three complexes $\mathrm{A},\mathrm{B},\mathrm{C} \in \mathrm{D}_c(\mathscr{X}_{\text{lis-ét}})$

(**3.5.ii**) $$\mathscr{R}hom(\mathrm{A},\mathscr{R}hom(\mathrm{B},\mathrm{C})) = \mathscr{R}hom(\mathrm{A}\overset{\mathbf{L}}{\otimes}\mathrm{B},\mathrm{C}) = \mathscr{R}hom(\mathrm{B},\mathscr{R}hom(\mathrm{A},\mathrm{C})).$$

One gets then

$$\begin{aligned}\mathscr{R}hom(\mathrm{D}(\mathrm{B}),\mathrm{D}(\mathrm{A})) &= \mathscr{R}hom(\mathrm{D}(\mathrm{B}),\mathscr{R}hom(\mathrm{A},\Omega_{\mathscr{X}}))\\ &\overset{(3.5.\mathrm{ii})}{=} \mathscr{R}hom(\mathrm{A},\mathrm{D}\circ\mathrm{D}(\mathrm{B}))\\ &= \mathscr{R}hom(\mathrm{A},\mathrm{B}).\end{aligned}$$

□

4. The 6 operations

4.1. *The functor* $\mathrm{R}f_*$. — Let $f : \mathscr{X} \to \mathscr{Y}$ be a finite type morphism between S-stacks locally of finite type. We then have the derived functor of f_*

$$\mathrm{R}f_* : \mathrm{D}(\mathscr{X}) \to \mathrm{D}(\mathscr{Y}).$$

Note that in general $\mathrm{R}f_*$ does not map $\mathrm{D}_c(\mathscr{X})$ to $\mathrm{D}_c(\mathscr{Y})$. For example consider $\mathrm{B}\mathbf{G}_m \to \mathrm{Spec}(k)$ and $\mathrm{A} = \bigoplus_{i\geq 0}\Lambda[i]$. However, by [23, 9.9], $\mathrm{R}f_*$ restricts to a functor

$$\mathrm{R}f_* : \mathrm{D}_c^{(+)}(\mathscr{X}) \to \mathrm{D}_c^{(+)}(\mathscr{Y}).$$

4.2. *The functor* $\mathscr{R}hom(-,-)$. — Let $\mathscr{X}$ be an S-stack locally of finite type.

4.2.1. *Lemma.* — *Let* $F \in D_c(\mathscr{X})$ *and* $G \in D_c(\mathscr{X})$*, and let* j *be an integer. Then the restriction of the sheaf* $\mathscr{H}^j(\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F,G))$ *to the étale topos of any object* $U \in \text{Lisse-Et}(\mathscr{X})$ *is canonically isomorphic to* $\mathscr{E}xt^j_{U_{\text{ét}}}(F_U, G_U)$*, where* F_U *and* G_U *denote the restrictions to* $U_{\text{ét}}$.

Proof. — The sheaf $\mathscr{H}^j(\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F,G))$ is the sheaf associated to the presheaf which to any smooth affine $\mathscr{X}$-scheme U associates $\text{Ext}^j_{\mathscr{X}_{\text{lis-ét}|U}}(F,G)$, where $\mathscr{X}_{\text{lis-ét}|U}$ denotes the localized topos. Let $\epsilon : \mathscr{X}_{\text{lis-ét}|U} \to U_{\text{ét}}$ be the morphism of topos induced by the inclusion of Étale(U) into $\text{Lisse-Et}(\mathscr{X})_{|U}$. Then since F and G have constructible cohomology, the natural maps $\epsilon^*\epsilon_*F \to F$ and $\epsilon^*\epsilon_*G \to G$ are isomorphisms in $D(\mathscr{X}_{\text{lis-ét}|U})$. Since the natural map $\text{id} \to \epsilon_*\epsilon^*$ is an isomorphism we get as in 3.4.1(i)

$$\begin{aligned}\text{Ext}^j_{\mathscr{X}_{\text{lis-ét}|U}}(F,G) &\simeq \text{Ext}^j_{\mathscr{X}_{\text{lis-ét}|U}}(\epsilon^*\epsilon_*F, \epsilon^*\epsilon_*G)\\ &\simeq \text{Ext}^j_{U_{\text{ét}}}(\epsilon_*F, \epsilon_*\epsilon^*\epsilon_*G)\\ &\simeq \text{Ext}^j_{U_{\text{ét}}}(\epsilon_*F, \epsilon_*G).\end{aligned}$$

Sheafifying this isomorphism we obtain the isomorphism in the lemma. □

4.2.2. *Corollary.* — *If* $F \in D_c^{(-)}(\mathscr{X})$ *and* $G \in D_c^{(+)}(\mathscr{X})$*, the complex* $\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F,G)$ *lies in* $D_c^{(+)}(\mathscr{X})$.

Proof. — By the previous lemma and the constructibility of the cohomology sheaves of F and G, it suffices to prove the following statement: Let $f : V \to U$ be a smooth morphism of schemes of finite type over S, and let $F \in D_c^-(U_{\text{ét}})$ and $G \in D_c^+(U_{\text{ét}})$. Then the natural map $f^*\mathscr{R}hom_{U_{\text{ét}}}(F,G) \to \mathscr{R}hom_{V_{\text{ét}}}(f^*F, f^*G)$ is an isomorphism as we saw in the proof of 3.5.2 (see [14, I.7.2]). □

4.2.3. *Proposition.* — *Let* $X_\bullet \to \mathscr{X}$ *be a smooth hypercover of* $\mathscr{X}$ *by a simplicial scheme* $X_\bullet$. *Let* $F, G \in D_c(\mathscr{X}_{\text{lis-ét}})$*, and let* $F_{\text{ét}}, G_{\text{ét}} \in D_c(X_{\bullet,\text{ét}})$ *denote the restrictions of* F *and* G *respectively to the étale topos of* $X_\bullet$. *Then there is a canonical isomorphism*

$$\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F,G)|_{X_\bullet,\text{ét}} \simeq \mathscr{R}hom_{X_{\bullet,\text{ét}}}(F_{\text{ét}}, G_{\text{ét}}). \tag{4.2.i}$$

In particular, if $F \in D_c^{(-)}(\mathscr{X})$ *and* $G \in D_c^{(+)}$ *then* $\mathscr{R}hom_{X_{\bullet,\text{ét}}}(F_{\text{ét}}, G_{\text{ét}})$ *maps under the equivalence of categories* $D_c(X_{\bullet,\text{ét}}) \simeq D_c(\mathscr{X}_{\text{lis-ét}})$ *to* $\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F,G)$.

Proof. — Let $\mathscr{X}_{\text{lis-ét}|X_\bullet}$ denote the strictly simplicial localized topos and consider the morphisms of topos

$$\mathscr{X}_{\text{lis-ét}} \xleftarrow{\pi} \mathscr{X}_{\text{lis-ét}|X_\bullet} \xrightarrow{\epsilon} X_{\bullet,\text{ét}}. \tag{4.2.ii}$$

4.2.4. *Lemma.* — *Let* $A, B \in D(\mathscr{X}_{\text{lis-ét}})$. *Then there is a canonical isomorphism*

$$\text{(4.2.iii)} \qquad \pi^* \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(A, B) \simeq \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, \pi^*B).$$

Proof. — Let $I^\cdot$ be a homotopically injective representative for B, and let $\pi^*I^\cdot \to J^\cdot$ be a quasi-isomorphism with $J^\cdot$ homotopically injective. Then by construction [16, 18.4.2], we have

$$\pi^* \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(A, B) = \pi^* \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}}(A, I^\cdot) = \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, \pi^*I^\cdot),$$

and

$$\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, \pi^*B) = \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, J^\cdot).$$

We define (4.2.iii) to be the map

$$\mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, \pi^*I^\cdot) \to \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*A, J^\cdot)$$

induced by the map $\pi^*I^\cdot \to J^\cdot$. To check that this map is a quasi-isomorphism, it suffices to show that for every n the map

$$\text{(4.2.iv)} \qquad \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_n}(\pi_n^*A, \pi_n^*I^\cdot) \to \mathscr{H}om^\cdot_{\mathscr{X}_{\text{lis-ét}}|X_n}(\pi_n^*A, J_n^\cdot)$$

is a quasi-isomorphism, where $\pi_n : \mathscr{X}_{\text{lis-ét}}|_{X_n} \to \mathscr{X}_{\text{lis-ét}}$ is the localization morphism and $J_n^\cdot$ is the restriction of $J^\cdot$ to $\mathscr{X}_{\text{lis-ét}}|_{X_n}$.

For this note that π_n^* is an exact functor on the category of abelian sheaves in $\mathscr{X}_{\text{lis-ét}}|_{X_n}$ and π_n^* admits an exact left adjoint $\pi_{n!}$ (as is the case for any localization morphism). Therefore $\pi_n^*I^\cdot$ is a homotopically injective resolution of π_n^*B. Also the restriction functor from abelian sheaves on $\mathscr{X}_{\text{lis-ét}}|_{X_\bullet}$ to abelian sheaves on $\mathscr{X}_{\text{lis-ét}}|_{X_n}$ admits an exact left adjoint (see for example [13, V^{bis}.1.2.11]) and is exact. It follows that $J_n^\cdot$ is also a homotopically injective resolution of π_n^*B and so the map (4.2.iv) is a quasi-isomorphism. □

By definition $F_{\text{ét}} = \epsilon_*\pi^*F$ and $G_{\text{ét}} = \epsilon_*\pi^*G$, and since $F, G \in D_c(\mathscr{X}_{\text{lis-ét}})$ the natural maps $\epsilon^*F_{\text{ét}} \to \pi^*F$ and $\epsilon^*G_{\text{ét}} \to \pi^*G$ are isomorphisms (2.2.3). Using 4.2.4 we then obtain

$$\begin{aligned}
\mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F, G)|_{X_\bullet,\text{ét}} &\simeq \epsilon_*\pi^* \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}}(F, G)\\
&\simeq \epsilon_* \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\pi^*F, \pi^*G)\\
&\simeq \epsilon_* \mathscr{R}hom_{\mathscr{X}_{\text{lis-ét}}|X_\bullet}(\epsilon^*F_{\text{ét}}, \epsilon^*G_{\text{ét}})\\
&\simeq \mathscr{R}hom_{X_\bullet,\text{ét}}(F_{\text{ét}}, G_{\text{ét}}),
\end{aligned}$$

where the last isomorphism is by trivial duality. □

4.3. *The functor f^*.* — The lisse-étale site is not functorial (cf. [1, 5.3.12]): a morphism of stacks does not induce in general a morphism between corresponding lisse-étale topos. In [23], a functor f^* is constructed on D_c^+ using cohomological descent. Using the results of 2.2.3 which imply that we have cohomological descent also for unbounded complexes, the construction of [23] can be used to define f^* on the whole category D_c.

Let us review the construction here. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic S-stacks locally of finite type. Choose a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathscr{Y} \end{array}$$

where the horizontal lines are presentations inducing a commutative diagram of strict simplicial spaces

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\eta_X} & \mathscr{X} \\ {\scriptstyle f_\bullet}\downarrow & & \downarrow{\scriptstyle f} \\ Y_\bullet & \xrightarrow{\eta_Y} & \mathscr{Y}. \end{array}$$

We get a diagram of topos

$$\begin{array}{ccccc} X_{\bullet,\text{ét}} & \xleftarrow{\Phi_X} & \mathscr{X}_{\text{lis-ét}}|_{X_\bullet} & \xrightarrow{\eta_X} & \mathscr{X}_{\text{lis-ét}} \\ {\scriptstyle f_\bullet}\downarrow & & & & \\ Y_{\bullet,\text{ét}} & \xleftarrow{\Phi_Y} & \mathscr{Y}_{\text{lis-ét}}|_{Y_\bullet} & \xrightarrow{\eta_Y} & \mathscr{Y}_{\text{lis-ét}}. \end{array}$$

By 2.2.6 the horizontal morphisms induce equivalences of topos

$$D_c(\mathscr{X}_{\text{lis-ét}}) \simeq D_c(X_{\bullet,\text{ét}}), \qquad D_c(\mathscr{Y}_{\text{lis-ét}}) \simeq D_c(Y_{\bullet,\text{ét}}).$$

We define the functor $f^* : D_c(\mathscr{Y}_{\text{lis-ét}}) \to D_c(\mathscr{X}_{\text{lis-ét}})$ to be the composite

(**4.3.i**) $$D_c(\mathscr{Y}_{\text{lis-ét}}) \simeq D_c(Y_{\bullet,\text{ét}}) \xrightarrow{f_\bullet^*} D_c(X_{\bullet,\text{ét}}) \simeq D_c(\mathscr{X}_{\text{lis-ét}}),$$

where $f_\bullet^*$ denotes the derived pullback functor induced by the morphism of topos $f_\bullet : X_{\bullet,\text{ét}} \to Y_{\bullet,\text{ét}}$. Note that f^* takes distinguished triangles to distinguished triangles since this is true for $f_\bullet^*$.

4.3.1. *Proposition.* — *Let* $A \in D_c(\mathscr{Y})$ *and let* $B \in D_c^{(+)}(\mathscr{X})$, *and assume* f *is of finite type. Then there is a canonical isomorphism*

(**4.3.ii**) $$\mathscr{R}hom(A, f_*B) \to f_*\mathscr{R}hom(f^*A, B).$$

Proof. — We first reduce to the case when $A \in D_c^-(\mathscr{Y})$. For this we use some standard properties of homotopy limits and colimits as discussed for example in [3].

4.3.2. *Lemma.* — *Let* $(T, \mathscr{O})$ *be a ringed topos, and let* $C \in D(\mathscr{O})$ *be a complex. Then* C *is isomorphic to* $\mathrm{hocolim}_n \tau_{\leq n} C$.

Proof. — By definition $\mathrm{hocolim}_n \tau_{\leq n} C$ is the cone of the morphism

$$\text{1-shift} : \bigoplus_n \tau_{\leq n} C \to \bigoplus_n \tau_{\leq n} C.$$

Let $\pi : \bigoplus_n \tau_{\leq n} C \to C$ be the map induced by the natural maps $\tau_{\leq n} C \to C$. Then π and $\pi \circ \text{shift}$ are equal, and therefore there exists a map from the cone

$$\mathrm{hocolim}\, \tau_{\leq n} C \to C.$$

We claim that this map is a quasi-isomorphism. For this note that by construction we have

$$\mathscr{H}^i(\mathrm{hocolim}\, \tau_{\leq n} C) = \varinjlim \mathscr{H}^i(\tau_{\leq n} C) = \mathscr{H}^i(C)$$

for all i. □

We therefore have a distinguished triangle

$$\bigoplus_n \tau_{\leq n} A \xrightarrow{\text{1-shift}} \bigoplus_n \tau_{\leq n} A \longrightarrow A$$

which induces a commutative diagram

$$\begin{array}{ccccc} \mathscr{R}hom(\bigoplus_n \tau_{\leq n} A, f_* B) & \xrightarrow{\text{1-shift}} & \mathscr{R}hom(\bigoplus_n \tau_{\leq n} A, f_* B) & \longrightarrow & \mathscr{R}hom(A, f_* B) \\ \simeq \downarrow & & \downarrow \simeq & & \\ \prod_n \mathscr{R}hom(\tau_{\leq n} A, f_* B) & \xrightarrow{\text{1-shift}} & \prod_n \mathscr{R}hom(\tau_{\leq n} A, f_* B). & & \end{array}$$

We conclude that

$$\mathscr{R}hom(A, f_* B) \simeq \mathrm{holim}\, \mathscr{R}hom(\tau_{\leq n} A, f_* B).$$

Similarly since $f^* \tau_{\leq n} = \tau_{\leq n} f^*$ we have

$$\mathscr{R}hom(f^* A, B) \simeq \mathrm{holim}\, \mathscr{R}hom(f^* \tau_{\leq n} A, B),$$

and since f_* commutes with homotopy limits (since f_* commutes with products) we conclude that

$$f_* \mathscr{R}hom(f^*\mathrm{A}, \mathrm{B}) \simeq \mathrm{holim} f_* \mathscr{R}hom(f^*\tau_{\leq n}\mathrm{A}, \mathrm{B}).$$

This reduces the proof of 4.3.1 to the case when $\mathrm{A} \in \mathrm{D}_c^-(\mathscr{Y})$. In this case both sides of (4.3.ii) have constructible cohomology, so it suffices to construct the isomorphism after restricting to $\mathrm{Y}_{\bullet,\text{ét}}$.

By 4.2.3 and [23, 9.8], we have

$$\mathrm{R}f_* \mathscr{R}hom(f^*\mathrm{A}, \mathrm{B})|_{\mathrm{Y}_{\bullet,\text{ét}}} \simeq \mathrm{R}f_{\bullet *} \mathscr{R}hom_{\mathrm{X}_{\bullet,\text{ét}}}(f_\bullet^*\mathrm{A}|_{\mathrm{Y}_{\bullet,\text{ét}}}, \mathrm{B}|_{\mathrm{X}_{\bullet,\text{ét}}}).$$

The result therefore follows from the usual adjunction

$$(\mathbf{4.3.iii}) \qquad \mathrm{R}f_{\bullet *} \mathscr{R}hom_{\mathrm{X}_{\bullet,\text{ét}}}(f_\bullet^*(\mathrm{A}|_{\mathrm{Y}_{\bullet,\text{ét}}}), \mathrm{B}|_{\mathrm{X}_{\bullet,\text{ét}}}) \simeq \mathscr{R}hom_{\mathrm{Y}_{\bullet,\text{ét}}}(\mathrm{A}|_{\bullet,\text{ét}}, f_*\mathrm{B}|_{\mathrm{X}_{\bullet,\text{ét}}}).$$

□

4.4. *Definition of* $\mathrm{R}f_!, f^!$. — Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks (locally of finite type over S) of finite type. Recall (4.1) that $\mathrm{R}f_*$ maps $\mathrm{D}_c^{(+)}(\mathscr{X}_{\text{lis-ét}})$ to $\mathrm{D}_c^{(+)}(\mathscr{Y}_{\text{lis-ét}})$.

4.4.1. *Definition. — We define*

$$\mathrm{R}f_! : \mathrm{D}_c^{(-)}(\mathscr{X}_{\text{lis-ét}}) \to \mathrm{D}_c^{(-)}(\mathscr{Y}_{\text{lis-ét}})$$

by the formula

$$\mathrm{R}f_! = \mathrm{D}_{\mathscr{Y}} \circ \mathrm{R}f_* \circ \mathrm{D}_{\mathscr{X}},$$

and

$$f^! : \mathrm{D}_c(\mathscr{Y}_{\text{lis-ét}}) \to \mathrm{D}_c(\mathscr{X}_{\text{lis-ét}})$$

by the formula

$$f^! = \mathrm{D}_{\mathscr{X}} \circ f^* \circ \mathrm{D}_{\mathscr{Y}}.$$

By construction, one has

$$(\mathbf{4.4.i}) \qquad f^!\Omega_{\mathscr{Y}} = \Omega_{\mathscr{X}}.$$

4.4.2. *Proposition. — Let* $\mathrm{A} \in \mathrm{D}_c^{(-)}(\mathscr{X}_{\text{lis-ét}})$ *and* $\mathrm{B} \in \mathrm{D}_c(\mathscr{Y}_{\text{lis-ét}})$. *Then there is a (functorial) adjunction formula*

$$\mathrm{R}f_* \mathscr{R}hom(\mathrm{A}, f^!\mathrm{B}) = \mathscr{R}hom(\mathrm{R}f_!\mathrm{A}, \mathrm{B}).$$

Proof. — We write D for $\mathrm{D}_{\mathscr{X}}, \mathrm{D}_{\mathscr{Y}}$ and $\mathrm{A}' = \mathrm{D}(\mathrm{A}) \in \mathrm{D}_c^{(+)}(\mathscr{X})$. One has

$$\begin{aligned}\mathscr{R}hom(\mathrm{R}f_!\mathrm{D}(\mathrm{A}'), \mathrm{B}) &= \mathscr{R}hom(\mathrm{D}(\mathrm{R}f_*\mathrm{A}'), \mathrm{B})\\ &= \mathscr{R}hom(\mathrm{D}(\mathrm{B}), \mathrm{R}f_*\mathrm{A}') \ \ (3.5.9)\\ &= \mathrm{R}f_* \mathscr{R}hom(f^*\mathrm{D}(\mathrm{B}), \mathrm{A}') \ \ (4.3.1)\\ &= \mathrm{R}f_* \mathscr{R}hom(\mathrm{D}(\mathrm{A}'), f^!\mathrm{B}) \ \ (3.5.9).\end{aligned}$$

□

4.5. *Projection formula*

4.5.1. *Lemma. — Let* $A, B \in D_c(\mathscr{X})$.

(i) One has

$$\mathscr{R}hom(A, B) = D_{\mathscr{X}}(A \overset{\mathbf{L}}{\otimes} D_{\mathscr{X}}(B)).$$

(ii) If $A, B \in D_c^{(-)}(\mathscr{X})$, *then* $A \overset{\mathbf{L}}{\otimes} B \in D_c^{(-)}(\mathscr{X})$.

Proof. — Let $\Omega_{\mathscr{X}}$ be the dualizing complex of $\mathscr{X}$.

$$\begin{aligned} \mathscr{R}hom(A, B) &= \mathscr{R}hom(D_{\mathscr{X}}(B), \mathscr{R}hom(A, \Omega_{\mathscr{X}})) \ \ (3.5.9) \\ &= \mathscr{R}hom(D_{\mathscr{X}}(B) \overset{\mathbf{L}}{\otimes} A, \Omega_{\mathscr{X}}) \ \ (3.5.ii) \\ &= D_{\mathscr{X}}(A \overset{\mathbf{L}}{\otimes} D_{\mathscr{X}}(B)) \end{aligned}$$

proving (i). For (ii), using truncations, one can assume that A, B are sheaves: the result is obvious in this case. □

4.5.2. *Corollary. — Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a morphism as in 4.4, and let* $B \in D_c^{(-)}(\mathscr{Y})$, $A \in D_c^{(-)}(\mathscr{X})$. *One has the projection formula*

$$Rf_!(A \overset{\mathbf{L}}{\otimes} f^*B) = Rf_!A \overset{\mathbf{L}}{\otimes} B.$$

Proof. — Notice that the left-hand side is well defined by 4.5.1. One has

$$\begin{aligned} Rf_!(A \overset{\mathbf{L}}{\otimes} f^*B) &= D_{\mathscr{Y}} \circ Rf_* \circ D_{\mathscr{X}}(A \overset{\mathbf{L}}{\otimes} D_{\mathscr{X}} f^! D_{\mathscr{Y}} B) \\ &= D_{\mathscr{Y}} \circ Rf_*(\mathscr{R}hom(A, f^! D_{\mathscr{Y}} B)) \ \ (4.5.1) \\ &= D_{\mathscr{Y}}(\mathscr{R}hom(Rf_!A, D_{\mathscr{Y}} B)) \ \ (4.4.2) \\ &= Rf_!A \overset{\mathbf{L}}{\otimes} B \ \ (4.5.1) \text{ and } (3.5.7). \end{aligned}$$

□

4.5.3. *Corollary. — For all* $A \in D_c^{(-)}(\mathscr{Y}), B \in D_c^{(+)}(\mathscr{Y})$, *one has*

$$f^! \, \mathscr{R}hom(A, B) = \mathscr{R}hom(f^*A, f^!B).$$

Proof. — By Lemma 4.5.1 and biduality, the formula reduces to the formula

$$f^*(A \overset{\mathbf{L}}{\otimes} D(B)) = f^*A \overset{\mathbf{L}}{\otimes} f^*D(B).$$

Using suitable presentation, one is reduced to the obvious formula

$$f_\bullet^*(A_\bullet \overset{\mathbf{L}}{\otimes} B_\bullet) = f_\bullet^* A_\bullet \overset{\mathbf{L}}{\otimes} f_\bullet^* B_\bullet$$

for a morphism $f_\bullet$ of simplicial étale topos. □

4.6. *Computation of $f^!$ for f smooth.* — Let $f : \mathscr{X} \to \mathscr{Y}$ be a smooth morphism of stacks of relative dimension d. Using 2.3.4, one gets immediately the formula

$$f^*\Omega_{\mathscr{Y}} = \Omega_{\mathscr{X}}\langle -d\rangle$$

(choose a presentation of $Y \to \mathscr{Y}$ and then a presentation $X \to \mathscr{X}_Y$; the morphism $X \to Y$ being smooth, one checks that these two complexes coincide on $\mathscr{X}_{\text{lis-ét}|X}$ and have zero negative $\mathscr{E}xt$'s).

4.6.1. *Lemma.* — *Let* $A \in D_c(\mathscr{Y})$. *Then, the canonical morphism*

$$f^* \mathscr{R}hom(A, \Omega_{\mathscr{Y}}) \to \mathscr{R}hom(f^*A, f^*\Omega_{\mathscr{Y}})$$

is an isomorphism.

Proof. — Using 3.4.1, one is reduced to the corresponding statement for étale sheaves on quasi-compact algebraic spaces. Because, in this case, both $\Omega_{\mathscr{Y}}$ and $f^*\Omega_{\mathscr{Y}}$ are of finite injective dimension, one can assume that A is bounded or even a sheaf. The assertion is well-known in this case (by dévissage, one reduces to $A = \Lambda_{\mathscr{Y}}$ in which case the assertion is trivial, cf. [14, exp. I]). □

4.6.2. *Corollary.* — *Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a smooth morphism of stacks of relative dimension* d. *One has* $f^! = f^*\langle d\rangle$.

Let $j : \mathscr{U} \to \mathscr{X}$ be an open immersion. Let us denote for a while by $j_¡$ the extension by zero functor: it is an exact functor on the category sheaves preserving constructibility and therefore passes to the derived category D_c.

4.6.3. *Proposition.* — *One has* $j^! = j^*$ *and* $j_! = j_¡$.

Proof. — The first equality is a particular case of 4.6.2. Because j^* has a left adjoint $j_¡$ which is exact, it preserves (homotopical) injectivity. Let A, B be constructible complexes on $\mathscr{U}, \mathscr{X}$ respectively and assume that B is homotopically injective. One has

$$\begin{aligned} \mathrm{Rhom}(j_¡A, B) &= \mathrm{Hom}(j_¡A, B) \\ &= \mathrm{Hom}(A, j^*B) \quad \text{(adjunction)} \\ &= \mathrm{Rhom}(A, j^*B). \end{aligned}$$

Taking H^0, one obtains that j^* is the right adjoint of $j_¡$ proving the lemma because $j^! = j^*$ is also the right adjoint of $j_!$. □

4.7. *Computation of* $Ri_!$ *for* i *a closed immersion.* — Let $i : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a closed immersion and $j : \mathcal{U} = \mathcal{Y} - \mathcal{X} \hookrightarrow \mathcal{Y}$ the open immersion of the complement: both are representable. We define the cohomology with support on $\mathcal{X}$ for any $F \in \mathcal{X}_{\text{lis-ét}}$ as follows. First, for any $Y \to \mathcal{Y}$ in Lisse-Et($\mathcal{Y}$), the pull-back $Y_{\mathcal{U}} \to \mathcal{U}$ is in Lisse-Et($\mathcal{U}$) and $Y_{\mathcal{U}} \to \mathcal{U} \to \mathcal{Y}$ is in Lisse-Et($\mathcal{Y}$). Then, we define $\underline{H}^0_{\mathcal{X}}(F)$

$$\Gamma\big(Y, \underline{H}^0_{\mathcal{X}}(F)\big) = \ker(\Gamma(Y, F) \to \Gamma(Y_{\mathcal{U}}, F)) \tag{4.7.i}$$

and $R\Gamma_{\mathcal{X}}$ is the total derived functor of the left exact functor $\underline{H}^0_{\mathcal{X}}$.

As usual if I is an injective Λ-module on $\mathcal{Y}$, then the adjunction map $I \to j_*j^*I$ is surjective. It follows that for any $F \in D^+(\mathcal{Y})$ there is a distinguished triangle

$$R\underline{H}^0_{\mathcal{X}}(F) \to F \to Rj_*j^*F \to R\underline{H}^0_{\mathcal{X}}(F)[1].$$

This implies in particular that if $F \in D_c^{(+)}(\mathcal{Y})$, then $R\underline{H}^0_{\mathcal{X}}(F)$ is also in $D_c^{(+)}(\mathcal{Y})$.

4.7.1. *Lemma.* — *One has* $\Omega_{\mathcal{X}} = i^*R\Gamma_{\mathcal{X}}(\Omega_{\mathcal{Y}})$.

Proof. — If i is a closed immersion of schemes (or algebraic spaces), one has a canonical (and functorial) isomorphism, simply because $i^*\underline{H}^0_{\mathcal{X}}$ is the right adjoint of i_*. If K denotes one of the objects on the two sides of the equality to be proven, one has therefore $\mathcal{E}xt^i(K, K) = 0$ for $i < 0$. Therefore, these isomorphisms glue by 2.3.3. □

4.7.2. *Proposition.* — *The functor* $B \mapsto i^*R\Gamma_{\mathcal{X}}(B)$ *is the right adjoint of* i_*, *and therefore coincides with* $i^!$. *More generally, one has*

$$\mathcal{R}hom(i_*A, B) = i_*\,\mathcal{R}hom\big(A, i^*R\underline{H}^0_{\mathcal{X}}(B)\big) \tag{4.7.ii}$$

for all $A \in D_c(\mathcal{X}), B \in D_c^{(+)}(\mathcal{Y})$. *Moreover, one has one has* $i_! = i_*$ *and* i_* *has a right adjoint, the sections with support on* $\mathcal{X}$.

Proof. — If A, B are sheaves on $\mathcal{Y}_{\text{lis-ét}}$, one has by definition of $\underline{H}^0_{\mathcal{X}}$ an isomorphism

$$\hom(i_*A, B) \simeq \hom\big(i_*A, \underline{H}^0_{\mathcal{X}}(B)\big).$$

This isomorphism induces a morphism of derived functors

$$\mathcal{R}hom(i_*A, B) \to \mathcal{R}hom\big(i_*A, R\underline{H}^0_{\mathcal{X}}(B)\big). \tag{4.7.iii}$$

We claim that if $A \in D_c(\mathscr{X})$ and $B \in D_c^{(+)}(\mathscr{Y})$ then (4.7.iii) is an isomorphism. To verify this it suffices to show that for any $(U \to \mathscr{Y}) \in \text{Lisse-Et}(\mathscr{Y})$, the map on $U_{ét}$

$$\mathscr{R}hom(i_*A, B)_U \to \mathscr{R}hom\left(i_*A, R\underline{H}^0_{\mathscr{X}}(B)\right)_U$$

is an isomorphism. Using 3.4.1 this reduces to the analogous statement in the étale topos of U for sheaves $A, B \in D_c(U_{ét})$. Now in this case we can define for any sheaf F on $U_{ét}$ the sheaf $i_U^*\underline{H}^0_{\mathscr{X}_U}(F)$, where $i_U : \mathscr{X}_U \hookrightarrow U$ denotes the inverse image of $\mathscr{X}$. The functor $i_U^*\underline{H}^0_{\mathscr{X}_U}(-)$ is right adjoint to the exact functor i_{U*}, and therefore $i_U^*\underline{H}^0_{\mathscr{X}_U}(-)$ sends homotopically injective complexes to homotopically injective complexes. This implies that

$$\mathscr{R}hom(i_{U*}A, B) \simeq i_{U*}\,\mathscr{R}hom\left(A, i_U^* R\underline{H}^0_{\mathscr{X}_U}(B)\right).$$

On the other hand, the adjunction maps $i_U^* i_{U*}A \to A$ and $R\underline{H}^0_{\mathscr{X}_U}(B) \to i_{U*}i_U^*R\underline{H}^0_{\mathscr{X}_U}(B)$ are isomorphisms, so we have

$$\begin{aligned} i_{U*}\,\mathscr{R}hom\left(A, i_U^*R\underline{H}^0_{\mathscr{X}_U}(B)\right) &\simeq i_{U*}\,\mathscr{R}hom\left(i_U^*i_{U*}A, i_U^*R\underline{H}^0_{\mathscr{X}_U}(B)\right)\\ &\simeq \mathscr{R}hom\left(i_{U*}A, i_{U*}i_U^*R\underline{H}^0_{\mathscr{X}_U}(B)\right)\\ &\simeq \mathscr{R}hom\left(i_{U*}A, R\underline{H}^0_{\mathscr{X}_U}(B)\right). \end{aligned}$$

This implies that (4.7.iii) is an isomorphism.

This also shows that the natural maps

$$B \to i_*i^*R\underline{H}^0_{\mathscr{X}}(B), \quad i^*i_*A \to A$$

are isomorphisms. From this and 4.3.1 we obtain

$$\begin{aligned} \mathscr{R}hom\left(i_*A, R\underline{H}^0_{\mathscr{X}}(B)\right) &\simeq \mathscr{R}hom\left(i_*A, i_*i^*R\underline{H}^0_{\mathscr{X}}(B)\right)\\ &\simeq i_*\,\mathscr{R}hom\left(i^*i_*A, i^*R\underline{H}^0_{\mathscr{X}}(B)\right)\\ &\simeq i_*\,\mathscr{R}hom\left(A, i^*R\underline{H}^0_{\mathscr{X}}(B)\right) \end{aligned}$$

proving (4.7.ii).

One gets therefore

$$\begin{aligned} i_!A &= \mathscr{R}hom(i_*\,\mathscr{R}hom(A, \Omega_{\mathscr{X}}), \Omega_{\mathscr{Y}})\\ &= i_*\,\mathscr{R}hom\left(\mathscr{R}hom(A, \Omega_{\mathscr{X}}), i^*R\underline{H}^0_{\mathscr{X}}(\Omega_{\mathscr{Y}})\right)\\ &= i_*\,\mathscr{R}hom(\mathscr{R}hom(A, \Omega_{\mathscr{X}}), \Omega_{\mathscr{X}}) \text{ (4.7.1)}\\ &= i_*A \text{ (3.5.1)}. \end{aligned}$$

□

4.8. *Computation of $f^!$ for a universal homeomorphism.* — By universal homeomorphism we mean a representable, radiciel and surjective morphism. By Zariski's main theorem, such a morphism is finite.

In the schematic situation, we know that such a morphism induces an isomorphism of the étale topos [13, VIII.1.1]. In particular, f^* is also a right adjoint of f_*. Being exact, one gets in this case an identification $f^* = f^!$. In particular, f^* identifies the corresponding dualizing complexes. Exactly as in the proof of 4.7.1, one gets

4.8.1. *Lemma.* — *Let $f : \mathscr{X} \to \mathscr{Y}$ be a universal homeomorphism of stacks. One has $f^*\Omega_{\mathscr{X}} = \Omega_{\mathscr{Y}}$.*

One gets therefore

4.8.2. *Corollary.* — *Let $f : \mathscr{X} \to \mathscr{Y}$ be a universal homeomorphism of stacks. One has $f^! = f^*$ and $\mathrm{R}f_! = \mathrm{R}f_*$.*

Proof. — By a similar argument to the one used in the proof of 4.6.1 the natural map

$$f^* \mathscr{R}hom(\mathrm{A}, \Omega_{\mathscr{Y}}) \to \mathscr{R}hom(f^*\mathrm{A}, f^*\Omega_{\mathscr{Y}})$$

is an isomorphism. We therefore have

$$\begin{aligned} f^!\mathrm{A} &= \mathscr{R}hom(f^* \mathscr{R}hom(\mathrm{A}, \Omega_{\mathscr{Y}}), \Omega_{\mathscr{X}}) \\ &= \mathscr{R}hom(\mathscr{R}hom(f^*\mathrm{A}, f^*\Omega_{\mathscr{Y}}), \Omega_{\mathscr{X}}) \\ &= \mathscr{R}hom(\mathscr{R}hom(f^*\mathrm{A}, \Omega_{\mathscr{X}}), \Omega_{\mathscr{X}}) \;\; (4.8.1) \\ &= f^*\mathrm{A} \;\; (3.5.1). \end{aligned}$$

The last formula follows by adjunction. □

4.9. *Computation of $\mathrm{R}f_!$ via hypercovers.* — Let Y be an S-scheme of finite type and $f : \mathscr{X} \to \mathrm{Y}$ a morphism of finite type from an algebraic stack $\mathscr{X}$. Let $\mathrm{X}_\bullet \to \mathscr{X}$ be a smooth hypercover by algebraic spaces, and for each n let d_n denote the locally constant function on X_n which is the relative dimension over $\mathscr{X}$. By the construction, the restriction $\mathrm{K}_{\mathrm{X}_n}$ of the dualizing complex $\Omega_{\mathscr{X}}$ to $\mathrm{X}_{n,\text{ét}}$ is canonically isomorphic to $\Omega_{\mathrm{X}_n}\langle -d_n\rangle$. Let $\mathrm{K}_{\mathrm{X}_\bullet}$ denote the restriction of $\Omega_{\mathscr{X}}$ to $\mathrm{X}_{\bullet,\text{ét}}$.

Let $\mathrm{L} \in \mathrm{D}_c^-(\mathscr{X})$, and let $\mathrm{L}|_{\mathrm{X}_\bullet}$ denote the restriction of L to $\mathrm{X}_{\bullet,\text{ét}}$. Then $\mathrm{D}_{\mathscr{X}}(\mathrm{L})|_{\mathrm{X}_\bullet}$ is isomorphic to $\mathrm{D}_{\mathrm{X}_\bullet}(\mathrm{L}|_{\mathrm{X}_\bullet}) := \mathscr{R}hom_{\mathrm{X}_{\bullet,\text{ét}}}(\mathrm{L}|_{\mathrm{X}_\bullet}, \mathrm{K}_{\mathrm{X}_\bullet})$. In particular, the restriction of $\mathrm{R}f_!\mathrm{L}$ to $\mathrm{Y}_{\text{ét}}$ is canonically isomorphic to

$$\mathscr{R}hom_{\mathrm{Y}_{\text{ét}}}(\mathrm{R}f_{\bullet *}\mathrm{D}_{\mathrm{X}_\bullet}(\mathrm{L}|_{\mathrm{X}_\bullet}), \mathrm{K}_{\mathrm{Y}}) \in \mathrm{D}_c(\mathrm{Y}_{\text{ét}}), \tag{4.9.i}$$

where $f_\bullet : \mathrm{X}_{\text{ét}} \to \mathrm{Y}_{\text{ét}}$ denotes the morphism of topos induced by f.

Let $Y_{\bullet,\text{ét}}$ denote the simplicial topos obtained by viewing Y as a constant simplicial scheme. Let $\epsilon : Y_{\bullet,\text{ét}} \to Y_{\text{ét}}$ denote the canonical morphism of topos, and let $\tilde{f} : X_{\bullet,\text{ét}} \to Y_{\bullet,\text{ét}}$ be the morphism of topos induced by f. We have $f_\bullet = \epsilon \circ \tilde{f}$. As in [23, 2.7], it follows that there is a canonical spectral sequence

$$(\mathbf{4.9.ii}) \qquad E_1^{pq} = R^q f_{p*} D_{X_p}(L|_{X_p}) \implies R^{p+q} f_{\bullet *} D_{X_\bullet}(L_{X_\bullet}).$$

On the other hand, we have

$$R^q f_{p*} D_{X_p}(L|_{X_p}) = R^q f_{p*} \mathscr{R}hom(L|_{X_p}, \Omega_{X_p}\langle -d_p\rangle) \simeq \mathscr{H}^q(D_Y(Rf_{p!}(L|_{X_p}\langle d_p\rangle)),$$

where the second isomorphism is by biduality 3.5.7. Combining all this we obtain

4.9.1. *Proposition. — There is a canonical spectral sequence*

$$(\mathbf{4.9.iii}) \qquad E_1^{pq} = \mathscr{H}^q(D_{Y_{\text{et}}}(Rf_{p!}L|_{X_p}\langle d_p\rangle)) \implies \mathscr{H}^{p+q}(D_{Y_{\text{et}}}(Rf_! L|_{Y_{\text{et}}})).$$

4.9.2. *Example.* — Let k be an algebraically closed field and G a finite group. We can then compute $H_c^*(BG, \Lambda)$ as follows. We first compute $\mathscr{R}hom(R\Gamma_!(BG, \Lambda), \Lambda)$. Let $\text{Spec}(k) \to BG$ be the surjection corresponding to the trivial G-torsor, and let $X_\bullet \to BG$ be the 0-coskeleton. Note that each X_n isomorphic to G^n and in particular is a discrete collection of points. Therefore $Rf_{!p}\Lambda \simeq \text{Hom}(G^n, \Lambda)$. From this it follows that $\mathscr{R}hom(R\Gamma_!(BG, \Lambda), \Lambda)$ is represented by the standard cochain complex computing the group cohomology of Λ, and hence $R\Gamma_!(BG, \Lambda)$ is the dual of this complex. In particular, this can be nonzero in infinitely many negative degrees. For example if $G = \mathbf{Z}/\ell$ for some prime ℓ and $\Lambda = \mathbf{Z}/\ell$ since in this case the group cohomology $H^i(G, \mathbf{Z}/\ell) \simeq \mathbf{Z}/\ell$ for all $i \geq 0$.

4.9.3. *Example.* — Let k be an algebraically closed field and P the affine line $\mathbf{A}^1$ with the origin doubled. By definition P is equal to two copies of $\mathbf{A}^1$ glued along $\mathbf{G}_m$ via the standard inclusions $\mathbf{G}_m \subset \mathbf{A}^1$. We can then compute $R\Gamma_!(P, \Lambda)$ as follows. Let $j_i : \mathbf{A}^1 \hookrightarrow P$ $(i = 1, 2)$ be the two open immersions, and let $h : \mathbf{G}_m \hookrightarrow P$ be the inclusion of the overlaps. We then have an exact sequence

$$0 \to h_!\Lambda \to j_{1!}\Lambda \oplus j_{2!}\Lambda \to \Lambda \to 0.$$

From this we obtain a long exact sequence

$$\cdots \to H_c^i(\mathbf{G}_m, \Lambda) \to H_c^i(\mathbf{A}^1, \Lambda) \oplus H_c^i(\mathbf{A}^1, \Lambda) \to H_c^i(P, \Lambda) \to \cdots.$$

From this sequence one deduces that $H_c^0(P, \Lambda) \simeq \Lambda$, $H_c^2(P, \Lambda) \simeq \Lambda(1)$, and all other cohomology groups vanish. In particular, the cohomology of P is isomorphic to the cohomology of $\mathbf{P}^1$.

4.10. *Purity and the fundamental distinguished triangle.* — We consider the usual situation of a closed immersion $i : \mathscr{X} \to \mathscr{Y}$ of S-stacks locally of finite type, the open immersion of the complement of $\mathscr{Y}$ being $j : \mathscr{U} = \mathscr{Y} - \mathscr{X} \to \mathscr{Y}$. For any bounded complex of sheaves A on $\mathscr{Y}$ with constructible cohomology sheaves, one has the exact sequence

$$0 \to j_! j^* \mathrm{A} \to \mathrm{A} \to i_* i^* \mathrm{A} \to 0.$$

Therefore, for any $\mathrm{A} \in \mathrm{D}_c^{(b)}(\mathscr{Y})$, one has the distinguished triangle (4.6.3)

$$(\mathbf{4.10.i}) \qquad j_! j^* \mathrm{A} \to \mathrm{A} \to i_* i^* \mathrm{A}$$

which by duality gives the distinguished triangle

$$(\mathbf{4.10.ii}) \qquad i_* i^! \mathrm{A} \to \mathrm{A} \to j_* j^* \mathrm{A}.$$

Recall (4.7.2) the formula $i_* i^! = \mathrm{R}\underline{\mathrm{H}}^0_{\mathscr{X}}$. The usual purity theorem for S-schemes gives

4.10.1. *Proposition.* — *Purity. Assume moreover that $i : \mathscr{X} \to \mathscr{Y}$ is a closed immersion of* S*-stacks which are regular*[6]. *Then, one has $i^! \Lambda = \Lambda\langle -c \rangle$ where c denotes the codimension of i (a locally constant function on $\mathscr{X}$).*

Proof. — For a closed immersion of excellent regular schemes $j : \mathrm{X} \to \mathrm{Y}$ the fundamental class of X in Y determines by Gabber's absolute purity theorem [24, 3.1], a *natural* isomorphism

$$(\mathbf{4.10.iii}) \qquad \gamma_j : \Lambda_{\mathrm{X}}\langle -c_j \rangle \xrightarrow{\sim} j^! \Lambda_{\mathrm{Y}},$$

where c_j is the codimension of j.

In particular, one has

$$\tau_{<0}\, \mathscr{R}hom(\Lambda_{\mathrm{X}}\langle -c \rangle, j^! \Lambda_{\mathrm{Y}}) = \tau_{<0}\, \mathscr{R}hom(\Lambda_{\mathrm{X}}\langle -c \rangle, \Lambda_{\mathrm{X}}\langle -c \rangle) = 0$$

because Λ_{X} is dualizing on the excellent regular scheme X. By 2.3.4, the isomorphism γ generalizes in the case where i is only a closed immersion of excellent regular algebraic space. Now, if one has a cartesian diagram with smooth vertical arrows

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{j} & \mathrm{Y} \\ \downarrow & & \downarrow \\ \mathscr{X} & \xrightarrow{i} & \mathscr{Y} \end{array}$$

[6] In fact, the purity statement remains valid for closed immersions of arbitrary locally noetherian excellent regular stacks provided $i^!$ is *defined* as cohomology with compact support. The proof given below generalizes in this situation without any change.

with Y (and therefore X) an algebraic space, one has $(i^!\Lambda)_{\mathrm{X}_{\text{ét}}} = j^!\Lambda_{\mathrm{Y}}$. If $\mathrm{Y}\to\mathscr{Y}$ is a presentation of $\mathscr{Y}$ and X^n (resp. Y^n) denotes the n-fold product of X (resp. Y) over $\mathscr{X}$ (resp. $\mathscr{Y}$), we get local functorial isomorphisms

$$\Lambda_{\mathrm{X}^n}\langle -c\rangle \xrightarrow{\sim} i^!\Lambda_{\mathrm{Y}^n}$$

inducing an isomorphism at the strict simplicial level

$$\Lambda_{\mathrm{X}_\bullet}\langle -c\rangle \xrightarrow{\sim} i^!\Lambda_{\mathrm{Y}_\bullet}.$$

Using 2.3.4, 2.2.6 and 4.2.1, one gets that the isomorphisms (4.10.iii) glue.

5. Base change

We start with a cartesian diagram of stacks locally of finite type over S

(**5.0.i**)
$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \\ {\scriptstyle\phi}\downarrow & \square & \downarrow{\scriptstyle f} \\ \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y}, \end{array}$$

with f of finite type, and we would like to construct a natural base change isomorphism

(**5.0.ii**) $$p^*\mathrm{R}f_! = \mathrm{R}\phi_!\pi^*$$

of functors $\mathrm{D}_c^{(-)}(\mathscr{X})\to\mathrm{D}_c^{(-)}(\mathscr{Y}')$. Note that this is equivalent to the dual version:

(**5.0.iii**) $$p^!\mathrm{R}f_* = \mathrm{R}\phi_*\pi^! : \mathrm{D}_c^{(+)}(\mathscr{X}) \to \mathrm{D}_c^{(+)}(\mathscr{Y}').$$

Unfortunately we are unable to construct this base change isomorphism in full generality. However, we construct the base change isomorphism in several special cases which suffice to define the base change isomorphism on the level of cohomology sheaves.

5.1. *Smooth base change.* — In this subsection we prove the base change isomorphism in the case when p (and hence also π) is smooth. Note that in this case p^* is defined on all of $\mathrm{D}(\mathscr{Y})$ since it is just restriction from $\mathscr{Y}_{\text{lis-ét}}$ to $\mathscr{Y}'_{\text{lis-ét}}$.

Proof. — Because the relative dimension of p and π are the same, by 4.6.2, one reduces the formula (5.0.iii) to

$$p^*\mathrm{R}f_* = \mathrm{R}\phi_*\pi^*.$$

By adjunction, one has a morphism $p^*\mathrm{R}f_*\to\mathrm{R}\phi_*\pi^*$ which is an isomorphism by the smooth base change theorem. This therefore gives the base change isomorphism in this case. □

5.1.1. *Remark.* — The base change isomorphism $p^*f_!\mathrm{F} \simeq \phi_!\pi^*$ can also be defined as follows. For a sheaf G on $\mathscr{Y}_{\text{lis-ét}}$ the pullback $p^*\mathrm{G}$ is simply the restriction of G to $\mathscr{Y}'_{\text{lis-ét}}$. It follows that there is a canonical map

$$p^* \mathscr{R}hom_{\mathscr{Y}_{\text{lis-ét}}}(\mathrm{G}, \Omega_{\mathscr{Y}}) \to \mathscr{R}hom_{\mathscr{Y}'_{\text{lis-ét}}}(p^*\mathrm{G}, p^*\Omega_{\mathscr{Y}})$$

which if G is cartesian is an isomorphism (this follows from the same argument proving 4.2.1). We therefore get $p^*\mathrm{D}_{\mathscr{Y}} = \mathrm{D}_{\mathscr{Y}'}p^*$ and $\pi^*\mathrm{D}_{\mathscr{X}} = \mathrm{D}_{\mathscr{X}'}\pi^*$. Similarly there is a canonical map $p^*f_* \to f'_*\pi^*$ so we obtain a canonical map

$$\mathrm{D}_{\mathscr{Y}'}f'_*\mathrm{D}_{\mathscr{X}'}\pi^* \simeq \mathrm{D}_{\mathscr{Y}'}f'_*\pi^*\mathrm{D}_{\mathscr{X}} \to \mathrm{D}_{\mathscr{Y}'}p^*f_*\mathrm{D}_{\mathscr{X}} \simeq p^*\mathrm{D}_{\mathscr{Y}}f_*\mathrm{D}_{\mathscr{X}}.$$

It is immediate from the constructions that these two base change morphisms agree. In summary, the base change isomorphism for a smooth morphism p is just the natural map defined by restriction.

From this remark one obtains immediately the following 5.1.2 and 5.1.3 which will be used later.

5.1.2. *Lemma.* — *With notation as in 5.1, let* $r : \mathscr{Y}'' \to \mathscr{Y}'$ *be a second smooth morphism so we obtain a commutative diagram*

$$\begin{array}{ccccc} \mathscr{X}'' & \xrightarrow{\rho} & \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'' & \xrightarrow{r} & \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y}. \end{array}$$

Let

$$bc_r : r^*\phi_! \simeq \psi_!\rho^*, \quad bc_p : p^*f_! \simeq \phi_!\pi^*, \quad bc_{pr} : (pr)^*f_! \simeq \psi_!(\pi\rho)^*$$

be the base change isomorphisms. Then for $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ *the diagram*

$$\begin{array}{ccccc} r^*p^*f_!\mathrm{F} & \xrightarrow{bc_p} & r^*\phi_!\pi^*\mathrm{F} & \xrightarrow{bc_r} & \psi_!\rho^*\pi^*\mathrm{F} \\ \downarrow{\scriptstyle\simeq} & & & & \downarrow{\scriptstyle\simeq} \\ (pr)^*f_!\mathrm{F} & & \xrightarrow{bc_{pr}} & & \psi_!(\pi\rho)^*\mathrm{F} \end{array}$$

commutes.

5.1.3. *Lemma.* — *Consider a diagram* (5.0.i) *with* p *smooth, and let* $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ *and* $\mathrm{G} \in \mathrm{D}_c^{(-)}(\mathscr{Y})$. *Let*

$$\alpha : (\phi_!\pi^*\mathrm{F}) \overset{\mathbf{L}}{\otimes} p^*\mathrm{G} \to \phi_!(\pi^*\mathrm{F} \overset{\mathbf{L}}{\otimes} \phi^*p^*\mathrm{G}), \quad \beta : f_!\mathrm{F} \overset{\mathbf{L}}{\otimes} \mathrm{G} \to f_!(\mathrm{F} \overset{\mathbf{L}}{\otimes} f^*\mathrm{G})$$

be the isomorphisms given by the projection formula 4.5.2. Then the diagram

$$\begin{array}{ccc}
p^*(f_!\mathrm{F}\overset{\mathbf{L}}{\otimes}\mathrm{G}) & \xrightarrow{\beta} & p^*f_!(\mathrm{F}\overset{\mathbf{L}}{\otimes}f^*\mathrm{G}) \\
\downarrow \simeq & & \downarrow bc \\
(p^*f_!\mathrm{F})\overset{\mathbf{L}}{\otimes}p^*\mathrm{G} & & \phi_!\pi^*(\mathrm{F}\overset{\mathbf{L}}{\otimes}f^*\mathrm{G}) \\
\downarrow bc & & \downarrow \simeq \\
(\phi_!\pi^*\mathrm{F})\overset{\mathbf{L}}{\otimes}p^*\mathrm{G} & \xrightarrow{\alpha} & \phi_!(\pi^*\mathrm{F}\overset{\mathbf{L}}{\otimes}\phi^*p^*\mathrm{G})
\end{array}$$

commutes, where we write bc for the base change isomorphisms.

5.2. *Computation of* $\mathrm{R}f_*$ *for proper representable morphisms*

5.2.1. *Proposition. — Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a proper representable morphism of* S*-stacks. Then the functor* $\mathrm{R}f_! : \mathrm{D}_c^{(-)}(\mathscr{X}) \to \mathrm{D}_c^{(-)}(\mathscr{Y})$ *is canonically isomorphic to* $\mathrm{R}f_* : \mathrm{D}_c^{(-)}(\mathscr{X}) \to \mathrm{D}_c^{(-)}(\mathscr{Y})$.

5.2.2. *Remark.* — Implicit in the proposition is the statement that $\mathrm{R}f_*$ takes $\mathrm{D}_c^{(-)}(\mathscr{X})$ to $\mathrm{D}_c^{(-)}(\mathscr{Y})$. This is because the morphism f is representable and of finite type. Indeed for any $\mathrm{K} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ and $\mathrm{Y} \in \text{Lisse-Et}(\mathscr{Y})$, we have $\mathrm{R}f_*\mathrm{K}|_\mathrm{Y} = \mathrm{R}f_{\mathrm{Y}*}\mathrm{K}|_{\mathscr{X}_\mathrm{Y}}$, where $\mathrm{K}|_{\mathscr{X}_\mathrm{Y}}$ denotes the restriction of K to the étale site of $\mathscr{X}_\mathrm{Y} := \mathscr{X}\times_{\mathscr{Y}}\mathrm{Y}$ and $f_\mathrm{Y} : \mathscr{X}_{\mathrm{Y},\text{ét}} \to \mathrm{Y}_{\text{ét}}$ is the natural morphism of topos. From 2.1.10 we conclude that for any j we have

$$\mathrm{R}^j f_*\mathrm{K}|_\mathrm{Y} \simeq \mathrm{R}^j f_{\mathrm{Y}*}\tau_{\geq -n}\mathrm{K}$$

for some $n \in \mathscr{Z}$. From this the assertion follows.

Proof. — The key point is the following lemma.

5.2.3. *Lemma. — There is a canonical morphism* $\mathrm{R}f_*\Omega_{\mathscr{X}} \to \Omega_{\mathscr{Y}}$.

Proof. — Using 2.3.4 and smooth base change, it suffices to construct a functorial morphism in the case of algebraic spaces, and to show that

$$\mathscr{E}xt^i(\mathrm{R}f_*\Omega_{\mathscr{X}}, \Omega_{\mathscr{Y}}) = 0$$

for $i < 0$. Now if $\mathscr{X}$ and $\mathscr{Y}$ are algebraic spaces, we have $\Omega_{\mathscr{X}} = f^!\Omega_{\mathscr{Y}}$ so we obtain by adjunction and the fact that $\mathrm{R}f_! = \mathrm{R}f_*$ a morphism $\mathrm{R}f_*\Omega_{\mathscr{X}} \to \Omega_{\mathscr{Y}}$. For the computation of $\mathscr{E}xt$'s note that

$$\mathscr{R}hom(\mathrm{R}f_*\Omega_{\mathscr{X}}, \Omega_{\mathscr{Y}}) = f_*\mathscr{R}hom(\Omega_{\mathscr{X}}, f^!\Omega_{\mathscr{Y}}) = f_*\Lambda.$$

□

We define a map $Rf_* \circ D_{\mathscr{X}} \to D_{\mathscr{Y}} \circ f_*$ by taking the composite

$$Rf_* \mathscr{R}hom(-, \Omega_{\mathscr{X}}) \to \mathscr{R}hom(Rf_*(-), Rf_*\Omega_{\mathscr{X}}) \to \mathscr{R}hom(Rf_*(-), \Omega_{\mathscr{Y}}).$$

To verify that this map is an isomorphism we may work locally on $\mathscr{Y}$. This reduces the proof to the case when $\mathscr{X}$ and $\mathscr{Y}$ are algebraic spaces in which case the result is standard. □

5.3. *Base change by an immersion.* — In this subsection we consider the case when p is an immersion.

By replacing $\mathscr{Y}$ by a suitable open substack, one is reduced to the case when p is a closed immersion. Then, (5.0.ii) follows from the projection formula 4.5.2 as in [6, p. 81]. Let us recall the argument. Let $A \in D_c^{(-)}(\mathscr{X})$. Because p is a closed immersion, one has $p^*p_* = \mathrm{Id}$. One has (projection formula 4.5.2 for p)

$$p_*p^*Rf_!A = p_*\Lambda \overset{\mathbf{L}}{\otimes} Rf_!A.$$

One has then

$$Rf_!A \overset{\mathbf{L}}{\otimes} p_*\Lambda = Rf_!(A \overset{\mathbf{L}}{\otimes} f^*p_*\Lambda)$$

(projection formula 4.5.2 for f). But, we have trivially the base change for p, namely

$$f^*p_* = \pi_*\phi^*.$$

Therefore, one gets

$$\begin{aligned} Rf_!(A \overset{\mathbf{L}}{\otimes} f^*p_*\Lambda) &= Rf_!(A \overset{\mathbf{L}}{\otimes} \pi_*\phi^*\Lambda) \\ &= Rf_!\pi_*(\pi^*A \overset{\mathbf{L}}{\otimes} \phi^*\Lambda) \text{ projection for } \pi \\ &= p_*\phi_!\pi^*A \text{ because } \pi_* = \pi_! \text{ (5.2.1).} \end{aligned}$$

Applying p^* gives the base change isomorphism.

5.3.1. *Remark.* — Alternatively one can prove (5.0.iii) as follows. Start with A on $\mathscr{X}$ a homotopically injective complex, with constructible cohomology sheaves. Because $R^0f_*A^i$ is flasque, it is $\Gamma_{\mathscr{Y}'}$-acyclic. Then, $p_*p^!Rf_*A$ can be computed using the complex $\underline{H}^0_{\mathscr{Y}'}(R^0f_*A^i)$. On the other hand, $\pi_*\pi^!A$ can be computed by the complex $\underline{H}^0_{\mathscr{X}'}(A^i)$ which is a flasque complex (formal, or [13, V.4.11]). Therefore,

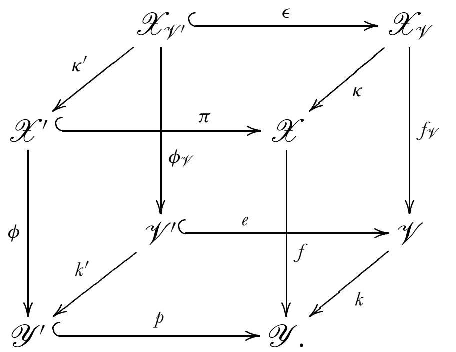
$$p_*R\phi_*\pi^!A = Rf_*\pi_*\pi^!A$$

is represented by $R^0f_*\underline{H}^0_{\mathscr{X}'}(A)$. The base change isomorphisms is therefore reduced to the formula

$$R^0f_*\underline{H}^0_{\mathscr{X}'} = \underline{H}^0_{\mathscr{Y}'}(R^0f_*).$$

For later use let us also note the following two propositions.

5.3.2. *Proposition. — Let $p : \mathscr{Y}' \hookrightarrow \mathscr{Y}$ be an immersion, and let $k : \mathscr{V} \to \mathscr{Y}$ be a smooth morphism. Let $\mathscr{V}'$ denote $\mathscr{V} \times_{\mathscr{Y}} \mathscr{Y}'$, and let $\mathscr{X}_{\mathscr{V}}$ (resp. $\mathscr{X}_{\mathscr{V}'}$) denote the base change of $\mathscr{X}$ to $\mathscr{V}$ (resp. $\mathscr{V}'$) so we have a commutative diagram*



Then for any $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ the diagram

$$\begin{array}{ccccc} k'^*p^*f_!\mathrm{F} & \xrightarrow{bc_p} & k'^*\phi_!\pi^*\mathrm{F} & \xrightarrow{bc_{k'}} & \phi_{\mathscr{V}'!}\kappa'^*\pi^*\mathrm{F} \\ \downarrow \simeq & & & & \downarrow \simeq \\ e^*k^*f_!\mathrm{F} & \xrightarrow{bc_k} & e^*f_{\mathscr{V}!}\kappa^*\mathrm{F} & \xrightarrow{bc_e} & \phi_{\mathscr{V}'!}\epsilon^*\kappa^*\mathrm{F} \end{array}$$

commutes, where bc_p and bc_e (resp. bc_k and $bc_{k'}$) are the base change isomorphisms for the immersions p and e (resp. the smooth morphisms k and k').

Proof. — This follows from the construction of the base change isomorphism for an immersion and 5.1.3. □

5.3.3. *Proposition. — Let $f : \mathscr{X} \to \mathscr{Y}$ be a finite type morphism of algebraic* S*-stacks, and consider a composite*

$$\mathscr{Y}'' \xrightarrow{r} \mathscr{Y}' \xrightarrow{p} \mathscr{Y},$$

with r an immersion, p smooth and representable, and pr an immersion. Let

$$\begin{array}{ccccc} \mathscr{X}'' & \xrightarrow{\rho} & \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \\ \downarrow \psi & & \downarrow \phi & & \downarrow f \\ \mathscr{Y}'' & \xrightarrow{r} & \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y} \end{array}$$

be the resulting commutative diagram with cartesian squares. Let

$$bc_p : p^*f_! \simeq \phi_!\pi^*$$

be the base change isomorphism defined in 5.1, and let

$$bc_r : r^*\phi_! \simeq \psi_!\rho^*, \quad bc_{pr} : (pr)^*f_! \simeq \psi_!(\pi\rho)^*$$

be the base change isomorphisms obtained above. Then for $F \in D_c^{(-)}(\mathscr{X})$ *the diagram*

(5.3.i)
$$\begin{array}{ccccc} r^*p^*f_!F & \xrightarrow{bc_p} & r^*\phi_!\pi^*F & \xrightarrow{bc_r} & \psi_!\rho^*\pi^*F \\ \downarrow\simeq & & & & \downarrow\simeq \\ (pr)^*f_!F & & \xrightarrow{bc_{pr}} & & \psi_!(\pi\rho)^*F \end{array}$$

commutes.

Proof

Special case. — Assume first that $\mathscr{Y}$ is a quasi-compact algebraic space in which case $\mathscr{Y}'$ and $\mathscr{Y}''$ are algebraic spaces. It then suffices to show that the diagram (5.3.i) restricted to the étale site of $\mathscr{Y}''$ commutes.

We may therefore assume that $\mathscr{Y}$, $\mathscr{Y}'$, and $\mathscr{Y}''$ are quasi-compact algebraic spaces, which we for the rest of the proof denote by roman letters Y, Y′, and Y″. Let

$$\widehat{bc}_p : p^!f_*F \simeq \phi_*\pi^!, \quad \widehat{bc}_r : r^!\phi_* \simeq \psi_*\rho^!, \quad \widehat{bc}_{pr} : (pr)^!f_* \simeq \psi_*(\pi\rho)^!$$

be the duals of the base change isomorphisms. Then by duality it suffices to show that for $G \in D_c^+(\mathscr{X})$ the diagram in $D_c(Y''_{\text{ét}})$

(5.3.ii)
$$\begin{array}{ccccc} r^!p^!f_*G & \xrightarrow{\widehat{bc}_p} & r^!\phi_*\pi^!G & \xrightarrow{\widehat{bc}_r} & \psi_*\rho^!\pi^!G \\ \downarrow\simeq & & & & \downarrow\simeq \\ (pr)^!f_*G & & \xrightarrow{\widehat{bc}_{pr}} & & \psi_*(\pi\rho)^!G \end{array}$$

commutes. Let

$$\alpha : (pr)^!f_*G \to (pr)^!f_*G$$

be the automorphism defined by going around the diagram, so we need to show that α is the identity.

Let $X \to \mathscr{X}$ be a smooth surjection with associated 0-coskeleton $X_\cdot \to \mathscr{X}$. Let $X'_\cdot$ (resp. $X''_\cdot$) be the base change of $X_\cdot$ to Y′ (resp. Y″) so that we have a commutative diagram of topos

(5.3.iii)
$$\begin{array}{ccccc} X''_{\cdot\text{ét}} & \xrightarrow{\rho_\cdot} & X'_{\cdot\text{ét}} & \xrightarrow{\pi_\cdot} & X_{\cdot\text{ét}} \\ \downarrow\psi_\cdot & & \downarrow\phi_\cdot & & \downarrow f_\cdot \\ Y''_{\text{ét}} & \xrightarrow{r} & Y'_{\text{ét}} & \xrightarrow{p} & Y_{\text{ét}}. \end{array}$$

By the classical theory $r^!$, $p^!$, and $(pr)^!$ are defined on the whole derived category $D(Y_{\text{ét}})$, and so we get a functor

$$(pr)^!f_* : D^+(X_{\cdot\text{ét}}) \to D^+(Y''_{\text{ét}}).$$

The automorphisms α define an automorphism of the restriction of this functor to $D^+_c(X_{\cdot\text{ét}})$. We extend this automorphism α to an automorphism of $(pr)^!_\cdot f_*$ on all of $D^+(X_{\cdot\text{ét}})$.

Let d be the relative dimension of Y' over Y. Define

$$\pi^!_\cdot : D(X_{\cdot\text{ét}}) \to D(X'_{\cdot\text{ét}})$$

to be the functor $\pi^*\langle d\rangle$.

There is a functor

$$\underline{H}^0_{X''_\cdot}(-) : (\Lambda\text{-modules on } X'_{\cdot\text{ét}}) \to (\Lambda\text{-modules on } X'_{\cdot\text{ét}})$$

sending a sheaf $F_\cdot$ on $X'_{\cdot\text{ét}}$ to the sheaf whose restriction to X'_n is $\underline{H}^0_{X''_n}(F_n)$. If $\sigma : [m] \to [n]$ is a morphism in the simplicial category, then the square

$$\begin{array}{ccc} X''_n & \longrightarrow & X'_n \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ X''_m & \longrightarrow & X'_m \end{array}$$

is cartesian so that there is a canonical morphism $\sigma^*\underline{H}^0_{X''_m}(F_m) \to \underline{H}^0_{X''_n}(\sigma^* F_m)$ and hence a natural map

$$\sigma^*\underline{H}^0_{X''_m}(F_m) \to \underline{H}^0_{X''_n}(F_n)$$

giving the $\underline{H}^0_{X''_n}(F_n)$ the structure of a sheaf on $X'_{\cdot\text{ét}}$. Note also that $\underline{H}^0_{X''_\cdot}(-)$ is a left exact functor. We define $\rho^!_\cdot$ to be the functor

$$\rho^* R\underline{H}^0_{X''_\cdot}(-) : D^+(X'_{\cdot\text{ét}}) \to D^+(X''_{\cdot\text{ét}}).$$

Similarly we define

$$(\pi_\cdot\rho_\cdot)^! := (\pi_\cdot\rho_\cdot)^* R\underline{H}^0_{X''_\cdot}(-) : D^+(X_{\cdot\text{ét}}) \to D^+(X''_{\cdot\text{ét}}).$$

As in the classical case the functor $\rho^!_\cdot$ (resp. $(\pi_\cdot\rho_\cdot)^!$) is right adjoint to the functor $\rho_{\cdot *}$ (resp. $(\pi_\cdot\rho_\cdot)_*$).

There is an isomorphism

(**5.3.iv**) $$(\pi_\cdot\rho_\cdot)^! \to \rho^!_\cdot\pi^!_\cdot$$

defined as follows. Let $F^\cdot$ be a bounded below complex of injectives on $X_\cdot$, and choose a morphism to a double complex

$$\pi^* F^\cdot\langle d\rangle \to I^{\cdot\cdot}$$

such that for all i the map $\pi^*F^i \to I^{i\cdot}$ is an injective resolution. Then $\rho_\cdot^!\pi^!(F^\cdot)$ is represented by the total complex of the bicomplex

$$J^{\cdot\cdot} := \rho_\cdot^* \underline{H}^0_{X''_\cdot}(I^{\cdot\cdot}).$$

Now for any fixed i and n, the restriction of $J^{i\cdot}$ to X''_n computes $\rho_n^!\pi_n^!(F^i_n)$ (defined in the classical way), and therefore $\mathscr{H}^k(J^{i\cdot})$ is zero for $k \neq 0$. Moreover, for every n the restriction of $\mathscr{H}^k(J^{i\cdot})$ to X''_n is by the classical theory canonically isomorphic to $(\pi_n\rho_n)^*\underline{H}^0_{X''_n}(F^i_n)$. Moreover these isomorphisms are compatible so we obtain an isomorphism $(\pi_\cdot\rho_\cdot)^*\underline{H}^0_{X''_\cdot}(F^i_\cdot) \simeq \mathscr{H}^0(J^{i\cdot})$. Let $\tilde{J}^{\cdot\cdot}$ denote the bicomplex with

$$\tilde{J}^{i\cdot} = \tau_{\geq 0}J^{i\cdot}.$$

We then have a diagram

$$\begin{array}{ccc} J^{\cdot\cdot} & \longrightarrow & \tilde{J}^{\cdot\cdot} \\ & & \uparrow \\ & & (\pi_\cdot\rho_\cdot)^*\underline{H}^0_{X''_\cdot}(F^\cdot) \end{array}$$

where all the morphisms induce quasi-isomorphisms upon passing to the total complexes. This defines the isomorphism (5.3.iv).

There is a canonical map of functors $D^+(X_{\cdot \text{ét}}) \to D^+(Y'_{\text{ét}})$

$$(\mathbf{5.3.v}) \qquad \widehat{bc}_p : p^! f_{\cdot *} \to \phi_{\cdot *}\pi_\cdot^!$$

defined by the canonical map $p^*f_{\cdot *} \to \phi_{\cdot *}\pi_\cdot^*$ and the isomorphisms $p^! \simeq p^*\langle d\rangle$ (by [13, XVIII.3.2.5]) and $\pi_\cdot^! = \pi_\cdot^*\langle d\rangle$ (by definition). For any $F \in D(X_\cdot)$ the map $p^*R^q f_{\cdot *}F \to R^q\phi_{\cdot *}\pi_\cdot^*F$ extends to a morphism of spectral sequences

$$\begin{array}{ccc} E_1^{st} = p^*R^s f_{t*}F_t & \Longrightarrow & p^*R^{s+t}f_{\cdot *}F \\ & \downarrow & \\ E_1^{st} = R^s\phi_{t*}\pi_t^*F_t & \Longrightarrow & R^{s+t}\phi_{\cdot *}\pi^*F, \end{array}$$

and hence by the smooth base change theorem [13, XVI.1.2], the map (5.3.v) is an isomorphism.

There is also an isomorphism of functors $D^+(X'_{\cdot \text{ét}}) \to D^+(Y''_{\text{ét}})$

$$\widehat{bc}_r : r^!\phi_{\cdot *} \to \psi_{\cdot *}\rho_\cdot^!$$

obtained by taking derived functors of the natural isomorphism

$$r^*\underline{H}^0_{Y''}(R^0\phi_{\cdot *}) \simeq R^0\psi_{\cdot *}\big(\rho^*\underline{H}^0_{X''_\cdot}(0)\big).$$

Similarly there is an isomorphism of functors

$$\widehat{bc}_{pr} : (pr)^! f_{\cdot *} \to \psi_{\cdot *}(\pi_\cdot\rho_\cdot)^!.$$

We therefore obtain for any $G \in D(X_\cdot)$ a diagram

$$\begin{array}{ccccc} r^!p^!f_{\cdot *}G & \xrightarrow{\widehat{bc}_p} & r^!\phi_{\cdot *}\pi_\cdot^!G & \xrightarrow{\widehat{bc}_r} & \psi_{\cdot *}\rho_\cdot^!\pi_\cdot^!G \\ \downarrow\simeq & & & & \downarrow\simeq \\ (pr)^!f_{\cdot *}G & & \xrightarrow{\widehat{bc}_{pr}} & & \psi_{\cdot *}(\pi_\cdot\rho_\cdot)^!G \end{array} \qquad (\mathbf{5.3.vi})$$

and therefore also an automorphism, which we again denote by α, of the functor

$$(pr)^!f_{\cdot *} : D^+(X_{\cdot \text{ét}}) \to D^+(Y''_{\text{ét}}).$$

It follows from the construction that the restriction of this automorphism to $D^+_c(X_{\cdot \text{ét}}) \simeq D^+_c(\mathscr{X})$ agrees with the earlier defined automorphisms (as the base change isomorphisms agree).

Since $(pr)^!f_{\cdot *}$ is the derived functor of the functor

$$(pr)^*\underline{H}^0_{Y''}R^0f_{\cdot *}(-) : (\Lambda\text{-modules on } X_\cdot) \to (\Lambda\text{-modules on } Y'')$$

it suffices to show that for any sheaf of Λ-modules F on $X_\cdot$, the automorphism of $(pr)^*\underline{H}^0_{Y''}R^0f_{\cdot *}(F)$ defined by α is the identity.

For this consider the natural inclusion $R^0f_{\cdot *}(F) \hookrightarrow R^0f_{0*}(F_0)$, where F_0 denotes the X_0-component of F. The usual base change theorems applied to the diagram

$$\begin{array}{ccccc} X''_0 & \xrightarrow{\rho_0} & X'_0 & \xrightarrow{\pi_0} & X_0 \\ \downarrow\psi_0 & & \downarrow\psi_0 & & \downarrow f_0 \\ Y'' & \xrightarrow{r} & Y' & \xrightarrow{p} & Y \end{array}$$

give a diagram

$$\begin{array}{ccccc} r^!p^!f_{0*}F_0 & \xrightarrow{\widehat{bc}_p} & r^!\phi_{0*}\pi_0^!F_0 & \xrightarrow{\widehat{bc}_r} & \psi_{0*}\rho_0^!\pi_0^!F_0 \\ \downarrow\simeq & & & & \downarrow\simeq \\ (pr)^!f_{0*}F_0 & & \xrightarrow{\widehat{bc}_{pr}} & & \psi_{0*}(\pi_0\rho_0)^!F_0 \end{array} \qquad (\mathbf{5.3.vii})$$

which commutes by the classical theory. Moreover, the restriction maps define a morphism from the diagram (5.3.vi) for F to the diagram (5.3.vii). In particular, the automorphism of $(pr)^*\underline{H}^0_{Y''}R^0f_{\cdot *}(F)$ defined by α is equal to the restriction of the identity automorphism on $(pr)^*\underline{H}^0_{Y''}R^0f_{0*}(F_0)$.

General case. — Let us chose a presentation $Y \to \mathscr{Y}$ and let $Y_\cdot$ be the 0-cosqueleton. It is a simplicial algebraic space. We get by pull-back a commutative diagram from the diagram (5.3.iii)

(**5.3.viii**)
$$\begin{array}{ccccc} X''_{..\text{ét}} & \xrightarrow{\rho_{..}} & X'_{..\text{ét}} & \xrightarrow{\pi_{..}} & X_{..\text{ét}} \\ \downarrow{\scriptstyle \psi_{..}} & & \downarrow{\scriptstyle \phi_{..}} & & \downarrow{\scriptstyle f_{..}} \\ Y''_{.\text{ét}} & \xrightarrow{r_{.}} & Y'_{.\text{ét}} & \xrightarrow{p_{.}} & Y_{.\text{ét}}. \end{array}$$

One defines the bisimplicial version of π, ρ as above ($\pi^!_{..} = \pi^*\langle d\rangle$) and $\rho_{..} = \rho^*_{..}\mathrm{R}\underline{\mathrm{H}}^0_{X''_{..}}$). As above (5.3.vi), we get an automorphism $\alpha_{.}$ of the functor $(p_{.}r_{.})^! f_{..}$. The theorem to be proved is the equality $\alpha_{.} = \mathrm{Id}$. As above, one observes that $\alpha_{.}$ is the derived morphism of the corresponding morphism $\alpha^0_{.}$ of $(p_{.}r_{.})^*\underline{\mathrm{H}}^0_{Y''_{.}}\mathrm{R}^0 f_{..*}$. But proving that the latter is the identity is now a local question on $Y''_{.}$. One can therefore assume that $\mathscr{Y}$ is an algebraic space, which has been done before. □

5.3.4. *Proposition. — Let $f : \mathscr{X} \to \mathscr{Y}$ be a finite type morphism of algebraic* S*-stacks, and consider a composite*

$$\mathscr{Y}'' \xrightarrow{r} \mathscr{Y}' \xrightarrow{p} \mathscr{Y},$$

with r and p immersions. Let

$$\begin{array}{ccccc} \mathscr{X}'' & \xrightarrow{\rho} & \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}'' & \xrightarrow{r} & \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y} \end{array}$$

be the resulting commutative diagram with cartesian squares. Let

$$bc_p : p^* f_! \simeq \phi_! \pi^*, \quad bc_r : r^*\phi_! \simeq \psi_!\rho^*, \quad bc_{pr} : (pr)^* f_! \simeq \psi_!(\pi\rho)^*$$

be the base change isomorphisms. Then for $\mathrm{F} \in \mathrm{D}^{(-)}_c(\mathscr{X})$ the diagram

$$\begin{array}{ccccc} r^*p^*f_!\mathrm{F} & \xrightarrow{bc_p} & r^*\phi_!\pi^*\mathrm{F} & \xrightarrow{bc_r} & \psi_!\rho^*\pi^*\mathrm{F} \\ \downarrow{\scriptstyle \simeq} & & & & \downarrow{\scriptstyle \simeq} \\ (pr)^*f_!\mathrm{F} & & \xrightarrow{bc_{pr}} & & \psi_!(\pi\rho)^*\mathrm{F} \end{array}$$

commutes.

Proof. — This follows from a similar argument to the one used in the proof of 5.3.3 reducing to the case of schemes. □

5.4. *Base change by a universal homeomorphism.* — If p is a universal homeomorphism, then $p^! = p^*$ and $\pi^! = \pi^*$. Thus in this case (5.0.iii) is equivalent to an isomorphism $p^*Rf_* \to R\phi_*\pi^*$. We define such a morphism by taking the usual base change morphism (adjunction).

Let $A \in D_c(\mathscr{X})$. Using a hypercover of $\mathscr{X}$ as in 5.1, one sees that to prove that the map $p^*Rf_*A \to R\phi_*\pi^*A$ is an isomorphism it suffices to consider the case when $\mathscr{X}$ is a scheme. Furthermore, by the smooth base change formula already shown, it suffices to prove that this map is an isomorphism after making a smooth base change $Y \to \mathscr{Y}$. We may therefore assume that $\mathscr{Y}$ is also a scheme in which case the result follows from the classical corresponding result for étale topology (see [12, IV.4.10]).

5.5. *Base change for smoothable morphisms*

5.5.1. *Definition. — A morphism* $p : \mathscr{Y}' \to \mathscr{Y}$ *is* smoothable *if there exists a factorization of* p

(**5.5.i**) $$\mathscr{Y}' \overset{i}{\hookrightarrow} \mathscr{V} \xrightarrow{q} \mathscr{Y},$$

where i *is an immersion and* q *is a smooth representable morphism.*

5.5.2. *Example.* — Let $Y' \to \mathscr{Y}$ be a locally of finite type morphism with Y' a scheme. Then for any geometric point $\bar{y}' \to Y'$ there is an étale neighborhood W' of $\bar{y}'$ such that the composite morphism $W' \to Y' \to \mathscr{Y}$ is smoothable. For this choose a smooth morphism $V \to \mathscr{Y}$ with V an affine scheme and with $\bar{y}' \times_{\mathscr{Y}} V$ nonempty. Then the morphism $V \times_{\mathscr{Y}} Y' \to Y'$ is smooth, and therefore there exists an étale neighborhood $W' \to Y'$ with W' affine and a lifting $W' \to V$ of the morphism $W' \to V$. Since the morphism $W' \to V$ is of finite type there exists an immersion $W' \hookrightarrow \mathbf{A}^r_V$ over V. We therefore obtain a factorization

$$W' \hookrightarrow \mathbf{A}^r_V \longrightarrow \mathscr{Y}$$

of $W' \to \mathscr{Y}$.

Let $p : \mathscr{Y}' \to \mathscr{Y}$ be a smoothable morphism, and choose a factorization (5.5.i). We then obtain a commutative diagram

$$\begin{array}{ccccc} \mathscr{X}' & \xrightarrow{\iota} & \mathscr{Z} & \xrightarrow{\epsilon} & \mathscr{X} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle f} \\ \mathscr{Y}' & \xrightarrow{i} & \mathscr{V} & \xrightarrow{q} & \mathscr{Y}, \end{array}$$

with cartesian squares. We then obtain for $F \in D_c(\mathscr{X})$ an isomorphism

(**5.5.ii**) $$p^*f_!F \simeq \phi_!\pi^*F$$

from the composite

$$\begin{aligned} p^*f_!\mathrm{F} &\simeq i^*q^*f_!\mathrm{F} \\ &\simeq i^*\psi_!\epsilon^*\mathrm{F} \qquad (q^*f_! = \psi_!\epsilon^* \text{ by } 5.1) \\ &\simeq \phi_!\iota^*\epsilon^*\mathrm{F} \qquad (i^*\psi_! = \phi_!\iota^* \text{ by } 5.3) \\ &\simeq \phi_!\pi^*\mathrm{F}. \end{aligned}$$

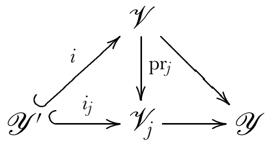
5.5.3. *Lemma.* — *The isomorphism* (5.5.ii) *is independent of the choice of the factorization* (5.5.i) *of* p.

Proof. — Consider two factorizations

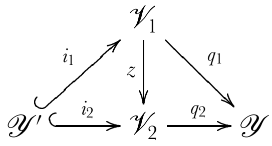
(5.5.iii) $$\mathscr{Y}' \xhookrightarrow{i_j} \mathscr{V}_i \xrightarrow{q_j} \mathscr{Y} \qquad j = 1, 2,$$

with the q_j smooth and representable.

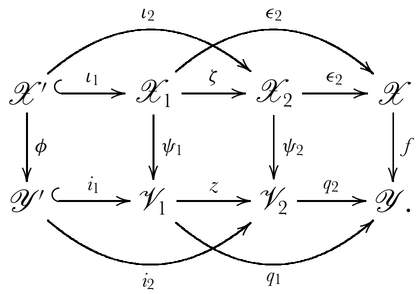
Let $\mathscr{V}$ denote the fiber product $\mathscr{V}_1 \times_{\mathscr{Y}} \mathscr{V}_2$ and let $i : \mathscr{Y}' \to \mathscr{V}$ denote the map $i_1 \times i_2$. Since the morphisms $\mathscr{V}_j \to \mathscr{Y}$ are representable, the map i is again an immersion. Consideration of the diagrams



then further reduces the proof to showing that two factorization (5.5.iii) define the same base change isomorphism under the further assumption that there exists a smooth morphism $z : \mathscr{V}_1 \to \mathscr{V}_2$ such that the diagram



commutes. Let $\mathscr{X}_i$ denote $\mathscr{X} \times_{\mathscr{Y}} \mathscr{V}_i$ so we have a commutative diagram



It then suffices to show that the following diagram commutes for $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$

$$\begin{tikzcd}
i_2^* q_2^* f_! \mathrm{F} \arrow[r, "bc_{q_2}"] \arrow[d, "\simeq"] & i_2^* \psi_{2!} \epsilon_2^* \mathrm{F} \arrow[r, "\simeq"] \arrow[rdd, bend left, "bc_{i_2}"] & i_1^* z^* \psi_{2!} \epsilon_2^* \mathrm{F} \arrow[d, "bc_z"] \\
i_1^* q_1^* f_! \mathrm{F} \arrow[rr, "bc_{q_1}"] & & i_1^* \psi_{1!} \zeta^* \epsilon_2^* \mathrm{F} \arrow[d, "bc_{i_1}"] \\
& & \phi_! \pi^* \mathrm{F},
\end{tikzcd}$$

where for a morphism ? which is either an immersion or smooth we write $bc_?$ for the base change isomorphism defined in either 5.1 or 5.3. This follows from 5.1.2 which shows that the inside pentagon commutes, and 5.3.3 which shows that the diagram

$$\begin{array}{ccc}
i_2^* \psi_{2!} \epsilon_2^* \mathrm{F} & \xrightarrow{bc_{i_2}} & \phi_! \pi^* \mathrm{F} \\
\downarrow \simeq & & \uparrow bc_{i_1} \\
i_1^* z^* \psi_{2!} \epsilon_2^* \mathrm{F} & \xrightarrow{bc_z} & i_1^* \psi_{1!} \zeta^* \epsilon_2^* \mathrm{F}
\end{array}$$

commutes. □

5.5.4. *Proposition. — Consider a diagram of algebraic stacks*

$$\mathscr{Y}'' \xrightarrow{h} \mathscr{Y}' \xrightarrow{p} \mathscr{Y}$$

with h, p, and ph smoothable and representable, and let

$$\begin{array}{ccccc}
\mathscr{X}'' & \xrightarrow{\eta} & \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \\
\downarrow \psi & & \downarrow \phi & & \downarrow f \\
\mathscr{Y}'' & \xrightarrow{h} & \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y}
\end{array}$$

be the resulting commutative diagram with cartesian squares. Let

$$bc_p : p^* f_! \to \phi_! \pi^*, \quad bc_h : h^* \phi_! \to \psi_! \eta^*, \quad bc_{ph} : (ph)^* f_! \to \psi_! (\pi\eta)^*$$

be the base change isomorphisms. Then for any $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ the diagram

$$\begin{array}{ccccc}
h^* p^* f_! \mathrm{F} & \xrightarrow{bc_p} & h^* \phi_! \pi^* \mathrm{F} & \xrightarrow{bc_h} & \psi_! \eta^* \pi^* \mathrm{F} \\
\downarrow \simeq & & & & \downarrow \simeq \\
(ph)^* f_! \mathrm{F} & & \xrightarrow{bc_{ph}} & & \psi_! (\pi\eta)^* \mathrm{F}
\end{array}$$

commutes.

5.5.5. *Remark.* — In fact the proof will show that if p and ph are smoothable then h is automatically smoothable.

Proof. — Note first that there exists a commutative diagram

$$\begin{array}{ccccc} & & & & \mathscr{V} \\ & & & \overset{j}{\nearrow} & \downarrow{\scriptstyle k} \\ & & \mathscr{W} & & \mathscr{V}' \\ & \overset{w}{\nearrow} & \downarrow{\scriptstyle s} & \overset{i}{\nearrow} & \downarrow{\scriptstyle t} \\ \mathscr{Y}'' & \xrightarrow{h} & \mathscr{Y}' & \xrightarrow{p} & \mathscr{Y}, \end{array} \tag{5.5.iv}$$

with i, w, j immersions and s, k, t smooth and representable and the square

$$\begin{array}{ccc} \mathscr{W} & \xrightarrow{j} & \mathscr{V} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle k} \\ \mathscr{Y}' & \xrightarrow{i} & \mathscr{V} \end{array}$$

cartesian. Indeed by assumption there exists factorizations of p and ph respectively

$$\mathscr{Y}' \xrightarrow{i} \mathscr{V}' \xrightarrow{t} \mathscr{Y}$$

and

$$\mathscr{Y}'' \xrightarrow{g} \mathscr{V}'' \xrightarrow{m} \mathscr{Y}$$

with t and m smooth and representable and g and i closed immersions. Now take $\mathscr{V}$ to be $\mathscr{V}' \times_{\mathscr{Y}} \mathscr{V}''$, let k be the projection, and define $\mathscr{W}$ to be the fiber product $\mathscr{Y}' \times_{i, \mathscr{V}'} \mathscr{V}$. Define w to be the natural map defined by g and ih.

Note that this construction gives a diagram with i and j closed immersions. After further replacing $\mathscr{V}$ by an open substack through which jw factors as a closed immersion, we may also assume that w is a closed immersion.

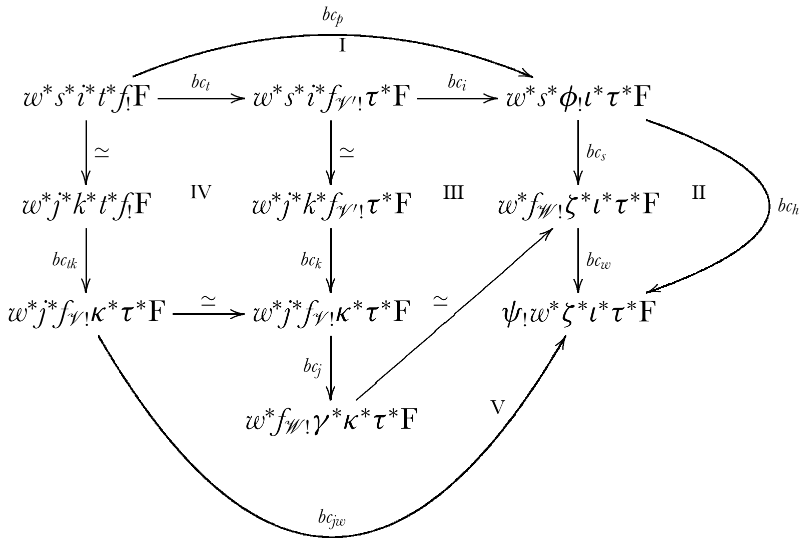
Base changing the diagram (5.5.iv) along $f : \mathscr{X} \to \mathscr{Y}$ we obtain a commutative diagram

$$\begin{array}{ccccc} & & & & \mathscr{X}_{\mathscr{V}} \\ & & & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle \kappa} \\ & & \mathscr{X}_{\mathscr{W}} & & \mathscr{X}_{\mathscr{V}'} \\ & \overset{\epsilon}{\nearrow} & \downarrow{\scriptstyle \zeta} & \overset{\iota}{\nearrow} & \downarrow{\scriptstyle \tau} \\ \mathscr{X}'' & \xrightarrow{\eta} & \mathscr{X}' & \xrightarrow{\pi} & \mathscr{X} \end{array}$$

over (5.5.iv). Let

$$f_{\mathscr{V}'} : \mathscr{X}_{\mathscr{V}'} \to \mathscr{V}', \quad f_{\mathscr{V}} : \mathscr{X}_{\mathscr{V}} \to \mathscr{V}, \quad f_{\mathscr{W}} : \mathscr{X}_{\mathscr{W}} \to \mathscr{W}$$

be the projections. We then obtain a diagram



The small inside diagrams I and II commute by the definition of bc_p and bc_h, and the composite $bc_{jw} \circ bc_{tk}$ is by definition bc_{ph}. The inside diagram labeled III commutes by 5.3.2. This reduces the proof to the case when both p and h are smooth (resp. closed immersions) as this case implies that the inside diagram IV (resp. V) commutes. In this case the result is 5.1.2 (resp. 5.3.4). □

5.5.6. *Corollary. — For any commutative diagram* (5.0.i) *of algebraic stacks,* $\mathrm{F} \in \mathrm{D}_c^{(-)}(\mathscr{X})$ *and* $q \in \mathbf{Z}$ *there is a canonical isomorphism of sheaves on* $\mathscr{Y}'$

$$p^*\mathrm{R}^q f_!(\mathrm{F}) \simeq \mathrm{R}^q\phi_!\pi^*\mathrm{F}.$$

Proof. — By 5.5.2 it suffices to construct for every smooth morphism $\mathrm{Y}' \to \mathscr{Y}'$ such that the composite $\mathrm{Y}' \to \mathscr{Y}' \to \mathscr{Y}$ is smoothable an isomorphism

$$\sigma_{\mathrm{Y}'} : p^*\mathrm{R}^q f_!\mathrm{F}|_{\mathrm{Y}'_{\text{ét}}} \simeq \mathrm{R}^q\phi_!\pi^*\mathrm{F}|_{\mathrm{Y}'_{\text{ét}}}$$

such that if $h : \mathrm{Y}'' \to \mathrm{Y}'$ is a morphism in Lis-Et($\mathscr{Y}'$) with the composite $\mathrm{Y}'' \to \mathscr{Y}' \to \mathscr{Y}$ smoothable, then the diagram of sheaves on $\mathrm{Y}''_{\text{ét}}$

$$(\mathbf{5.5.v}) \qquad \begin{array}{ccc} h^*(p^*\mathrm{R}^q f_!\mathrm{F}|_{\mathrm{Y}'_{\text{ét}}}) & \xrightarrow{\sigma_{\mathrm{Y}'}} & h^*(\mathrm{R}^q\phi_!\pi^*\mathrm{F}|_{\mathrm{Y}'_{\text{ét}}}) \\ \downarrow \simeq & & \downarrow \simeq \\ p^*\mathrm{R}^q f_!\mathrm{F}|_{\mathrm{Y}''_{\text{ét}}} & \xrightarrow{\sigma_{\mathrm{Y}''}} & \mathrm{R}^q\phi_!\pi^*\mathrm{F}|_{\mathrm{Y}''_{\text{ét}}} \end{array}$$

commutes. For this take $\sigma_{\mathrm{Y}'}$ to be the map induced by the base change isomorphism for the smoothable morphisms $\mathrm{Y}' \to \mathscr{Y}$. The commutativity (5.5.v) follows from 5.5.4. □

5.6. *Kunneth formula.* — Throughout this section we assume that S is regular so that $\Omega_S \simeq \Lambda$.

Let $\mathscr{Y}_1$ and $\mathscr{Y}_2$ be S-stacks locally of finite type, and set $\mathscr{Y} := \mathscr{Y}_1 \times \mathscr{Y}_2$. Let $p_i : \mathscr{Y} \to \mathscr{Y}_i$ ($i = 1, 2$) be the projection and for two complexes $L_i \in D_c^{(-)}(\mathscr{Y}_i)$ let $L_1 \overset{\mathbf{L}}{\otimes}_S L_2 \in D_c^{(-)}(\mathscr{Y})$ denote $p_1^* L_1 \overset{\mathbf{L}}{\otimes}_\Lambda p_2^* L_2$.

5.6.1. *Lemma.* — *There is a natural isomorphism* $\Omega_{\mathscr{Y}} \simeq \Omega_{\mathscr{Y}_1} \overset{\mathbf{L}}{\otimes}_S \Omega_{\mathscr{Y}_2}$.

Proof. — For any smooth morphisms $U_i \to \mathscr{Y}_i$ ($i = 1, 2$) with U_i a scheme, there is a canonical isomorphism

$$\text{(5.6.i)} \qquad \Omega_{\mathscr{Y}}|_{U_1 \times_S U_2} \simeq \Omega_{\mathscr{Y}_1}|_{U_1} \overset{\mathbf{L}}{\otimes}_S \Omega_{\mathscr{Y}_2}|_{U_2},$$

and this isomorphism is functorial with respect to morphisms $V_i \to U_i$ (in the case when $\dim(S) \leq 1$ this is [14, III.1.7.6]). It follows that the sheaf $\Omega_{\mathscr{Y}_1} \overset{\mathbf{L}}{\otimes}_S \Omega_{\mathscr{Y}_2}$ also satisfies the $\mathscr{E}xt$-condition (2.3.3), and hence to give an isomorphism as in the lemma it suffices to give an isomorphism in the derived category of $U_1 \times_S U_2$ for all smooth morphisms $U_i \to \mathscr{Y}_i$. □

5.6.2. *Lemma.* — *Let* $(\mathscr{T}, \Lambda)$ *be a ringed topos. Then for any*

$$P_1, P_2, M_1, M_2 \in D(\mathscr{T}, \Lambda),$$

there is a canonical morphism

$$\mathscr{R}hom(P_1, M_1) \overset{\mathbf{L}}{\otimes} \mathscr{R}hom(P_2, M_2) \to \mathscr{R}hom(P_1 \overset{\mathbf{L}}{\otimes} P_2, M_1 \overset{\mathbf{L}}{\otimes} M_2).$$

Proof. — It suffices to give a morphism

$$\mathscr{R}hom(P_1, M_1) \overset{\mathbf{L}}{\otimes} \mathscr{R}hom(P_2, M_2) \overset{\mathbf{L}}{\otimes} P_1 \overset{\mathbf{L}}{\otimes} P_2 \to M_1 \overset{\mathbf{L}}{\otimes} M_2.$$

This we get by tensoring the two evaluation morphisms

$$\mathscr{R}hom(P_i, M_i) \overset{\mathbf{L}}{\otimes} P_i \to M_i.$$

□

5.6.3. *Lemma.* — *Let* $A, B \in D(\mathscr{X})$. *Then we have*

$$A \overset{\mathbf{L}}{\otimes} B = \operatorname{hocolim} \tau_{\leq n} A \overset{\mathbf{L}}{\otimes} \tau_{\leq n} B.$$

Proof. — By 4.3.2, we have $A \simeq \operatorname{hocolim}_n \tau_{\le n}A$, and therefore we have a distinguished triangle

$$\bigoplus \tau_{\le n}A \xrightarrow{\text{1-shift}} \bigoplus \tau_{\le n}A \to A.$$

Tensoring this triangle with B we get a distinguished triangle

$$\bigoplus \tau_{\le n}A \overset{\mathbf{L}}{\otimes} B \xrightarrow{\text{1-shift}} \bigoplus \tau_{\le n}A \overset{\mathbf{L}}{\otimes} B \to A \overset{\mathbf{L}}{\otimes} B$$

proving

$$\operatorname{hocolim} \tau_{\le n}A \overset{\mathbf{L}}{\otimes} B = A \overset{\mathbf{L}}{\otimes} B.$$

Applying this process again we find

$$\operatorname{hocolim} \tau_{\le n}A \overset{\mathbf{L}}{\otimes} \tau_{\le m}B = A \overset{\mathbf{L}}{\otimes} B.$$

Because the diagonal is cofinal in $\mathbf{N} \times \mathbf{N}$, the lemma follows. □

5.6.4. *Proposition.* — *For* $L_i \in D_c^{(-)}(\mathscr{Y}_i)$ $(i = 1, 2)$, *there is a canonical isomorphism*

$$D_{\mathscr{Y}_1}(L_1)\overset{\mathbf{L}}{\otimes}_S D_{\mathscr{Y}_2}(L_2) \simeq D_{\mathscr{Y}}(L_1\overset{\mathbf{L}}{\otimes}_S L_2). \tag{5.6.ii}$$

Proof. — By 5.6.1 and 5.6.2 there is a canonical morphism

$$D_{\mathscr{Y}_1}(L_1)\overset{\mathbf{L}}{\otimes}_S D_{\mathscr{Y}_2}(L_2) \to D_{\mathscr{Y}}(L_1\overset{\mathbf{L}}{\otimes}_S L_2). \tag{5.6.iii}$$

To verify that this map is an isomorphism, it suffices to show that for every $j \in \mathbf{Z}$ the map

$$\mathscr{H}^j(D_{\mathscr{Y}_1}(L_1)\overset{\mathbf{L}}{\otimes}_S D_{\mathscr{Y}_2}(L_2)) \to \mathscr{H}^j(D_{\mathscr{Y}}(L_1\overset{\mathbf{L}}{\otimes}_S L_2)). \tag{5.6.iv}$$

Because $\overset{\mathbf{L}}{\otimes}$ commutes with homotopy colimits (5.6.3), we deduce from $D(A) = \operatorname{hocolim} D(\tau_{\ge m}A)$ (use 4.3.2) that to prove this we may replace L_i by $\tau_{\ge m}L_i$ for m sufficiently negative, and therefore it suffices to consider the case when $L_i \in D_c^b(\mathscr{Y}_i)$. Furthermore, we may work locally in the smooth topology on $\mathscr{Y}_1$ and $\mathscr{Y}_2$, and therefore it suffices to consider the case when the stacks $\mathscr{Y}_i$ are schemes. In this case the result is [13, XVII, 5.4.3]. □

Now consider finite type morphisms of S-stacks $f_i : \mathscr{X}_i \to \mathscr{Y}_i$ $(i = 1, 2)$, and let $f : \mathscr{X} := \mathscr{X}_1 \times_S \mathscr{X}_2 \to \mathscr{Y} := \mathscr{Y}_1 \times_S \mathscr{Y}_2$ be the morphism obtained by taking fiber products. Let $L_i \in D_c^{(-)}(\mathscr{X}_i)$.

5.6.5. *Theorem.* — *There is a canonical isomorphism in* $D_c(\mathscr{Y})$

$$Rf_!(L_1 \overset{\mathbf{L}}{\otimes}_S L_2) \to Rf_{1!}(L_1) \overset{\mathbf{L}}{\otimes}_S Rf_{2!}(L_2). \tag{5.6.v}$$

Proof. — We define the morphism (5.6.v) as the composite

$$\begin{aligned} Rf_!(L_1 \overset{\mathbf{L}}{\otimes}_S L_2) &\xrightarrow{\simeq} D_{\mathscr{Y}}(f_* D_{\mathscr{X}}(L_1 \overset{\mathbf{L}}{\otimes}_S L_2)) \\ &\xrightarrow{\simeq} D_{\mathscr{Y}}(f_*(D_{\mathscr{X}_1}(L_1) \overset{\mathbf{L}}{\otimes}_S D_{\mathscr{X}_2}(L_2))) \\ &\longrightarrow D_{\mathscr{Y}}(f_{1*} D_{\mathscr{X}_1}(L_1) \overset{\mathbf{L}}{\otimes}_S (f_{2*} D_{\mathscr{X}_2}(L_2))) \\ &\xrightarrow{\simeq} D_{\mathscr{Y}_1}(f_{1*} D_{\mathscr{X}_1}(L_1)) \overset{\mathbf{L}}{\otimes}_S D_{\mathscr{Y}_2}(f_{2*} D_{\mathscr{X}_2}(L_2)) \\ &\xrightarrow{\simeq} Rf_{1!}(L_1) \overset{\mathbf{L}}{\otimes}_S Rf_{2!}(L_2). \end{aligned}$$

That this map is an isomorphism follows from a standard reduction to the case of schemes using hypercovers of $\mathscr{X}_i$, biduality, and the spectral sequences 4.9.1. □

REFERENCES

1. K. A. Behrend, *Derived l-Adic Categories for Algebraic Stacks*, Mem. Amer. Math. Soc., vol. 163, no. 774, Amer. Math. Soc., Providence, RI, 2003.
2. A. A. Beĭlinson, J. Bernstein, and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces, I (Luminy, 1981)*, Astérisque, vol. 100, pp. 5–171, Soc. Math. France, Paris, 1982.
3. M. Bokstedt and A. Neeman, Homotopy limits in triangulated categories, *Compos. Math.*, **86** (1993), 209–234.
4. P. Deligne, Cohomologie étale, in *Séminaire de Géométrie Algébrique du Bois-Marie SGA* $(4\frac{1}{2})$, Lect. Notes Math., vol. 569, Springer, Berlin, 1977.
5. J. Dieudonné and A. Grothendieck, Éléments de géométrie algébrique, *Publ. Math., Inst. Hautes Étud. Sci.*, **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1961–1967).
6. E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, Ergeb. Math. Grenzgeb., 3. Folge, vol. 13, Springer, Berlin, 1988.
7. E. Frenkel, D. Gaitsgory, and K. Vilonen, On the geometric Langland's conjecture, *J. Amer. Math. Soc.*, **15** (2002), 367–417.
8. K. Fujiwara, A proof of the absolute purity conjecture (after Gabber), in S. Usui, M. Green, L. Illusie, K. Kato, E. Looijenga, S. Mukai, S. Saito (eds.), *Algebraic Geometry 2000, Azumino*, Adv. Stud. Pure Math., vol. 36, pp. 153–183, Math. Soc. Japan, Tokyo, 2002.
9. O. Gabber, A finiteness theorem for non abelian H^1 of excellent schemes, in *Conférence en l'honneur de L. Illusie, Orsay (2005)*, http://www.math.polytechnique.fr/~laszlo/gdtgabber/non-abelien.pdf.
10. O. Gabber, Finiteness theorems for étale cohomology of excellent schemes, in *Conference in Honor of P. Deligne on the Occasion of his 61st Birthday, IAS, Princeton (2005)*, http://www.math.polytechnique.fr/~laszlo/gdtgabber/abelien.pdf.
11. P.-P. Grivel, Catégories dérivées et foncteurs dérivés, in A. Borel (ed.), *Algebraic* D-*Modules*, Perspect. Math., vol. 2, Academic Press, Boston, MA, 1987.
12. A. Grothendieck, Revétements étales et groupe fondamental, in *Séminaire de Géométrie Algébrique du Bois-Marie (SGA 1)*, Doc. Math., vol. 3, Société Mathématique de France, Paris, 2003.
13. M. Artin, A. Grothendieck, and J.-L. Verdier, Théorie des topos et cohomologie étale des schémas, in *Séminaire de Géométrie Algébrique du Bois-Marie (SGA 4)*, Lect. Notes Math. vols. 269, 270, 305, Springer, Berlin, 1972.
14. A. Grothendieck, Cohomologie *l*-adique et fonctions L, in L. Illusie (ed.), *Séminaire de Géometrie Algébrique du Bois-Marie (SGA 5)*, Lect. Notes Math., vol. 589, Springer, Berlin, 1977.

15. L. Illusie, Y. Laszlo, and F. Orgogozo, *Théorème de finitude en cohomologie étale, d'après Gabber*, in preparation, École Polytechnique (2007), http://www.math.polytechnique.fr/~laszlo/gdtgabber/gdtgabber.html.
16. M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren Math. Wiss., vol. 332, Springer, Berlin, 2006.
17. Y. Laszlo and M. Olsson, The six operations for sheaves on Artin stacks II: Adic Coefficients, *Publ. Math., Inst. Hautes Étud. Sci.*, (2008).
18. Y. Laszlo and M. Olsson, Perverse sheaves on Artin stacks, Math. Z., to appear.
19. G. Laumon, Transformation de Fourier homogène, *Bull. Soc. Math. Fr.*, **131** (2003), 527–551.
20. G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergeb. Math. Grenzgeb., 3. Folge, vol. 39, Springer, Berlin, 2000.
21. G. Laumon and B. C. Ngo, Le lemme fondamental pour les groupes unitaires, preprint (2004), http://arxiv.org/abs/math/0404454.
22. A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, *J. Amer. Math. Soc.*, **9** (1996), 205–236.
23. M. Olsson, Sheaves on Artin stacks, *J. Reine Angew. Math.*, **603** (2007), 55–112.
24. J. Riou, Pureté (d'après Ofer Gabber), in *Théorèmes de finitude en cohomogie étale d'après Ofer Gabber*, in preparation, preprint (2007), http://www.math.u-psud.fr/~riou/doc/gysin.pdf.
25. C. Serpét, Resolution of unbounded complexes in Grothendieck categories, *J. Pure Appl. Algebra*, **177** (2003), 103–112.
26. J.-P. Serre, *Cohomologie galoisienne*, 5th edn., Lect. Notes Math., vol. 5, Springer, Berlin, 1994.
27. N. Spaltenstein, Resolutions of unbounded complexes, *Compos. Math.*, **65** (1988), 121–154.

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