THREE RESULTS ON THE REGULARITY OF THE CURVES THAT ARE INVARIANT BY AN EXACT SYMPLECTIC TWIST MAP

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ABSTRACT

A theorem due to G. D. Birkhoff states that every essential curve which is invariant under a symplectic twist map of the annulus is the graph of a Lipschitz map. We prove: if the graph of a Lipschitz map $h : \mathbf{T} \to \mathbf{R}$ is invariant under a symplectic twist map, then h is a little bit more regular than simply Lipschitz (Theorem 1); we deduce that there exists a Lipschitz map $h : \mathbf{T} \to \mathbf{R}$ whose graph is invariant under no symplectic twist map (Corollary 2).

Assuming that the dynamic of a twist map restricted to a Lipschitz graph is bi-Lipschitz conjugate to a rotation, we obtain that the graph is even C^1 (Theorem 3).

Then we consider the case of the C^0 integrable symplectic twist maps and we prove that for such a map, there exists a dense G_δ subset of the set of its invariant curves such that every curve of this G_δ subset is C^1 (Theorem 4).

CONTENTS

1. Introduction

The exact symplectic twist maps were studied for a long time because they represent (via a symplectic change of coordinates) the dynamic of the generic symplectic diffeomorphisms of surfaces near their elliptic periodic points (see [3]).

Let us introduce some notations and definition:

Notations.

- $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the circle.
- $\mathbf{A} = \mathbf{T} \times \mathbf{R}$ is the annulus and an element of $\mathbf{A}$ is denoted by (θ, r).
- $\mathbf{A}$ is endowed with its usual symplectic form, $\omega = d\theta \wedge dr$.
- $\pi : \mathbf{T} \times \mathbf{R} \to \mathbf{T}$ is the projection and $\tilde{\pi} : \mathbf{R}^2 \to \mathbf{R}$ its lift.

Definition. — A C^1 *diffeomorphism* $f : \mathbf{A} \to \mathbf{A}$ *of the annulus which is isotopic to identity is a* positive twist map *if, for any given lift* $\tilde{f} : \mathbf{R}^2 \to \mathbf{R}^2$ *and for every* $x \in \mathbf{R}$, *the maps* $y \mapsto \tilde{\pi} \circ \tilde{f}(x, y)$ *and* $y \mapsto \tilde{\pi} \circ \tilde{f}^{-1}(x, y)$ *are both diffeomorphisms, the first one increasing and the second one decreasing. If* f *is a positive twist map,* f^{-1} *is a* negative *twist map. A* twist map *may be positive or negative.*

Moreover, f *is* exact symplectic *if the 1-form* $f^*(rd\theta) - rd\theta$ *is exact.*

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Notations. — $\mathcal{M}_\omega^+$ is the set of exact symplectic positive C^1 twist maps of $\mathbf{A}$, $\mathcal{M}_\omega^-$ is the set of exact symplectic negative C^1 twist maps of $\mathbf{A}$ and $\mathcal{M}_\omega = \mathcal{M}_\omega^+ \cup \mathcal{M}_\omega^-$ is the set of exact symplectic C^1 twist maps of $\mathbf{A}$.

It is obvious that if the graph C of a continuous map is invariant by $f \in \mathcal{M}_\omega$, then there exists no orbit for f joining one of the connected component of $\mathbf{A}\backslash C$ to the other one.

Birkhoff's theory states a kind of converse result (see [2, 6, 10, 12]):

Criterion (Birkhoff). — Let $\eta_1, \eta_2 : \mathbf{T} \to \mathbf{R}$ be two continuous maps such that $\eta_1 < \eta_2$. Let $f \in \mathcal{M}_\omega$. The three following properties are equivalent:

1. *there exists no orbit under f joining $S_-(\eta_1) = \{(\theta, r); r < \eta_1(\theta)\}$ to $S_+(\eta_2) = \{(\theta, r); r > \eta_2(\theta)\}$;*
2. *there exists no orbit under f joining $S_+(\eta_2)$ to $S_-(\eta_1)$;*
3. *there exists $\eta : \mathbf{T} \to \mathbf{R}$ continuous whose graph is invariant under f such that: $\eta_1 \leq \eta \leq \eta_2$.*

Moreover, Birkhoff proved that if the graph of a continuous map η is invariant under $f \in \mathcal{M}_\omega$, then η is Lipschitz.

Having explained the link between the existence of invariant continuous (even Lipschitz) graphs of symplectic twist maps and the dynamic of such a twist map, we will now study the Lipschitz maps $\eta : \mathbf{T} \to \mathbf{R}$ whose graphs are invariant under an exact symplectic C^1 twist map.

We easily see that for every C^1 map $\eta : \mathbf{T} \to \mathbf{R}$, there exists a C^1 exact symplectic twist map $f : (\theta, r) \to (\theta + \varepsilon(r - \eta(\theta)), r - \eta(\theta) + \eta(\theta + \varepsilon(r - \eta(\theta))))$ (where $\varepsilon > 0$ is small enough) which preserves the graph of η. But very few examples of Lipschitz but not C^1 maps whose graph is invariant under an exact symplectic C^1 twist map are known. The most classical example is the time T map Φ of the pendulum for $T > 0$ small enough (see [4]): any separatrix of the hyperbolic fixed point is a Lipschitz graph which is invariant under Φ, and this graph is not differentiable at the fixed point. Of course, we can contruct similar examples:

Example. — Let $\alpha, \beta \in \mathbf{Z}^*$ be some integers and let $\mathcal{V} : \mathbf{T} \to \mathbf{R}$ be a small C^2 function having a strict non degenerate global maximum at $0 \in \mathbf{T}$ (for example $\mathcal{V}(t) = \varepsilon \cos t$ with $\varepsilon > 0$ small enough). Let $V : \mathbf{T}^2 \to \mathbf{R}$ be defined by: $V(x, t) = \mathcal{V}(\beta x - \alpha t)$ and let $H : \mathbf{A} \times \mathbf{R} \to \mathbf{R}$ be the time dependent Hamiltonian function defined by: $H(x, p; t) = \frac{1}{2}p^2 + V(x, t)$. The time 1 map of this Hamiltonian function is then a twist map, and if (x, p) is a solution, if we define: $X = \beta x - \alpha t$ and $P = \beta p - \alpha$, then (X, P) is a solution for the (time-independent) Hamiltonian $\mathcal{H}(X, P) = \frac{1}{2}P^2 + \beta^2 \mathcal{V}(X)$. Therefore, there exists for the Hamiltonian flow of $\mathcal{H}$ (as for the pendulum) an invariant Lipschitz graph which is not differentiable at 0; then, the time 1 map of H leaves a Lipschitz graph invariant, and this Lipschitz graph is not differentiable at β points of $\mathbf{T}$ (they correspond to a periodic orbit with period β); the rotation number of this graph is then $\frac{\alpha}{\beta}$.

Questions.

1. *Is it possible to construct less regular examples of invariant curves (which have at some points no left or right derivative)?*
2. *Does there exist an example of an invariant curve which is not* C^1 *and has an irrational rotation number?*

In this article, we don't answer these questions. We study the regularity of the curves invariant by exact symplectic C^1 twist maps and prove that they are in general more regular than simply Lipschitz:

Theorem **1.** — *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an exact symplectic positive* C^1 *twist map and let* $\gamma : \mathbf{T} \to \mathbf{R}$ *be a Lipschitz map whose graph is invariant by* f. *Then there exists a dense* G_δ *subset* U *of* $\mathbf{T}$ *whose Lebesgue measure is 1 and such that every* t *of* U *is a point of differentiability of* γ *and a point of continuity of* γ'.

We endow the set of the Lipschitz maps $\gamma : \mathbf{T} \to \mathbf{R}$ with the metric d_ℓ defined by: $d_\ell(\gamma_1, \gamma_2) = d_\infty(\gamma_1, \gamma_2) + \mathrm{Lip}(\gamma_1 - \gamma_2)$ where $\mathrm{Lip}(\gamma)$ is the Lipschitz constant of γ. This metric space $(\mathcal{L}, d_\ell)$ is then complete.

Corollary **2.** — *There exists a dense open subset* $\mathcal{U}$ *of* $(\mathcal{L}, d_\ell)$ *such that no* $\gamma \in \mathcal{U}$ *is invariant by an exact symplectic positive* C^1 *twist map.*

We obtain a stronger regularity if we can specify the dynamic of the restriction of the twist map to the curve:

Theorem **3.** — *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an exact symplectic positive* C^1 *twist map and let* $\gamma : \mathbf{T} \to \mathbf{R}$ *be a Lipschitz map whose graph is invariant by* f. *Let* g *be the restriction of* f *to the graph of* γ. *We assume that there exist two sequences of integers* $(n_i)_{i\in\mathbf{N}}$ *and* $(m_i)_{i\in\mathbf{N}}$ *tending to* $+\infty$ *such that* $(g^{m_i})_{i\in\mathbf{N}}$ *and* $(g^{-n_i})_{i\in\mathbf{N}}$ *are equi-Lipschitz.*

Then γ *is* C^1.

Using a theorem of Michel Herman concerning the diffeomorphisms of the circle (see [9]), we deduce:

Corollary **4.** — *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an exact symplectic positive* C^1 *twist map and let* $\gamma : \mathbf{T} \to \mathbf{R}$ *be a Lipschitz map whose graph is invariant by* f. *Let us assume that the restriction of* f *to the graph of* γ *is bi-Lipschitz conjugate to a rotation.*

Then γ *is* C^1 *and the restriction of* f *to the graph of* γ *is* C^1 *conjugate to a rotation.*

Now we are interested in studying the regularity of the exact symplectic C^1 twist maps having many invariant curves: the C^0 integrable ones.

Definition. — *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an exact symplectic positive* C^1 *twist map. Then* f *is* C^0-integrable *if* $\mathbf{A} = \bigcup_{\gamma\in\Gamma} G(\gamma)$ *where:*

1. Γ *is a subset of* $C^0(\mathbf{T}, \mathbf{R})$ *and* $G(\gamma)$ *is the graph of* γ*;*
2. $\forall \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2 \Rightarrow G(\gamma_1) \cap G(\gamma_2) = \emptyset$*;*
3. $\forall \gamma \in \Gamma, f(G(\gamma)) = G(\gamma)$.

Remark. — The general reference for this remark is [10].

A theorem of Birkhoff states that under the hypothesis of this definition, every $\gamma \in C^0(\mathbf{T}, \mathbf{R})$ whose graph is invariant by f is Lipschitz and that the set $\mathcal{I}(f)$ of those invariant graphs is closed for the C^0-topology.

If we fix a lift $\tilde{f}$ of f, we can associate to every $\gamma \in \mathcal{I}(f)$ its rotation number $\rho(\gamma)$. Then, if $\gamma_1, \gamma_2 \in \mathcal{I}(f)$, we have: $G(\gamma_1) \cap G(\gamma_2) \neq \emptyset \Rightarrow \rho(\gamma_1) = \rho(\gamma_2)$ and $G(\gamma_1) \cap G(\gamma_2) = \emptyset \Rightarrow \rho(\gamma_1) \neq \rho(\gamma_2)$. We deduce that $\mathcal{I}(f) = \Gamma$ and therefore Γ is closed for the C^0 topology.

Theorem **5.** — *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an exact symplectic positive* C^1 *twist map which is* C^0 *integrable. Let* Γ *be the set of* $\gamma \in C^0(\mathbf{T}, \mathbf{R})$ *whose graph is invariant under* f. *Then there exists a dense* G_δ *subset* $\mathcal{G}$ *of* (Γ, d_∞) *such that: every* $\gamma \in \mathcal{G}$ *is* C^1. *Moreover, in* $\mathcal{G}$, *the* C^0*-topology is equal to the* C^1*-topology.*

There exists a common argument to the proof of all these results: the existence of two invariant (non continuous) subbundles along the invariant curves, the so-called "Green bundles".

The original Green bundles were introduced by L. W. Green in [8] for Riemannian geodesic flows; then P. Foulon extended this construction to Finsler metrics in [7] and G. Contreras and R. Iturriaga extended it in [5] to optical Hamiltonian flows; in [1], M. Bialy and R. S. Mackay give an analogous construction for the dynamics of sequence of symplectic twist maps of $T^*\mathbf{T}^d$ without conjugate point. Let us cite also a very short survey [11] of R. Iturriaga on the various uses of these bundles (problems of rigidity, measure of hyperbolicity...).

The way we use the Green bundles in our article is different: the two Green bundles will bound the "derivative" below and above (this derivative is in fact the accumulation points of the slope between a given point and a variable one tending to the fixed one) of the invariant curve: therefore, if the two Green bundles are equal at one point, the curve has a derivative at this point.

2. Construction of the Green bundles along an invariant curve

Notations. — $\pi : \mathbf{T} \times \mathbf{R} \to \mathbf{T}$ is the projection.

If $x \in \mathbf{A}$, $V(x) = \ker D\pi(x) \subset T_x\mathbf{A}$ is the vertical at x.

If $x \in \mathbf{A}$ and $n \in \mathbf{N}$, $G_n^+(x) = Df^n(f^{-n}(x))V(f^{-n}(x))$ and $G_n^-(x) = Df^{-n}(f^n(x))V(f^n(x))$ are two 1-dimensional linear subspaces (or lines) of $T_x\mathbf{A}$.

Definition. — *If we identify* $T_x\mathbf{A}$ *with* $\mathbf{R}^2$ *by using the standard coordinates* $(\theta, r) \in \mathbf{R}^2$, *we may deal with the* slope $s(L)$ *of any line* L *of* $T_x\mathbf{A}$ *which is transverse to the vertical* $V(x)$*: it means that* $L = \{(t, s(L)t);\ t \in \mathbf{R}\}$.

If $x \in \mathbf{A}$ *and if* L_1, L_2 *are two lines of* $T_x\mathbf{A}$ *which are transverse to the vertical* $V(x)$, L_2 *is* above *(resp.* strictly above*)* L_1 *if* $s(L_2) \geq s(L_1)$ *(resp.* $s(L_2) > s(L_1)$*). In this case, we write:* $L_1 \preceq L_2$ *(resp.* $L_1 \prec L_2$*).*

A sequence $(L_n)_{n\in\mathbf{N}}$ *of lines of* $T_x\mathbf{A}$ *is* non decreasing *(resp.* increasing*) if for every* $n \in \mathbf{N}$, L_n *is transverse to the vertical and* L_{n+1} *is above (resp. strictly above)* L_n. *We define the* non increasing *and* decreasing *sequences of lines of* T_xM *in a similar way.*

Remark. — A decreasing sequence of lines corresponds to a decreasing sequence of slopes.

Definition. — *If* K *is a subset of* $\mathbf{A}$ *or of its universal covering* $\mathbf{R} \times \mathbf{R}$, *if* F *is a 1-dimensional subbundle of* $T_K\mathbf{A}$ *(resp.* $T_K\mathbf{R}^2$*) transverse to the vertical, we say that* F *is upper (resp. lower) semi-continuous if the map which maps* $x \in K$ *onto the slope* $s(F(x))$ *of* $F(x)$ *is upper (resp. lower) semi-continuous.*

Notations. — If the graph of a continuous map $\gamma : \mathbf{T} \to \mathbf{R}$ is invariant by f, Birkhoff's theorem (see [10]) states that γ is Lipschitz. Therefore, at every $t \in \mathbf{T}$, we can define:

$$\gamma'_-(t) = \liminf_{u\to t} \frac{\gamma(u) - \gamma(t)}{u - t} \quad \text{and} \quad \gamma'_+(t) = \limsup_{u\to t} \frac{\gamma(u) - \gamma(t)}{u - t}$$

which are two real numbers. We will use too:

$$\gamma'_{+,r}(t) = \limsup_{u\to t^+} \frac{\gamma(u) - \gamma(t)}{u - t} \quad \text{and} \quad \gamma'_{+,l} = \limsup_{u\to t^-} \frac{\gamma(u) - \gamma(t)}{u - t}$$

and in a similar way $\gamma'_{-,r}$ and $\gamma'_{-,l}$.

(If u is close enough to t, the difference $u - t$ is the unique real number of $]-0.5; 0.5[$ which represents $u - t$.)

Proposition **6.** — *Let* $f : \mathbf{T} \times \mathbf{R} \to \mathbf{T} \times \mathbf{R}$ *be an exact symplectic positive* C^1 *twist map and let* $\gamma : \mathbf{T} \to \mathbf{R}$ *be a Lipschitz map whose graph is invariant by* f.

Then for every $t \in \mathbf{T}$ *and every* $n \in \mathbf{N}$, *we have:*

$$G_n^-(t, \gamma(t)) \prec G_{n+1}^-(t, \gamma(t)) \prec \mathbf{R}(1, \gamma'_-(t)) \preceq \mathbf{R}(1, \gamma'_+(t)) \\ \prec G_{n+1}^+(t, \gamma(t)) \prec G_n^+(t, \gamma(t)).$$

Notations. — If $(x_1, x_2) \in \mathbf{R}^2$, we will denote by $\mathcal{V}^+(x)$ the set: $\mathcal{V}^+(x) = \{(x_1, y) \in \mathbf{R}^2;\ y \geq x_2\}$.

Proof of Proposition 6. — Let $\tilde{f} : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ be a lift of f and $\tilde{\gamma} : \mathbf{R} \to \mathbf{R}$ be defined by: $\tilde{\gamma}(\theta) = \gamma(\bar{\theta})$ where $\bar{\theta}$ is the projection of θ on $\mathbf{T}$. Then the graph $G(\tilde{\gamma})$ is invariant by $\tilde{f}$ and every connected component of $\mathbf{R}^2 \backslash G(\tilde{\gamma})$ is invariant by $\tilde{f}$.

Let $x = (t, \tilde{\gamma}(t))$ be any point of $G(\tilde{\gamma})$. We denote by $Q(x)$ the connected component of $\mathbf{R}^2 \backslash (G(\tilde{\gamma}) \cup \{t\} \times \mathbf{R})$ which is above $G(\tilde{\gamma})$ and in $]t, +\infty[\times \mathbf{R}$; moreover we denote by $R(x) = \overline{Q(x)} = Q(x) \cup (\{(\tau, \tilde{\gamma}(\tau)); \tau \geq t\} \cup \mathcal{V}^+(x))$ the closure of $Q(x)$. The diffeomorphism f being an exact symplectic positive C^1 twist map, we have: $\forall x \in G(\tilde{\gamma})$, $\tilde{f}(R(x)) \subset R(\tilde{f}(x))$. Therefore:

$$\forall n \in \mathbf{N}^*,\ \forall x \in G(\tilde{\gamma}), \quad \tilde{f}^n(R(\tilde{f}^{-n}(x))) \subset \tilde{f}^{n-1}(R(\tilde{f}^{-(n-1)}(x))).$$

We deduce that for every $n \in \mathbf{N}^*$ and every $x \in G(\tilde{\gamma})$, the curve $\tilde{f}^n(\mathcal{V}^+(\tilde{f}^{-n}(x)))$ is a subset of $\tilde{f}^{n-1}(R(\tilde{f}^{-(n-1)}(x)))$. Therefore, its tangent space at x, which is $G_n^+(x)$ is under $G_{n-1}^+(x)$ and above $\mathbf{R}(1, \gamma'_{+,r}(t))$. The fact that $G_{n-1}^+(x)$ is strictly above $G_n^+(x)$ follows from the fact that this subspaces have to be transverse because $V(f^{-n-1}(x))$ and $Df(V(f^{-n}(x)))$ are transverse (f being an exact symplectic positive C^1 twist map). The fact that $G_n^+(x)$ is strictly above $\mathbf{R}(1, \gamma'_{+,r}(t))$ comes then from the fact that the sequence $(G_n^+(x))$ is strictly decreasing.

The proof of the other inequalities is similar. □

Remark. — In the last proof, we have noticed that if $x \in G(\tilde{\gamma})$ the curve $\tilde{f}^n(\mathcal{V}^+(\tilde{f}^{-n}(x)))$ is a subset of $R(x)$ which is transverse to the vertical at x. Therefore, the first (or "horizontal") coordinate of $D\tilde{f}^n(\tilde{f}^{-n}(x))(0, 1)$ is strictly positive.

Then $(G_n^+(x))$ is a strictly decreasing sequence of lines of $T_x\mathbf{A}$ which is bounded below. Hence it tends to a limit $G^+(x)$. In a similar way, the sequence $(G_n^-(x))$ tends to a limit, $G^-(x)$.

Definition. — *If $x \in \mathbf{A}$ belongs to a continuous graph invariant under $f \in \mathcal{M}_\omega^+$, the bundles $G^-(x)$ and $G^+(x)$ are called the* Green bundles *at x associated to f.*

Example. — Let us assume that $x \in G(\gamma)$ is a periodic hyperbolic periodic point of f; then $G^+(x) = E^u(x)$ is the tangent space to the unstable manifold of x and $G^-(x) = E^s(x)$ is the tangent space to the stable manifold.

Proposition **7.** — *Let $\gamma : \mathbf{T} \to \mathbf{R}$ be a continuous map whose graph is invariant by an exact symplectic positive C^1 twist map $f : \mathbf{A} \to \mathbf{A}$. Then the Green bundles, defined at every point of $G(\gamma)$, are invariant by Df and for every $t \in \mathbf{T}$, we have:* $G^-(t, \gamma(t)) \preceq \mathbf{R}(1, \gamma'_-(t)) \preceq$

$\mathbf{R}(1,\gamma'_+(t)) \preceq G^+(t,\gamma(t))$. *Moreover, the map* $t \to G^+(t,\gamma(t))$ *is upper semi-continuous and the map* $t \to G^-(t,\gamma(t))$ *is lower semi-continuous. Therefore, the set:*

$$\mathcal{G}(\gamma) = \{t \in \mathbf{T};\, G^-(t,\gamma(t)) = G^+(t,\gamma(t))\}$$

is a G_δ *set and for every* $t_0 \in \mathcal{G}(\gamma)$, γ *is differentiable,* γ' *is continuous at* t_0 *and* $\mathbf{R}(1,\gamma'(t_0)) = G^+(t_0,\gamma'(t_0)) = G^-(t_0,\gamma'(t_0))$. *Moreover,* G^- *and* G^+ *are continuous at* $(t_0,\gamma(t_0))$ *too.*

This proposition is a corollary of Proposition 6 and of usual properties of real functions (the fact that the (simple) limit of a decreasing sequence of continuous functions is upper semi-continuous).

Corollary **8.** — *Let* $\gamma : \mathbf{T} \to \mathbf{R}$ *be a continuous map whose graph is invariant by an exact symplectic positive* C^1 *twist map* $f : \mathbf{A} \to \mathbf{A}$. *We assume that:*

$$\forall t \in \mathbf{T}, \quad G^-(t,\gamma(t)) = G^+(t,\gamma(t)).$$

Then γ *is* C^1.

Moreover, in this case, the sequences $(s(G_n^-(t,\gamma(t))))_{n\in\mathbf{N}}$ *and* $(s(G_n^+(t,\gamma(t))))_{n\in\mathbf{N}}$ *converge uniformly to* $\gamma'(t)$.

Everything in this corollary is a consequence of Proposition 7; the fact that the convergence is uniform comes from Dini's theorem: if an increasing or decreasing sequence of real valued continuous functions defined on a compact set converges simply to a continuous function, then the convergence is uniform.

Example. — We may ask ourselves: if the graph of a C^1 map $\gamma : \mathbf{T} \to \mathbf{R}$ is invariant under an exact symplectic positive C^1 twist map f, do we necessarily have along the graph of γ the equality $G^+ = G^-$? We will show that the answer is no.

In fact, if $g : \mathbf{T} \to \mathbf{T}$ is any orientation preserving C^1 diffeomorphism, we may "immerse" g into an exact symplectic C^1 twist map f. Let us explain this fact: let $\tilde{g} : \mathbf{R} \to \mathbf{R}$ be any lift of g. We define $\tilde{f} : \mathbf{R}^2 \to \mathbf{R}^2$ by:

$$\tilde{f}(x,r) = (\tilde{g}(x) + r, \tilde{g}^{-1}(r + \tilde{g}(x)) - x).$$

Then $\tilde{f}$ is a lift of an exact symplectic positive C^1 twist map f and we have: $\forall t \in \mathbf{T}$, $f(t,0) = (g(t),0)$.

If now we assume that g has a hyperbolic periodic point $x_0 \in \mathbf{T}$ (then x_0 is attracting or repulsing), $(x_0,0)$ is a hyperbolic periodic point for f and therefore $G^-(x_0,0) \neq G^+(x_0,0)$.

Using Proposition 7, we will prove in the next section that if the graph of a continuous map γ is invariant by an exact symplectic C^1 twist map $f : \mathbf{A} \to \mathbf{A}$, then there exists a dense G_δ subset G of $\mathbf{T}$ such that every $x \in G$ is a point of differentiability of γ and a point of continuity of γ'.

3. Regularity of the invariant graphs

We begin by giving a criterion to determine if a given vector is in one of the two Green bundles.

Proposition **9.** — *Let f be an exact symplectic positive C^1 twist map and let $\gamma : \mathbf{T} \to \mathbf{R}$ be a Lipschitz map whose graph $G(\gamma)$ is invariant by f.*

Let us assume that $x \in G(\gamma)$ and that $v \in T_x\mathbf{A}$ is such that the sequence $(|D(\pi \circ f^n)(x)v|)_{n\in\mathbf{N}}$ doesn't tends to $+\infty$. Then $v \in G^-(x)$. In a similar way, if the sequence $(|D(\pi \circ f^{-n})(x)v|)_{n\in\mathbf{N}}$ doesn't tends to $+\infty$, then $v \in G^+(x)$.

Proof of Proposition 9. — We use the standard symplectic coordinates (θ, r) of $\mathbf{A}$ and we define for every $k \in \mathbf{Z}$: $x_k = f^k(x)$.

In these coordinates, the line $G_n^+(x_k)$ is the graph of $(t \to s_n^+(x_k)t)$ ($s_n^+(x_k)$ is the slope of $G_n^+(x_k)$) and the line $G_n^-(x_k)$ is the graph of $(t \to s_n^-(x_k)t)$. Moreover, the matrix $M_n(x_k)$ of $Df^n(x_k)$ (for $n \geq 1$) is a symplectic matrix:

$$M_n(x_k) = \begin{pmatrix} a_n(x_k) & b_n(x_k) \\ c_n(x_k) & d_n(x_k) \end{pmatrix}$$

with $\det M_n(x_k) = 1$. We have noticed just after the proof of Proposition 6 that the coordinate $D(\pi \circ f^n)(x_k)(0, 1) = b_n(x_k)$ is strictly positive. Using the definition of $G_n^+(x_{k+n})$, we obtain: $d_n(x_k) = s_n^+(x_{k+n})b_n(x_k)$.

The matrix $M_n(x_k)$ being symplectic, we have:

$$M_n(x_k)^{-1} = \begin{pmatrix} d_n(x_k) & -b_n(x_k) \\ -c_n(x_k) & a_n(x_k) \end{pmatrix}$$

we deduce from the definition of $G_n^-(x_k)$ that: $a_n(x_k) = -b_n(x_k)s_n^-(x_k)$. Finally, if we use the fact that $\det M_n(x_k) = 1$, we obtain:

$$M_n(x_k) = \begin{pmatrix} -b_n(x_k)s_n^-(x_k) & b_n(x_k) \\ -b_n(x_k)^{-1} - b_n(x_k)s_n^-(x_k)s_n^+(x_{k+n}) & s_n^+(x_{k+n})b_n(x_k) \end{pmatrix}.$$

Lemma **10.** — *There exists a constant* $M > 0$ *such that:*

$$\forall x \in G(\gamma),\ \forall n \in \mathbf{N}^*, \quad \max\{|s_n^+(x)|, |s_n^-(x)|\} \leq M.$$

Proof of Lemma 10. — We deduce from Proposition 6 that: $\forall x \in G(\gamma)$, $\forall n \in \mathbf{N}^*$, $s_1^-(x) \leq s_n^-(x) < s_n^+(x) \leq s_1^+(x)$. Therefore, we only have to prove the inequalities of the lemma for $n = 1$.

The real number $s_1^-(x)$, which is the slope of $Df^{-1}(f(x))V(f(x))$, depends continuously on x, and is defined for every x belonging to the compact subset $G(\gamma)$. Hence it is uniformly bounded. The same argument proves that s_1^+ is uniformly bounded on $G(\gamma)$ and concludes the proof of Lemma 10. □

Lemma **11.** — *If* $x \in G(\gamma)$*, we have:* $\lim_{n\to\infty} b_n(x) = +\infty$.

Proof of Lemma 11. — We have: $\forall n, m \in \mathbf{N}^*$, $\forall i \in \mathbf{Z}$, $M_{n+m}(x_i) = M_n(x_{i+m})M_m(x_i)$. It implies: $b_{n+m}(x_i) = b_n(x_{i+m})b_m(x_i)(s_m^+(x_{i+m}) - s_n^-(x_{i+m}))$ and: $-b_{n+m}(x_i)s_{n+m}^-(x_i) = b_n(x_{i+m})s_n^-(x_{i+m})b_m(x_i)s_m^-(x_i) - b_n(x_{i+m})(b_m(x_i))^{-1} - b_n(x_{i+m})b_m(x_i)s_m^+(x_{i+m})s_m^-(x_i)$.

Hence:

$$\begin{aligned}&-b_{n+m}(x_i)s_{n+m}^-(x_i)\\&\quad= -b_{n+m}(x_i)s_m^-(x_i) - b_{n+m}(x_i)\,(b_m(x_i))^{-2}\frac{1}{s_m^+(x_{i+m}) - s_n^-(x_{i+m})}.\end{aligned}$$

Therefore:

$$s_{n+m}^-(x_i) = s_m^-(x_i) + (b_m(x_i))^{-2}\frac{1}{s_m^+(x_{i+m}) - s_n^-(x_{i+m})}.$$

In particular:

$$s_{1+m}^-(x_i) = s_m^-(x_i) + (b_m(x_i))^{-2}\frac{1}{s_m^+(x_{i+m}) - s_1^-(x_{i+m})}.$$

Using the constant M found via Lemma 10, we have:

$$s_{1+m}^-(x_i) \geq s_m^-(x_i) + \frac{1}{2M\,(b_m(x_i))^2}.$$

Hence:

$$s_{1+m}^-(x_i) \geq s_1^-(x_i) + \frac{1}{2M}\sum_{k=2}^{m}\frac{1}{(b_k(x_i))^2}.$$

The sequence $(s_m^-(x_i))_{m\in\mathbf{N}^*}$ being convergent, we must have:

$$\sum_{k=2}^{\infty}\frac{1}{(b_k(x_i))^2} < \infty$$

and thus: $\lim_{k\to\infty} b_k(x_i) = +\infty$. □

Let us now prove Proposition 9. Let us assume that $v \in T_x\mathbf{A}$ is such that the sequence $(|D(\pi\circ f^n)(x)v|)_{n\in\mathbf{N}}$ doesn't tends to $+\infty$. Then there is a sequence $(k_n)_{n\in\mathbf{N}}$ of integers tending to $+\infty$ such that the sequence $(|D(\pi\circ f^{k_n})(x)v|)_{n\in\mathbf{N}}$ is bounded. If $v = (v_1, v_2)$, we have: $D(\pi\circ f^{k_n})(x)v = b_{k_n}(x)(v_2 - s_{k_n}^-(x)v_1)$ and $\lim_{n\to\infty} b_{k_n}(x) = +\infty$. We deduce: $\lim_{n\to\infty}(v_2 - s_{k_n}^-(x)v_1) = 0$. The sequence $(s_{k_n}^-(x))_{n\in\mathbf{N}}$ tends to the slope $s^-(x)$ of $G^-(x)$, and then $v \in G^-(x)$. □

Example. — Let us assume that the exact symplectic positive C^1 twist map $f : \mathbf{A} \to \mathbf{A}$ has a regular and proper integral, i.e. that there exists a C^1 regular and proper function $H : \mathbf{A} \to \mathbf{R}$ such that: $\forall x \in \mathbf{A}$, $H(f(x)) = H(x)$. Then, for every $n \in \mathbf{Z}$, we have: $H(f^n(x)) = H(x)$ and then: $DH(f^n(x))Df^n(x) = DH(x)$ and $\|\operatorname{grad} H(x)\|^2 = DH(x)\operatorname{grad} H(x) = DH(f^n(x))Df^n(x)\operatorname{grad} H(x)$ i.e. if we denote by $(.|.)$ the usual scalar product and if $\|.\| = \sqrt{(.|.)}$:

$$(*) \qquad (\operatorname{grad} H(f^n(x))|Df^n(x).\operatorname{grad} H(x)) = \|\operatorname{grad} H(x)\|^2.$$

Let $\Gamma = H^{-1}(c)$ be a curve invariant by f. If we use a good parametrization $\gamma : \mathbf{R}/T\mathbf{Z} \to \mathbf{A}$ of Γ, the base $(\dot\gamma(t), \operatorname{grad} H(\gamma(t)))$ is symplectic: the base is orthogonal, oriented, and $\|\dot\gamma(t)\| = \frac{1}{\|\operatorname{grad} H(\gamma(t))\|}$.

The image of this symplectic base by Df^n is symplectic too. This new symplectic base is: $(Df^n(\gamma(t))\dot\gamma(t), Df^n(\gamma(t))\operatorname{grad} H(\gamma(t))) = (\lambda\dot\gamma(\tau_n), Df^n(\gamma(t))\operatorname{grad} H(\gamma(t)))$ where $\gamma(\tau_n) = f^n(\gamma(t))$ and $\lambda \in \mathbf{R}$. Because this base is symplectic, we have: $1 = \lambda\omega(\dot\gamma(\tau_n), Df^n(\gamma(t))\operatorname{grad} H(\gamma(t)))$, this last value being equal to:

$$1 = \frac{\lambda}{\|\operatorname{grad} H(\gamma(\tau_n))\|^2}(\operatorname{grad} H(\gamma(\tau_n))|Df^n(\gamma(t))\operatorname{grad} H(\gamma(t))).$$

Using $(*)$, we obtain: $\lambda = \frac{\|\operatorname{grad} H(\gamma(\tau_n))\|^2}{\|\operatorname{grad} H(\gamma(t))\|^2}$; hence the sequence $(Df^n(\gamma(t))\dot\gamma(t))_{n\in\mathbf{Z}} = (\frac{\|\operatorname{grad} H(\gamma(\tau_n)))\|^2}{\|\operatorname{grad} H(\gamma(t))\|^2}\dot\gamma(\tau_n))_{n\in\mathbf{N}}$ is bounded (and even uniformly bounded in $t \in \mathbf{R}$). Using Proposition 9, we deduce that:

$$\forall x \in \Gamma, \quad G^-(x) = G^+(x).$$

Hence, if f has a regular and proper integral, the two Green bundles are equal at every point.

Let us notice too that for every $c \in \mathbf{R}$, the restriction of f to $H^{-1}(c)$ is C^1 conjugate to a rotation: indeed, with the notations introduced before, the sequence $((Df^n \circ \gamma)\dot\gamma)_{n\in\mathbf{Z}}$ is uniformly bounded. Let $g : \mathbf{R}/T\mathbf{Z} \to \mathbf{R}/T\mathbf{Z}$ be the unique C^1-diffeomorphism such that: $\forall t \in \mathbf{R}/T\mathbf{Z}$, $f(\gamma(t)) = \gamma(g(t))$. Then: $\forall n \in \mathbf{Z}$, $f^n(\gamma(t)) = \gamma(g^n(t))$ and $Dg^n(t) = (D\gamma(g^n(t)))^{-1}(Df^n \circ \gamma(t))\dot\gamma(t)$ is uniformly bounded in $n \in \mathbf{Z}$ and $t \in \mathbf{R}/T\mathbf{Z}$. By a theorem of Michel Herman (Theorem 6.1.1 of [9]), it implies that g and then $f_{|\Gamma}$ is C^1 conjugate to a rotation.

Proposition **12.** — *Let $\gamma : \mathbf{T} \to \mathbf{R}$ be a Lipschitz map whose graph is invariant by an exact symplectic positive C^1 twist map $f : \mathbf{A} \to \mathbf{A}$. Then for almost every $t \in \mathbf{T}$, the sequences $(|D(\pi \circ f^n)(t, \gamma(t))(1, \gamma'(t))|)_{n\in\mathbf{N}}$ and $(|D(\pi \circ f^{-n})(t, \gamma(t))(1, \gamma'(t))|)_{n\in\mathbf{N}}$ don't tend to $+\infty$.*

Proof of Proposition 12. — We define: $f(t, \gamma(t)) = (f_1(t, \gamma(t)), f_2(t, \gamma(t))) = (g(t), \gamma(g(t)))$. Then $g : \mathbf{T} \to \mathbf{T}$ is a (bi)-Lipschitz homeomorphism of $\mathbf{T}$ which is homotopic to $\mathrm{Id}_{\mathbf{T}}$. There exists a set $U \subset \mathbf{T}$ whose Lebesgue measure is one and such that γ is differentiable at every $x \in \mathrm{U}$. Then, for every $k \in \mathbf{Z}$ and every $x \in \mathrm{U}$, the map $g^k = \pi \circ f^k(., \gamma(.))$ is differentiable at x; we have then: $\forall t \in \mathrm{U}$, $\forall k \in \mathbf{Z}$, $(g^k)'(t) \geq 0$. Let $\tilde{g} : \mathbf{R} \to \mathbf{R}$ be a lift of g. Then: $\forall k \in \mathbf{Z}$, $\forall t \in \mathbf{R}$, $\tilde{g}^k(t+1) = \tilde{g}^k(t) + 1$. Therefore:

$$\forall k \in \mathbf{Z}, \quad 1 = \tilde{g}^k(t+1) - \tilde{g}^k(t) = \int_{\mathrm{U}} (g^k)'(s)ds.$$

Using Fatou's theorem, we obtain: $1 \geq \int_{\mathrm{U}} \liminf_{n\to\infty} (g^n)'(s)ds$ and then for almost $t \in \mathrm{U}$, the sequence $(|(g^n)'(t)|)_{n\in\mathbf{N}}$ doesn't tend to $+\infty$. As we have:

$$\forall t \in \mathrm{U},\ \forall n \in \mathbf{N}, \quad (g^n)'(t) = \mathrm{D}(\pi \circ f^n)(t, \gamma(t))(1, \gamma'(t)),$$

we obtain the Proposition 12. □

End of the Proof of Theorem 1. — We can now finish the proof of Theorem 1. Let $\gamma : \mathbf{T} \to \mathbf{R}$ be a Lipschitz map whose graph is invariant by an exact symplectic positive C^1 twist map $f : \mathbf{A} \to \mathbf{A}$. By Proposition 12, there exists a subset U of $\mathbf{T}$ with Lebesgue measure 1 such that for every $t \in \mathrm{U}$, the sequences $(|\mathrm{D}(\pi \circ f^n)(t, \gamma(t))(1, \gamma'(t))|)_{n\in\mathbf{N}}$ and $(|\mathrm{D}(\pi \circ f^{-n})(t, \gamma(t))(1, \gamma'(t))|)_{n\in\mathbf{N}}$ are well defined and doesn't tend to $+\infty$. By Proposition 9, for every $t \in \mathrm{U}$, we have: $(1, \gamma'(t)) \in \mathrm{G}^-(t, \gamma(t)) \cap \mathrm{G}^+(t, \gamma(t))$. By Proposition 7, $\mathcal{G}(\gamma)$ is a dense G_δ subset which contains U and thus the Lebesgue measure of $\mathcal{G}(\gamma)$ is 1, and every point of $\mathcal{G}(\gamma)$ is a point of derivability of γ and a point of continuity of γ'. □

Proof of Theorem 3. — Let $f : \mathbf{A} \to \mathbf{A}$ be an exact symplectic positive C^1 twist map and let $\gamma : \mathbf{T} \to \mathbf{R}$ be a Lipschitz map whose graph is invariant by f. Let g be the restriction of f to the graph of γ. We assume that there exist two sequences of integers $(n_i)_{i\in\mathbf{N}}$ and $(m_i)_{i\in\mathbf{N}}$ tending to $+\infty$ such that $(g^{m_i})_{i\in\mathbf{N}}$ and $(g^{-n_i})_{i\in\mathbf{N}}$ are equi-Lipschitz with constant K. We assume that K is a Lipschitz constant for $(t \to (t, \gamma(t))$ too. Then:

$$\begin{aligned}\forall t, u \in \mathbf{R},\ \forall i \in \mathbf{N}, \quad & d(f^{m_i}(t, \gamma(t)), f^{m_i}(u, \gamma(u))) \\ & \leq \mathrm{K}d((t, \gamma(t)), (u, \gamma(u))) \leq \mathrm{K}^2 d(u, t); \\ \forall t, u \in \mathbf{R},\ \forall i \in \mathbf{N}, \quad & d(f^{-n_i}(t, \gamma(t)), f^{-n_i}(u, \gamma(u))) \\ & \leq \mathrm{K}d((t, \gamma(t)), (u, \gamma(u))) \leq \mathrm{K}^2 d(u, t).\end{aligned}$$

Let us now consider $t \in \mathbf{T}$; as γ is Lipschitz, there exists a sequence $(t_n)_{n\in\mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ such that $\lim_{n\to\infty} t_n = t$ and the sequence $(\frac{\gamma(t)-\gamma(t_n)}{t-t_n})_{n\in\mathbf{N}}$ tends to $\delta \in \mathbf{R}$. Then we have ($\tilde{f}$ is any lift of f):

$$\forall i \in \mathbf{N}, \quad \mathrm{D}f^{m_i}(t, \gamma(t))(1, \delta) = \lim_{n\to\infty} \frac{1}{t - t_n}\left(\tilde{f}^{m_i}(t, \tilde{\gamma}(t)) - \tilde{f}^{m_i}(t_n, \tilde{\gamma}(t_n))\right)$$

and

$$\forall i \in \mathbf{N}, \quad \mathrm{D}f^{-n_i}(t, \gamma(t))(1, \delta) = \lim_{n\to\infty} \frac{1}{t - t_n}\left(\tilde{f}^{-n_i}(t, \tilde{\gamma}(t)) - \tilde{f}^{-n_i}(t_n, \tilde{\gamma}(t_n))\right).$$

Hence: $\forall i \in \mathbf{N}$, $\max\{\|Df^{m_i}(t,\gamma(t))(1,\delta)\|, \|Df^{-n}{}^{}_i(t,\gamma(t))(1,\delta)\| \leq K^2$. Therefore, by Proposition 9, $G^-(t,\gamma(t)) = G^+(t,\gamma(t)) = \mathbf{R}(1,\gamma'(t))$. We deduce from Corollary 8 that γ is C^1. □

Proof of Corollary 4. — Let $f : \mathbf{A} \to \mathbf{A}$ be an exact symplectic positive C^1 twist map and let $\gamma : \mathbf{T} \to \mathbf{R}$ be a Lipschitz map whose graph is invariant by f. Let us assume that the restriction g of f to the graph of γ is bi-Lipschitz conjugate to a rotation: there exists $\varphi : \mathbf{T} \to G(\gamma)$ such that φ and φ^{-1} are Lipschitz and a rotation $R : \mathbf{T} \to \mathbf{T}$ such that $\varphi \circ R \circ \varphi^{-1} = g$. Then: $\forall n \in \mathbf{N}$, $g^n = \varphi \circ R^n \circ \varphi^{-1}$; therefore, if K is a common Lipschitz constant of γ, φ and φ^{-1}, as R is an isometry, we have:

$$\begin{aligned}\forall t, u \in \mathbf{R}, \quad & d(f^n(t,\gamma(t)), f^n(u,\gamma(u))) \\ & = d(\varphi \circ R^n \circ \varphi^{-1}(t,\gamma(t)), \varphi \circ R^n \circ \varphi^{-1}(u,\gamma(u))) \\ & \leq K^3 d(t,u) \leq K^3 d((t,\gamma(t)),(u,\gamma(u))).\end{aligned}$$

Hence $(g^k)_{k\in\mathbf{Z}}$ is equi-Lipschitz.

We deduce from Theorem 3 that γ is C^1.

Moreover, for every $k \in \mathbf{Z}$, we have: $\|D(\pi \circ f^k)(t,\gamma(t))(1,\gamma'(t))\| \leq K^3$; hence, $\pi \circ f(.,\gamma(.))$ is a C^1 diffeomorphism of $\mathbf{T}$ which satisfies the assumptions of Theorem 6.1.1 of [9]: therefore it is C^1 conjugate to a rotation, and $f_{|G(\gamma)}$ too. □

4. A generic property of Lipschitz functions

We think that the results contained in this section should be known as folklore.

If $\theta \in \mathbf{R}$, its projection on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is denoted by $\bar{\theta}$. We define on $\mathbf{T}$ a metric d by:

$$\forall (\bar{\alpha},\bar{\beta}) \in \mathbf{T}^2, \quad d(\bar{\alpha},\bar{\beta}) = \min_{\bar{x}=\bar{\alpha},\bar{y}=\bar{\beta}} |x-y|.$$

Moreover, λ is the Lebesgue measure on $\mathbf{T}$.

Let $\mathcal{L}$ be the vector space of Lipschitz maps from $\mathbf{T}$ to $\mathbf{R}$. We define on $\mathcal{L}$ a map Lip by:

$$\mathrm{Lip}(\gamma) = \sup_{\bar{x}\neq\bar{y}} \frac{|\gamma(\bar{x}) - \gamma(\bar{y})|}{d(\bar{x},\bar{y})}.$$

We define on $\mathcal{L}$ a norm $\|.\| = \|.\|_\infty + \mathrm{Lip}$. Then $(\mathcal{L}, \|.\|)$ is a Banach space.

Lemma **13.** — *There exists a subset* A *of* $\mathbf{T}$ *such that, for every open and non-empty subset* U *of* $\mathbf{T}$, $\lambda(U \cap A) > 0$ *and* $\lambda(U \cap (\mathbf{T}\backslash A)) > 0$.

Proof of Lemma 13. — Let us introduce a notation: if J is a closed interval which is not a point and $\mu \in]0,1[$, $C_\mu(J)$ is a Cantor subset of J such that: $\lambda(C(J)) = \mu\lambda(J)$.

We define $\lambda_0 = \frac{1}{3}$ and construct $C_0 = C_{\lambda_0}([0,1])$. Then $\lambda(C_0) = \frac{1}{3}$ and $[0,1]\backslash C_0$ is the union of a countable family $(J_n^0)_{n\in\mathbf{N}}$ of open intervals. Let us notice that the measure of each of these intervals is less than $\frac{1}{2}$.

We define $\lambda_1 = \frac{1}{6}$ and for every $n \in \mathbf{N}$, we build $C_n^1 = C_{\lambda_1}(\bar{J}_n^0)$, a Cantor subset built in $\bar{J}_n^0$. We define: $C_1 = \bigcup_{n\in\mathbf{N}} C_n^1$. Then $\lambda(C_0 \cup C_1) = \frac{1}{3} + \frac{1}{6}.\frac{2}{3} = \frac{1}{3} + \frac{1}{3^2}$ and $[0,1]\backslash(C_0 \cup C_1)$ is the union of a countable family $(J_n^1)_{n\in\mathbf{N}}$ of open intervals. Let us notice that the measure of each of these intervals is less than $\frac{1}{4}$.

We repeat this construction: for every $n \in \mathbf{N}$, $\lambda_n(1 - \frac{1}{2}(1 - \frac{1}{3^n})) = \frac{1}{3^{n+1}}$ is such that $C_j^n = C_{\lambda_n}(\bar{J}_j^n)$, we have: $C_n = \bigcup_{j\in\mathbf{N}} C_j^n$, $\lambda(C_0 \cup \cdots \cup C_n) = \frac{1}{2}(1 - \frac{1}{3^{n+1}})$ and $[0,1]\backslash(C_0 \cup \cdots \cup C_n)$ is the union of a countable family $(J_j^n)_{j\in\mathbf{N}}$ of open intervals. The measure of each of these intervals is less than $\frac{1}{2^n}$.

We define: $C = \bigcup_{n\in\mathbf{N}} C_n$.

Let now $J =]a,b[$ be an open interval in $[0,1]$. We choose $n \in \mathbf{N}$ such that $\frac{1}{2^n} < \frac{b-a}{4}$. As the measure of each J_j^n is less than $\frac{1}{2^n}$, the set $C_0 \cup \cdots \cup C_n$ meets $]\frac{b+a}{2} - \frac{b-a}{4}, \frac{b+a}{2} + \frac{b-a}{4}[$; the set $C_0 \cup \cdots \cup C_n$ being totally discontinuous, one open set J_j^n meets $]\frac{b+a}{2} - \frac{b-a}{4}, \frac{b+a}{2} + \frac{b-a}{4}[$ and therefore is contained in J. We know that:

$$\lambda(J_j^n\backslash C) = \lambda\Big(J_j^n \backslash \bigcup_{k\geq n+1} C_k\Big) = \Big(\prod_{i=n+1}^{\infty}(1-\lambda_i)\Big)\lambda(J_j^n)$$

$$= \Big(\prod_{i=n+1}^{\infty}\Big(1 - \frac{2}{3^{i+1}+3}\Big)\Big)\lambda(J_j^n).$$

Therefore $\lambda(J_j^n\backslash C) \in]0, \lambda(J_j^n)[$ and J meets C and $[0,1]\backslash C$ in subsets which have a non zero measure. □

Proposition **14.** — *There exists a dense and open subset* $\mathcal{U}$ *of* $\mathcal{L}$ *such that, for every* $\gamma \in \mathcal{U}$*, there exists a subset* $U_\gamma \subset \mathbf{T}$ *such that* $\lambda(U_\gamma) > 0$ *and every* $t \in U_\gamma$ *is a point of differentiability of* γ *and a point of discontinuity of* γ'.

Notations. — If A is a subset of $\mathbf{R}$ (resp. $\mathbf{T}$), χ_A is the characteristic function of A, i.e.: $\forall x \in A$, $\chi_A(x) = 1$ and $\forall x \notin A$, $\chi_A(x) = 0$.

Proof of Proposition 14. — We begin by exhibiting one example of $\eta : \mathbf{T} \to \mathbf{R}$ in $\mathcal{L}$ such that the derivative of η has no point of continuity. Let $A \subset \mathbf{T}$ be chosen as in Lemma 13: A is a set such that for every open and non empty subset U of $\mathbf{T}$, $\lambda(U\cap A) > 0$ and $\lambda(U \cap (\mathbf{T}\backslash A)) > 0$. Then the map: $\alpha : \mathbf{T} \to \mathbf{R}$ defined by: $\alpha(t) = \lambda(\mathbf{T}\backslash A)\chi_A(t) - \lambda(A)\chi_{\mathbf{T}\backslash A}(t)$ is such that: $\int_{\mathbf{T}} \alpha = 0$. Hence, α has a primitive $\eta : \mathbf{T} \to \mathbf{R}$ defined by: $\forall \theta \in [0,1[$, $\eta(\bar{\theta}) = \int_{[0,\bar{\theta}]} \alpha$.

The function α being Lebesgue integrable, we have: for almost every $t \in \mathbf{T}$, η is differentiable at t and $\eta'(t) = \alpha(t)$. Moreover, α being bounded, the map η is Lipschitz. We denote by D the set of $t \in \mathbf{T}$ such that η is differentiable at t and $\eta'(t) = \alpha(t)$. We have noticed that $\lambda(\mathrm{D}) = 1$. Moreover, if J is any open non empty interval of $\mathbf{T}$, by Lemma 13, $\mu(\mathrm{D} \cap \mathrm{J} \cap \mathrm{A}) > 0$ and $\mu(\mathrm{D} \cap \mathrm{J} \cap (\mathbf{T} \backslash \mathrm{A})) > 0$. If $t \in \mathrm{D} \cap \mathrm{J} \cap \mathrm{A}$, η is differentiable at t and $\eta'(t) = \alpha(t) = \lambda(\mathbf{T} \backslash \mathrm{A}) = a > 0$; if $t \in \mathrm{D} \cap \mathrm{J} \cap (\mathbf{T} \backslash \mathrm{A})$, then η is differentiable at t and $\eta'(t) = \alpha(t) = -\lambda(\mathrm{A}) = -b < 0$. Then in every neighbourhood of any point of differentiability of η, there exists t_1, t_2 points of differentiability of η such that $\eta'(t_1) = a$ and $\eta'(t_2) = -b$. It implies that η' is nowhere continuous.

Before going on with the proof, let us notice that the set of the point of continuity of any function is a G_δ subset, and then measurable.

We consider a Lipschitz map $\gamma : \mathbf{T} \to \mathbf{R}$ and an open subset $\mathcal{U}$ of $\mathcal{L}$ which contains γ; there are two cases:

1. either for every $\gamma_1 \in \mathcal{U}$, there exists $\mathrm{U} \subset \mathbf{T}$ such that $\lambda(\mathrm{U}) > 0$ and every $t \in \mathrm{U}$ is a point of differentiability of γ_1 and a point of discontinuity of γ_1';
2. or there exists $\gamma_1 \in \mathcal{U}$ and $\mathrm{U} \subset \mathbf{T}$ such that $\lambda(\mathrm{U}) = 1$ and every $t \in \mathrm{U}$ is a point of differentiability of γ_1 and a point of continuity of γ_1'.

In this last case, we will prove that there exists an open non empty subset $\mathcal{V} \subset \mathcal{U}$ such that: for every $\gamma_2 \in \mathcal{V}$, there exists $\mathrm{U} \subset \mathbf{T}$ such that $\lambda(\mathrm{U}) > 0$ and every $t \in \mathrm{U}$ is a point of differentiability of γ_2 and a point of discontinuity of γ_2'. If we succeed in proving that, the Proposition 14 is proved.

Let us now build $\mathcal{V}$. Let $\mathrm{D}(\gamma_1)$ be the set of the points of continuity of γ_1' and let $d(\gamma_1)$ be the set of the points of differentiability of γ_1: we know that $\lambda(\mathrm{D}(\gamma_1)) = 1$. Let $\varepsilon \in]0, 1[$ be such that the ball centered at $\gamma_1 + \varepsilon\eta$ with radius equal to $\varepsilon\frac{b+a}{8}$ is contained in $\mathcal{U}$: this ball is then denoted by $\mathcal{V}$. As at the beginning of the proof, we denote by D the set of $t \in \mathbf{T}$ such that η is differentiable at t and $\eta'(t) = \alpha(t)$. Let now $t_0 \in \mathrm{D}(\gamma_1)$. As t_0 is a point of continuity of γ_1', there exists a neighbourhood U_0 of t_0 in $\mathbf{T}$ such that: $\forall t \in \mathrm{U}_0 \cap d(\gamma_1)$, $|\gamma_1'(t) - \gamma_1'(t_0)| < \varepsilon\frac{b+a}{16}$. Let now $\gamma_2 \in \mathcal{V}$: then $\gamma_2 = \gamma_1 + \varepsilon\eta + u$ with $\|u\|_\infty + \mathrm{Lip}(u) < \varepsilon\frac{b+a}{8}$. Let $d(u)$ be the set of points of differentiability of u and let $\mathrm{V}_0 = \mathrm{U}_0 \cap d(\gamma_1) \cap \mathrm{D} \cap d(u)$. Then $\lambda(\mathrm{V}_0) = \lambda(\mathrm{U}_0) > 0$ and:

1. u, γ_1 and η are differentiable at every $t \in \mathrm{V}_0$;
2. for every $t \in \mathrm{V}_0$, we have $|u'(t)| < \varepsilon\frac{a+b}{8}$ because $\mathrm{Lip}(u) < \varepsilon\frac{a+b}{8}$;
3. for every $t, t' \in \mathrm{V}_0$, $|\gamma_1'(t) - \gamma_1'(t')| < \varepsilon\frac{b+a}{8}$ because $t, t' \in \mathrm{U}_0$;
4. if $t \in \mathrm{V}_0 \cap \mathrm{A}$, then $\eta'(t) = a$ and if $t \in \mathrm{V}_0 \cap (\mathbf{T} \backslash \mathrm{A})$, then $\eta'(t) = -b$.

We deduce:

1. if $t \in \mathrm{V}_0 \cap \mathrm{A}$, then:

$$\gamma_2'(t) = \gamma_1'(t) + \varepsilon\eta'(t) + u'(t) > \varepsilon a + \gamma_1'(t_0) - 2\varepsilon\frac{b+a}{8}$$

$$= \gamma_1'(t_0) + \varepsilon\left(a - \frac{b+a}{4}\right);$$

2. if $t \in V \cap (\mathbf{T}\backslash A)$, then:

$$\gamma_2'(t) < -\varepsilon b + \gamma_1'(t_0) + 2\varepsilon\frac{b+a}{8} = \gamma_1'(t_0) + \varepsilon\left(-b + \frac{b+a}{4}\right).$$

We have: $-b + \frac{b+a}{4} < a - \frac{b+a}{4}$. Hence γ_2' is discontinuous at every point of U_0. Finally, we have proved that for every $\gamma_2 \in \mathcal{V}$, the Lebesque measure of the set of the points of discontinuity of γ_2' is non zero. This ends the proof. □

Of course, Corollary 2 is a consequence of the last proposition and Theorem 1.

5. The C^0 integrability

In this section, we will prove Theorem 5. We consider an exact symplectic C^1 twist map $f : \mathbf{A} \to \mathbf{A}$ which is C^0 integrable and denote by Γ the set of the C^0-maps $\gamma : \mathbf{T} \to \mathbf{R}$ whose graph is invariant under f. Using the remark given in the introduction, we notice that: $\forall \gamma_1, \gamma_2 \in \Gamma$, either $\gamma_1 < \gamma_2$ or $\gamma_1 > \gamma_2$. We endow Γ with the order $\leq$ and the metric d_∞ of the uniform convergence.

Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of f. For every $\gamma \in C^0(\mathbf{T}, \mathbf{R})$, $\tilde{\gamma}$ is defined by: $\tilde{\gamma}(\theta) = \gamma(\bar{\theta})$. Then $\tilde{\Gamma} = \{\tilde{\gamma}; \gamma \in \Gamma\}$ is also an ordered set, and the graph of every $\tilde{\gamma} \in \tilde{\Gamma}$ is invariant by $\tilde{f}$. For every $\gamma \in \Gamma$ we will denote by $\rho(\gamma)$ the rotation number of $\tilde{f}_1(., \tilde{\gamma}(.)) : \mathbf{R} \to \mathbf{R}$ (see [9] for the definition). Then it is proved in [10] (2.4.2) that $\rho : \Gamma \to \mathbf{R}$ is increasing; moreover, it is continuous.

Proposition **15.** — *Let $f : \mathbf{A} \to \mathbf{A}$ be an exact symplectic positive C^1 twist map which is C^0 integrable. If the graph of a continuous map $\gamma : \mathbf{T} \to \mathbf{R}$ is invariant by f and if its rotation number $\rho(\gamma) = \frac{p}{q}$ is rational, then: $\forall \theta \in \mathbf{T}, f^q(\theta, \gamma(\theta)) = (\theta, \gamma(\theta))$.*

Proof of Proposition 15. — Let $(\gamma_n)_{n\in\mathbf{N}}$ be a decreasing sequence of elements of Γ which tends to γ. Then: $\forall n \in \mathbf{N}$, $\rho(\gamma_n) > \frac{p}{q} = \rho(\gamma)$ and $\lim_{n\to\infty} \rho(\gamma_n) = \frac{p}{q}$. We may also choose γ_n in such a way that: $\forall n \in \mathbf{N}$, $\rho(\gamma_n) \in \mathbf{R}\backslash\mathbf{Q}$.

Then, we have: $\forall k \in \mathbf{N}$, $\forall \theta \in \mathbf{R}, \tilde{f}_1^q(\theta, \tilde{\gamma}_k(\theta)) \neq \theta + p$. We deduce that for every $k \in \mathbf{N}$: either $(*)_1$ $\forall \theta \in \mathbf{R}, \tilde{f}_1^q(\theta, \tilde{\gamma}_k(\theta)) > \theta + p$ or $(*)_2$ $\forall \theta \in \mathbf{R}, \tilde{f}_1^q(\theta, \tilde{\gamma}_k(\theta)) < \theta + p$. Using the fact that $\tilde{f}_1(., \tilde{\gamma}_k())$ is increasing and the fact that $\tilde{f}_1(\theta+1, \tilde{\gamma}_k(\theta+1)) = \tilde{f}_1(\theta, \tilde{\gamma}_k(\theta)) + 1$: we deduce:

$(*)_1$ either: $\forall n \in \mathbf{N}^*, \forall \theta \in \mathbf{R}, \tilde{f}_1^{nq}(\theta, \tilde{\gamma}_k(\theta)) > \theta + np$;
$(*)_2$ or: $\forall n \in \mathbf{N}^*, \forall \theta \in \mathbf{R}, \tilde{f}_1^{nq}(\theta, \tilde{\gamma}_k(\theta)) < \theta + np$;

and then:

$(*)_1$ either: $\forall\theta\in\mathbf{R}$, $\rho(\gamma_k)=\lim_{n\to\infty}\frac{\tilde{f}_1^{nq}(\theta,\tilde{\gamma}_k(\theta))-\theta}{nq}\geq\frac{p}{q}$;

$(*)_2$ or: $\forall\theta\in\mathbf{R}$, $\rho(\gamma_k)=\lim_{n\to\infty}\frac{\tilde{f}_1^{nq}(\theta,\tilde{\gamma}_k(\theta))-\theta}{nq}\leq\frac{p}{q}$.

But we know that $\rho(\gamma_k)>\frac{p}{q}$; therefore, the case $(*)_2$ is impossible and we have: $\forall k\in\mathbf{N}$, $\forall\theta\in\mathbf{R}, \tilde{f}_1^q(\theta,\tilde{\gamma}_k(\theta))>\theta+p$. We deduce that: $\forall\theta\in\mathbf{R}, \tilde{f}_1^q(\theta,\tilde{\gamma}(\theta))\geq\theta+p$.

Using now a increasing sequence of elements of Γ tending to γ, we obtain, similarly: $\forall\theta\in\mathbf{R}, \tilde{f}_1^q(\theta,\tilde{\gamma}(\theta))\leq\theta+p$. □

Proof of Theorem 5. — Let $\mathcal{C}=\{\gamma\in\Gamma;\ \gamma\in C^1(\mathbf{T},\mathbf{R})$ and $\forall\theta\in\mathbf{T}$, $G^-(\theta,\gamma(\theta))=G^+(\theta,\gamma(\theta))\}$. By Corollary 8, we know that the condition $\gamma\in C^1(\mathbf{T},\mathbf{R})$ is redundant.

Lemma **16.** — *If $\gamma\in\Gamma$ is such that $\rho(\gamma)=\frac{p}{q}\in\mathbf{Q}$, then $\gamma\in\mathcal{C}$.*

Proof of Lemma 16. — Let $\gamma\in\Gamma$ be such that $\rho(\gamma)\in\mathbf{Q}$. We deduce from Proposition 15 that every $(\theta,\gamma(\theta))$ is q-periodic for f.

Hence if g is the restriction of f to the graph $G(\gamma)$ of γ, the family $(g^{nq})_{n\in\mathbf{Z}}=(Id_{G(\gamma)})_{n\in\mathbf{Z}}$ is equi-Lipschiz. We deduce from Theorem 3 and from its proof that $\gamma\in\mathcal{C}$. □

We define: $\Gamma_0=\{\gamma\in\Gamma;\ \rho(\gamma)\in\mathbf{Q}\}$; then Γ_0 is dense in Γ; Lemma 16 implies that: $\Gamma_0\subset\mathcal{C}$. Hence $\mathcal{C}$ is dense in Γ.

Let us now prove:

Lemma **17.** — *The map:* $F:\mathbf{T}\times\Gamma\to\mathbf{A}$ *defined by:* $F(\theta,\gamma)=(\theta,\gamma(\theta))$ *is a homeomorphism.*

Proof of Lemma 17. — This map is continuous, one-to-one and onto. Moreover, a result due to Birkhoff states that for every compact set K of $\mathbf{A}$, the set $\{\gamma\in\Gamma;\ G(\gamma)\cap K\neq\emptyset\}$ is compact. Therefore the map F is proper; hence, F is a homeomorphism. □

Lemma **18.** — *The set* $\mathcal{G}=\{x\in\mathbf{A};\ G^+(x)=G^-(x)\}$ *is a* G_δ *subset of* $\mathbf{A}$.

The proof is the same as in Proposition 7: G^+ is upper semi-continuous and G^- is lower semi-continuous.

The map F being a homeomorphism, we deduce from Lemma 18 that $\mathcal{G}=\{(\theta,\gamma)\in\mathbf{T}\times\Gamma;\ G^-(\theta,\gamma(\theta))=G^+(\theta,\gamma(\theta))\}$ is a G_δ subset of $\mathbf{T}\times\Gamma$. Moreover, it contains $\mathbf{T}\times\mathcal{C}$ which is dense in $\mathbf{T}\times\Gamma$. Hence $\mathcal{G}$ is a dense G_δ subset of $\mathbf{T}\times\Gamma$. Therefore there exists a sequence $(U_n)_{n\in\mathbf{N}}$ of open subsets of $\mathbf{T}\times\Gamma$ such that $\mathcal{G}=\bigcap_{n\in\mathbf{N}}U_n$. If $n\in\mathbf{N}$, then $\mathbf{T}\times\mathcal{C}\subset U_n$. As every set $\mathbf{T}\times\{\gamma\}$ is compact, the set $V_n=\{\gamma\in\Gamma;\ \mathbf{T}\times\{\gamma\}\subset U_n\}$ contains an open subset W_n of Γ which contains $\mathcal{C}$; then $\mathbf{T}\times W_n$ is a dense and open subset of $\mathbf{T}\times\Gamma$ such that: $\mathbf{T}\times\mathcal{C}\subset\mathbf{T}\times W_n\subset U_n$. We deduce that $G=\bigcap_{n\in\mathbf{N}}W_n$ is a G_δ of Γ such that: $\mathbf{T}\times\mathcal{C}\subset\mathbf{T}\times G\subset\mathcal{G}$. Hence G is a dense G_δ subset of Γ such that: $\forall\gamma\in G$, $\mathbf{T}\times\{\gamma\}$ is a subset in $\mathcal{G}$; using Corollary 8, we deduce that every $\gamma\in G$ is C^1. □

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