WEAKLY COMMENSURABLE ARITHMETIC GROUPS AND ISOSPECTRAL LOCALLY SYMMETRIC SPACES

by Gopal PRASAD *and* Andrei S. RAPINCHUK

Dedicated to the memory of A.R.'s mother Izabella B. Rapinchuk

CONTENTS

1. Introduction

The goal of this paper is two-fold. First, we introduce and analyze a new relationship between (Zariski-dense) abstract subgroups of the groups of F-rational points of two connected semi-simple algebraic groups defined over a field F, which we call *weak commensurability*. This relationship is expressed in terms of the eigenvalues of individual elements, and does not involve any structural connections between the subgroups. Nevertheless, it turns out that weakly commensurable S-arithmetic subgroups always split into finitely many commensurability classes, and that in certain types of groups, any two weakly commensurable S-arithmetic subgroups are actually commensurable. Second, we use results and conjectures in transcendental number theory to relate weak commensurability with interesting differential geometric problems on length-commensurable, and isospectral, locally symmetric spaces, and to settle a series of open questions in this area by applying our results on weakly commensurable arithmetic (and more general) subgroups. These applications lead us to believe that the notion of weak commensurability is likely to become a useful tool in the theory of Lie groups and related areas.

We begin with the definition of weak commensurability.

Definitions **1.1.** — *Let* G_1 *and* G_2 *be two semi-simple groups defined over a field* F.

DOI 10.1007/s10240-009-0019-6

1. Elements $g_i \in G_i(F)$, where $i = 1, 2$, are *weakly commensurable* if there exist maximal F-tori T_i of G_i such that $g_i \in T_i(F)$, and for some characters χ_i of T_i (defined over an algebraic closure $\overline{F}$ of F), we have

$$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$[1]

2. (Zariski-dense) subgroups Γ_i of $G_i(F)$, for $i = 1, 2$, are *weakly commensurable* if given a semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order, there is a semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order which is weakly commensurable to γ_1, and given a semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, there is a semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order which is weakly commensurable to γ_2.

The following theorems (1 and 2) provide two basic results about weakly commensurable Zariski-dense subgroups.

Theorem **1.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero. Assume that for* $i = 1, 2$, *there exist finitely generated Zariski-dense subgroups* Γ_i *of* $G_i(F)$ *which are weakly commensurable. Then either* G_1 *and* G_2 *are of the same Killing-Cartan type, or one of them is of type* B_n *and the other is of type* C_n.

(We notice that split groups G_1 and G_2 of types B_n and C_n respectively indeed contain weakly commensurable arithmetic subgroups, cf. Example 6.7.)

Theorem **2.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero. For* $i = 1, 2$, *let* Γ_i *be a finitely generated Zariski-dense subgroup of* $G_i(F)$, *and* K_{Γ_i} *be the subfield of* F *generated by the traces* $\mathrm{Tr\,Ad}\,\gamma$, *in the adjoint representation, of* $\gamma \in \Gamma_i$. *If* Γ_1 *and* Γ_2 *are weakly commensurable, then* $K_{\Gamma_1} = K_{\Gamma_2}$.

Most of the results of this paper are on arithmetic subgroups. In fact, the central issue for us is what can be said about two connected absolutely simple groups defined over number fields given that these groups contain weakly commensurable Zariski-dense S-arithmetic subgroups. To give the precise statements (see Theorems 3–6), we need to describe our set-up more carefully. Let G be a connected absolutely almost simple algebraic group defined over a field F of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi: G \to \overline{G}$ be the natural isogeny. Now, suppose we are given a number field K, an embedding $K \hookrightarrow F$, and an algebraic K-group $\mathscr{G}$ such that the F-group ${}_F\mathscr{G}$ obtained from it by extension of scalars $K \hookrightarrow F$, is F-isomorphic to $\overline{G}$ (in other words, $\mathscr{G}$ is an F/K-form of $\overline{G}$). Then we have an embedding $\iota: \mathscr{G}(K) \hookrightarrow \overline{G}(F)$, which is well-defined up to an F-automorphism of $\overline{G}$. Next, let S be a finite set of places of K which contains the set V_∞^K of all archimedean places, but does not contain any nonarchimedean

[1] In other words, the subgroup of $\overline{F}^{\times}$ generated by the eigenvalues(in a faithful representation of G) of g_1 intersects the subgroup generated by the eigenvalues of g_2 nontrivially.

place where $\mathscr{G}$ is anisotropic. We let $\mathcal{O}_K(S)$ denote the ring of S-integers in K (with $\mathcal{O}_K = \mathcal{O}_K(V_\infty^K)$ denoting the ring of algebraic integers in K), and let $\mathscr{G}(\mathcal{O}_K(S))$ be the S-arithmetic subgroup defined in terms of a fixed K-embedding $\mathscr{G} \hookrightarrow GL_n$, i.e., $\mathscr{G}(\mathcal{O}_K(S)) = \mathscr{G}(K) \cap GL_n(\mathcal{O}_K(S))$. A subgroup Γ of G(F) such that $\pi(\Gamma)$ is commensurable with $\sigma(\iota(\mathscr{G}(\mathcal{O}_K(S))))$, for some F-automorphism σ of $\overline{G}$, will be called a $(\mathscr{G}, K, S)$-*arithmetic subgroup*.[2] As usual, $(\mathscr{G}, K, V_\infty^K)$-arithmetic subgroups will simply be called $(\mathscr{G}, K)$-arithmetic.

Now, let G_i, for $i = 1, 2$, be a connected absolutely almost simple F-group, and let $\pi_i\colon G_i \to \overline{G}_i$ be the isogeny onto the corresponding adjoint group. We will say that the subgroups Γ_i of $G_i(F)$ are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if there exists an F-isomorphism $\sigma\colon \overline{G}_1 \to \overline{G}_2$ such that $\sigma(\pi_1(\Gamma_1))$ is commensurable with $\pi(\Gamma_2)$ in the usual sense, i.e., their intersection is of finite index in both of them. (If $G_1 = G_2 =: G$, which was the situation considered in earlier versions of this paper, then we talk about commensurability up to an F-automorphism of G or $\overline{G}$.) Let now Γ_i be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$. The key question for us is when does the fact that Γ_1 and Γ_2 are weakly commensurable imply that they are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$, i.e., $K_1 = K_2$, $S_1 = S_2$ and $\mathscr{G}_1$ and $\mathscr{G}_2$ are K-isomorphic (cf. Proposition 2.5)? Theorems 3–5 address this question.

Theorem **3.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero. If Zariski-dense* $(\mathscr{G}_i, K_i, S_i)$*-arithmetic subgroups* Γ_i *of* $G_i(F)$ *are weakly commensurable for* $i = 1, 2$, *then* $K_1 = K_2$ *and* $S_1 = S_2$.

Examples 6.5, 6.6 and 6.7 show that the existence of weakly commensurable S-arithmetic subgroups does not guarantee that $\mathscr{G}_1$ and $\mathscr{G}_2$ are always isomorphic over $K := K_1 = K_2$. In the next theorem we list the cases where it can be asserted that $\mathscr{G}_1$ and $\mathscr{G}_2$ are K-isomorphic, and then give a general finiteness result for the number of K-isomorphism classes.

Theorem **4.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero, of the same type different from* A_n, D_{2n+1}, *with* $n > 1$, D_4 *and* E_6. *If for* $i = 1, 2$, $G_i(F)$ *contain Zariski-dense weakly commensurable* $(\mathscr{G}_i, K, S)$*-arithmetic subgroups* Γ_i, *then* $\mathscr{G}_1 \simeq \mathscr{G}_2$ *over* K, *and hence* Γ_1 *and* Γ_2 *are commensurable up to an* F*-isomorphism between* $\overline{G}_1$ *and* $\overline{G}_2$.

In earlier versions of this paper, the case of groups of type D_{2n} in Theorem 4 was left open. This case was recently settled in [35] using techniques of the current paper in

[2] This notion of arithmetic subgroups coincides with that in Margulis' book [21] for absolutely simple adjoint groups. Notice that if $\mathscr{G}$ is anisotropic over K_v, where v is a nonarchimedean place of K, then $\mathscr{G}(\mathcal{O}_K(S))$ is commensurable with $\mathscr{G}(\mathcal{O}_K(S \cup \{v\}))$, so the classes of S- and $(S \cup \{v\})$-arithmetic subgroups coincide. Thus, the above assumption on S is necessary if one wants to recover S from a given S-arithmetic subgroup.

conjunction with new results on embedding of fields with involutive automorphisms into simple algebras with involutions.

Theorem **5.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple groups defined over a field* F *of characteristic zero. Let* Γ_1 *be a Zariski-dense* $(\mathscr{G}_1, K, S)$*-arithmetic subgroup of* $G_1(F)$. *Then the set of* K*-isomorphism classes of* K*-forms* $\mathscr{G}_2$ *of* $\overline{G}_2$ *such that* $G_2(F)$ *contains a Zariski-dense* $(\mathscr{G}_2, K, S)$*-arithmetic subgroup weakly commensurable to* Γ_1 *is finite. In other words, the set of all Zariski-dense* (K, S)*-arithmetic subgroups of* $G_2(F)$ *which are weakly commensurable to a given Zariski-dense* (K, S)*-arithmetic subgroup of* $G_1(F)$ *is a union of finitely many commensurability classes.*

A noteworthy fact about weak commensurability is that it has the following implication for the existence of unipotent elements in arithmetic subgroups (even though it is formulated entirely in terms of semi-simple ones).

Theorem **6.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero. For* $i = 1, 2$, *let* Γ_i *be a Zariski-dense* $(\mathscr{G}_i, K, S)$*-arithmetic subgroup of* $G_i(F)$. *Assume that* Γ_1 *and* Γ_2 *are weakly commensurable. Then* $\mathrm{rk}_K\, \mathscr{G}_1 = \mathrm{rk}_K\, \mathscr{G}_2$ *(in particular, if* $\mathscr{G}_1$ *is* K*-isotropic, then so is* $\mathscr{G}_2$*). If* G_1 *and* G_2 *are of the same type, then the Tits indices of* $\mathscr{G}_1/K$ *and* $\mathscr{G}_2/K$, *and for every place* v *of* K, *the Tits indices of* $\mathscr{G}_1/K_v$ *and* $\mathscr{G}_2/K_v$, *are isomorphic.*

(For a description of Tits index of a simple algebraic group, see §7.)

The following result asserts that a lattice[3] which is weakly commensurable with an S-arithmetic group is arithmetic.

Theorem **7.** — *Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field* F *of characteristic zero, and for* $i = 1, 2$, *let* Γ_i *be a Zariski-dense lattice in* $G_i(F)$. *Assume that* Γ_1 *is a* (K, S)*-arithmetic subgroup of* $G_1(F)$. *If* Γ_1 *and* Γ_2 *are weakly commensurable, then* Γ_2 *is a* (K, S)*-arithmetic subgroup of* $G_2(F)$.

The proofs of these theorems use a variety of algebraic and number-theoretic techniques. One of the key ingredients is a new method for constructing elements with special properties in a given Zariski-dense subgroup of a semi-simple algebraic group developed in our papers [31–33] to answer questions of Y. Benoist, G.A. Margulis, R. Spatzier et al. arising in geometry. This method is described in Section 3 below in a considerably modified form required for the proofs of Theorems 1–7. Among other important ingredients of our proofs are Tits' classification of semi-simple algebraic groups over nonalgebraically closed fields (cf. [44]), and results on Galois cohomology of semi-simple groups over local and global fields. As a by-product of our argument, we obtain an almost complete solution of the old problem whether an absolutely simple group over a number field is

[3] A discrete subgroup Γ of a locally compact topological group $\mathcal{G}$ is said to be a lattice in $\mathcal{G}$ if $\mathcal{G}/\Gamma$ carries a *finite* $\mathcal{G}$-invariant Borel measure, see [36].

determined by the set of isomorphism classes of its maximal tori (cf. Theorem 7.5). It is our belief that the notion of weak commensurability, and the techniques involved in its analysis, in conjunction with the results of [31–33], will have numerous applications in the theory of Lie groups, ergodic theory, and (differential) geometry. In fact, the results on weak commensurability stated above were motivated by, and actually enabled us to settle, some problems about the lengths of closed geodesics in, and isospectrality of, arithmetically defined locally symmetric spaces. We now proceed to describe these geometric applications.

For a Riemannian manifold M, the *length spectrum* $\mathcal{L}(M)$ (resp., the *weak length spectrum* $L(M)$) is defined to be the set of lengths of closed geodesics in M with multiplicities (resp., without multiplicities), cf. [19]. The following question has received considerable attention: to what extent do $\mathcal{L}(M)$, $L(M)$, or the spectrum of the Laplace-Beltrami operator on M, determine M? It turns out that all these sets are interrelated: for example, two compact hyperbolic 2-manifolds are isospectral[4] if and only if they have the same length spectrum, cf. [22]; two hyperbolic 3-manifolds are isospectral if and only if they have the same *complex-length* spectrum, cf. [13]. Furthermore, it is known that isospectral compact locally symmetric spaces, with nonpositive sectional curvatures, have the same weak length spectrum, see Theorem 10.1 below. The first examples of isospectral but not isometric (although commensurable[5]) compact hyperbolic 2- and 3-manifolds were given in [45]. Recently, in [20], noncommensurable isospectral locally symmetric spaces have been constructed. On the other hand, in 1985 Sunada [42] described a general method for producing examples of nonisometric (but commensurable) isospectral manifolds. A variant of Sunada's construction has been used in [19] to give examples of hyperbolic manifolds with equal weak length spectra but different volumes. Earlier, in [38], the same approach was used to produce nonisometric hyperbolic 3-manifolds with equal weak length spectra. It should be pointed out that Sunada's construction, which is the only known general method for constructing manifolds with the same (weak) length, or Laplace-Beltrami operator, spectra, *always* produces commensurable manifolds (in particular, the examples in [19] and [38] are commensurable). So the following question was raised (cf., for example, [38]):

(1) *Let* M_1 *and* M_2 *be two (hyperbolic) manifolds (of finite volume or even compact). Suppose* $L(M_1) = L(M_2)$. *Are* M_1 *and* M_2 *necessarily commensurable?*

One may generalize this question by introducing the notion of length-commensurability, which in particular allows us to replace the manifolds under consideration with commensurable ones: we say that M_1 and M_2 are *length-commensurable* if $\mathbf{Q} \cdot L(M_1) = \mathbf{Q} \cdot L(M_2)$. Now, (1) can be reformulated as follows:

---

[4] Two compact Riemannian manifolds are said to be *isospectral* if their Laplace-Beltrami operators have the same eigenvalues with the same multiplicities, cf. Section 10.

[5] Two manifolds are called *commensurable* if they admit a common finite-sheeted cover.

(2) *Suppose* M_1 *and* M_2 *are length-commensurable. Are they commensurable?*

In [38], an affirmative answer (to (1)) was given for arithmetically defined hyperbolic 2-manifolds, and very recently in [8] a similar result has been obtained for hyperbolic 3-manifolds. The results of this paper, combined with those of [35], §9, for groups of type D_{2n}, provide an affirmative answer to (2) for arithmetically defined hyperbolic manifolds of dimensions $2n$ and $4n+7$ for $n \geqslant 1$, but a negative answer for hyperbolic manifolds of dimension $4n+1$, and for complex hyperbolic manifolds. In fact, we analyze the problem in the general context of arithmetically defined locally symmetric spaces.

For $i = 1, 2$, let G_i be a connected adjoint semi-simple real algebraic subgroup of SL_n, $\mathcal{G}_i = G_i(\mathbf{R})$ considered as a Lie group, and $\mathcal{K}_i$ be a maximal compact subgroup of $\mathcal{G}_i$. Then $\mathfrak{X}_i = \mathcal{K}_i \backslash \mathcal{G}_i$ is the symmetric space of $\mathcal{G}_i$. Given a discrete torsion-free subgroup Γ_i of $\mathcal{G}_i$, the quotient $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}_i / \Gamma_i$ is a locally symmetric space. For $i = 1$ or 2, we say that $\mathfrak{X}_{\Gamma_i}$ is *arithmetically defined* if Γ_i is an arithmetic subgroup of $\mathcal{G}_i$ (cf. [21], Chap. IX). According to the following theorem, length-commensurability of locally symmetric spaces is closely related to weak commensurability of the corresponding discrete subgroups.

Theorem **8.12.** — *If* Γ_1 *and* Γ_2 *are not weakly commensurable, then, possibly after interchanging them, the following assertions hold.*

(i) *If* G_1 *and* G_2 *are of real rank* 1, *and either there exists a number field* $K \subset \mathbf{R}$ *such that both* Γ_1 *and* Γ_2 *can be conjugated into* $SL_n(K)$, *or* $G_1 \simeq G_2$, *then there exists* $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ *such that for* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$, *the ratio* λ_1/λ_2 *is irrational.*

(ii) *If there exists a number field* $K \subset \mathbf{R}$ *such that both* Γ_1 *and* Γ_2 *can be conjugated into* $SL_n(K)$, *and Schanuel's conjecture holds, then there exists* $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ *which is algebraically independent from* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$.

In either case, (under the above assumptions) $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are not length-commensurable.*

We would like to emphasize that while our results for rank one locally symmetric spaces (which include hyperbolic spaces of all types) are unconditional, the results for spaces of higher rank depend on the validity of the well-known conjecture in transcendental number theory due to Schanuel (see Section 8 for the statement); needless to say that the results in Sections 2–7, 9 on weak commensurability (in particular, Theorems 1–7) do not involve any transcendental number theory.

In the sequel, we will refer to the following situation as the *exceptional case:*

($\mathcal{E}$) One of the locally symmetric spaces, say, $\mathfrak{X}_{\Gamma_1}$, is 2-dimensional and the corresponding discrete subgroup Γ_1 cannot be conjugated into $PGL_2(K)$, for any number field $K \subset \mathbf{R}$, and the other space, $\mathfrak{X}_{\Gamma_2}$, has dimension > 2.

Theorem 8.12 implies the following.

Corollary **8.14.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_i}$ *be a locally symmetric space of finite volume, of* $\mathcal{G}_i = G_i(\mathbf{R})$, *for* $i = 1, 2$. *Assume that we are* *not in the exceptional case* $(\mathcal{E})$. *If* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable, then* Γ_1 *and* Γ_2 *are weakly commensurable.*

Now, applying Theorems 1, 2 and 7 we obtain the following:

Theorem **8.15.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_i}$ *be a locally symmetric space of finite volume, of* $\mathcal{G}_i$, *for* $i = 1, 2$. *Assume that* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable, and we are not in the exceptional case* $(\mathcal{E})$. *Then* (i) *either* G_1 *and* G_2 *are of same Killing-Cartan type, or one of them is of type* B_n *and the other is of type* C_n, (ii) $K_{\Gamma_1} = K_{\Gamma_2}$.

Combining Corollary 8.14 with Theorems 4 and 5, we obtain

Theorem **8.16.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathcal{G}_i = G_i(\mathbf{R})$, *for* $i = 1, 2$. *Then the set of arithmetically defined locally symmetric spaces* $\mathfrak{X}_{\Gamma_2}$ *of* $\mathcal{G}_2$, *which are length-commensurable to a given arithmetically defined locally symmetric space* $\mathfrak{X}_{\Gamma_1}$ *of* $\mathcal{G}_1$, *is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if* G_1 *and* G_2 *have the same type different from* A_n, D_{2n+1}, *with* $n > 1$, D_4 *and* E_6.

Furthermore, Theorems 6 and 7 imply the following rather surprising result which has so far defied attempts to prove it purely geometrically.

Theorem **8.19.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be length-commensurable locally symmetric spaces of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined and that we are not in the exceptional case* $(\mathcal{E})$. *Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

In Section 9, we present a general cohomological construction which, in particular, enables us to give examples of length-commensurable, but not commensurable, arithmetically defined locally symmetric spaces associated to an absolutely simple Lie group of any of the following types: A_n, D_{2n+1} $(n > 1)$, or E_6, see Construction 9.15 (thus, the second assertion of Theorem 8.16 definitely cannot be extended to these types). Towards this end, we establish a new local-global principle for the existence of an embedding of a given K-torus as a maximal torus in an absolutely simple simply connected K-group (for the precise assertion, see Theorem 9.5). Using this local-global principle, we show that there exist nonisomorphic K-forms G_1 and G_2 of an absolutely simple K-group of each of the types A_n, D_{2n+1} $(n > 1)$, and E_6, such that (i) G_1 is isomorphic to G_2 over K_v, for all places v of K (so $G_1(\mathbf{A}_K)$ is isomorphic to $G_2(\mathbf{A}_K)$ as a topological group, where $\mathbf{A}_K$ is the adèle ring of K), and (ii) given a maximal K-torus T_i of G_i, there is an isomorphism $G_i \to G_{3-i}$ whose restriction to T_i is defined over K. Such K-forms are likely to be of interest in the Langlands program. Given nonisomorphic K-forms G_1 and G_2 with the above properties, any arithmetic subgroup Γ_1 of $G_1(K)$ is weakly commensurable, but not commensurable, to any arithmetic subgroup Γ_2 of $G_2(K)$, and the associated locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable but not commensurable (see Proposition 9.14 and Construction 9.15).

We assume now that both G_1 and G_2 are connected absolutely simple adjoint real algebraic groups, and for $i = 1, 2$, Γ_i is a torsion-free discrete cocompact subgroup of $\mathcal{G}_i$, and $\mathfrak{X}_{\Gamma_i}$ is the associated compact locally symmetric space. As a consequence of our previous results, and Theorem 10.1, we obtain the following theorem, which answers Mark Kac's famous question "Can one hear the shape of a drum?" for arithmetically defined compact locally symmetric spaces.

Theorem **10.4.** — *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and at least one of the subgroups Γ_1 and Γ_2 is arithmetic. Then $G_1 = G_2 =: G$. Moreover, unless G is of type A_n, D_{2n+1} $(n > 1)$, D_4 or E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.*

Notation and conventions. Unless stated otherwise, all our fields will be of characteristic zero. For a number field K, we let V^K (resp., V^K_∞ and V^K_f) denote the set of all places (resp., the subsets of archimedean and nonarchimedean places). For a torus T, we let X(T) denote the character group, and for a morphism $\pi: T_1 \to T_2$ between two tori, we let $\pi^*: X(T_2) \to X(T_1)$ denote the induced homomorphism of the character groups. If T is defined over K, then K_T will denote the (minimal) splitting field of T over K and X(T) will be considered as a module over the Galois group $\mathrm{Gal}(K_T/K)$.

In the sequel, all number fields are assumed to be contained in the field $\mathbf{C}$ of complex numbers. For a subfield K (resp., K_i) of $\mathbf{C}$, $\overline{K}$ (resp., $\overline{K}_i$) will denote its algebraic closure in $\mathbf{C}$. For a place v of a number field K (resp., K_i), $\overline{K}_v$ (resp., $\overline{K}_{iv}$) will denote an algebraic closure of the completion K_v (resp., K_{iv}) of K (resp., K_i) at v. In particular, $\overline{\mathbf{Q}}_p$ will denote an algebraic closure of $\mathbf{Q}_p$.

Every algebraic K-group occurring in this paper will be assumed to be linear in terms of some K-embedding of the group in GL_n. We will use the adjoint representation to realize a semi-simple adjoint group as a linear group. For an algebraic K-subgroup G of GL_n, and a subring R of a commutative K-algebra C, G(R) will denote the group $G(C) \cap \mathrm{GL}_n(R)$.

2. Preliminaries

Let G_1 and G_2 be two semi-simple algebraic groups defined over a field F. We begin with a simple comment on the notion of weak commensurability of semi-simple elements.

Lemma **2.1.** — *For $i = 1, 2$, let $\gamma_i \in G_i(F)$ be a semi-simple element. The following conditions are equivalent:*

(1) γ_1 *and* γ_2 *are weakly commensurable, i.e.,* there exist *maximal* F*-tori* T_i *of* G_i *for* $i = 1, 2$ *such that* $\gamma_i \in T_i(F)$ *and* $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ *for some characters* $\chi_i \in X(T_i)$*;*
(2) for any *maximal* F*-tori* T_i *of* G_i *with* $\gamma_i \in T_i(F)$*, there exist characters* $\chi_i \in X(T_i)$ *such that* $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$.

While (2) trivially implies (1), the opposite implication follows from the fact that if C_i is the Zariski-closure in G_i of the subgroup generated by γ_i, then for *any* torus T_i containing C_i, the restriction map $X(T_i) \to X(C_i)$ is surjective (cf. [3], 8.2).

Corollary **2.2.** — *For* $i = 1, 2$*, let* K_i *be a subfield of* F*,* $\mathscr{G}_i$ *be an* F/K_i*-form of* G_i*, and* $\gamma_i \in \mathscr{G}_i(K_i) \hookrightarrow G_i(F)$ *be a semi-simple element. Then* γ_1 *and* γ_2 *are weakly commensurable if and only if there exist maximal* K_i*-tori* $\mathscr{T}_i$ *of* $\mathscr{G}_i$ *such that* $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ *for some* $\chi_i \in X(\mathscr{T}_i)$.

This follows from the above lemma because every semi-simple $\gamma_i \in \mathscr{G}_i(K_i)$ is contained in a maximal K_i-torus of $\mathscr{G}_i$.

We will now prove two elementary lemmas on weak commensurability of subgroups. The first lemma enables one to replace each of the two weakly commensurable subgroups with a commensurable subgroup.

Lemma **2.3.** — *For* $i = 1, 2$*, let* Γ_i *and* Γ_2 *be a finitely generated Zariski-dense subgroups of* $G_i(F)$. *We assume that* Γ_1 *and* Γ_2 *are weakly commensurable. If for* $i = 1, 2$*,* Δ_i *is a subgroup of* $G_i(F)$ *commensurable with* Γ_i*, then the subgroups* Δ_1 *and* Δ_2 *are weakly commensurable.*

Proof. — We recall that a subgroup Δ of $GL_n(K)$ is *neat* if for every $\delta \in \Delta$, the subgroup of $\overline{K}^{\times}$ generated by the eigenvalues of Δ is torsion-free. According to a result proved by Borel (cf. [36], Theorem 6.11) every finitely generated subgroup of $GL_n(K)$ contains a neat subgroup of finite index. We fix a neat subgroup Θ of $\Gamma_1 \cap \Delta_1$ of finite index, then $[\Delta_1 : \Theta] < \infty$. Given a semi-simple element $\delta_1 \in \Delta_1$ of infinite order, we can pick $n_1 \geqslant 1$ so that $\gamma_1 := \delta_1^{n_1} \in \Theta$. Since Γ_1 and Γ_2 are weakly commensurable, there exists a semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order so that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1,$$

where for $i = 1, 2$, χ_i is a character of a maximal F-torus T_i of G_i such that $\delta_1 \in T_1(F)$ and $\gamma_2 \in T_2(F)$. Now, pick $n_2 \geqslant 1$ so that $\delta_2 := \gamma_2^{n_2} \in \Gamma_2 \cap \Delta_2$. Then

$$(n_2\chi_1)(\gamma_1) = ((n_1 n_2)\chi_1)(\delta_1) = \chi_2(\delta_2). \tag{1}$$

It remains to observe that since $\chi_1(\gamma_1) \neq 1$ belongs to the subgroup generated by the eigenvalues of γ_1, which is torsion-free, it is not a root of unity. This implies that the common value in (1) is $\neq 1$, and therefore δ_1 and δ_2 are weakly commensurable. Thus, every semi-simple $\delta_1 \in \Delta_1$ of infinite order is weakly commensurable to some semi-simple $\delta_2 \in \Delta_2$ of infinite order, and by symmetry, every semi-simple $\delta_2 \in \Delta_2$ of infinite order is weakly commensurable to some semi-simple $\delta_1 \in \Delta_1$ of infinite order, which makes Δ_1 and Δ_2 weakly commensurable. □

The next lemma shows that in the analysis of weak commensurability of subgroups, one can replace the ambient algebraic groups with isogenous groups.

Lemma **2.4.** — *For $i = 1, 2$, let $\pi_i \colon G_i \to G'_i$ be an* F*-isogeny of connected semi-simple algebraic* F*-groups, and let Γ_i be a finitely generated Zariski-dense subgroup of* $G_i(F)$. *Then Γ_1 and Γ_2 are weakly commensurable if and only if their images $\Gamma'_1 = \pi_1(\Gamma_1)$ and $\Gamma'_2 = \pi_2(\Gamma_2)$ are weakly commensurable.*

Proof. — One direction is almost immediate. Namely, suppose Γ'_1 and Γ'_2 are weakly commensurable. Then for a given semi-simple element γ_1 of Γ_1 of infinite order, there exists a semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order so that for $i = 1, 2$, there exist a maximal F-torus T'_i of G'_i, and a character χ'_i of T'_i, such that $\pi_i(\gamma_i) \in T'_i(F)$, and

$$\chi'_1(\pi_1(\gamma_1)) = \chi'_2(\pi_2(\gamma_2)) \neq 1.$$

Then, for $i = 1, 2$, $T_i := \pi_i^{-1}(T'_i)$ is a maximal F-torus of G_i, $\gamma_i \in T_i(F)$, and for their characters $\chi_i = \pi_i^*(\chi'_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This, combined with a "symmetric" argument, implies that Γ_1 and Γ_2 are weakly commensurable.

Conversely, suppose that Γ_1 and Γ_2 are weakly commensurable, and for $i = 1, 2$, pick neat subgroups Δ_i of Γ_i of finite index. By Lemma 2.3, it is enough to show that $\pi_1(\Delta_1)$ and $\pi_2(\Delta_2)$ are weakly commensurable. Let δ_1 be a nontrivial semi-simple element of Δ_1. Then there exists a semi-simple element $\delta_2 \in \Delta_2$ such that for $i = 1, 2$, there exist a maximal F-torus T_i of G_i, with $\delta_i \in T_i(F)$, and a character χ_i of T_i, so that

$$\chi_1(\delta_1) = \chi_2(\delta_2) \neq 1. \tag{2}$$

Set $T'_i = \pi_i(T_i)$. Then $\pi_i(\delta_i) \in T'_i(F)$. If $m = |\ker \pi_1| \cdot |\ker \pi_2|$, then there exist characters $\chi'_i \in X(T'_i)$ such that $m\chi_i = \pi_i^*(\chi'_i)$. Since Δ_1 is neat, the common value in (2) is not an m-th root of unity, and then

$$\chi'_1(\pi_1(\delta_1)) = \chi'_2(\pi_2(\delta_2)) = \chi_1(\delta_1)^m \neq 1.$$

This, together with a "symmetric" argument, implies that $\pi_1(\Delta_1)$ and $\pi_2(\Delta_2)$ are weakly commensurable. □

Next, we prove the following (known) proposition which characterizes commensurable S-arithmetic subgroups. Since we have not been able to find a reference for its proof, we give a complete argument.

Proposition **2.5.** — *Let* G_1 *and* G_2 *be connected absolutely almost simple algebraic groups defined over a field* F *of characteristic zero, and for* $i = 1, 2$, *let* Γ_i *be a Zariski-dense* $(\mathscr{G}_i, K_i, S_i)$-*arithmetic subgroup of* $G_i(F)$. *Then* Γ_1 *and* Γ_2 *are commensurable up to an* F-*isomorphism between* $\overline{G}_1$ *and* $\overline{G}_2$ *if and only if* $K_1 = K_2 =: K$, $S_1 = S_2$, *and* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *are* K-*isomorphic.*

Proof. — It is obvious from the definition of various terms involved in the proposition that to prove it we can replace G_i by its adjoint group $\overline{G}_i$, which enables us to assume in the rest of the proof that for $i = 1, 2$, $G_i = \overline{G}_i$. Then, for $i = 1, 2$, we can fix an F-isomorphism $\iota_i{:}_F\, \mathscr{G}_i \to G_i$ so that Γ_i is commensurable with $\iota_i(\mathscr{G}_i(\mathcal{O}_{K_i}(S_i)))$. One implication is obvious. Namely, suppose $K_1 = K_2 =: K$, $S_1 = S_2 =: S$, and let $\tau: \mathscr{G}_1 \to \mathscr{G}_2$ be a K-isomorphism. Then $\tau(\mathscr{G}_1(\mathcal{O}_K(S)))$ is commensurable with $\mathscr{G}_2(\mathcal{O}_K(S))$, and $\sigma := \iota_2 \circ \tau \circ \iota_1^{-1}$ is an F-isomorphism between $G_1 = \overline{G}_1$ and $G_2 = \overline{G}_2$. Clearly, $\sigma(\iota_1(\mathscr{G}_1(\mathcal{O}_K(S))))$ is commensurable with $\iota_2(\mathscr{G}_2(\mathcal{O}_K(S)))$, implying that $\sigma(\Gamma_1)$ is commensurable with Γ_2, as required.

Conversely, suppose $\sigma: G_1 \to G_2$ is an F-isomorphism such that $\sigma(\Gamma_1)$ and Γ_2 are commensurable. Set $\Gamma'_1 = \Gamma_1 \cap \sigma^{-1}(\Gamma_2)$ and $\Gamma'_2 = \sigma(\Gamma_1) \cap \Gamma_2$. Then $\sigma(\Gamma'_1) = \Gamma'_2$, so the subfield $K_{\Gamma'_1}$ of F generated by $\mathrm{Tr}\,\mathrm{Ad}_{G_1}\gamma$ for $\gamma \in \Gamma'_1$ coincides with the subfield $K_{\Gamma'_2}$ generated by $\mathrm{Tr}\,\mathrm{Ad}_{G_2}\gamma$ for $\gamma \in \Gamma'_2$. Since Γ'_i is $(\mathscr{G}_i, K_i, S_i)$-arithmetic for $i = 1, 2$, the assertion that $K_1 = K_2$ is an immediate consequence of the following lemma.

Lemma **2.6.** — *Let* G *be a connected absolutely almost simple algebraic group defined over a field* F *of characteristic zero, and* Γ *be a Zariski-dense* $(\mathscr{G}, K, S)$-*arithmetic subgroup of* $G(F)$. *Then the subfield* K_Γ *of* F *generated by* $\mathrm{Tr}\,\mathrm{Ad}_G\gamma$ *for* $\gamma \in \Gamma$ *coincides with* K.

Proof. — We will assume (as we may) that the group G is adjoint. By definition, there exists an F-isomorphism $\iota{:}_F\, \mathscr{G} \simeq G$ such that Γ is commensurable with $\iota(\mathscr{G}(\mathcal{O}_K(S)))$. Set

$$\Delta = \iota^{-1}(\Gamma) \subset \mathscr{G}(F).$$

Then Δ is a Zariski-dense S-arithmetic subgroup of $\mathscr{G}(F)$. As $\mathscr{G}$ is of adjoint type, Δ is contained in $\mathscr{G}(K)$ (see, for example, Proposition 1.2 of [4]). This implies that $\mathrm{Tr}\,\mathrm{Ad}_G(\Gamma) = \mathrm{Tr}\,\mathrm{Ad}_{\mathscr{G}}(\Delta) \subset K$, hence the inclusion $K_\Gamma \subset K$. To prove the reverse inclusion, we observe that according to Theorem 1 of Vinberg [46], there exists a basis of the Lie algebra $\mathfrak{g}$ of $\mathscr{G}$ (which we fix) with respect to which Δ is represented by matrices with entries in K_Γ, and $\mathscr{G}$ admits K_Γ as its field of definition. Let $A \subset \mathrm{End}\,\mathfrak{g}$ be the linear span of Δ. (Note that as the action of $\mathscr{G}$, and so of Δ, on $\mathfrak{g}$ is absolutely irreducible, A in fact equals $\mathrm{End}\,\mathfrak{g}$.) Then A is invariant under conjugation by Δ, hence by $\mathscr{G}$, so we can consider the corresponding (faithful) representation $\rho: \mathscr{G} \to \mathrm{GL}(A)$. Let $\mathscr{A}$ be the K_Γ-linear span of Δ. Any subgroup Δ' of Δ of finite index has the same Zariski-closure as Δ (viz., $\mathscr{G}$), and hence the same K_Γ-linear span (viz., $\mathscr{A}$). Since for any $g \in \mathscr{G}(K)$, the intersection $\Delta \cap (g^{-1}\Delta g)$ is of finite index in Δ, we see that $\mathscr{A}$ is invariant under conjugation by $\mathscr{G}(K)$, and therefore, in terms of a basis of A contained in $\mathscr{A}$, $\rho(\mathscr{G}(K))$

is represented by matrices with entries in K_Γ. Hence, $\mathscr{G}(K_\Gamma) = \mathscr{G}(K)$, and the lemma is implied by the following.

Lemma **2.7.** — *Let $\mathscr{G}$ be a reductive algebraic group of positive dimension defined over an infinite field $\mathscr{K}$. Then for any nontrivial extension $\mathscr{L}/\mathscr{K}$, we have $\mathscr{G}(\mathscr{K}) \neq \mathscr{G}(\mathscr{L})$.*

Proof. — We may (and do) assume that $\mathscr{G}$ is connected. It is known that $\mathscr{G}$ is unirational over $\mathscr{K}$ (cf. [3], Theorem 18.2), i.e., there exists a dominant $\mathscr{K}$-rational map $f: \mathbf{A}^n \to \mathscr{G}$ from the affine space $\mathbf{A}^n$. We pick a line ℓ in $\mathbf{A}^n$, defined over $\mathscr{K}$, such that f restricts to a nonconstant map on ℓ. Let $\mathscr{C}$ be the Zariski-closure of $f(\ell(\mathscr{K}))$. Then $\mathscr{C}$ is a curve defined over $\mathscr{K}$; furthermore, by Lüroth's theorem, $\mathscr{C}$ is rational over $\mathscr{K}$, i.e., it is $\mathscr{K}$-isomorphic to an open subvariety of $\mathbf{A}^1$. This immediately implies that $\mathscr{C}(\mathscr{K}) \neq \mathscr{C}(\mathscr{L})$, and our claim follows. □

To complete the proof of Proposition 2.5, consider the F-isomorphism $\tau = \iota_2^{-1} \circ \sigma \circ \iota_1$ between ${}_F\mathscr{G}_1$ and ${}_F\mathscr{G}_2$. We can obviously choose subgroups Δ_i of $\mathscr{G}_i(\mathcal{O}_{K_i}(S_i))$ of finite indices so that $\sigma(\iota_1(\Delta_1)) = \iota_2(\Delta_2)$, and then $\tau(\Delta_1) = \Delta_2$. Since Δ_i is a Zariski-dense subgroup of $\mathscr{G}_i(K)$, where $K := K_1 = K_2$, we see that τ is in fact defined over K. Next, take any $v \notin S_1$. Since the closure of Δ_1 in $\mathscr{G}_1(K_v)$ is compact, we obtain that the closure of $\Delta_2 = \tau(\Delta_1)$ in $\mathscr{G}_2(K_v)$ is also compact. If we assume that $v \in S_2$, then $\mathscr{G}_2(K_v)$ is noncompact, and the fact that Δ_2 is a lattice in $\prod_{w \in S_2} \mathscr{G}_2(K_w)$, yields a contradiction. Thus, $v \notin S_2$, proving the inclusion $S_2 \subset S_1$. The opposite inclusion is proved similarly, so $S_1 = S_2$. □

Remark **2.8.** — The assertion of Lemma 2.7 remains true also over a finite field $\mathscr{K}$ for any connected reductive group $\mathscr{G}$ which is not a torus. Indeed, in this case $\mathscr{G}$ is quasi-split over $\mathscr{K}$ (cf. [3], Proposition 16.6), and therefore it contains a 1-dimensional split torus $\mathscr{C}$. Clearly, $\mathscr{C}(\mathscr{K}) \neq \mathscr{C}(\mathscr{L})$, implying that $\mathscr{G}(\mathscr{K}) \neq \mathscr{G}(\mathscr{L})$.

Let now $\mathscr{G} = \mathscr{T}$ be a torus over $\mathscr{K} = \mathbf{F}_q$, and let $\mathscr{L} = \mathbf{F}_{q^m}$ with $m > 1$. It follows from ([47], 9.1) that

$$|\mathscr{T}(\mathscr{K})| = \prod_{i=1}^{d} (q - \lambda_i) \quad \text{and} \quad |\mathscr{T}(\mathscr{L})| = \prod_{i=1}^{d} (q^m - \lambda_i^m),$$

where λ_i are certain complex roots of unity and $d = \dim \mathscr{T}$. We have

$$|q - \lambda_i| \leqslant q + 1 \quad \text{and} \quad |q^m - \lambda_i^m| \geqslant q^m - 1,$$

so if $q^m - q > 2$, which is always the case unless $q = 2 = m$, then $|\mathscr{T}(\mathscr{L})| > |\mathscr{T}(\mathscr{K})|$. Suppose now that $q = 2 = m$. Clearly, $|\mathscr{T}(\mathscr{K})| = |\mathscr{T}(\mathscr{L})|$ is possible only if $|q - \lambda_i| = q + 1$, i.e., $\lambda_i = -1$, for all i. This means that $\mathscr{T} \simeq (R^{(1)}_{\mathscr{L}/\mathscr{K}}(GL_1))^d$, where $R^{(1)}_{\mathscr{L}/\mathscr{K}}(GL_1)$ is the norm one torus associated with the extension $\mathscr{L}/\mathscr{K} = \mathbf{F}_4/\mathbf{F}_2$. For these tori we have $\mathscr{T}(\mathscr{K}) = \mathscr{T}(\mathscr{L})$, and our argument shows that these are the only exceptions to Lemma 2.7 over finite fields.

3. Results on irreducible tori

A pivotal role in the proof of Theorems 1–7 is played by a reformulation of Theorem 3 of [32]. To explain this reformulation, we need to introduce some additional notation.

Let K be an infinite field and G be a connected absolutely almost simple algebraic K-subgroup of GL_n. Let T be a maximal K-torus of G. As usual, $\Phi = \Phi(G, T)$ will denote the root system of G with respect to T, and $W(\Phi)$, or $W(G, T)$, the Weyl group of Φ. We shall denote by K_T the (minimal) splitting field of T in a fixed separable closure $\overline{K}$ of K. Then there exists a natural injective homomorphism $\theta_T\colon \mathrm{Gal}(K_T/K) \to \mathrm{Aut}(\Phi)$. The following result is a strengthening of Theorem 3(i) of [32], which does not require any significant changes in the proof.

Theorem **3.1.** — *Let* G *be a connected absolutely almost simple algebraic group defined over a finitely generated field* K *of characteristic zero, and* L *be a finitely generated field containing* K. *Let* r *be the number of nontrivial conjugacy classes of the Weyl group of* G, *and suppose that we are given* r *inequivalent nontrivial discrete valuations* $v_1, \ldots, v_r$ *of* K *such that the completion* K_{v_i} *is locally compact and contains* L, *and* G *splits over* K_{v_i}, *for* $i = 1, \ldots, r$. *There exist maximal* K_{v_i}*-tori* $T(v_i)$ *of* G, *one for each* $i \in \{1, \ldots, r\}$, *with the property that for any maximal* K*-torus* T *of* G *which is conjugate to* $T(v_i)$ *by an element of* $G(K_{v_i})$ *for all* $i = 1, \ldots, r$, *we have*

$$(\mathbf{3}) \qquad \theta_T(\mathrm{Gal}(L_T/L)) \supset W(G, T),$$

where $L_T = K_T L$ *is the splitting field of* T *over* L *so that* $\mathrm{Gal}(L_T/L)$ *can be identified with a subgroup of* $\mathrm{Gal}(K_T/K)$.

We will now derive two corollaries that will be used in the subsequent sections.

Corollary **3.2.** — *Let* G, K *and* L *be as in Theorem 3.1, and let* V *be a finite set of nontrivial valuations of* K *such that for every* $v \in V$, *the completion* K_v *is locally compact. Suppose that for each* $v \in V$ *we are given a maximal* K_v*-torus* $T(v)$ *of* G. *Then there exists a maximal* K*-torus* T *of* G *for which* (3) *holds and which is conjugate to* $T(v)$ *by an element of* $G(K_v)$, *for all* $v \in V$.

Proof. — Let r denote the number of nontrivial conjugacy classes in the Weyl group of G. Enlarging L if necessary, we assume that G splits over L. By Proposition 1 of [32], there exists an infinite set Π of rational primes such that for each $p \in \Pi$ there exists an embedding $\iota_p\colon L \to \mathbf{Q}_p$. It follows that one can pick r distinct primes $p_1, \ldots, p_r \in \Pi$ so that for the valuations v_i of K obtained as pullbacks of the p_i-adic valuations v_{p_i} on $\mathbf{Q}_{p_i}$, the set $R = \{v_1, \ldots, v_r\}$ is disjoint from V. Now, let $T(v_i)$, for $i = 1, \ldots, r$, be the tori as in Theorem 3.1. Since the completions K_v for $v \in R \cup V$ are locally compact, it follows from the Implicit Function Theorem that the tori in the $G(K_v)$-conjugacy class of $T(v)$ correspond to points of an open subset of $\mathscr{T}(K_v)$, where $\mathscr{T}$ is the variety of maximal tori of G. Since $\mathscr{T}$ has the weak approximation property (cf. [26], Corollary 3 in §7.2), there exists a maximal K-torus T of G which is conjugate to $T(v)$ by an element of $G(K_v)$ for all $v \in R \cup V$. It follows from our construction that this torus has the desired properties. □

To reformulate the above results for individual elements instead of tori, we need the following lemma.

Definition **3.3.** — *A subset of a topological group is called solid if it intersects every open subgroup of that group.*

Lemma **3.4.** — *Let* v *be a nontrivial valuation of* K *with locally compact completion* K_v, *and let* T *be a maximal* K_v*-torus of* G. *Consider the map*

$$\varphi\colon G \times T \longrightarrow G, \qquad (g, t) \mapsto gtg^{-1}.$$

Then

$$\mathscr{U}(T, v) := \varphi(G(K_v), T_{\mathrm{reg}}(K_v)),$$

where T_{reg} *is the Zariski-open subvariety of* T *of regular elements, is a solid open subset of* $G(K_v)$.

Proof. — Indeed, one easily verifies that the differential $d_{(g,t)}\varphi$ is surjective for any $(g, t) \in G(K_v) \times T_{\mathrm{reg}}(K_v)$, so the openness of $\mathscr{U}(T, v)$ follows from the Implicit Function Theorem. Furthermore, for any open subgroup Ω of $G(K_v)$, the set $T(K_v) \cap \Omega$ is Zariski-dense in T (cf. [26], Lemma 3.2), and therefore it contains an element of $T_{\mathrm{reg}}(K_v)$. So, $\mathscr{U}(T, v) \cap \Omega \neq \emptyset$. □

Corollary **3.5.** — *Let* G, K, L *and* r *be as in Theorem 3.1. Furthermore, let* $v_1, \ldots, v_r$ *be* r *valuations of* K *with the properties specified in Theorem 3.1, and let*

$$\delta\colon G(K) \hookrightarrow \prod_{i=1}^{r} G(K_{v_i}) =: \mathcal{G}$$

be the diagonal embedding. Then there exists a solid open subset $\mathscr{U} \subset \mathcal{G}$ *such that any* $\gamma \in G(K)$ *with* $\delta(\gamma) \in \mathscr{U}$ *is regular semi-simple, and for the torus* $T = Z_G(\gamma)^{\circ}$, *condition* (3) *holds.*

Indeed, let $T(v_i)$, where $i = 1, \ldots, r$, be the tori given by Theorem 3.1. Then it is easy to see that the set

$$\mathscr{U} = \prod_{i=1}^{r} \mathscr{U}(T(v_i), v_i)$$

(notations as in Lemma 3.4), satisfies all our requirements.

In [31], a K-torus T was called K-*irreducible* if it has no proper K-subtori, which is equivalent to the condition that the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ acts irreducibly on the **Q**-vector space $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. It follows that, in our previous notation, a maximal K-torus T of G such that $\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G,T)$ is K-irreducible (cf. [7], Chap. VI, §1, n° 2). If T is a K-irreducible torus, then the cyclic group generated by any element of T(K) of infinite order is Zariski-dense in T.

We will use the following general fact about K-irreducible tori.

Lemma **3.6.** — *Let* T *be a* K*-irreducible torus, and* K_T *be its splitting field over* K. *Let* $t \in T(K)$ *be an element of infinite order, and* $\chi \in X(T)$ *be a nontrivial character. Then for* $\lambda := \chi(t)$, *the Galois conjugates* $\sigma(\lambda)$, *with* $\sigma \in \mathrm{Gal}(K_T/K)$, *generate* K_T *over* K.

Proof. — We need to show that if $\tau \in \mathrm{Gal}(K_T/K)$ is such that

$$\tau(\sigma(\lambda)) = \sigma(\lambda) \quad \text{for all } \sigma \in \mathrm{Gal}(K_T/K),$$

then $\tau = \mathrm{id}$. For such a τ we have

$$\sigma^{-1}\tau\sigma(\chi(t)) = (\sigma^{-1}\tau\sigma(\chi))(t) = \chi(t).$$

Hence, the character $\sigma^{-1}\tau\sigma(\chi) - \chi$ takes the value 1 at t, and therefore, $\tau(\sigma(\chi)) = \sigma(\chi)$ because t generates a Zariski-dense subgroup of T. But the fact that T is K-irreducible implies that the characters $\sigma(\chi)$, for $\sigma \in \mathrm{Gal}(K_T/K)$, span $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$, so $\tau = \mathrm{id}$. □

4. Proof of Theorem 1 and the Isogeny Theorem

In this section, K will be a field of arbitrary characteristic and $K^{\mathfrak{s}}$ a fixed separable closure of K. Let G be a connected absolutely almost simple algebraic K-group. Let T and T′ be two maximal tori of G, and L any field extension of K such that both the tori are defined and split over it. Given systems $\Delta \subset \Phi(G,T)$ and $\Delta' \subset \Phi(G,T')$ of simple roots, there exists $g \in G(L)$ such that the corresponding inner automorphism i_g of G maps T onto T′, and the induced homomorphism $i_g^*: X(T') \to X(T)$ of the character groups maps Δ' onto Δ. Such a g is determined uniquely up to an element of T(L), which implies that the identification $\Delta \simeq \Delta'$ induced by i_g^* is independent of the choice of g. We will always employ this identification of Δ with Δ' in the sequel.

Now let T be a maximal K-torus of G. Fix a system $\Delta \subset \Phi(G,T)$ of simple roots. Then for any $\sigma \in \mathrm{Gal}(K^{\mathfrak{s}}/K)$, there exists a unique $w_\sigma \in W(G,T)$ such that $w_\sigma(\sigma(\Delta)) = \Delta$. The correspondence $\alpha \mapsto w_\sigma(\sigma(\alpha))$ defines an action of $\mathrm{Gal}(K^{\mathfrak{s}}/K)$ on Δ, which is called the $*$-*action* (cf. [43]).

The following lemma describes some properties of the $*$-action, and of the aforementioned identification of Δ with Δ', which will be used later in the paper.

Lemma **4.1.**

(a) *Let* T *and* T' *be two maximal* K*-tori of* G*, and let* $\Delta \subset \Phi(G, T)$ *and* $\Delta' \subset \Phi(G, T')$ *be two systems of simple roots. Pick* $g \in G(K^s)$ *so that* $i_g(T) = T'$ *and* $i_g^*(\Delta') = \Delta$*. Then* i_g^* *intertwines the* $*$*-action of* $\mathrm{Gal}(K^s/K)$ *on* Δ' *and* Δ *respectively. In particular, it carries the orbits of the* $*$*-action on* Δ' *to the orbits of the* $*$*-action on* Δ*.*

(b) *The following conditions are equivalent:*
  (i) G *is an inner form (i.e., an inner twist of a split group) over* K*;*
  (ii) $*$*-action is trivial for some (equivalently, any) maximal* K*-torus* T *and a system of simple roots* $\Delta \subset \Phi(G, T)$*;*
  (iii) $\theta_T(\mathrm{Gal}(K_T/K)) \subset W(G, T)$ *for some (equivalently, any) maximal* K*-torus* T *of* G*.*

(c) *The minimal Galois extension* L *of* K *over which* G *becomes an inner form admits the following (equivalent) characterizations:*
  (i) $L = (K^s)^{\mathcal{H}}$*, where* $\mathcal{H}$ *is the kernel of the* $*$*-action;*
  (ii) $L = (K_T)^{\mathcal{H}_T}$*, where* $\mathcal{H}_T = \theta_T^{-1}(\theta_T(\mathrm{Gal}(K_T/K)) \cap W(G, T))$*.*

Proof. — (a): Let $\sigma \in \mathrm{Gal}(K^s/K)$, and pick $w_\sigma \in W(G, T)$ and $w'_\sigma \in W(G, T')$ so that $w_\sigma(\sigma(\Delta)) = \Delta$ and $w'_\sigma(\sigma(\Delta')) = \Delta'$. We need to show that

$$(\mathbf{4}) \qquad i_g^*(w'_\sigma(\sigma(\alpha'))) = w_\sigma(\sigma(i_g^*(\alpha'))) \quad \text{for all } \alpha' \in \Delta'.$$

Since both T and T' are defined over K, we have $g^{-1}\sigma(g) \in N_G(T)$, and we let u_σ denote the corresponding element of $W(G, T)$. Then

$$\sigma(i_g^*(\alpha')) = u_\sigma(i_g^*(\sigma(\alpha'))).$$

Now, we observe that both $i_g^* \circ w'_\sigma \circ \sigma$ and $w_\sigma \circ \sigma \circ i_g^* = w_\sigma \circ u_\sigma \circ i_g^* \circ \sigma$ take Δ' to Δ. This means that

$$\widetilde{w} := (i_g^*)^{-1} \circ u_\sigma^{-1} \circ w_\sigma^{-1} \circ i_g^* \circ w'_\sigma$$

leaves the system of simple roots $\sigma(\Delta')$ invariant. On the other hand, $\widetilde{w} \in W(G, T')$. So, $\widetilde{w} = 1$, and (4) follows.

(b): It follows from (a) that if the $*$-action is trivial on some $\Delta \subset \Phi(G, T)$ for some maximal K-torus T, then it is trivial on any $\Delta' \subset \Phi(G, T')$ for any maximal K-torus T'. On the other hand, it follows from the description of the $*$-action on $\Delta \subset \Phi(G, T)$ that its triviality is equivalent to the following:

$$(\mathbf{5}) \qquad \theta_T(\mathrm{Gal}(K_T/K)) \subset W(G, T).$$

This shows that (ii) and (iii) are equivalent. It remains to show that (i) is equivalent to the inclusion (5). For this, we assume, as we clearly may, that G is adjoint. Let G_0 be the

K-split adjoint group of the same type as G, and T_0 be a K-split maximal torus of G_0. Pick an isomorphism $\varphi\colon G_0 \to G$ such that $\varphi(T_0) = T$. Then

$$\alpha_\sigma = \varphi^{-1} \circ \sigma(\varphi) \quad \text{for } \sigma \in \mathrm{Gal}(K^s/K),$$

defines a 1-cocycle $\alpha \in Z^1(K, \mathrm{Aut}\, G_0)$ associated to G. For any $\chi \in X(T)$ we have $\chi \circ \varphi \in X(T_0)$, and therefore, $\sigma(\chi \circ \varphi) = \chi \circ \varphi$ as T_0 is K-split. An easy computation then shows that

$$\sigma(\chi) = \chi \circ (\varphi \circ \alpha_\sigma^{-1} \circ \varphi^{-1}). \tag{6}$$

Next, (i) amounts to the assertion that α is cohomologous to a $\mathrm{Int}\, G_0(K^s)$-valued Galois-cocycle $\beta : \sigma \mapsto \beta_\sigma$, $\sigma \in \mathrm{Gal}(K^s/K)$, i.e., there exists $\gamma \in \mathrm{Aut}\, G_0$ such that $\alpha_\sigma = \gamma^{-1} \circ \beta_\sigma \circ \sigma(\gamma)$, for all $\sigma \in \mathrm{Gal}(K^s/K)$. Let us show that then in fact

$$\alpha_\sigma \in \mathrm{Int}\, G_0 \quad \text{for all } \sigma \in \mathrm{Gal}(K^s/K). \tag{7}$$

Indeed, it is well-known that

$$\mathrm{Aut}\, G_0 = \mathrm{Int}\, G_0 \rtimes \Psi(T_0, B_0),$$

where $\Psi(T_0, B_0)$ is a subgroup of the group of all K-rational automorphisms of G_0 that leave invariant T_0 and a Borel K-subgroup B_0 containing T_0. Since all the elements of $\Psi(T_0, B_0)$ are K-rational, by writing γ in the form $\gamma = \delta \circ \psi$ with $\delta \in \mathrm{Int}\, G_0$ and $\psi \in \Psi(T_0, B_0)$, we obtain that

$$\alpha_\sigma = \psi^{-1} \circ (\delta^{-1} \circ \beta_\sigma \circ \sigma(\delta)) \circ \psi.$$

So, since $\mathrm{Int}\, G_0 \lhd \mathrm{Aut}\, G_0$, we obtain (7). In addition, since both T_0 and T are defined over K, we have $\alpha_\sigma(T_0) = T_0$, and therefore for $\varkappa_\sigma := \varphi \circ \alpha_\sigma^{-1} \circ \varphi^{-1} (\in \mathrm{Aut}\, G)$, $\varkappa_\sigma(T) = T$. Thus, if G is an inner form, then $\varkappa_\sigma$ is an inner automorphism of G which leaves T invariant. Then its restriction $\varkappa_\sigma | T$ is given by an element of the Weyl group $W(G, T)$, so (6) yields the inclusion (5). Conversely, (5) in conjunction with (6) implies that $\varkappa_\sigma | T$ is induced by an element of $W(G, T)$. But then $\varkappa_\sigma$ itself is inner, which implies that G is an inner form.

(c): Characterization (i) immediately follows from part (b). For (ii), let $F = (K_T)^{\mathcal{H}_T}$. Since G is an inner form over L and splits over K_T, by (*b*), for $L_T = K_T$ we have

$$\theta_T(\mathrm{Gal}(L_T/L)) \subset W(G, T),$$

implying that $F \subset L$. On the other hand, using the definition of F we see that

$$\theta_T(\mathrm{Gal}(F_T/F)) \subset W(G, T).$$

Then, again by (*b*), G is an inner form over F, and therefore $L \subset F$. Thus, $L = F$, as claimed. □

Proof of Theorem 1. — Pick a finitely generated field K so that both G_1 and G_2 are defined and split over K, and in addition $\Gamma_i \subset G_i(K)$ for $i = 1, 2$. Using the proof of Theorem 2 in [32] we can see that there exists a regular semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order with the following property: if T_1 is the maximal K-torus of G_1 containing γ_1 and K_{T_1} is its splitting field over K, then

$$(\mathbf{8}) \qquad \theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supset W(G_1, T_1)$$

(we notice that the inclusion in (8) is in fact an equality as G_1 splits over K, hence is an inner form, cf. Lemma 4.1). Our assumption that Γ_1 and Γ_2 are weakly commensurable implies the existence of a semi-simple element $\gamma_2 \in \Gamma_2$ such that if T_2 is a maximal K-torus of G_2 containing γ_2, then for some characters χ_i of T_i,

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) := \lambda \neq 1.$$

It follows from Lemma 3.6 that K_{T_1} is generated over K by all the Galois conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(\overline{K}/K)$. On the other hand, since $\gamma_2 \in T_2(K)$, all these conjugates belong to K_{T_2}, hence the inclusion $K_{T_1} \subset K_{T_2}$. Since G_2 is an inner form over K, by Lemma 4.1, we have the inclusion

$$\theta_{T_2}(\mathrm{Gal}(K_{T_2}/K)) \subset W(G_2, T_2),$$

comparing which with (8) we obtain that $|W(G_1, T_1)|$ divides $|W(G_2, T_2)|$. By symmetry, we obtain that actually $|W(G_1, T_1)| = |W(G_2, T_2)|$. Our claim now follows since, as is well-known, the order of the Weyl group of a reduced and irreducible root system determines the pair consisting of the root system and its dual. □

Theorem **4.2** *(Isogeny Theorem). — Let* G_1 *and* G_2 *be two connected absolutely almost simple algebraic groups defined over an infinite field* K*, and let* L_i *be the minimal Galois extension of* K *over which* G_i *becomes an inner form of a split group. Suppose that for* $i = 1, 2$*, we are given a semi-simple element* $\gamma_i \in G_i(K)$ *contained in a maximal* K*-torus* T_i *of* G_i*. Assume that* (i) G_1 *and* G_2 *are either of the same Killing-Cartan type, or one of them is of type* B_n *and the other is of type* C_n*,* (ii) γ_1 *has infinite order,* (iii) T_1 *is* K*-irreducible, and* (iv) γ_1 *and* γ_2 *are weakly commensurable. Then*

(1) *there exists a* K*-isogeny* $\pi : T_2 \to T_1$ *which carries* $\gamma_2^{m_2}$ *to* $\gamma_1^{m_1}$ *for some integers* $m_1, m_2 \geqslant 1$*;*
(2) *if* $L_1 = L_2$[6]*, then* $\pi^* : X(T_1) \otimes_{\mathbf{Z}} \mathbf{Q} \to X(T_2) \otimes_{\mathbf{Z}} \mathbf{Q}$ *has the property that* $\pi^*(\mathbf{Q} \cdot \Phi(G_1, T_1)) = \mathbf{Q} \cdot \Phi(G_2, T_2)$*. Moreover if* G_1 *and* G_2 *are of the same Killing-Cartan type different from* $B_2 = C_2$*,* F_4 *or* G_2*, then a suitable rational multiple of* π^* *maps* $\Phi(G_1, T_1)$ *onto* $\Phi(G_2, T_2)$*, and if* G_1 *is of type* B_n *and* G_2 *is of type* C_n*, with* $n > 2$*, then a suitable rational multiple* λ *of* π^* *takes the long roots in* $\Phi(G_1, T_1)$ *to the short roots in* $\Phi(G_2, T_2)$ *while* 2λ *takes the short roots in* $\Phi(G_1, T_1)$ *to the long roots in* $\Phi(G_2, T_2)$*.*

[6] Cf. Theorem 6.3(2).

Proof. — By Lemma 2.1, there exist characters $\chi_i \in X(T_i)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) =: \lambda \neq 1.$$

Now to prove the first assertion of the theorem, we observe that the argument given in the proof of Theorem 1 yields the inclusion $K_{T_1} \subset K_{T_2}$, and hence a surjective homomorphism

$$\mathcal{G} := \mathrm{Gal}(K_{T_2}/K) \longrightarrow \mathrm{Gal}(K_{T_1}/K),$$

which allows us to view $X(T_1)$ as a $\mathcal{G}$-module. We wish to show that there is an isomorphism

$$\rho \colon X(T_2) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow X(T_1) \otimes_{\mathbf{Z}} \mathbf{Q}$$

of $\mathbf{Q}[\mathcal{G}]$-modules that takes χ_2 to χ_1. For this, we consider the maps

$$\nu_i \colon \mathbf{Q}[\mathcal{G}] \to X(T_i) \otimes_{\mathbf{Z}} \mathbf{Q}, \qquad \sum a_\sigma \sigma \mapsto \sum a_\sigma \sigma(\chi_i);$$

clearly, $\nu_i(\mathbf{Z}[\mathcal{G}]) \subset X(T_i)$. The K-irreducibility of T_1 implies that ν_1 is surjective. Now we assert that

$$(\mathbf{9}) \qquad \mathrm{Ker}\,\nu_1 \supset \mathrm{Ker}\,\nu_2.$$

To prove this assertion, we observe that given $a = \sum a_\sigma \sigma \in \mathbf{Z}[\mathcal{G}]$, we have

$$\nu_1(a)(\gamma_1) = \prod \sigma(\chi_1)(\gamma_1)^{a_\sigma} = \prod \sigma(\lambda)^{a_\sigma} = \prod \sigma(\chi_2)(\gamma_2)^{a_\sigma} = \nu_2(a)(\gamma_2),$$

and as γ_1 generates a Zariski-dense subgroup of $T_1(K)$, the above computation shows that if $\nu_2(a) = 0$, then $\nu_1(a) = 0$, and (9) follows. In turn, (9) implies that there exists a natural surjective homomorphism

$$\rho : \mathrm{Im}\,\nu_2 \to \mathrm{Im}\,\nu_1 = X(T_1) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Since $\dim T_1 = \dim T_2$, we conclude that $\mathrm{Im}\,\nu_2 = X(T_2) \otimes_{\mathbf{Z}} \mathbf{Q}$, and hence ρ is an isomorphism. Clearly, $\rho(\chi_2) = \chi_1$.

The subgroup $\Theta := \nu_1(\mathbf{Z}[\mathcal{G}])$ has finite index, say d, in $X(T_1)$. Then multiplication by d followed by ρ^{-1} gives a homomorphism

$$\pi^* \colon X(T_1) \to \nu_2(\mathbf{Z}[\mathcal{G}]) \subset X(T_2)$$

of $\mathcal{G}$-modules such that $\pi^*(\chi_1) = d\chi_2$. Let $\pi \colon T_2 \to T_1$ be the K-isogeny corresponding to π^*. Then $\chi_1(\pi(t)) = \chi_2(t)^d$ for every $t \in T_2$, and in particular,

$$\chi_1(\pi(\gamma_2)) = \chi_2(\gamma_2)^d = \chi_1(\gamma_1^d).$$

Applying the elements of $\mathcal{G}$, we see that $\chi(\pi(\gamma_2)) = \chi(\gamma_1^d)$ for all $\chi \in \Theta$, and therefore,

$$\chi(\pi(\gamma_2)^d) = \chi(\gamma_1^{d^2}) \quad \text{for every } \chi \in X(T_1).$$

Thus, $\pi(\gamma_2)^d = \gamma_1^{d^2}$, so the first assertion of the theorem holds with $m_1 = d^2$, $m_2 = d$.

The second assertion of Theorem 4.2 will be deduced from:

Lemma **4.3.** — *For $i = 1, 2$, let Φ_i be an irreducible reduced root system contained in, and spanning, the $\mathbf{Q}$-vector space V_i. We assume that either Φ_1 and Φ_2 are isomorphic, or Φ_1 is of type B_n and Φ_2 is of type C_n for some $n > 2$ so that there exists an isomorphism $\mu\colon W(\Phi_1) \to W(\Phi_2)$ of the corresponding Weyl groups. If $\lambda\colon V_1 \to V_2$ is a linear isomorphism compatible with μ (i.e., $\lambda(w(v)) = \mu(w)(\lambda(v))$ for all $v \in V_1$ and $w \in W(\Phi_1)$), then*

(1) $\lambda(\mathbf{Q} \cdot \Phi_1) = \mathbf{Q} \cdot \Phi_2$.

Moreover,

(2) *if Φ_1 and Φ_2 are isomorphic but not of type $B_2 = C_2$, F_4 or G_2, then a suitable rational multiple of λ maps Φ_1 onto Φ_2;*
(3) *if Φ_1 is of type B_n and Φ_2 is of type C_n with $n > 2$, then a suitable rational multiple λ' of λ maps the long roots of Φ_1 to the short roots of Φ_2, while $2\lambda'$ maps the short roots of Φ_1 to the long roots of Φ_2.*

Proof. — We equip V_i with a positive definite $W(\Phi_i)$-invariant inner product, and note that as V_i is an absolutely irreducible $W(\Phi_i)$-module, any two $W(\Phi_i)$-invariant inner products on V_i are multiples of each other, see [7], Chap. VI, §1, Proposition 7. This implies, in particular, that λ is a multiple of an isometry. For a root $\alpha \in \Phi_1$, let $w_\alpha \in W(\Phi_1)$ be the corresponding reflection. Then $\mu(w_\alpha)$ is the reflection of V_2 with respect to $\lambda(\alpha)$. On the other hand, $\mu(w_\alpha) \in W(\Phi_2)$, so it follows from ([7], Chap. V, §3, Cor. in n° 2) that $\mu(w_\alpha) = w_{\bar{\alpha}}$ for some $\bar{\alpha} \in \Phi_2$. So, $\lambda(\alpha) = t_\alpha \bar{\alpha}$ for some $t_\alpha \in \mathbf{Q}$, and our first assertion follows.

To prove the second assertion in the case where Φ_1 and Φ_2 are isomorphic, we scale the inner products on V_1 and V_2 so that the short (long) roots in Φ_1 and Φ_2 have the same length in the respective spaces. Next, we fix an arbitrary (resp., an arbitrary long) root $\alpha_0 \in \Phi_1$ if all roots have the same length (resp., if Φ_1 contains roots of unequal lengths). Replacing λ with $t_{\alpha_0}^{-1}\lambda$, we assume that $\lambda(\alpha_0) = \bar{\alpha}_0$. If all roots have the same length, then $W(\Phi_1) \cdot \alpha_0 = \Phi_1$ and $W(\Phi_2) \cdot \bar{\alpha}_0 = \Phi_2$ ([7], Chap. VI, §1, Proposition 11), yielding $\lambda(\Phi_1) = \Phi_2$. It remains to consider the situation where both Φ_1 and Φ_2 are of type either B_n or C_n with $n > 2$. Then $W(\Phi_1) \cdot \alpha_0$ is the subset Φ_1^{long} of all long roots, and $W(\Phi_2) \cdot \bar{\alpha}_0$ is either Φ_2^{long} or Φ_2^{short} depending on whether $\bar{\alpha}_0$ is long or short (cf. *loc. cit.*). But for the types under consideration, $|\Phi_1^{\mathrm{long}}| \neq |\Phi_2^{\mathrm{short}}|$, and therefore, $\lambda(\Phi_1^{\mathrm{long}}) = \Phi_2^{\mathrm{long}}$. Since α_0 and $\lambda(\alpha_0)$ have the same length, λ is an isometry. If $\beta_0 \in \Phi_1^{\mathrm{short}}$ and

$\lambda(\beta_0) = t_{\beta_0}\bar{\beta}_0$, then $\bar{\beta}_0$ cannot be a long root (as λ is a linear isomorphism). So, $\bar{\beta}_0$ is short, $t_{\beta_0} = \pm 1$, and it follows that $\lambda(\Phi_1) = \Phi_2$.

Finally, let Φ_1 be of type B_n and Φ_2 be of type C_n with $n > 2$. We scale the inner products so that the squared-length of long roots in Φ_1 equals the squared-length of short roots in Φ_2. We fix $\alpha_0 \in \Phi_1^{\mathrm{long}}$ and scale λ so that $\bar{\alpha}_0 = \lambda(\alpha_0) \in \Phi_2$. Since $|\Phi_1^{\mathrm{long}}| \neq |\Phi_2^{\mathrm{long}}|$, we conclude that $\bar{\alpha}_0$ is short, hence $\lambda(\Phi_1^{\mathrm{long}}) = \Phi_2^{\mathrm{short}}$ and λ is an isometry. Let now $\beta_0 \in \Phi_1^{\mathrm{short}}$ and $\lambda(\beta_0) = t_{\beta_0}\bar{\beta}_0$ where, obviously, $\bar{\beta}_0 \in \Phi_2^{\mathrm{long}}$. Since the ratio of the squared-lengths of α_0 and β_0, and of $\bar{\beta}_0$ and $\bar{\alpha}_0$ is 2, we see that $t_{\beta_0} = \pm 1/2$, and therefore, $2\lambda(\Phi_1^{\mathrm{short}}) = \Phi_2^{\mathrm{long}}$, as required. □

We will now prove the second assertion of Theorem 4.2. Set $L := L_1 = L_2$. Then it follows from Lemma 4.1 that

$$\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) = W(G_1, T_1) \quad \text{and} \quad \theta_{T_2}(\mathrm{Gal}(L_{T_2}/L)) \subset W(G_2, T_2).$$

Since $K_{T_1} = K_{T_2}$, $L_{T_1} = L_{T_2}$, and we see that the composite map

$$\mu \colon W(G_1, T_1) \xrightarrow{\theta_{T_1}^{-1}} \mathrm{Gal}(L_{T_1}/L) = \mathrm{Gal}(L_{T_2}/L) \xrightarrow{\theta_{T_2}} W(G_2, T_2)$$

is an isomorphism of the Weyl groups compatible with $\pi^*\colon \mathbf{Q} \otimes_{\mathbf{Z}} X(T_1) \to \mathbf{Q} \otimes_{\mathbf{Z}} X(T_2)$. Now, the second assertion of Theorem 4.2 follows from Lemma 4.3. □

Remark **4.4.** — Let us assume that π^* from Theorem 4.2(2) can be, and has been, scaled so that $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$. Then it induces a K-isomorphism $\bar{\pi}\colon \overline{T}_2 \to \overline{T}_1$ of the corresponding tori in the adjoint groups $\overline{G}_i$, which still has the property $\bar{\pi}(\bar{\gamma}_2^{\bar{m}_2}) = \bar{\gamma}_1^{\bar{m}_1}$ for some integers $\bar{m}_1, \bar{m}_2 \geqslant 1$, where $\bar{\gamma}_i$ is the image of γ_i in $\overline{T}_i(K)$. Furthermore, if Y_i is the dual in V_i (where V_i is as in Lemma 4.3) of the lattice X_i spanned by Φ_i, then Y_i is the character group of the maximal K-torus $\widetilde{T}_i$, corresponding to the maximal torus T_i, of the simply connected cover $\widetilde{G}_i$ of G_i, and π^* induces an isomorphism $Y_1 \to Y_2$, which in turn induces a K-isomorphism $\tilde{\pi}\colon \widetilde{T}_2 \to \widetilde{T}_1$. Both $\tilde{\pi}$ and $\bar{\pi}$ extend to $K^{\mathfrak{s}}$-isomorphisms $\widetilde{G}_2 \to \widetilde{G}_1$ and $\overline{G}_2 \to \overline{G}_1$. Also, if Δ_1 is a system of simple roots in $\Phi(G_1, T_1)$, and $\Delta_2 = \pi^*(\Delta_1)$, then π^* intertwines the $*$-action of $\mathrm{Gal}(K^{\mathfrak{s}}/K)$ on Δ_1 and Δ_2 respectively.

We also note the "symmetric" version of the concluding part of Theorem 4.2(2): if G_1 is of type C_n and G_2 is of type B_n with $n > 2$, then a suitable rational multiple λ of π^* takes the short roots in $\Phi(G_1, T_1)$ to the long roots in $\Phi(G_2, T_2)$, while $(1/2)\lambda$ takes the long roots in $\Phi(G_1, T_1)$ to the short roots in $\Phi(G_2, T_2)$.

5. Proofs of Theorems 2, 3 and 7

We begin this section with the following two auxiliary propositions, the first of which is a variant of Proposition 1 of [32].

Proposition **5.1.** — *Let* $\mathscr{F}_1 \subsetneqq \mathscr{F}_2 \subset \mathscr{E}$ *be a tower of finitely generated fields of characteristic zero, and let* $\mathscr{R} \subset \mathscr{E}$ *be a finitely generated subring. Then there exists an infinite set of rational primes* Π *such that for each* $p \in \Pi$*, there are embeddings* $\iota^{(1)}, \iota^{(2)}: \mathscr{E} \to \mathbf{Q}_p$ *with the following properties:*

(1) *both* $\iota^{(1)}(\mathscr{R})$ *and* $\iota^{(2)}(\mathscr{R})$ *are contained in* $\mathbf{Z}_p$*;*
(2) $\iota^{(1)}|\mathscr{F}_1 = \iota^{(2)}|\mathscr{F}_1$*, but* $\iota^{(1)}|\mathscr{F}_2 \neq \iota^{(2)}|\mathscr{F}_2$*.*

Proof. — First, we observe that there exists a transcendence basis $t_1, \ldots, t_n$ of $\mathscr{E}$ over $\mathbf{Q}$ such that for $\mathscr{K} := \mathbf{Q}(t_1, \ldots, t_n)$ we have $\mathscr{K}\mathscr{F}_1 \neq \mathscr{K}\mathscr{F}_2$. Indeed, let $t_1, \ldots, t_{n_1}$ be an arbitrary transcendence basis of $\mathscr{F}_1$ over $\mathbf{Q}$, and $t_{n_1+1}, \ldots, t_{n_2}$ be a transcendence basis of $\mathscr{F}_2$ over $\mathscr{F}_1$ such that

$$\mathscr{F}_2 \neq \mathscr{F}_1(t_{n_1+1}, \ldots, t_{n_2}).$$

Then, for $\mathscr{K}_0 := \mathbf{Q}(t_1, \ldots, t_{n_2})$, we have

$$\mathscr{K}_0\mathscr{F}_1 = \mathscr{F}_1(t_{n_1+1}, \ldots, t_{n_2}) \neq \mathscr{F}_2.$$

Now, let $t_{n_2+1}, \ldots, t_n$ be a transcendence basis of $\mathscr{E}$ over $\mathscr{F}_2$. Then, of course, $(\mathscr{K}_0\mathscr{F}_1)(t_{n_2+1}, \ldots, t_n) \neq \mathscr{F}_2(t_{n_2+1}, \ldots, t_n)$, and therefore,

$$\mathscr{K}\mathscr{F}_1 = (\mathscr{K}_0\mathscr{F}_1)(t_{n_2+1}, \ldots, t_n) \neq \mathscr{F}_2(t_{n_2+1}, \ldots, t_n) = \mathscr{K}\mathscr{F}_2,$$

as required.

Obviously, $\mathscr{E}$ is a finite extension of $\mathscr{K}\mathscr{F}_1$. Let $\mathscr{M}$ denote the Galois closure of $\mathscr{E}$ over $\mathscr{K}\mathscr{F}_1$. Then there exists $\sigma \in \mathrm{Gal}(\mathscr{M}/\mathscr{K}\mathscr{F}_1)$ which acts nontrivially on $\mathscr{K}\mathscr{F}_2$, and hence on $\mathscr{F}_2$. Let $\mathscr{R}_0$ be the subring generated by $\mathscr{R}$ and $\sigma(\mathscr{R})$. Since $\mathscr{M}$ is a finitely generated field and $\mathscr{R}_0$ is a finitely generated ring, by Proposition 1 of [32], one can find an infinite set of rational primes Π such that, for every $p \in \Pi$, there exists an embedding $\iota_p: \mathscr{M} \to \mathbf{Q}_p$ with the property $\iota_p(\mathscr{R}_0) \subset \mathbf{Z}_p$. Then, for $p \in \Pi$, the embeddings

$$\iota^{(1)} = \iota_p|\mathscr{E} \quad \text{and} \quad \iota^{(2)} = (\iota_p \circ \sigma)|\mathscr{E}$$

satisfy both of our requirements. □

Proposition **5.2.** — *Let* G *be a connected absolutely simple adjoint algebraic group defined over a field* L *of characteristic zero. Suppose* Γ *is a Zariski-dense subgroup of* G(L)*, and let* K_Γ *denote the subfield of* L *generated by the traces* $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$*, in the adjoint representation, of all* $\gamma \in \Gamma$*. For* $i = 1, 2$*, let* $\iota^{(i)}: L \to \mathbf{Q}_p$ *be an embedding,* $G^{(i)}$ *be the algebraic* $\mathbf{Q}_p$*-group obtained by the extension of scalars given by* $\iota^{(i)}$*, and* $\rho^{(i)}: G(L) \to G^{(i)}(\mathbf{Q}_p)$ *be the homomorphism induced by* $\iota^{(i)}$*. If*

(a) $\rho^{(i)}(\Gamma)$ *is relatively compact for* $i = 1, 2$*,*
(b) $\iota^{(1)}|K_\Gamma \neq \iota^{(2)}|K_\Gamma$*,*

then the closure of the image of the diagonal homomorphism

$$\rho\colon \Gamma \to \mathrm{G}^{(1)}(\mathbf{Q}_p) \times \mathrm{G}^{(2)}(\mathbf{Q}_p), \qquad \gamma \mapsto (\rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is open.

We begin by showing that the image of ρ is Zariski-dense in $\mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$.

Lemma **5.3.** — *For $i = 1, 2$, let $\mathrm{G}^{(i)}$ be a connected simple adjoint algebraic group defined over a field K of characteristic zero, and let $\rho^{(i)}\colon \Gamma \to \mathrm{G}^{(i)}(\mathrm{K})$ be a homomorphism of a group Γ with Zariski-dense image. Then either*

$$(\mathbf{10}) \qquad \mathrm{G}^{(1)} \simeq \mathrm{G}^{(2)} \quad \textit{and} \quad \mathrm{Tr}\,\mathrm{Ad}_{\mathrm{G}^{(1)}}\,\rho^{(1)}(\gamma) = \mathrm{Tr}\,\mathrm{Ad}_{\mathrm{G}^{(2)}}\,\rho^{(2)}(\gamma) \quad \textit{for all } \gamma \in \Gamma,$$

where $\mathrm{Ad}_{\mathrm{G}^{(i)}}$ is the adjoint representation of $\mathrm{G}^{(i)}$, or the image of the homomorphism

$$\rho\colon \Gamma \to \mathrm{G}^{(1)}(\mathrm{K}) \times \mathrm{G}^{(2)}(\mathrm{K}), \qquad \gamma \mapsto (\rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is Zariski-dense in $\mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$.

Proof. — Let H be the Zariski-closure of $\rho(\Gamma)$ in $\mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$, and assume that $\mathrm{H} \neq \mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$. Since both $\rho^{(1)}$ and $\rho^{(2)}$ have Zariski-dense images, for the corresponding projections we have

$$\mathrm{pr}_i(\mathrm{H}) = \mathrm{G}^{(i)}, \quad i = 1, 2.$$

Let $\mathrm{H}_i = \mathrm{H} \cap \mathrm{G}^{(i)}$. Then H_1 is a normal subgroup of $\mathrm{G}^{(1)}$, and therefore it either equals $\mathrm{G}^{(1)}$ or is trivial. Furthermore, if it equals $\mathrm{G}^{(1)}$, then as $\mathrm{pr}_2(\mathrm{H}) = \mathrm{G}^{(2)}$, we easily see that $\mathrm{H} = \mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$. Similarly, H_2 either equals $\mathrm{G}^{(2)}$ or is trivial, and in the former case $\mathrm{H} = \mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$. Thus, since $\mathrm{H} \neq \mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$, we see that H_i is trivial for $i = 1, 2$. Since the ground field is of characteristic zero, this means that pr_i induces an isomorphism $\epsilon_i\colon \mathrm{H} \to \mathrm{G}^{(i)}$ for $i = 1, 2$. Then $\sigma := \epsilon_2 \circ \epsilon_1^{-1}$ is an isomorphism between $\mathrm{G}^{(1)}$ and $\mathrm{G}^{(2)}$, and

$$\mathrm{H} = \{(g, \sigma(g)) \mid g \in \mathrm{G}^{(1)}\}.$$

It follows that $\rho_2 = \sigma \circ \rho_1$, which implies (10). □

We will now prove Proposition 5.2. Denote by $\mathscr{H}$ the closure of $\rho(\Gamma)$ in $\mathrm{G}^{(1)}(\mathbf{Q}_p) \times \mathrm{G}^{(2)}(\mathbf{Q}_p)$ in the p-adic topology. Then $\mathscr{H}$ is a p-adic Lie group (cf. [7], Chap. III, §8, Théorème 2), and we let $\mathfrak{h}$ denote its Lie algebra. It follows from condition (b) that (10) does not hold, and hence by Lemma 5.3, $\rho(\Gamma)$ is Zariski-dense in $\mathrm{G}^{(1)} \times \mathrm{G}^{(2)}$. This immediately implies (cf. [26], Proposition 3.4) that $\mathfrak{h}$ is an ideal of $\mathfrak{g}^{(1)} \times \mathfrak{g}^{(2)}$, where $\mathfrak{g}^{(i)}$ is the Lie algebra of $\mathrm{G}^{(i)}(\mathbf{Q}_p)$ as a p-adic Lie group. If the projection of $\mathfrak{h}$ to, say, $\mathfrak{g}^{(1)}$, is zero, then the image of $\rho^{(1)}$ would be discrete, hence finite (in view of condition (a)), which is impossible. Thus, $\mathfrak{h}$ has nonzero projections to both components, and therefore, being an ideal of $\mathfrak{g}^{(1)} \times \mathfrak{g}^{(2)}$, must coincide with $\mathfrak{g}^{(1)} \times \mathfrak{g}^{(2)}$ since $\mathfrak{g}^{(i)}$ is simple for $i = 1, 2$. But this means that $\mathscr{H}$ is open in $G^{(1)}(\mathbf{Q}_p) \times G^{(2)}(\mathbf{Q}_p)$. □

We are now in a position to prove Theorem 2. We assume (as we may, see Lemma 2.4) that the groups G_1 and G_2 are adjoint. Since Γ_i is finitely generated, it is contained in $GL_n(F_i)$ for some finitely generated subfield F_i of F. Then the field $K_i := K_{\Gamma_i}$ is a subfield of F_i, and therefore it is finitely generated, for $i = 1, 2$. By symmetry, it is enough to establish the inclusion $K_1 \subset K_2$. Assume the contrary, and set $K = K_1K_2$. According to the results of Vinberg [46], the group G_i is defined over K_i and $\Gamma_i \subset G_i(K_i)$, for $i = 1, 2$. Now, pick a finite extension L of K over which both G_1 and G_2 split, and a finitely generated subring R of L such that $\Gamma_1 \subset G_1(R)$. Let r be the number of nontrivial conjugacy classes of the Weyl group of G_1. Since L is finitely generated, by Proposition 1 of [32], there exist rational primes $p_1, \ldots, p_r$ and embeddings $\iota_j\colon L \to \mathbf{Q}_{p_j}$ such that $\iota_j(R) \subset \mathbf{Z}_{p_j}$. Let $\rho_j\colon \Gamma_1 \to G_1(\mathbf{Z}_{p_j})$ be the corresponding homomorphisms. Then according to Lemma 2 of [32], the closure of the image of the homomorphism

$$\delta\colon \Gamma_1 \to G_1(\mathbf{Z}_{p_1}) \times \cdots \times G_1(\mathbf{Z}_{p_r}), \qquad \gamma \mapsto (\rho_1(\gamma), \ldots, \rho_r(\gamma)),$$

is open. Moreover, by Corollary 3.5, there exists a solid open subset $U \subset G_1(\mathbf{Z}_{p_1}) \times \cdots \times G_1(\mathbf{Z}_{p_r})$ such that any $\gamma \in \Gamma_1$ $(\subset G_1(K_1))$, with $\delta(\gamma) \in U$, is regular semi-simple, and for the L-torus $T = Z_{G_1}(\gamma)^\circ$, we have

$$\theta_T(\mathrm{Gal}(L_T/L)) \supset W(G_1, T), \tag{11}$$

where L_T is the splitting field of T over L.

Next, applying Proposition 5.1 to the tower

$$K_2 \subsetneqq K \subset L$$

we find a prime $p \notin \{p_1, \ldots, p_r\}$ such that there exists a pair of embeddings $\iota^{(1)}, \iota^{(2)}\colon L \to \mathbf{Q}_p$ that have the same restriction to K_2, but different restrictions to K, hence to K_1, and also satisfy $\iota^{(i)}(R) \subset \mathbf{Z}_p$, for $i = 1, 2$. Now, we let $G_1^{(i)}$ denote the algebraic $\mathbf{Q}_p$-group obtained from the K_1-group G_1 by extension of scalars $\iota^{(i)}|_{K_1}\colon K_1 \to \mathbf{Q}_p$, and let $\rho^{(i)}\colon \Gamma_1 \to G_1^{(i)}(\mathbf{Z}_p)$ be the resulting homomorphism. Since $\iota^{(1)}$ and $\iota^{(2)}$ have different restrictions to $K_1 = K_{\Gamma_1}$, by Proposition 5.2, the closure of the image of the homomorphism

$$\Gamma_1 \to G_1^{(1)}(\mathbf{Z}_p) \times G_1^{(2)}(\mathbf{Z}_p), \quad \gamma \mapsto (\rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is open in $G_1^{(1)}(\mathbf{Z}_p) \times G_1^{(2)}(\mathbf{Z}_p)$. Since $p \notin \{p_1, \ldots, p_r\}$, it easily follows then that the closure of the image of

$$\rho\colon \Gamma_1 \to G_1(\mathbf{Z}_{p_1}) \times \cdots \times G_1(\mathbf{Z}_{p_r}) \times G_1^{(1)}(\mathbf{Z}_p) \times G_1^{(2)}(\mathbf{Z}_p),$$

$$\gamma \mapsto \rho(\gamma) := (\rho_1(\gamma), \ldots, \rho_r(\gamma), \rho^{(1)}(\gamma), \rho^{(2)}(\gamma))$$
$$= (\delta(\gamma), \rho^{(1)}(\gamma), \rho^{(2)}(\gamma)),$$

is open as well (cf. the proof of Lemma 2 in [32]). Since $L \subset \mathbf{Q}_p$, the group $G_1^{(1)}$ splits over $\mathbf{Q}_p$, and we fix a $\mathbf{Q}_p$-split maximal $\mathbf{Q}_p$-torus $T^{(1)}$ of $G_1^{(1)}$. According to [26], Theorem 6.21 (for a different proof, see [9], §2.4), $G_1^{(2)}$ contains a $\mathbf{Q}_p$-anisotropic maximal $\mathbf{Q}_p$-torus $T^{(2)}$. Let $U^{(i)} = \mathscr{U}(T^{(i)}, v_p)$ in the notation of Lemma 3.4 for $(G, T) = (G_1^{(i)}, T^{(i)})$, where v_p is the p-adic valuation on $\mathbf{Q}_p$ and $i = 1, 2$. Since the open sets U, $U^{(1)}$ and $U^{(2)}$ are solid in the corresponding groups, it follows from our preceding observation about the openness of the closure of $\operatorname{Im}\rho$ that there exists $\gamma_1 \in \Gamma_1$ such that

$$\rho(\gamma_1) \in U \times U^{(1)} \times U^{(2)}. \tag{12}$$

Then $T_1 := Z_{G_1}(\gamma_1)^\circ$ is a maximal K_1-torus of G_1 as $\gamma_1 \in \Gamma_1 \subset G_1(K_1)$. Since Γ_1 and Γ_2 are weakly commensurable, there exist a maximal K_2-torus T_2 of G_2, and $\gamma_2 \in \Gamma_2 \cap T_2(K_2)$ such that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) =: \lambda \neq 1$$

for some characters $\chi_i \in X(T_i)$. As $\gamma_i \in T_i(K_i)$ for $i = 1, 2$, the element λ is algebraic over both K_1 and K_2. For $i = 1, 2$, let $\mathscr{K}_i$ be the field generated over K_i by the conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(\overline{K}_i/K_i)$, and let $\mathscr{L}$ be the field generated over L by the conjugates $\sigma(\lambda)$ with $\sigma \in \mathrm{Gal}(\overline{L}/L)$. We claim that

$$\mathscr{K}_1 L = \mathscr{L} = \mathscr{K}_2 L. \tag{13}$$

By looking at the minimal polynomials of λ over K_i and L, we immediately see that $\mathscr{L} \subset \mathscr{K}_i L$ for $i = 1, 2$. For the opposite inclusion, we first observe that as $\delta(\gamma_1) \in U$, it follows from (11) that T_1 is L-irreducible, and therefore by Lemma 3.6, $\mathscr{L}$ coincides with L_{T_1}, the splitting field of T_1 over L. Hence, we conclude again from (11) that

$$|\mathrm{Gal}(\mathscr{L}/L)| \geqslant |W(G_1, T_1)|. \tag{14}$$

On the other hand, for both $i = 1, 2$, the field $\mathscr{K}_i L$ is contained in the splitting field L_{T_i} of T_i over L, and since G_i is of inner type over L (as it splits over L), we obtain from Lemma 4.1(b) that $\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) \subset W(G_i, T_i)$. Theorem 1 implies that $W(G_1, T_1) \cong W(G_2, T_2)$. So

$$|\mathrm{Gal}(\mathscr{K}_i L/L)| \leqslant |W(G_1, T_1)|;$$

combining this with (14), we obtain (13).

To complete the argument, we let v_1 and v_2 denote the valuations of K_1 obtained by pulling back the p-adic valuation on $\mathbf{Q}_p$ under the embeddings $\iota^{(1)}|K_1$ and $\iota^{(2)}|K_1$, of K_1 into $\mathbf{Q}_p$, respectively. Then, of course, the completion $K_{1 v_i}$ can be identified with $\mathbf{Q}_p$

for $i = 1, 2$. It follows from (12) and the construction of the open sets $U^{(i)}$ (cf. Lemma 3.4) that over K_{1v_1}, the torus T_1 is isomorphic to $T^{(1)}$, hence is split, and over K_{1v_2}, it is isomorphic to $T^{(2)}$, hence is anisotropic. Therefore, given a nontrivial character $\chi \in X(T_1)$, there exists $\sigma \in \mathrm{Gal}(\overline{K}_{1v_2}/K_{1v_2})$ such that $\sigma(\chi) \neq \chi$. Then, in view of the Zariski-density of the subgroup generated by γ_1, we have

$$\sigma(\chi)(\gamma_1) = \sigma(\chi(\gamma_1)) \neq \chi(\gamma_1),$$

and consequently,

$$(\mathbf{15}) \qquad \chi(\gamma_1) \notin K_{1v_2} \quad \text{for any nontrivial } \chi \in X(T_1).$$

Now, we extend our original embeddings $\iota^{(1)}, \iota^{(2)}: L \to \mathbf{Q}_p$ to embeddings $\tilde{\iota}^{(1)}, \tilde{\iota}^{(2)}: \mathscr{L} \to \overline{\mathbf{Q}}_p$. As T_1 splits over K_{1v_1},

$$\sigma(\lambda) = \sigma(\chi_1)(\gamma_1) \in K_{1v_1} \quad \text{for all } \sigma \in \mathrm{Gal}(\overline{K}_1/K_1),$$

and therefore, $\tilde{\iota}^{(1)}(\mathscr{K}_1) \subset \mathbf{Q}_p$. Then $\tilde{\iota}^{(1)}(\mathscr{L}) \subset \mathbf{Q}_p$, which, in view of (13), implies that $\tilde{\iota}^{(1)}(\mathscr{K}_2) \subset \mathbf{Q}_p$. On the other hand, it follows from (15) that $\tilde{\iota}^{(2)}(\mathscr{K}_1) \not\subset \mathbf{Q}_p$, so $\tilde{\iota}^{(2)}(\mathscr{K}_2) \not\subset \mathbf{Q}_p$. But $\tilde{\iota}^{(1)}$ and $\tilde{\iota}^{(2)}$ have the same restriction to K_2, and since $\mathscr{K}_2/K_2$ is a Galois extension, the restrictions $\tilde{\iota}^{(1)}|\mathscr{K}_2$ and $\tilde{\iota}^{(2)}|\mathscr{K}_2$ differ by an element of $\mathrm{Gal}(\mathscr{K}_2/K_2)$, which shows that the assertions

$$\tilde{\iota}^{(1)}(\mathscr{K}_2) \subset \mathbf{Q}_p \quad \text{and} \quad \tilde{\iota}^{(2)}(\mathscr{K}_2) \not\subset \mathbf{Q}_p$$

are incompatible. A contradiction, which shows that our assumption that $K_1 \not\subset K_2$ is false, and therefore, $K_1 \subset K_2$. This proves Theorem 2. □

Remark **5.4.** — As we will prove soon, weakly commensurable Zariski-dense S-arithmetic subgroups share not only the field of definition, but also many other important characteristics (cf. Theorems 3, 4 and 6). For arbitrary finitely generated Zariski-dense subgroups, however, we cannot say much beyond Theorem 2. One of the reasons is that at this point, classification results for semi-simple groups over general fields are quite scarce. Here is one intriguing basic question in this direction: *Let* D_1 *and* D_2 *be two quaternion division algebras over a field* K. *Assume that* D_1 *and* D_2 *are* weakly isomorphic, *i.e., have the same maximal subfields. Are they isomorphic?* The answer is easily seen to be positive when K is a global field. On the other hand, M. Rost informed us that over large fields (like those used in the proof of the Merkurjev-Suslin theorem), the answer can be negative (as we learned later, this was also observed by A. Wadsworth and some others). Recently, Saltman [40] has shown that if the unramified Brauer group $\mathrm{Br}_u(K)$ is trivial, then the answer to the above question is in the affirmative. This result (and its variants) yield an affirmative answer for $K = k(x_1, \ldots, x_r)$, a purely transcendental extension of a number field k. However, for finitely generated fields (and the fields that arise in the context of the present paper are finitely generated), the question remains wide open. If the answer turns out to be negative in general, we would like to know if every class of weakly isomorphic quaternion algebras splits into finitely many isomorphism classes (for a finitely generated field K). Of course, we can ask similar questions for other types of algebraic groups (defining two K-forms of the same group to be *weakly isomorphic* if they have the same maximal K-tori).

Proof of Theorem 3. — For $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$, and assume that Γ_1 and Γ_2 are weakly commensurable. Since the Γ_i's are finitely generated (cf. [26], Theorem 6.1), we can use Theorem 1 to conclude that either G_1 and G_2 are of same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n.

By Lemma 2.6, the field K_{Γ_i} generated by $\mathrm{Tr}\,\mathrm{Ad}_{G_i}\gamma$ for $\gamma \in \Gamma_i$, coincides with K_i. From Theorem 2 we now deduce that

$$K_1 = K_{\Gamma_1} = K_{\Gamma_2} = K_2 =: K.$$

In view of the obvious symmetry, to prove that $S_1 = S_2$, it is enough to prove the inclusion $S_1 \subset S_2$. Suppose there exists $v_0 \in S_1 \setminus S_2$. Our restrictions on S_1 implies that the group $\mathscr{G}_1$ is K_{v_0}-isotropic, so there exists a maximal K_{v_0}-torus $\mathscr{T}(v_0)$ of $\mathscr{G}_1$ which is K_{v_0}-isotropic. Then by Corollary 3.2, there exists a maximal K-torus $\mathscr{T}_1$ of $\mathscr{G}_1$ for which

$$(\mathbf{16}) \qquad \theta_{\mathscr{T}_1}(\mathrm{Gal}(K_{\mathscr{T}_1}/K)) \supset W(\mathscr{G}_1, \mathscr{T}_1),$$

and which is conjugate to $\mathscr{T}(v_0)$ under an element of $\mathscr{G}_1(K_{v_0})$, hence is K_{v_0}-isotropic.

Clearly, $\mathscr{T}_1$ is K-anisotropic, so the quotient $\mathscr{T}_{1S_1}/\mathscr{T}_1(\mathcal{O}_K(S_1))$ is compact, where $\mathscr{T}_{1S_1} = \prod_{v\in S_1} \mathscr{T}_1(K_v)$ (cf. [26], Theorem 5.7), which implies that the quotient of $\mathscr{T}_1(K_{v_0})$ by the closure $\mathscr{C}$ of $\mathscr{T}_1(\mathcal{O}_K(S_1))$ in $\mathscr{T}_1(K_{v_0})$ is also compact. But as $\mathscr{T}_1$ is K_{v_0}-isotropic, the group $\mathscr{T}_1(K_{v_0})$ is noncompact, and we conclude that $\mathscr{C}$ is noncompact as well. Since $\mathscr{T}_1(\mathcal{O}_K(S_1))$ is a finitely generated abelian group (cf. [26], Theorem 5.12), this implies that there exists $\gamma_1 \in \mathscr{T}_1(\mathcal{O}_K(S_1))$ such that the closure of the cyclic group $\langle\gamma_1\rangle$ in $\mathscr{T}_1(K_{v_0})$ is noncompact. We can in fact assume that $\gamma_1 \in \Gamma_1 \cap \mathscr{T}_1(\mathcal{O}_K(S_1))$. By our assumption, γ_1 is weakly commensurable to a semi-simple element γ_2 of Γ_2. Let $\mathscr{T}_2$ be a maximal K-torus containing γ_2. Then according to Theorem 4.2, there exists a K-isogeny $\pi\colon \mathscr{T}_2 \to \mathscr{T}_1$ such that $\pi(\gamma_2^{m_2}) = \gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$, which induces a continuous homomorphism $\pi_{v_0}\colon \mathscr{T}_2(K_{v_0}) \to \mathscr{T}_1(K_{v_0})$. But since $v_0 \notin S_2$ and Γ_2 is S_2-arithmetic, the subgroup $\langle\gamma_2\rangle$ has compact closure in $\mathscr{T}_2(K_{v_0})$, and we obtain that $\langle\gamma_1^{m_1}\rangle$, and hence $\langle\gamma_1\rangle$, has compact closure in $\mathscr{T}_1(K_{v_0})$; a contradiction. □

Proof of Theorem 7[7]. — We will assume (as we may) that for $i = 1, 2$, G_i is adjoint and is realized as a linear group via the adjoint representation on its Lie algebra $\mathfrak{g}_i$. Suppose that Γ_1 is $(\mathscr{G}_1, K, S)$-arithmetic; then $K(\subset F)$ is a number field, $\mathscr{G}_1$ is a K-form of

[7] Of course, if $\mathrm{rk}_F\, G \geqslant 2$, then Γ_2 is automatically arithmetic by Margulis' Arithmeticity Theorem (cf. [21], Chap. IX), so we only need to consider the case $\mathrm{rk}_F\, G = 1$. Our argument, however, does not depend on $\mathrm{rk}_F\, G$.

G_1, and as $\mathscr{G}_1$ is adjoint, $\Gamma_1 \subset \mathscr{G}_1(K)$ (see, for example, [4], Proposition 1.2). Let v_0 be the valuation of K obtained as the pullback of the normalized valuation on F using the embedding $K \hookrightarrow F$. Then of course $K_{v_0} \subset F$. Furthermore, $v_0 \in S$. Indeed, if $v_0 \notin S$, then v_0 is nonarchimedean and the group $\mathscr{G}_1(\mathcal{O}_K(S))$ is relatively compact in $\mathscr{G}_1(K_{v_0})$. Since Γ_1 is commensurable with $\mathscr{G}_1(\mathcal{O}_K(S))$, it would then be relatively compact in $\mathscr{G}_1(K_{v_0})$, and so in $G_1(F)$. However, as Γ_1 is discrete, it would be finite, which would contradict its Zariski-density. Moreover, being commensurable with Γ_1, the subgroup $\mathscr{G}_1(\mathcal{O}_K(S))$ is discrete in $\mathscr{G}_1(K_{v_0})$. Combining this with the fact that $\mathscr{G}_1(\mathcal{O}_K(S))$ is a lattice in $\mathscr{G}_{1S} := \prod_{v \in S} \mathscr{G}_1(K_{1v})$, we obtain that the group $\mathscr{G}_1(K_v)$ is compact for all $v \in S \setminus \{v_0\}$ (so, in particular, $K_{1v} = \mathbf{R}$ for all archimedean $v \in S \setminus \{v_0\}$). Because of our convention regarding S, we see that there are in fact only two possibilities: (1) $S = V_\infty^K$, or (2) v_0 is nonarchimedean, and $S = V_\infty^K \cup \{v_0\}$. Furthermore, as we have seen above, Γ_1 is relatively compact in $G_1(K_v)$ for all $v \notin S$. Thus, for any $\gamma_1 \in \Gamma_1$, the cyclic subgroup $\langle \gamma_1 \rangle$ is relatively compact in $\mathscr{G}_1(K_v)$ for all $v \in V^K \setminus \{v_0\}$.

Let K_{Γ_i} denote the field generated by the traces of all elements $\gamma \in \Gamma_i$. Being lattices, Γ_1 and Γ_2 are finitely generated (cf. [36], 13.21, for the real case), and therefore Theorem 2 applies. Combining the latter with Lemma 2.6, we conclude that

$$K_{\Gamma_1} = K = K_{\Gamma_2}.$$

By Vinberg's theorem [45], there exists a basis of $\mathfrak{g}_2$ in which Γ_2 is represented by matrices with entries in K; we fix such a basis for the rest of the proof. Then G_2 has a K-form $\mathscr{G}_2$ such that $\Gamma_2 \subset \mathscr{G}_2(K)$. In the sequel, the groups of points of $\mathscr{G}_2$ over subrings of K will be understood in terms of the realization of $\mathscr{G}_2$ as a matrix group using the basis of $\mathfrak{g}_2$ fixed above. We will show that Γ_2 is commensurable with $\mathscr{G}_2(\mathcal{O}_K(S))$, which will prove the theorem. For this it is enough to establish the following two assertions:

(a) $\mathscr{G}_2(K_v)$ is compact for all $v \in V_\infty^K \setminus \{v_0\}$;
(b) Γ_2 is bounded in $\mathscr{G}_2(K_v)$ for all $v \in V_f^K \setminus \{v_0\}$.

Indeed, since Γ_2 is finitely generated, and therefore it is contained in $\mathscr{G}_2(\mathcal{O}_v)$ for all but finitely many $v \in V_f^K$, we derive from (b) that, in either possibility for S,

$$[\Gamma_2 : \Gamma_2 \cap \mathscr{G}_2(\mathcal{O}_K(S))] < \infty;$$

in particular, $\Gamma_2 \cap \mathscr{G}_2(\mathcal{O}_K(S))$ is a lattice in $G_2(F)$, and hence in $\mathscr{G}_2(K_{v_0})$. On the other hand, it follows from (a) that, in either of the two possibilities for S, the subgroup $\mathscr{G}_2(\mathcal{O}_K(S))$ is a lattice in $\mathscr{G}_2(K_{v_0})$, implying that $[\mathscr{G}_2(\mathcal{O}_K(S)) : \Gamma_2 \cap \mathscr{G}_2(\mathcal{O}_K(S))] < \infty$.

Let $v \in V^K \setminus \{v_0\}$ be such that at least one of the assertions (a) and (b) fails. We will then find a regular semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order such that the closure of $\langle \gamma_2 \rangle$ in $\mathscr{G}_2(K_v)$ is noncompact and for the unique maximal K-torus $\mathscr{T}_2$ of $\mathscr{G}_2$ containing γ_2 we have

$$\theta_{\mathscr{T}_2}(\mathrm{Gal}(K_{\mathscr{T}_2}/K)) \supset W(\mathscr{G}_2, \mathscr{T}_2). \qquad (\mathbf{17})$$

Let us assume for a moment that such a γ_2 exists. Then since Γ_1 and Γ_2 are weakly commensurable, there exists a semi-simple element $\gamma_1 \in \Gamma_1$ which is weakly commensurable to γ_2. Then, if $\mathscr{T}_1$ is a maximal K-torus of $\mathscr{G}_1$ containing γ_1, by Theorem 4.2 there exists a K-isogeny $\pi\colon \mathscr{T}_1 \to \mathscr{T}_2$ which carries $\gamma_1^{m_1}$ to $\gamma_2^{m_2}$ for some integers $m_1, m_2 \geqslant 1$. The isogeny π induces a continuous group homomorphism $\overline{\langle\gamma_1^{m_1}\rangle} \to \overline{\langle\gamma_2^{m_2}\rangle}$ of the closures of the cyclic subgroups generated by $\gamma_1^{m_1}$ and $\gamma_2^{m_2}$ in $\mathscr{G}_1(\mathrm{K}_v)$ and $\mathscr{G}_2(\mathrm{K}_v)$ respectively. As we observed above, $\overline{\langle\gamma_1^{m_1}\rangle}$ is compact, so $\overline{\langle\gamma_2^{m_2}\rangle}$ must also be compact, a contradiction.

To find a $\gamma_2 \in \Gamma_2$ with the desired properties we will use the results of [32]. Let us consider first the case where $v \in \mathrm{V}_\infty^\mathrm{K} \setminus \{v_0\}$ and $\mathscr{G}_2(\mathrm{K}_v)$ is noncompact (or, equivalently, $\mathrm{rk}_{\mathrm{K}_v}\mathscr{G}_2 > 0$). It has been shown in [32] (cf. the proof of Theorem 2) that there exists a regular **R**-regular[8] semi-simple element $\gamma_2 \in \Gamma_2$ for which the corresponding torus $\mathscr{T}_2$ satisfies (17). Since the fact that γ_2 is **R**-regular clearly implies that the closure of $\langle\gamma_2\rangle$ is noncompact, we see that γ_2 has the required properties.

Assuming now that $v \in \mathrm{V}_f^\mathrm{K} \setminus \{v_0\}$ and Γ_2 is unbounded in $\mathscr{G}_2(\mathrm{K}_v)$, we will prove the existence of a $\gamma_2 \in \Gamma_2$ with the desired properties. For this, we will use the results of [32] in conjunction with the following result of Weisfeiler ([48], Theorem 10.5): there exists a finite subset $\mathcal{S}$ of V^K containing $\mathrm{V}_\infty^\mathrm{K}$ such that (i) the subgroup $\widetilde{\Gamma}_2 := \Gamma_2 \cap \mathscr{G}_2(\mathcal{O}(\mathcal{S}))$ is Zariski-dense in $\mathscr{G}_2$, (ii) for every $v \in \mathrm{V}^\mathrm{K} \setminus \mathcal{S}$, the closure of $\widetilde{\Gamma}_2$ in $\mathscr{G}_2(\mathrm{K}_v)$ is open, and (iii) for any $v \in \mathcal{S} \setminus \mathrm{V}_\infty^\mathrm{K}$, the subgroup $\widetilde{\Gamma}_2$ is discrete in $\mathscr{G}_2(\mathrm{K}_v)$. Pick such a set $\mathcal{S}$, and first consider the case where $v \in \mathcal{S} \setminus \mathrm{V}_\infty^\mathrm{K}$. Since $\widetilde{\Gamma}_2$ is Zariski-dense, by [32], there exists a regular semi-simple element $\gamma_2 \in \widetilde{\Gamma}_2$ of infinite order such that the corresponding torus $\mathscr{T}_2$ satisfies (17). But since $\widetilde{\Gamma}_2$ is discrete in $\mathscr{G}_2(\mathrm{K}_v)$, the subgroup $\langle\gamma_2\rangle$ is automatically unbounded. Now, let $v \in \mathrm{V}^\mathrm{K} \setminus \mathcal{S}$, and suppose that Γ_2 is unbounded in $\mathscr{G}_2(\mathrm{K}_v)$. Then $\mathscr{G}_2$ is K_v-isotropic and the closure of Γ_2 in $\mathscr{G}_2(\mathrm{K}_v)$ is unbounded and open, so it contains the normal subgroup $\mathscr{G}_2(\mathrm{K}_v)^+$ of $\mathscr{G}_2(\mathrm{K}_v)$ generated by the unipotent elements (cf. [28]), which is known to be an open subgroup of $\mathscr{G}_2(\mathrm{K}_v)$ of finite index (cf. [26], Theorem 3.3 and Proposition 3.17). Now we fix a maximal K_v-torus $\mathscr{T}_2^v$ of $\mathscr{G}_2$ which contains a maximal K_v-split torus of the latter. Consider the solid open subset $\mathscr{U} = \mathscr{U}(\mathscr{T}_2^v, v)$ of $\mathscr{G}_2(\mathrm{K}_v)$ provided by Lemma 3.4 for $\mathrm{G} = \mathscr{G}_2$ and $\mathrm{T} = \mathscr{T}_2^v$. Then $\Omega_2^v := \mathscr{U} \cap \mathscr{G}_2(\mathrm{K}_v)^+$ is a non-empty open subset of $\mathscr{G}_2(\mathrm{K}_v)^+$. Hence, $\Gamma_2 \cap \Omega_2^v$ is dense in Ω_2^v. Pick a $y \in \Gamma_2 \cap \Omega_2^v$. Then an argument similar to the one used to prove Theorem 2 in [32] (where instead of using Lemma 3.5 of [29], we use Proposition 2.6 of [27]) shows that there exists $x \in \Gamma_2$ such that, for a suitable large positive integer n, $\gamma_2 := xy^n$ is regular K_v-regular, and for the unique maximal K-torus $\mathscr{T}_2$ of $\mathscr{G}_2$ containing γ_2, (17) holds. At the same time, since γ_2 is K_v-regular, the subgroup $\langle\gamma_2\rangle$ is unbounded in $\mathscr{G}_2(\mathrm{K}_v)$. Thus, γ_2 has the required properties. This completes the proof of Theorem 7. □

Remark **5.5.** — Let Γ be a torsion-free Zariski-dense subgroup of G(F). For any positive integer m, the normal subgroup $\Gamma^{(m)}$ of Γ, generated by the m-th powers of the

[8] Given a connected semi-simple algebraic group G defined over a local field L, an element $x \in \mathrm{G}(\mathrm{L})$ is called L-regular if the number of eigenvalues, counted with multiplicity, of modulus 1 of $\mathrm{Ad}\,x$ is minimum possible.

elements in Γ, is weakly commensurable with Γ. On the other hand, it is known that in some situations there exists an integer m such that $\Gamma^{(m)}$ is of infinite index in Γ: this is the case when, for example, Γ is of finite index in $\mathrm{SL}_2(\mathbf{Z})$ or $\mathrm{SL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of algebraic integers in the imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$, $d \geqslant 1$, or Γ is a cocompact lattice in a real semi-simple Lie group of real rank 1, see [24]. This shows that the requirement that Γ_2 be a lattice in Theorem 7 cannot be omitted in case G is of F-rank 1.[9] The question whether or not a (discrete) subgroup weakly commensurable to an irreducible lattice (which is, of course, automatically arithmetic) in a real semi-simple Lie group of real rank > 1, is itself a lattice, remains open. (Of course, the same question can be asked for irreducible higher rank lattices in the products of real and p-adic semi-simple Lie groups; for example, we do not know whether a subgroup Δ of $\Gamma = \mathrm{SL}_2(\mathbf{Z}[\frac{1}{p}])$, which is weakly commensurable to Γ, is necessarily of finite index.) We would like to point out, however, that no variant of the above method for constructing counter-examples is likely to work in the higher rank case.

More precisely, let again Γ be a torsion-free Zariski-dense subgroup of G(F). Given a map $\varphi\colon \Gamma \to \mathbf{N}$, we let Γ_φ denote the subgroup of Γ generated by $\gamma^{\varphi(\gamma)}$ for all $\gamma \in \Gamma$. This subgroup is obviously weakly commensurable to Γ for *any* choice of φ. However, in contrast to the case of cocompact lattices in rank one groups discussed in the previous paragraph, or even finite index subgroups of $\mathrm{SL}_2(\mathbf{Z})$, where the subgroup $\Gamma^{(m)}$ (which corresponds to $\varphi \equiv m$) has infinite index in Γ for a suitable m, the subgroup Γ_φ always has finite index in Γ if Γ is "boundedly generated" (this fact was pointed out to us by Thomas Delzant). Several non-cocompact arithmetic lattices in the higher rank case are known to be boundedly generated (see [12], [23] for the definition of, and most recent results on, "bounded generation"), and for them considering subgroups of the form Γ_φ will never lead to a weakly commensurable subgroup of infinite index.

We conclude this section with a discussion of the following question: *Let* F *be a nondiscrete locally compact field, and let* G_1 *and* G_2 *be two semi-simple algebraic group over* F. *Given*

[9] In the rank one case, the group $\Gamma^{(m)}$ is typically infinitely generated for sufficiently large m. So, we outline a construction of a *finitely generated* subgroup of infinite index in $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ which is weakly commensurable to Γ (a similar construction works for $\Gamma = \mathrm{SL}_2(\mathcal{O}_d)$). Set $u^+(\alpha) = \left(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}\right)$, $u^-(\beta) = \left(\begin{smallmatrix}1 & 0\\ \beta & 1\end{smallmatrix}\right)$, and let $\Delta_{m,n}$ denote the subgroup of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ generated by $u^+(m)$ and $u^-(n)$, where m, n are even integers $\geqslant 4$. It is well-known that $\Delta_{m,n}$ is of infinite index in Γ. To show that $\Delta_{m,n}$ is weakly commensurable to Γ it is enough to show that

$$\Gamma(mn) \subset \bigcup_{g \in \mathrm{GL}_2(\mathbf{Q})} g\Delta_{m,n}g^{-1}, \tag{18}$$

where $\Gamma(mn)$ is the congruence subgroup of Γ of level mn. To prove $(*)$, we first observe that for $\gamma \in \Gamma(mn)$ we have $\mathrm{Tr}\,\gamma \equiv 2 (\mathrm{mod}\ mn)$; in particular $\mathrm{Tr}\,\gamma \neq -2$. If $\mathrm{Tr}\,\gamma = 2$ and $\gamma \neq 1$, then γ is conjugate in $\mathrm{GL}_2(\mathbf{Q})$ to $u^+(m)$ (and $u^-(n)$). Hence, we may assume that $\mathrm{Tr}\,\gamma \neq \pm 2$. Then it is enough to find $\delta \in \Delta_{m,n}$ with $\mathrm{Tr}\,\delta = \mathrm{Tr}\,\gamma$, for which it suffices to show that $\mathrm{Tr}(\Delta_{m,n})$ contains all $t \in \mathbf{Z}$ satisfying $t \equiv 2 (\mathrm{mod}\ mn)$. A direct computation shows that

$$\mathrm{Tr}\, u^+(\alpha)u^-(\beta) = 2 + \alpha\beta.$$

So, if $t = 2 + mns$, then $\mathrm{Tr}\, u^+(m)^s u^-(n) = t$, as required. Finally, we would like to mention in connection with $(*)$ that D. Morris showed us a construction of an infinitely generated subgroup of the free group $\mathscr{F}_\ell$ (which then necessarily has infinite index) whose conjugates in $\mathscr{F}_\ell$ fill up all of $\mathscr{F}_\ell$.

weakly commensurable Zariski-dense subgroups Γ_i *of* $G_i(F)$ *for* $i = 1, 2$*, is it true that discreteness of one of them implies that of the other?* The following proposition provides an affirmative answer to this question in some situations.

Proposition **5.6.** — *Let* G_1 *and* G_2 *be connected absolutely almost simple algebraic groups defined over a nonarchimedean locally compact field* F *of characteristic zero, and let* Γ_i *be a finitely generated Zariski-dense subgroups of* $G_i(F)$ *for* $i = 1, 2$*. Assume that* Γ_1 *is discrete. If* Γ_1 *and* Γ_2 *are weakly commensurable, then* Γ_2 *is also discrete.*

Proof. — We may (and we will) assume that for $i = 1, 2$, G_i is adjoint and is realized as a matrix group via the adjoint representation on its Lie algebra $\mathfrak{g}_i$. By Theorem 1, either both G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other of type C_n. Furthermore, by Theorem 2, the traces of the elements of Γ_1 and of those of Γ_2 generate the same subfield K of F. Then, according to the results of Vinberg [46], for $i = 1, 2$, the group G_i is defined over K and $\Gamma_i \subset G_i(K)$. Moreover, since Γ_1 and Γ_2 are finitely generated, we can find a finitely generated subring R of K such that $\Gamma_i \subset G_i(R)$, for $i = 1, 2$. Finally, F is a finite extension of $\mathbf{Q}_p$ for some prime p, and after replacing it with the closure of K, we assume that F is generated over $\mathbf{Q}_p$ by the traces of the elements of Γ_2. Let v_0 be the restriction to K of the natural valuation on F; then F coincides with the completion K_{v_0}.

The main observation is that if Γ_2 is not discrete, then its closure $\overline{\Gamma}_2$ in $G_2(F)$ is open. In order to apply the theory of p-adic Lie groups, we introduce the group $H = R_{F/\mathbf{Q}_p}(G_2)$ obtained by restriction of scalars, so that $G_2(F) \simeq H(\mathbf{Q}_p)$ as topological groups. Then Γ_2 considered as a subgroup of $H(\mathbf{Q}_p)$ is Zariski-dense in H. This fact is implicitly contained in the proof of Theorem 10.5 in [48], but for the reader's convenience we will give a complete argument. We need the following.

Lemma **5.7.** — *Let* F *be a finite separable field extension of a field* E*, and let* Δ *be a multiplicative subsemi-group of the matrix algebra* $M_n(F)$*, where* $n \geqslant 1$*, that spans* $M_n(F)$ *over* F*. If the traces* $\operatorname{Tr}\delta$ *for* $\delta \in \Delta$ *generate* F *over* E *then* Δ *spans* $M_n(F)$ *over* E.

Proof. — It is enough to show that Δ spans $M_n(F) \otimes_E \overline{E} =: A$ over $\overline{E}$. We have $F \otimes_E \overline{E} \simeq \bigoplus_{i=1}^m E_i$, where $m = [F : E]$, $E_i = \overline{E}$ for all i, and $A \simeq \bigoplus_{i=1}^m M_n(E_i)$. Let B be the $\overline{E}$-subalgebra of A generated by Δ (which, of course, coincides with the $\overline{E}$-span of Δ). Since Δ spans $M_n(F)$ over F, the projection of B to each $M_n(E_i)$ is surjective. Let I be a maximal subset of $\{1, 2, \ldots, m\}$ for which the projection $\mathrm{pr}_I \colon B \to \bigoplus_{i\in I} M_n(E_i) =: A_I$ is surjective. If $I = \{1, 2, \ldots, m\}$, then $B = A$, as claimed. So, assume that I is a proper subset. Then for any $j \in \{1, 2, \ldots, m\} \setminus I$, the image $\mathrm{pr}_j(B \cap (\ker \mathrm{pr}_I))$ is a proper two-sided ideal of $M_n(E_j)$, hence is zero. This shows that pr_I is an isomorphism between B and A_I, and we let ϵ denote its inverse. Fix $j \in \{1, 2, \ldots, m\} \setminus I$ and set $\theta = \mathrm{pr}_j \circ \epsilon$. Since $\mathrm{pr}_j \colon B \to A_j$ is surjective, there exists a unique $i \in I$ for which the restriction of θ yields an isomorphism between $M_n(E_i)$ and $M_n(E_j)$, and then for any $i' \in I \setminus \{i\}$, the restriction of θ to $M_n(E_{i'})$ is zero. So, for all $b \in B$, $\mathrm{pr}_j(b) = \theta(\mathrm{pr}_i(b))$, and hence,

$$\mathrm{Tr}\,\mathrm{pr}_i(b) = \mathrm{Tr}\,\theta(\mathrm{pr}_i(b)) = \mathrm{Tr}\,\mathrm{pr}_j(b).$$

This contradicts the fact that the traces of elements of Δ generate F over E, hence $F \otimes_E \overline{E}$ over $\overline{E}$. □

Now, since G_2 is absolutely simple and Γ_2 is Zariski-dense in $G_2(F)$, we see that Γ_2 acts (absolutely) irreducibly on $\mathfrak{g}_2$, and hence it spans $\mathrm{End}_F\,\mathfrak{g}_2(F)$ over F. Since the traces of the elements of Γ_2 generate F over $\mathbf{Q}_p$, by Lemma 5.7, the group Γ_2 spans $\mathrm{End}_F\,\mathfrak{g}_2(F)$ over $\mathbf{Q}_p$. It follows that $\mathfrak{g}_2(F)$ ($\simeq \mathfrak{h}(\mathbf{Q}_p)$, where $\mathfrak{h}$ is the Lie algebra of H) does not contain a proper Γ_2-invariant $\mathbf{Q}_p$-subspace as such a subspace would be an F-subspace, hence would coincide with $\mathfrak{g}_2(F)$. Thus, Γ_2 acts on $\mathfrak{h}(\mathbf{Q}_p)$ irreducibly. Letting C denote the Zariski-closure of Γ_2 in H and observing that for the Lie algebra $\mathfrak{c}$ of C, the subspace $\mathfrak{c}(\mathbf{Q}_p) \subset \mathfrak{h}(\mathbf{Q}_p)$ is Γ_2-invariant, we see that $\mathfrak{c} = \mathfrak{h}$ and hence $C = H$ since H is connected, as required. Let now $\mathfrak{l}$ denote the Lie algebra of $\overline{\Gamma}_2$ as a p-adic Lie group. Then, since Γ_2 is Zariski-dense in H, $\mathfrak{l}$ is an ideal of $\mathfrak{h}(\mathbf{Q}_p)$ (cf. [26], Proposition 3.4), hence $\mathfrak{l} = \mathfrak{h}(\mathbf{Q}_p)$ as H is $\mathbf{Q}_p$-simple. Thus, $\overline{\Gamma}_2$ is open in $H(\mathbf{Q}_p) \simeq G(F)$.

According to [26], Theorem 6.21, or [9], §2.4, there exists a maximal F-torus T_0 of G which is anisotropic over F. We let $\mathscr{U}_0 := \mathscr{U}(T_0, v_0)$ denote the solid open set as in Lemma 3.4 for $(T, v) = (T_0, v_0)$. Let r be the number of nontrivial conjugacy classes of the Weyl group of G. By Proposition 1 of [32], we can find r distinct rational primes $p_1, \dots, p_r$, different from p, such that there exist embeddings $\iota_i \colon K \to \mathbf{Q}_{p_i}$ satisfying $\iota_i(R) \subset \mathbf{Z}_{p_i}$. Let v_i be the pull-back of the p_i-adic valuation under ι_i. By Lemma 2 of [32], the closure of the image under the diagonal embedding $\Gamma_2 \hookrightarrow \prod_{i=1}^{r} G(K_{v_i}) =: \mathcal{G}$ is open. Since $p \notin \{p_1, \dots, p_r\}$ and $\overline{\Gamma}_2$ is open, the closure of the image under the diagonal embedding $\delta \colon \Gamma_2 \hookrightarrow G(F) \times \mathcal{G}$ is also open (cf. the proof of Lemma 2 in [32]). Let $\mathscr{U} \subset \mathcal{G}$ be the solid open subset, constructed in Corollary 3.5, such that every $\gamma \in \Gamma_2 \cap \mathscr{U}$ is regular semi-simple, and for $T = Z_G(\gamma)^\circ$, we have $\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G, T)$. There exists $\gamma_2 \in \Gamma_2$ of infinite order for which $\delta(\gamma_2) \in \mathscr{U}_0 \times \mathscr{U}$, and we let $T_2 = Z_G(\gamma_2)^\circ$. By our assumption, γ_2 is weakly commensurable to a semi-simple $\gamma_1 \in \Gamma_1$ of infinite order. Let T_1 be a maximal K-torus of G_1 containing γ_1. It follows from the Isogeny Theorem 4.2 that T_1 and T_2 are K-isogenous. But by our construction T_2 is F-anisotropic, which forces T_1 to be F-anisotropic. Then $T_1(F)$ is compact, so invoking the discreteness of Γ_1, we see that the intersection $T_1(F) \cap \Gamma_1$ is finite. Thus, $\gamma_1 \in T_1(F) \cap \Gamma_1$ has finite order, a contradiction. □

With obvious modifications, the proof of Proposition 5.6 applies to the situation where $F = \mathbf{R}$ and G_2 contains an $\mathbf{R}$-anisotropic maximal $\mathbf{R}$-torus. For general real semi-simple groups, the above question remains open.

6. The invariance of rank and the proofs of Theorems 4 and 5

In view of Theorem 3, weakly commensurable Zariski-dense S-arithmetic subgroups necessarily have the same field of definition K and correspond to the same set of places of K. So now the focus of our study of such subgroups shifts to identifying common characteristics of the K-forms $\mathscr{G}_i$ used to construct them.

In the rest of this paper G_1 and G_2 will denote two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero.

Proposition **6.1.** — *Let* V_0 *be a finite set of places of* K. *Let* Γ_i *be a Zariski-dense* $(\mathscr{G}_i, K, S)$*-arithmetic subgroup of* $G_i(F)$ *for* $i = 1, 2$. *Let* L_i *be the smallest Galois extension of* K *over which* $\mathscr{G}_i$ *is inner. If* Γ_1 *and* Γ_2 *are weakly commensurable, then there exists a maximal* K*-torus* $\mathscr{T}_1$ *of* $\mathscr{G}_1$ *which contains a maximal* K_{v_0}*-split torus of* $\mathscr{G}_1$ *for all* $v_0 \in V_0$, *a maximal* K*-torus* $\mathscr{T}_2$ *of* $\mathscr{G}_2$, *and a* K*-isogeny* $\pi : \mathscr{T}_2 \to \mathscr{T}_1$. *Moreover, if* $L_1 = L_2$ *and both* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *are of the same type different from* $B_2 = C_2$, F_4 *or* G_2, *then we can assume that* π *is an isomorphism, and* $\pi^*(\Phi(\mathscr{G}_1, \mathscr{T}_1)) = \Phi(\mathscr{G}_2, \mathscr{T}_2)$.

Proof. — Using Corollary 3.2, we can find a maximal K-torus $\mathscr{T}_1$ of $\mathscr{G}_1$ which contains a maximal K_v-split torus of $\mathscr{G}_1$ for every $v \in S \cup V_0$, and for which

$$\theta_{\mathscr{T}_1}(\mathrm{Gal}(K_{\mathscr{T}_1}/K)) \supset W(\mathscr{G}_1, \mathscr{T}_1).$$

Then the group $\mathscr{T}_{1S} := \prod_{v \in S} \mathscr{T}_1(K_v)$ is noncompact, and since the quotient $\mathscr{T}_{1S}/\mathscr{T}_1(\mathcal{O}_K(S))$ is compact as $\mathscr{T}_1$ is K-anisotropic, we infer that $\mathscr{T}_1(\mathcal{O}_K(S))$ is infinite. Therefore, $\Gamma_1 \cap \mathscr{T}_1(K)$ contains an element γ_1 of infinite order. By our assumption, γ_1 is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2 \cap \mathscr{G}_2(K)$. Let $\mathscr{T}_2$ be a maximal K-torus of $\mathscr{G}_2$ that contains γ_2. According to Theorem 4.2, there exists a K-isogeny $\pi : \mathscr{T}_2 \to \mathscr{T}_1$. The second assertion of the proposition follows from Theorem 4.2 and Remark 4.4. □

Theorem **6.2.** — *Let* Γ_i *be a Zariski-dense* $(\mathscr{G}_i, K, S)$*-arithmetic subgroup of* $G_i(F)$ *for* $i = 1, 2$. *If* Γ_1 *and* Γ_2 *are weakly commensurable, then*

$$\mathrm{rk}_{K_v}\, \mathscr{G}_1 = \mathrm{rk}_{K_v}\, \mathscr{G}_2 \quad \textit{for all } v \in V^K.$$

Proof. — Fix $v_0 \in V^K$. By symmetry, it is enough to show that

$$\mathrm{rk}_{K_{v_0}}\, \mathscr{G}_1 \leqslant \mathrm{rk}_{K_{v_0}}\, \mathscr{G}_2.$$

Applying the preceding proposition to $V_0 = \{v_0\}$, we can find a maximal K-torus $\mathscr{T}_i$ of $\mathscr{G}_i$, for $i = 1, 2$, such that $\mathscr{T}_1$ contains a maximal K_{v_0}-split torus of $\mathscr{G}_1$, and there is a K-isogeny $\pi : \mathscr{T}_2 \to \mathscr{T}_1$. From this we see that

$$\mathrm{rk}_{K_{v_0}}\, \mathscr{G}_1 = \mathrm{rk}_{K_{v_0}}\, \mathscr{T}_1 = \mathrm{rk}_{K_{v_0}}\, \mathscr{T}_2 \leqslant \mathrm{rk}_{K_{v_0}}\, \mathscr{G}_2.$$

□

For a connected absolutely simple algebraic group $\mathscr{G}$ defined over a number field K, we let $\Sigma(\mathscr{G}, \mathrm{K})$ (resp., $\Sigma^q(\mathscr{G}, \mathrm{K})$) be the set of places v of K such that $\mathscr{G}$ is split (resp., is quasi-split but not split) over K_v (of course, $\Sigma^q(\mathscr{G}, \mathrm{K})$ is empty if $\mathscr{G}$ is an inner form of a split group over K).

Theorem **6.3.** — *Let* Γ_i *be a Zariski-dense* $(\mathscr{G}_i, \mathrm{K}, \mathrm{S})$*-arithmetic subgroup of* $\mathrm{G}_i(\mathrm{F})$ *for* $i = 1, 2$. *If* Γ_1 *and* Γ_2 *are weakly commensurable, then*

(1) $\Sigma(\mathscr{G}_1, \mathrm{K}) = \Sigma(\mathscr{G}_2, \mathrm{K})$*;*
(2) *if* L_i *is the minimal Galois extension of* K *over which* $\mathscr{G}_i$ *becomes an inner form (of a split group), then* $\mathrm{L}_1 = \mathrm{L}_2$*;*
(3) $\Sigma^q(\mathscr{G}_1, \mathrm{K}) = \Sigma^q(\mathscr{G}_2, \mathrm{K})$.

Proof. — By Theorem 1, the groups G_1 and G_2 have the same absolute rank, so assertion (1) immediately follows from the preceding theorem. To prove (2), by symmetry it is enough to show that $\mathrm{L}_1 \subset \mathrm{L}_2$. Assume, if possible, that L_1 is not contained in L_2. Then $\mathrm{L}_1\mathrm{L}_2$ is a Galois extension of K that properly contains L_2. It follows from Chebotarev's Density Theorem that there are infinitely many $v \in \mathrm{V}_f^{\mathrm{K}}$ which split completely in L_2 but not in L_1. Also, $\mathscr{G}_2$ is quasi-split over K_v for all but finitely many $v \in \mathrm{V}_f^{\mathrm{K}}$, cf. [26], Theorem 6.7. So there exists a $v \in \mathrm{V}_f^{\mathrm{K}}$ which splits completely in L_2 but not in L_1, and $\mathscr{G}_2$ is quasi-split over K_v. Then $\mathscr{G}_2$ actually splits over K_v, i.e., $v \in \Sigma(\mathscr{G}_2, \mathrm{K})$, but since v does not split in L_1, we have $v \notin \Sigma(\mathscr{G}_1, \mathrm{K})$, this contradicts assertion (1). Since assertion (3) is vacuous if one of the groups is of type B_n and the other of type C_n, we can assume, in view of Theorem 1, that G_1 and G_2 are of the same type. Then, as the equality $\mathrm{L}_1 = \mathrm{L}_2$ has already been established, assertion (3) follows at once from Theorem 6.2. □

Remark **6.4.** — In the case where G_1 and G_2 are of the same type, Theorem 6.2 and Theorem 6.3, parts (1) and (3), can be viewed as formal consequences of the assertion in Theorem 6 (to be proved in the next section) that in the situation at hand, the Tits indices of $\mathscr{G}_1$ and $\mathscr{G}_2$ over K_v are identical, for all $v \in \mathrm{V}^{\mathrm{K}}$. So, we would like to point out that Theorems 6.2 and 6.3 are actually used in the proof of Theorem 6, and, on the other hand, all theorems *except* Theorem 6 can be obtained without using the Tits index. At the same time, in contrast to Theorem 6, Theorems 6.2 and 6.3 apply in the situation where one of the groups is of type B_n and the other of type C_n.

Before we proceed to the proofs of Theorems 4 and 5, we briefly recall the classification of absolutely simple algebraic groups of a given type over a field K (cf. [39], [44]). Any such group is an inner twist of a K-quasi-split group of the given type. So, fix a K-quasi-split group $\mathscr{G}$. Notice that $\mathscr{G}$ is completely determined by specifying (in addition to its type) the minimal Galois extension L/K over which it splits; this extension necessarily has degree 1 (which means that $\mathscr{G}$ splits over K) if the type is different from A_n $(n > 1)$, D_n $(n \geqslant 4)$, or E_6, can have degree 1 or 2 for the types A_n, D_n and E_6, and can also be either a cyclic extension of degree 3 or a Galois extension with the Galois group S_3 for type D_4. Furthermore, the K-isomorphism classes of inner twists of $\mathscr{G}$ correspond bijectively to the elements lying in the image of the natural map

$$H^1(K, \overline{\mathscr{G}}) \longrightarrow H^1(K, \mathrm{Aut}\,\overline{\mathscr{G}}),$$

where $\overline{\mathscr{G}}$ is the adjoint group of $\mathscr{G}$ identified with its group of inner automorphisms. When K is a number field, one considers the natural "global-to-local" map

$$H^1(K, \overline{\mathscr{G}}) \xrightarrow{\omega} \bigoplus_{v \in V^K} H^1(K_v, \overline{\mathscr{G}}),$$

and also the truncated maps

$$H^1(K, \overline{\mathscr{G}}) \xrightarrow{\omega_S} \bigoplus_{v \notin S} H^1(K_v, \overline{\mathscr{G}}),$$

for every finite subset S of V^K. It is known that ω is injective (cf. [26], Theorem 6.22) and $\mathrm{Ker}\,\omega_S$ is finite (cf. [39], Theorem 7 in Chap. III, §4.6).

Proof of Theorem 4. — When G_1 and G_2 are of type D_{2n}, $n > 2$, Theorem 4 is Theorem 9.1 of [35],[10] therefore to prove the theorem we assume that G_1 and G_2 are not of type A, D or E_6.

Let $\mathscr{G}$ be the K-split form of G. For the groups of the types under consideration we have $\mathrm{Aut}\,\mathscr{G} = \overline{\mathscr{G}}$, so the group $\mathscr{G}_i$, for $i = 1, 2$, is obtained from $\mathscr{G}$ by twisting with a Galois-cocycle representing an appropriate element ξ_i of $H^1(K, \overline{\mathscr{G}})$. We need to show that $\xi_1 = \xi_2$. For this we notice that according to Theorem 6.2, we have $\mathrm{rk}_{K_v}\,\mathscr{G}_1 = \mathrm{rk}_{K_v}\,\mathscr{G}_2$, for all $v \in V^K$. But for the types currently being considered this implies that

$$\mathscr{G}_1 \simeq \mathscr{G}_2 \quad \text{over } K_v. \tag{18}$$

Indeed, for v real, this follows from the classification of real forms of absolutely simple Lie algebras/real algebraic groups (cf. [17], Chap. X, §6, or [44]). Now let v be nonarchimedean. For the groups under consideration, the center $\mathscr{Z}$ of the corresponding simply connected group is a subgroup of μ_2, the kernel of the endomorphism $x \mapsto x^2$ of GL_1. In view of the bijection between $H^1(K_v, \overline{\mathscr{G}})$ and $H^2(K_v, \mathscr{Z})$ (cf. [26], Corollary to Theorem 6.20), we see that $H^1(K_v, \overline{\mathscr{G}}) \simeq H^2(K_v, \mathscr{Z}) = \mathrm{Br}(K_v)_2$, and hence, $|H^1(K_v, \overline{\mathscr{G}})| \leqslant 2$, which means that there exists at most one nonsplit form, and therefore the equality of ranks implies the isomorphism between the forms. If we now let

$$\omega_v \colon H^1(K, \overline{\mathscr{G}}) \longrightarrow H^1(K_v, \overline{\mathscr{G}})$$

[10] When G_1 and G_2 are of the same type, by passing to the corresponding adjoint groups and enlarging the field F, we can actually assume that $G_1 = G_2 =: G$, which is the case considered in [35].

denote the restriction map, then the isomorphism (18) implies that $\omega_v(\xi_1) = \omega_v(\xi_2)$, for all $v \in V^K$. Then, $\omega(\xi_1) = \omega(\xi_2)$, and therefore, $\xi_1 = \xi_2$, as required. □

Proof of Theorem 5. — Let L_i be the minimal Galois extension of K over which $\mathscr{G}_i$ becomes an inner form, and let

$$V_i = V^K \setminus (\Sigma(\mathscr{G}_i, K) \cup \Sigma^q(\mathscr{G}_i, K))$$

be the set of places v of K where $\mathscr{G}_i$ is not quasi-split. It is well-known that V_i is finite (cf. [26], Theorem 6.7). By Theorem 6.3, the fact the $(\mathscr{G}_i, S, K)$-arithmetic subgroups, for $i = 1, 2$, are Zariski-dense and weakly commensurable, implies that

$$L_1 = L_2 := L \quad \text{and} \quad V_1 = V_2 := V.$$

Thus, by fixing $\mathscr{G}_1$, we get a Galois extension L and a finite set V of places such that any K-form $\mathscr{G}_2$ of $\overline{G}_2$ as in the statement of the theorem has L as the minimal Galois extension (of K) over which it becomes an inner form, and is quasi-split over K_v for all $v \notin V$. Let $\mathscr{G}$ be the quasi-split K-form of $\overline{G}_2$ associated with L. Then any $\mathscr{G}_2$ can be obtained from $\mathscr{G}$ by twisting it by a Galois-cocycle that represents some $\xi \in H^1(K, \mathscr{G})$. Furthermore, for $v \notin V$, the group $\mathscr{G}_2$ is quasi-split over K_v, hence it is K_v-isomorphic to $\mathscr{G}$, which means that $\omega_v(\xi)$ is trivial. (Here we use the fact that for a connected absolutely simple quasi-split adjoint group $\mathscr{G}$ over any field F, the map $H^1(F, \mathscr{G}) \to H^1(F, \mathrm{Aut}\,\mathscr{G})$ has trivial kernel, which follows from the observation that $\mathrm{Aut}\,\mathscr{G}$ is a semi-direct product over F, of $\mathscr{G}$ and a finite F-group of automorphisms corresponding to the symmetries of the Dynkin diagram.) Thus, $\xi \in \mathrm{Ker}\,\omega_V$, so the finiteness of this kernel yields the finiteness of the number of K-isomorphism classes of possible K-groups $\mathscr{G}_2$ with the properties described in the theorem. □

We conclude this section with three examples. The first two demonstrate, by means of explicit constructions, that in groups of both, inner and outer, type A_n, $n > 1$, the collection of weakly commensurable arithmetic subgroups may consist of more than one commensurability classes. Later, in Section 9, the idea underlying these examples will be developed into a new general technique for constructing nonisomorphic K-groups of type A_n, D_{2n+1} $(n > 1)$ and E_6 which contain weakly commensurable arithmetic subgroups. The third example provides weakly commensurable S-arithmetic groups in groups G_1 and G_2 of type B_n and C_n respectively. So, this possibility in Theorem 1 cannot be ruled out.

Example **6.5.** — Take $G_1 = G_2 = SL_d$, where $d > 2$, over $F = \mathbf{R}$ (so that G_1 and G_2 are of type A_n with $n = d - 1 > 1$), and fix a number field K contained in F. Pick four arbitrary nonarchimedean places $v_1, v_2, v_3, v_4 \in V_f^K$. Let D_1 and D_2 be the central division algebras of degree d over K whose local invariants ($\in \mathbf{Q}/\mathbf{Z}$) are respectively

$$n_v^{(1)} = \begin{cases} 0, & v \neq v_i,\ i \leqslant 4, \\ 1/d, & v = v_1 \text{ or } v_2, \\ -1/d, & v = v_3 \text{ or } v_4 \end{cases} \quad \text{and} \quad n_v^{(2)} = \begin{cases} 0, & v \neq v_i,\ i \leqslant 4, \\ 1/d, & v = v_1 \text{ or } v_3, \\ -1/d, & v = v_2 \text{ or } v_4. \end{cases}$$

Then as $d > 2$, the algebras D_1 and D_2 are neither isomorphic nor anti-isomorphic. So the algebraic K-groups $\mathscr{G}_1 = SL_{1,D_1}$ and $\mathscr{G}_2 = SL_{1,D_2}$, which are inner K-forms of G_1 and G_2, are not K-isomorphic. Thus, for any finite $S \subset V^K$, containing V_∞^K, the corresponding $(\mathscr{G}_i, K, S)$-arithmetic subgroups $\Gamma_i \subset G_i(F)$ are not commensurable (cf. Proposition 2.5). On the other hand, if D is a central division algebra of degree d over K, then an extension L/K of degree d is isomorphic to a maximal subfield of D if and only if for every $v \in V^K$, and any extension $w|v$, the local degree $[L_w : K_v]$ annihilates the corresponding local invariant n_v of D (cf. [25], Corollary b in §18.4). It follows that the maximal subfields of either D_1 or D_2 are characterized as those extensions L/K of degree d for which $[L_{w_i} : K_{v_i}] = d$ for $i = 1, 2, 3, 4$. Thus, D_1 and D_2 have the *same* maximal subfields, which easily implies that Γ_1 and Γ_2 are weakly commensurable. Indeed, let $\gamma_1 \in \Gamma_1$ be a semi-simple element of infinite order, and let T_1 be a maximal K-torus of $\mathscr{G}_1$ that contains γ_1. Since D_1 and D_2 have the same maximal subfields, there exists a K-isomorphism $T_1 \overset{\varphi}{\simeq} T_2$ with a maximal K-torus T_2 of $\mathscr{G}_2$. Then the subgroup $\varphi(T_1(K) \cap \Gamma_1)$ is an S-arithmetic subgroup of $T_2(K)$, so there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n \in \Gamma_2$. Let $\chi_1 \in X(T_1)$ be a character such that $\chi_1(\gamma_1)$ is not a root of unity. Then for $\chi_2 = (\varphi^*)^{-1}(\chi_1) \in X(T_2)$ we have:

$$(n\chi_1)(\gamma_1) = \chi_1(\gamma_1)^n = \chi_2(\gamma_2) \neq 1,$$

which implies that Γ_1 and Γ_2 are weakly commensurable.

This example can be refined in two ways. First, by picking a sufficiently large number of nonarchimedean places and modifying the above construction accordingly, one can construct an arbitrarily large number of noncommensurable weakly commensurable S-arithmetic subgroups of the group $G_1(F) = G_2(F) = SL_d(\mathbf{R})$. Second, suppose $d > 2$ is even, and take for G_1 and G_2 the real algebraic group $G = SL_{d/2,\mathbf{H}}$, where $\mathbf{H}$ is the division algebra of Hamiltonian quaternions. Assume that K is a number field that admits a real embedding $K \hookrightarrow \mathbf{R} =: F$, and we let v_∞ denote the real place corresponding to this embedding. In addition to the four places $v_1, v_2, v_3, v_4 \in V_f^K$ fixed in the above example, we pick a fifth place $v_5 \in V_f^K \setminus \{v_1, v_2, v_3, v_4\}$, and consider the central division algebras D_1 and D_2 of degree d over K with the same local invariants at v_1, v_2, v_3, v_4 as above, and having the invariant 1/2 at v_∞ and v_5, and 0 everywhere else. Then for any finite $S \subset V^K$ containing V_∞^K (in particular, for $S = V_\infty^K$ itself), the corresponding $(\mathscr{G}_i, K, S)$-arithmetic subgroups are weakly commensurable, but not commensurable, and in addition are contained in $G_1(F) = G_2(F) = SL_{d/2}(\mathbf{H})$. Furthermore, by increasing the number of places picked, we can construct an arbitrarily large number of noncommensurable weakly commensurable S-arithmetic subgroups of $\mathrm{SL}_{d/2}(\mathbf{H})$.

The above construction implemented for $K = \mathbf{Q}$ and $d = 4$ has the following geometric significance. Over $\mathbf{R}$, the group G is isomorphic to the spinor group of a real quadratic form with signature $(5, 1)$, and therefore the associated symmetric space is the real hyperbolic 5-space. So, the noncommensurable arithmetic subgroups constructed above give rise to noncommensurable length-commensurable compact hyperbolic 5-manifolds (cf. Remark 8.18). We will elaborate on this observation in Section 9, where, in particular, noncommensurable length-commensurable compact hyperbolic manifolds of dimension $4n + 1$ ($n \geqslant 1$) will be constructed.

Example **6.6.** — Let K be a number field and L be a quadratic extension of K. For $i = 1, 2$, let v_i be a nonarchimedean place of K which splits in L, and v_i', v_i'' be the places of L lying over v_i. Let $d > 1$ be an odd integer. Let D_1 and D_2 be the division algebra over L of degree d whose local invariants are respectively

$$n_v^{(1)} = \begin{cases} 1/d, & v = v_1' \text{ or } v_2', \\ -1/d, & v = v_1'' \text{ or } v_2'', \end{cases} \quad \text{and} \quad n_v^{(2)} = \begin{cases} 1/d, & v = v_1' \text{ or } v_2'', \\ -1/d, & v = v_1'' \text{ or } v_2', \end{cases}$$

and whose local invariant at every other place of L is zero. Then for $i = 1, 2$, the algebra D_i admits an involution σ_i of the second kind such that the fixed field L^{σ_i} coincides with K. Let $\mathscr{G}_i$ be the absolutely almost simple K-group with

$$\mathscr{G}_i(K) = \{x \in D_i^\times \mid x\sigma_i(x) = 1,\ \mathrm{Nrd}_{D_i/L}(x) = 1\}.$$

Then $\mathscr{G}_i$ is an outer form of type A_n with $n = d - 1 > 1$. For simplicity, let us assume that the involutions are chosen so that $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-split at every real place of K which does not split in L; notice that the local invariants of the algebras are trivial at all archimedean places, G_1 and G_2 automatically split at all other real places of K. Furthermore, since d is odd, G_1 and G_2 are quasi-split at every nonarchimedean place of K which does not split in L. Thus, it follows from Proposition A.2 of Appendix A in [30] and the subsequent discussion (see [35], §4, for more general results which apply also to the case of even d) that for an extension P/L of degree d provided with an automorphism τ of order two which induces the nontrivial automorphism of L/K, an embedding $(P, \tau) \to (D_i, \sigma_i)$ as algebras with involution exists if and only if $[P_w : K_{v_j}] = d$ for $j = 1, 2$ and $w|v_j$. This easily implies that the maximal σ_1-invariant subfields in D_1 are the same as the maximal σ_2-invariant subfields in D_2, and therefore $\mathscr{G}_1$ and $\mathscr{G}_2$ have the same maximal K-tori. Then as in the previous example, we conclude that for any S, the S-arithmetic subgroups of $\mathscr{G}_1(K)$ and $\mathscr{G}_2(K)$ are weakly commensurable. On the other hand, it follows from our choice of local invariants that $\mathscr{G}_1$ and $\mathscr{G}_2$ are not isomorphic even over L, so the above S-arithmetic subgroups are not commensurable. A suitable variant of this construction (applied to $K = \mathbf{Q}$, $L = \mathbf{Q}(i)$) enables one to construct length-commensurable, but not commensurable, compact complex hyperbolic $(d-1)$-manifolds, providing thereby a negative answer to Question (2) of the introduction for complex hyperbolic manifolds of any even dimension. We will not give the details here as the general construction described in Section 9 yields counter-examples in *all* dimensions.

Example **6.7.** — Let K be any field, and let G_1 and G_2 be the split K-groups of type B_n and C_n respectively which are either both simply connected or both adjoint. The purpose of this example is to show that G_1 and G_2 have the same isomorphism classes of maximal K-tori. The argument given in Example 6.5 shows that if K is a number field then for any subset S of V^K containing V^K_∞, the (K, S)-arithmetic subgroups of G_1 and G_2 are weakly commensurable. One can derive the fact that G_1 and G_2 have the same maximal K-tori from the results of Kariyama [18] who gave an explicit description of maximal tori in classical groups. However, a more efficient way to see this is to use a cohomological parametrization of the conjugacy classes of a semi-simple K-group G and the results of Gille [15] and Raghunathan [37]. We refer the reader to Section 9 for a systematic treatment of the aforementioned cohomological parametrization; all we need at this point is the following. Given a semi-simple K-group G, we fix a maximal K-torus T of G, and let $N = N_G(T)$ and $W = N/T$ denote its normalizer and the corresponding Weyl group. Then there exists a natural bijection between the set of G(K)-conjugacy classes of maximal K-tori of G, and $\mathscr{C}_K := \mathrm{Ker}(H^1(K, N) \to H^1(K, G))$ (cf. Lemma 9.1). Furthermore, the natural homomorphisms induce the following maps on Galois cohomology:

$$\theta_K : H^1(K, N) \longrightarrow H^1(K, W) \quad \text{and} \quad \nu_K : H^1(K, W) \longrightarrow H^1(K, \mathrm{Aut}\, T).$$

Recall that $H^1(K, \mathrm{Aut}\, T)$ parametrizes classes of K-isomorphism of K-tori of dimension equal to dim T. So, if $x_1, x_2 \in \mathscr{C}_K$ are such that $\nu_K(\theta_K(x_1)) = \nu_K(\theta_K(x_2))$ (in particular, if $\theta_K(x_1) = \theta_K(x_2)$), then the corresponding maximal K-tori of G are K-isomorphic. This leads to the following.

Lemma **6.8.** — *For $i = 1, 2$, let G_i be a semi-simple K-group, and let T_i be a fixed maximal K-torus of G_i. Let $N_i, W_i, \mathscr{C}_K^{(i)}, \theta_K^{(i)}, \ldots$ denote the above attributes attached to G_i. Assume that*

(1) *there exists compatible K-isomorphisms $\varphi : T_1 \to T_2$ and $\psi : W_1 \to W_2$ (i.e., $\varphi(w \cdot t) = \psi(w) \cdot \varphi(t)$ for any $t \in T_1$, $w \in W_1$, for the action induced by conjugation);*
(2) *ψ induces a bijection between $\theta_K^{(1)}(\mathscr{C}_K^{(1)})$ and $\theta_K^{(2)}(\mathscr{C}_K^{(2)})$.*

Then G_1 and G_2 have the same K-isomorphism classes of maximal K-tori.

We will now apply Lemma 6.8 in the situation described in the beginning of this example, i.e., where G_1 and G_2 are either both simply connected or adjoint split K-groups of type B_n and C_n respectively. Let T_i be a maximal K-split torus of G_i. Then it easily follows from the description of the corresponding root systems that there exist compatible K-isomorphisms $\varphi : T_1 \to T_2$ and $\psi : W_1 \to W_2$. Now, it was proved by Gille [15] and

Raghunathan [37] that if G is quasi-split over K, and T is a maximal K-torus contained in a Borel subgroup defined over K, then $\theta_K(\mathscr{C}_K) = H^1(K, W)$. It follows that in the situation at hand ψ induces a bijection between $\theta_K^{(1)}(\mathscr{C}_K^{(1)}) = H^1(K, W_1)$ and $\theta_K^{(2)}(\mathscr{C}_K^{(2)}) = H^1(K, W_2)$. Invoking Lemma 6.8, we see that G_1 and G_2 have the same K-isomorphism classes of maximal K-tori, as claimed.

7. Proof of Theorem 6

7.1. Tits index of a semi-simple algebraic group (cf. [44], or [41], §15.5). Let G be a connected semi-simple algebraic K-group. To describe the Tits index of G/K, we pick a maximal K-split torus T_s of G and a maximal K-torus T of G containing T_s. Furthermore, we choose coherent orderings on the vector spaces $X(T_s) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$, and let $\Delta \subset \Phi(G, T)$ denote the system of simple roots associated with the ordering on the latter. Then the *Tits index* of G/K is the data consisting of Δ (or the corresponding Dynkin diagram), the subset of *distinguished* roots, and the $*$-action (see Section 4). We recall that a root $\alpha \in \Delta$ (or the corresponding vertex in the Dynkin diagram) is *distinguished* if its restriction to T_s is nontrivial. If $\alpha \in \Delta$ is distinguished, then every root in the orbit Ω of α, under the $*$-action, is distinguished; this is indicated by circling together all the vertices corresponding to the roots in Ω, and the latter is referred to as a *distinguished orbit*. We note that $\mathrm{rk}_K\, G$ equals the number of distinguished orbits, and G is quasi-split over K if and only if every root in Δ is distinguished.

For a subset Θ of Δ, we let P_Θ denote the corresponding standard parabolic subgroup which contains the centralizer of $(\bigcap_{\beta\in\Theta} \ker\beta)^\circ$ as a Levi subgroup. Then for a subset Ω of Δ, the subgroup $P_{\Delta\setminus\Omega}$ is defined over K if and only if Ω is $*$-invariant and consists entirely of distinguished roots (in other words, it is a union of distinguished orbits). In particular, a root $\alpha \in \Delta$ is distinguished if and only if for its $*$-orbit Ω the subgroup $P_{\Delta\setminus\Omega}$ is defined over K.

In the proof of Theorem 6, we will need to work with the Tits indices of a given connected absolutely simple algebraic K-group G over various completions of K. For this purpose, we fix a maximal K-torus T of G and a system of simple roots $\Delta \subset \Phi(G, T)$. Given a field extension L/K, we choose a maximal L-torus T' containing a maximal L-split torus T'_s of G, and a system of simple roots $\Delta' \subset \Phi(G, T')$ determined by some coherent orderings on $X(T'_s) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(T') \otimes_{\mathbf{Z}} \mathbf{R}$. We say that $\alpha \in \Delta$ *corresponds to a distinguished vertex in the Tits index of* G/L if the root $\alpha' \in \Delta'$ corresponding to α, under the identification of Δ with Δ' described at the beginning of Section 4, is distinguished. The set of all $\alpha \in \Delta$ which correspond to distinguished vertices in the Tits index of G/L will be denoted $\Delta^{(d)}(L)$. It follows from Lemma 4.1(a), and the above discussion, that $\alpha \in \Delta^{(d)}(L)$ if and only if for the orbit Ω of α under the $*$-action of $\mathrm{Gal}(\overline{L}/L)$, a suitable conjugate of $P_{\Delta\setminus\Omega}$ is defined over L. More generally, for an arbitrary subset Ω of Δ, a suitable conjugate of $P_{\Delta\setminus\Omega}$ is defined over L if and only if Ω is invariant under the $*$-action of $\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})$ and contained in $\Delta^{(d)}(\mathrm{L})$. Thus, $\mathrm{rk}_{\mathrm{L}}\,\mathrm{G}$ equals the number of orbits of the $*$-action of $\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})$ on $\Delta^{(d)}(\mathrm{L})$, and G is quasi-split over L if and only if $\Delta^{(d)}(\mathrm{L}) = \Delta$.

Let G be a connected absolutely simple algebraic group over a number field K. Fix a maximal K-torus T of G, and a system of simple roots $\Delta \subset \Phi(\mathrm{G}, \mathrm{T})$. We will say that an orbit in Δ, under the $*$-action of $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, is *distinguished everywhere* if it is contained in $\Delta^{(d)}(\mathrm{K}_v)$ for all $v \in \mathrm{V}^{\mathrm{K}}$. The following proposition, which is proved using some results of [34], will not only play a crucial role in the proof of Theorem 6, but is also of independent interest.

Proposition **7.2.** — *An orbit under the $*$-action of* $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ *on Δ is contained in* $\Delta^{(d)}(\mathrm{K})$*, i.e., it is a distinguished orbit in the Tits index of* G/K*, if and only if it is distinguished everywhere. Therefore,* $\mathrm{rk}_{\mathrm{K}}\,\mathrm{G} = r$*, where r is the number of orbits which are distinguished everywhere.*

Proof. — Without any loss of generality, we may (and we do) assume that G is adjoint and T contains a maximal K-split torus of G. Clearly, the distinguished orbits in the Tits index of G/K are distinguished everywhere, yielding the inequality $\mathrm{rk}_{\mathrm{K}}\,\mathrm{G} \leqslant r$. To prove the opposite inequality, we can assume that $r \geqslant 1$. Let $\Omega_{i_1}, \ldots, \Omega_{i_r}$ be the orbits in Δ which are distinguished everywhere. We will prove that these are precisely the distinguished orbits in the Tits index of G/K. For this, we set

$$\Omega = \Omega_{i_1} \cup \cdots \cup \Omega_{i_r},$$

and let $\mathrm{P}_{\Delta\setminus\Omega}$ be the corresponding parabolic subgroup. It will suffice to prove that the conjugacy class of $\mathrm{P}_{\Delta\setminus\Omega}$ contains a subgroup defined over K. The group G is an inner twist of a unique quasi-split K-group G_0. Let T_0 be the centralizer of a maximal K-split torus T_0^s of G_0. Furthermore, let $\Delta_0 \subset \Phi(\mathrm{G}_0, \mathrm{T}_0)$ be the system of simple roots with respect to some coherent orderings on $\mathrm{X}(\mathrm{T}_0^s) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathrm{X}(\mathrm{T}_0) \otimes_{\mathbf{Z}} \mathbf{R}$ (then, in particular, all the roots in Δ_0 are distinguished). Since G is an inner twist of G_0, we can pick a $\overline{\mathrm{K}}$-isomorphism $f : \mathrm{G}_0 \to \mathrm{G}$ so that the associated Galois-cocycle

$$\sigma \mapsto \xi_\sigma := f^{-1} \circ \sigma(f), \quad \sigma \in \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K}),$$

is of the form

$$\xi_\sigma = i_{g_\sigma},$$

where i_z denotes the inner automorphism of G_0 corresponding to $z \in \mathrm{G}_0(\overline{\mathrm{K}})$, and $g : \sigma \mapsto g_\sigma$ is a Galois-cocycle with values in $\mathrm{G}_0(\overline{\mathrm{K}})$. After modifying f by a suitable inner automorphism, we assume that $f(\mathrm{T}_0) = \mathrm{T}$ and $f^*(\Delta) = \Delta_0$. We set $\Omega_0 = f^*(\Omega)$. Then for the parabolic K-subgroup $\mathrm{P}_{\Delta_0\setminus\Omega_0}$ of G_0, we have $f(\mathrm{P}_{\Delta_0\setminus\Omega_0}) = \mathrm{P}_{\Delta\setminus\Omega}$. Let H_0 be a Levi K-subgroup of $\mathrm{P}_{\Delta_0\setminus\Omega_0}$, and $\omega : \mathrm{H}^1(\mathrm{K}, \mathrm{H}_0) \to \mathrm{H}^1(\mathrm{K}, \mathrm{G}_0)$ be the Galois-cohomology map induced by the inclusion $\mathrm{H}_0 \hookrightarrow \mathrm{G}_0$.

Take an arbitrary $v \in V^K$. Then as Ω is a union of orbits in $\Delta^{(d)}(K_v)$, there exists $a_v \in G(\overline{K}_v)$ such that $P^{(v)}_{\Delta\setminus\Omega} := a_v P_{\Delta\setminus\Omega} a_v^{-1}$ is defined over K_v. Set $b_v = f^{-1}(a_v)$ and $f_v = f \circ i_{b_v}$. Then $f_v(P_{\Delta_0\setminus\Omega_0}) = P^{(v)}_{\Delta\setminus\Omega}$, and since both $P_{\Delta_0\setminus\Omega_0}$ and $P^{(v)}_{\Delta\setminus\Omega}$ are defined over K_v, for any $\sigma \in \mathrm{Gal}(\overline{K}_v/K_v)$, the automorphism

$$\xi^{(v)}_\sigma := f_v^{-1} \circ \sigma(f_v) = i_{b_v}^{-1} \circ \xi_\sigma \circ i_{\sigma(b_v)} = i_{b_v^{-1} g_\sigma \sigma(b_v)}$$

leaves $P_{\Delta_0\setminus\Omega_0}$ invariant. As $P_{\Delta_0\setminus\Omega_0}$ coincides with its normalizer in G_0 (cf. [3], Theorem 11.16), we conclude that $b_v^{-1} g_\sigma \sigma(b_v)$ lies in $P_{\Delta_0\setminus\Omega_0}(\overline{K}_v)$. Furthermore, since the unipotent radical of any parabolic K_v-subgroup of a reductive K_v-group has trivial Galois cohomology, we conclude that the cocycle $\sigma \mapsto b_v^{-1} g_\sigma \sigma(b_v)$ is cohomologous to a $H_0(\overline{K}_v)$-valued Galois-cocycle $h^{(v)}$. Thus, the image of the cohomology class x corresponding to the cocycle g, under the restriction map $\rho_v \colon H^1(K, G_0) \to H^1(K_v, G_0)$, is equal to the image of the cohomology class in $H^1(K_v, H_0)$, corresponding to $h^{(v)}$, under the map $H^1(K_v, H_0) \to H^1(K_v, G_0)$.

Now, let L be the minimal Galois extension of K over which G_0 splits, and set $P = L$ if $[L : K] \neq 6$, and let P be any cubic extension of K contained in L otherwise. Pick $v_0 \in V_f^K$ which does not split in P (i.e., $P \otimes_K K_{v_0}$ is a field). We will assume for the moment that $\Omega_0 \neq \Delta_0$ (or, equivalently, $\Omega \neq \Delta$). Then using Theorem 2 of [34], we easily conclude that there exists $y \in H^1(K, H_0)$ which maps to $(\rho_v(x))$ under the composite of the following two maps

$$H^1(K, H_0) \xrightarrow{\ \omega\ } H^1(K, G_0) \xrightarrow{\rho=(\rho_v)} \bigoplus_{v\neq v_0} H^1(K_v, G_0).$$

But according to Theorem 3 in [34], ρ is injective, so $x = \omega(y)$. This means that there exists $c \in G_0(\overline{K})$ such that

$$(\mathbf{19}) \qquad c^{-1} g_\sigma \sigma(c) \in H_0(\overline{K}) \quad \text{for all } \sigma \in \mathrm{Gal}(\overline{K}/K).$$

We claim that the subgroup $f(c) P_{\Delta\setminus\Omega} f(c)^{-1} = f(c P_{\Delta_0\setminus\Omega_0} c^{-1})$ is defined over K. Indeed, for $\sigma \in \mathrm{Gal}(\overline{K}/K)$ we have

$$\begin{aligned}\sigma(f(cP_{\Delta_0\setminus\Omega_0}c^{-1})) &= \sigma(f)(\sigma(c)P_{\Delta_0\setminus\Omega_0}\sigma(c)^{-1}) \\ &= f(g_\sigma\sigma(c)P_{\Delta_0\setminus\Omega_0}\sigma(c)^{-1}g_\sigma^{-1}) = f(cP_{\Delta_0\setminus\Omega_0}c^{-1}),\end{aligned}$$

in view of (19), proving our claim. This proves the proposition if $\Omega \neq \Delta$. If $\Omega = \Delta$, then, for all $v \in V^K$, G is quasi-split over K_v, and hence is isomorphic to G_0 over K_v, which implies that $\rho_v(x)$ is trivial for all v. From the Hasse principle for G_0 (Theorem 6.6 of [26]) we infer that x is trivial, so G is isomorphic to G_0, and hence every $*$-orbit in Δ is distinguished. □

Remark **7.3.** — One can give an alternative proof of Proposition 7.2 using techniques involving homogeneous spaces. Indeed, it is known (cf. [5], 5.24) that if $\Omega \subset \Delta$ is a subset invariant under the $*$-action, then the parabolic subgroups of G conjugate to $P_{\Delta\setminus\Omega}$ naturally correspond to the points of a projective homogeneous space $\mathscr{P}_\Omega$ of G defined over K. Now, as in the proof of Proposition 7.2, we let Ω denote the union of all orbits of the $*$-action which are distinguished everywhere. Then $\mathscr{P}_\Omega(K_v) \neq \emptyset$ for all $v \in V^K$. However, it was shown in [16] and reproved in [6] that projective homogeneous spaces satisfy the Hasse principle, which yields $\mathscr{P}_\Omega(K) \neq \emptyset$. Thus, there exists a parabolic K-subgroup conjugate to $P_{\Delta\setminus\Omega}$, as required.

Corollary **7.4.** — *Let* G *be an absolutely simple* K*-group of one of the following types:* B_n $(n \geqslant 2)$, C_n $(n \geqslant 2)$, E_7, E_8, F_4 *or* G_2. *If* G *is isotropic over* K_v *for all real* $v \in V^K_\infty$, *then* G *is isotropic over* K. *Additionally, if* G *is as above, but not of type* E_7, *then*

$$\mathrm{rk}_K\, G = \min_{v \in V^K} \mathrm{rk}_{K_v}\, G. \tag{20}$$

Proof. — The groups of these types do not have outer automorphisms, so given any two maximal K-tori T and T′ of G, and systems of simple roots $\Delta \subset \Phi(G, T)$ and $\Delta' \subset \Phi(G, T')$, there is a unique isomorphism between $\Phi(G, T)$ and $\Phi(G, T')$ that carries Δ to Δ'. It necessarily coincides with the canonical identification as defined at the beginning of Section 4. Using this remark and inspecting Table II in [44], we see that for the types listed in the statement, if for every real place v of K, G is isotropic over K_v, then there is a vertex in the Tits index of G/K which corresponds to a distinguished vertex in the Tits index of G/K_v, for all $v \in V^K$. Then it follows from the proposition that this vertex is distinguished in the Tits index of G/K, and therefore G is K-isotropic. Moreover, if G is not of type E_7, then it follows from the tables in [44] that the total number of vertices which are distinguished in the Tits index of G/K_v for all $v \in V^K$ is $\min_{v\in V^K} \mathrm{rk}_{K_v} G$, so (20) follows from the proposition. □

Proof of Theorem 6. — According to Theorem 6.2, $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2$ for every place v of K. Moreover, as S-arithmetic groups are finitely generated, Theorem 1 implies that either G_1 and G_2 are of same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n. In the latter case, using Corollary 7.4, we obtain

$$\mathrm{rk}_K\, \mathscr{G}_1 = \min_{v\in V^K} \mathrm{rk}_{K_v}\, \mathscr{G}_1 = \min_{v\in V^K} \mathrm{rk}_{K_v}\, \mathscr{G}_2 = \mathrm{rk}_K\, \mathscr{G}_2.$$

In particular, if $\mathscr{G}_1$ is K-isotropic, then so is $\mathscr{G}_2$.[11]

Now to prove the rest of Theorem 6, we can assume that F is algebraically closed and both G_1, G_2 are adjoint of the same type, so in effect $G_1 = G_2 =: G$. If G is of type

[11] Since for groups of type B_n and C_n over local and global fields, the relative rank completely determines the corresponding Tits index (cf. [44]), we see that in the case at hand, the Tits index of $\mathscr{G}_1/K$ determines that of $\mathscr{G}_2/K$, and for any $v \in V^K$, the Tits index of $\mathscr{G}_1/K_v$ determines that of $\mathscr{G}_2/K_v$.

$B_2 = C_2$, F_4, or G_2, then its Tits index over any extension L/K is uniquely determined by its L-rank. Therefore, since $\mathrm{rk}_{K_v}\mathscr{G}_1 = \mathrm{rk}_{K_v}\mathscr{G}_2$ according to Theorem 6.2, and consequently $\mathrm{rk}_K\mathscr{G}_1 = \mathrm{rk}_K\mathscr{G}_2$ by Corollary 7.4, all our assertions follow. So, we assume that G is not of any of the above three types.

We pick a finite set V_0 of places of K such that for every $v \notin V_0$, both $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-split over K_v. By Theorem 6.3(2), we have $L_1 = L_2$, so we can use Proposition 6.1 to find maximal K-tori $\mathscr{T}_i$ of $\mathscr{G}_i$ such that $\mathscr{T}_1$ contains a maximal K_v-split torus $\mathscr{T}_{1s}^v$ of $\mathscr{G}_1$ for all $v \in V_0$, and a K-isogeny (actually, a K-isomorphism) $\pi\colon \mathscr{T}_2 \to \mathscr{T}_1$ such that $\pi^*(\Phi(\mathscr{G}_1, \mathscr{T}_1)) = \Phi(\mathscr{G}_2, \mathscr{T}_2)$. Since $\mathrm{rk}_{K_v}\mathscr{G}_1 = \mathrm{rk}_{K_v}\mathscr{G}_2$ for all v, we see that $\mathscr{T}_2$ also contains a maximal K_v-split torus $\mathscr{T}_{2s}^v$ of $\mathscr{G}_2$, for all $v \in V_0$. Notice that if we choose any system of simple roots Δ_1 in $\Phi(\mathscr{G}_1, \mathscr{T}_1)$ and set $\Delta_2 = \pi^*(\Delta_1)$, then as π^* intertwines the action of $\mathrm{Gal}(\overline{K}/K)$ and the corresponding Weyl groups, it also intertwines the $*$-action of $\mathrm{Gal}(\overline{F}/F)$ for any extension F/K. Now, let $v \in V_0$, and let Δ_1^v be a system of simple roots in $\Phi(\mathscr{G}_1, \mathscr{T}_1)$ that corresponds to a coherent choice of orderings on $X(\mathscr{T}_{1s}^v) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(\mathscr{T}_1) \otimes_{\mathbf{Z}} \mathbf{R}$. Then $\Delta_2^v = \pi^*(\Delta_1^v)$ corresponds to the coherent orderings on $X(\mathscr{T}_{2s}^v) \otimes_{\mathbf{Z}} \mathbf{R}$ and $X(\mathscr{T}_2) \otimes_{\mathbf{Z}} \mathbf{R}$. Furthermore, since π induces an isomorphism between $\mathscr{T}_{2s}^v$ and $\mathscr{T}_{1s}^v$, we see that $\alpha \in \Delta_1^v$ has nontrivial restriction to $\mathscr{T}_{1s}^v$, i.e., it is distinguished in the Tits index of $\mathscr{G}_1/K_v$ if and only if $\pi^*(\alpha)$ has nontrivial restriction to $\mathscr{T}_{2s}^v$, i.e., it is distinguished in the Tits index of $\mathscr{G}_2/K_v$. This shows that the Tits indices of $\mathscr{G}_2/K_v$ and $\mathscr{G}_2/K_v$ are isomorphic for all $v \in V_0$. They are also isomorphic for any $v \in V^K \setminus V_0$ because then $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-split, which completes the proof of the "local" part of Theorem 6.

It remains to prove that the Tits indices of $\mathscr{G}_1/K$ and $\mathscr{G}_2/K$ are isomorphic. For this, we fix a system of simple roots Δ_1 of $\Phi(\mathscr{G}_1, \mathscr{T}_1)$ and set $\Delta_2 = \pi^*(\Delta_1)$. If $\Delta_1' \subset \Phi(\mathscr{G}_1, \mathscr{T}_1)$ is another system of simple roots and $\Delta_2' = \pi^*(\Delta_1')$, then the fact that π^* commutes with the action of the corresponding Weyl groups implies that π^* transports the canonical identification $\Delta_1 \simeq \Delta_1'$ to the canonical identification $\Delta_2 \simeq \Delta_2'$ (another way to see this is to observe that according to Remark 4.4, π extends to a $\overline{K}$-isomorphism $f\colon \mathscr{G}_2 \to \mathscr{G}_1$). So, by symmetry, it is enough to prove that if $\Omega \subset \Delta_1$ is an orbit of the $*$-action of $\mathrm{Gal}(\overline{K}/K)$ which corresponds to a distinguished orbit in the Tits index of $\mathscr{G}_1/K$, then $\pi^*(\Omega)$ (which is also a $*$-orbit) corresponds to a distinguished orbit in the Tits index of $\mathscr{G}_2/K$. According to Proposition 7.2, it is enough to show that

$$\textbf{(21)} \qquad \pi^*(\Omega) \subset \Delta_2^{(d)}(K_v)$$

for all $v \in V^K$. As $\Delta_2^{(d)}(K_v) = \Delta_2$ for all $v \in V^K \setminus V_0$, we only need to establish (21) for $v \in V_0$. But since π^* induces a bijection between distinguished vertices in Δ_1^v and Δ_2^v in the above notations, we see that

$$\Delta_2^{(d)}(K_v) = \pi^*(\Delta_1^{(d)}(K_v)),$$

and (21) follows. This completes the proof of Theorem 6. □

The following interesting result is an immediate consequence of Theorems 1, 4, 5, 6, and 6.3(2).

Theorem **7.5.** — *Let* G_1 *and* G_2 *be connected absolutely simple algebraic groups defined over a number field* K. *For* $i = 1, 2$, *let* L_i *be the smallest Galois extension of* K *over which* G_i *is an inner form of a split group. If the set of isomorphism classes of maximal* K*-tori in* G_1 *equals that in* G_2, *then either* G_1 *and* G_2 *are of same Killing-Cartan type, or one of them is of type* B_n *and the other is of type* C_n, *and moreover,* $L_1 = L_2$.

Now let us assume that $G_1 = G_2 =: G$. *Let* $\mathfrak{F}$ *be a collection of* K*-forms* G' *of* G *such that the set of* K*-isomorphism classes of maximal* K*-tori of* G' *equals the set of* K*-isomorphism classes of maximal* K*-tori of* G. *Then*

(1) *For any* $G' \in \mathfrak{F}$, *the Tits indices of* G/K *and* G'/K, *and for every place* v *of* K, *the Tits indices of* G/K_v *and* G'/K_v, *are isomorphic.*
(2) *If* G *is not of type* A_n, D_{2n+1}, D_4 *or* E_6, *then every* $G' \in \mathfrak{F}$ *is* K*-isomorphic to* G.
(3) $\mathfrak{F}$ *consists of finitely many* K*-isomorphism classes.*

Proof. — We pick a finite set S of places of K containing all the archimedean ones so that $\prod_{v\in S} G_1(K_v)$ and $\prod_{v\in S} G_2(K_v)$ are noncompact. For $i = 1, 2$ let Γ_i be an S-arithmetic subgroup of $G_i(K)$. As G_1 and G_2 have the same K-tori, it immediately follows from the definition of weak commensurability that Γ_1 and Γ_2 are weakly commensurable (cf. Example 6.5). Now, the assertion that either G_1 and G_2 are of same type, or one of them is of type B_n and the other is of type C_n, follows from Theorem 1, and the assertion that $L_1 = L_2$ is a consequence of Theorem 6.3(2).

Let us now assume that $G_1 = G_2 = G$, and fix a $G' \in \mathfrak{F}$. Let Γ and Γ' be S-arithmetic groups of $G(K)$ and $G'(K)$ respectively, where S is chosen so that both $\prod_{v\in S} G(K_v)$ and $\prod_{v\in S} G'(K_v)$ are noncompact. Then Γ and Γ' are weakly commensurable and Zariski-dense in the respective groups. Now the assertions (1), (2) of the theorem follow from Theorems 4 and 6. To prove (3), we observe that if we choose a finite set S containing all the archimedean places of K so that $\prod_{v\in S} G(K_v)$ is noncompact, then by (1), $\prod_{v\in S} G'(K_v)$ is automatically noncompact for all $G' \in \mathfrak{F}$. Thus, the set S as above can be chosen independent of $G' \in \mathfrak{F}$, and then Theorem 5 yields (3). □

Remark **7.6.** — In Section 9 we will show that assertion (2) of the preceding theorem is false in general if G is of type A_n, D_{2n+1}, or E_6. We recall that two simply connected (or adjoint) split K-groups G_1 and G_2 of type B_n and C_n respectively have the same maximal K-tori (Example 6.7). It would be interesting to determine precisely all the pairs G_1 and G_2 of simply connected (or adjoint) groups of type B_n and C_n respectively over a number field K that have the same maximal K-tori. We observe that given such a pair, for any $v \in V_f^K$ we have $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2$ (Theorem 6.2), and at the same time, $\mathrm{rk}_{K_v} G_1 \geqslant n - 1$ and $\mathrm{rk}_{K_v} G_2 \leqslant n/2$ if G_2 is not K_v-split. Clearly, these conditions are incompatible for $n > 2$, which means that G_1 and G_2 split over K_v for all $v \in V_f^K$. In particular, for a given number field K, there are only finitely many pairs G_1 and G_2 with the above properties (cf. the proof of Theorem 5).

8. Lengths of closed geodesics, length-commensurable locally symmetric spaces and Schanuel's conjecture

Let G be a connected semi-simple real algebraic group, $\mathcal{G} = \mathrm{G}(\mathbf{R})$, and let $\mathcal{K}$ be a maximal compact subgroup of $\mathcal{G}$. We let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $\mathcal{G}$ and $\mathcal{K}$ respectively, and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form $\langle\ ,\ \rangle$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The corresponding symmetric space $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ is a Riemannian manifold with the metric induced by the restriction of the Killing form to $\mathfrak{p}$ (see [17] for the details).

8.1. Positive characters. — A character χ of an $\mathbf{R}$-torus T is said to be *positive* if for every $x \in \mathrm{T}(\mathbf{R})$, the value $\chi(x)$ is a positive real number. Any positive character of T is defined over $\mathbf{R}$. Given an arbitrary character $\chi \in \mathrm{X}(\mathrm{T})$, the character $\chi + \bar{\chi}$, where $\bar{\chi}$ is the character obtained by applying the complex conjugation to χ, satisfies

$$(\chi + \bar{\chi})(x) = \chi(x)\overline{\chi(x)} = |\chi(x)|^2$$

for all $x \in \mathrm{T}(\mathbf{R})$. Thus, for any character χ and any $x \in \mathrm{T}(\mathbf{R})$, the square of the absolute value of $\chi(x)$ is the value assumed by the positive character $\chi + \bar{\chi}$ of T at x.

Let S be an $\mathbf{R}$-split torus and T be a $\mathbf{R}$-torus containing S. Then every character of S is defined over $\mathbf{R}$. Given a character α of S, let χ be a complex character of T whose restriction to S equals α. Then the restriction of the positive character $\chi + \overline{\chi}$ to S is 2α. Thus every character lying in the subgroup $2\mathrm{X}(\mathrm{S})$ of the character group $\mathrm{X}(\mathrm{S})$ of S extends to a positive character of any $\mathbf{R}$-torus containing S.

Let $\mathfrak{a}$ be a Cartan subspace contained in $\mathfrak{p}$, and $\mathcal{A} = \exp\mathfrak{a}$ be the connected abelian subgroup of $\mathcal{G}$ with Lie algebra $\mathfrak{a}$. Let S be the Zariski-closure of $\mathcal{A}$. Then S is a maximal $\mathbf{R}$-split torus of G and $\mathcal{A} = \mathrm{S}(\mathbf{R})^{\circ}$. We fix a closed Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$. Let $\{\alpha_1, \ldots, \alpha_r\}$, where $r = \mathrm{rk}_{\mathbf{R}}\,\mathrm{G} = \dim \mathrm{S}$, be the basis of the root system of G, with respect to S, determined the Weyl chamber $\mathfrak{a}^+$, and let $\beta_i = 2\alpha_i$. Then $\beta_1, \ldots, \beta_r$ are linearly independent positive characters. In the sequel, we will identify $\mathfrak{a}$ with $\mathbf{R}^r$ by identifying $\mathrm{X} \in \mathfrak{a}$ with $(d\beta_1(\mathrm{X}), \ldots, d\beta_r(\mathrm{X}))$, where, for $i \in \{1, \ldots, r\}$, $d\beta_i$ denotes the differential of β_i at the identity.

We will now make some brief comments on the Lyapunov map and its relations with weak commensurability, and will then proceed to the core issue of the lengths of closed geodesics and length-commensurable locally symmetric spaces.

8.2. Lyapunov map. — For an element $g \in \mathcal{G}$, we let $g = g_s g_u$ be its Jordan decomposition. For simplicity, we denote the semi-simple component g_s by s. Let T be a maximal $\mathbf{R}$-torus of G containing s. Let $\mathcal{C}$ be the maximal compact subgroup of $T(\mathbf{R})$ and T_s be the maximal $\mathbf{R}$-split subtorus of T. Then $T(\mathbf{R})$ is a direct product of $\mathcal{C}$ and $T_s(\mathbf{R})^\circ$, so we can write $s = s_e \cdot s_h$, with $s_e \in \mathcal{C}$, and $s_h \in T_s(\mathbf{R})^\circ$. The elements s_e and s_h are called the *elliptic* and the *hyperbolic* components of s (or of g). There is an element $z \in \mathcal{G}$ which conjugates $\mathcal{C}$ into $\mathcal{K}$ and $T_s(\mathbf{R})^\circ$ into $\mathcal{A}$ such that $zs_hz^{-1} = \exp X$, with $X \in \mathfrak{a}^+$. The element X is the unique element of $\mathfrak{a}^+$ such that the hyperbolic component s_h of g is a conjugate of $\exp X$, and we will denote it by $\ell(g)$. Thus we get a map (the Lyapunov map) $\ell : \mathcal{G} \to \mathfrak{a}^+$. Clearly, for any $g \in \mathcal{G}$ we have $\ell(g) = \ell(g_s)$, and moreover, for any positive integer n, $\ell(g^n) = n\ell(g)$.

Continuing with the above notations, we let χ_i, for $i \in \{1, \ldots, r\}$, be the unique positive character of T extending the character $\operatorname{Int} z^{-1} \cdot \beta_i|_{T_s}$, and let $d\chi_i$ denote its differential at the identity. Since $\chi_i(s) = \chi_i(s_h)$, we have

$$\ell(s) = (d\chi_1(\operatorname{Ad} z^{-1}(X)), \ldots, d\chi_r(\operatorname{Ad} z^{-1}(X))) = (\log\chi_1(s), \ldots, \log\chi_r(s)).$$

For a subgroup Γ of $\mathcal{G}$, let Γ^{ss} denote the set of semi-simple elements of Γ. From the above description of the Lyapunov map, the following proposition is obvious.

Proposition **8.3.** — *If* Γ_1 *and* Γ_2 *are two discrete subgroups of* $\mathcal{G}$ *such that*

$$\mathbf{Q} \cdot \ell(\Gamma_1^{ss}) = \mathbf{Q} \cdot \ell(\Gamma_2^{ss}),$$

then Γ_1 *and* Γ_2 *are weakly commensurable.*

If Γ is an arithmetic subgroup of $\mathcal{G}$ and $g \in \Gamma$, then there exists an integer $n = n(g)$ such that $g_u^n \in \Gamma$. Then g_s^n lies in Γ. On the other hand, if Γ is an irreducible nonarithmetic lattice of $\mathcal{G}$ (then $\mathcal{G}$ is of $\mathbf{R}$-rank 1), then it can be shown that there exists a positive integer $n = n(\Gamma)$ such that for every non-semi-simple element g of Γ, g^n is unipotent. We conclude that if Γ is lattice (arithmetic or not) of $\mathcal{G}$, then $\mathbf{Q} \cdot \ell(\Gamma) = \mathbf{Q} \cdot \ell(\Gamma^{ss})$.

8.4. Lengths of closed geodesics on locally symmetric spaces. — Given a discrete torsion-free subgroup Γ of $\mathcal{G}$, the quotient $\mathfrak{X}_\Gamma := \mathfrak{X}/\Gamma$ is a Riemannian locally symmetric space. We first need to recall some facts about closed (or periodic) geodesics in $\mathfrak{X}_\Gamma$, and in particular the formula for their length, given in [33]. Closed geodesics in $\mathfrak{X}_\Gamma$ correspond to semi-simple elements in Γ, and are obtained by a construction similar to the one used to define the Lyapunov map. More precisely, let γ be a fixed semi-simple element of Γ, and let T be a maximal $\mathbf{R}$-torus of G containing γ. As we mentioned above, $T(\mathbf{R})$ is a direct product of $\mathcal{C}$ and $T_s(\mathbf{R})^\circ$, where $\mathcal{C}$ is the maximal compact subgroup of $T(\mathbf{R})$ and T_s is the maximal $\mathbf{R}$-split subtorus of T. Take *any* $z \in \mathcal{G}$ such that zTz^{-1} is invariant under the Cartan involution associated with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and consequently

$$z\mathcal{C}z^{-1} \subset \mathcal{K} \quad \text{and} \quad zT_s(\mathbf{R})^\circ z^{-1} \subset \exp \mathfrak{p}. \tag{22}$$

Here we do not require the inclusion $z\mathrm{T}_S(\mathbf{R})^\circ z^{-1} \subset \exp \mathfrak{a}$. Now if we write $\gamma = \gamma_e \cdot \gamma_h$, with $\gamma_e \in \mathcal{C}$ and $\gamma_h \in \mathrm{T}_S(\mathbf{R})^\circ$, then $\gamma_h = z^{-1}\exp(\mathrm{X})z$ for some $\mathrm{X} \in \mathfrak{p}$ that commutes with $z\gamma_e z^{-1}$; moreover, it is obvious that X is a conjugate of $\ell(\gamma)$ under an element of Ad$\mathcal{K}$. With these notations, the curve $\tilde{c}_\gamma$ parametrized by $\widetilde{\varphi}: t \mapsto \mathcal{K}\exp(t\mathrm{X})z$, for $t \in \mathbf{R}$, is a geodesic on X which passes through the point $\mathcal{K}z$. Furthermore,

$$\widetilde{\varphi}(t)\cdot\gamma = \mathcal{K}\exp(t\mathrm{X})\cdot z\gamma_e z^{-1}\cdot z\gamma_h z^{-1}\cdot z = \mathcal{K}\exp(t\mathrm{X})\cdot\exp(\mathrm{X})\cdot z$$
$$= \widetilde{\varphi}(t+1),$$

implying that the map $\varphi: \mathbf{R} \to \mathfrak{X}_\Gamma$, obtained by composing $\widetilde{\varphi}$ with the natural map $\pi: \mathfrak{X} \to \mathfrak{X}_\Gamma$, is periodic with period 1, and hence its smallest period is of the form $1/n_\gamma$ for some integer $n_\gamma \geqslant 1$. It follows that the image c_γ of $\tilde{c}_\gamma$ in $\mathfrak{X}_\Gamma$ is a closed geodesic, and since

$$\langle\varphi'(t), \varphi'(t)\rangle = \langle\widetilde{\varphi}'(t), \widetilde{\varphi}'(t)\rangle = \langle \mathrm{X}, \mathrm{X}\rangle,$$

for all $t \in \mathbf{R}$, we see that the length of c_γ is $(1/n_\gamma)\langle \mathrm{X}, \mathrm{X}\rangle^{1/2}$.

Proposition **8.5.**

(i) *Every closed geodesic in $\mathfrak{X}_\Gamma$ is of the form c_γ for some semi-simple $\gamma \in \Gamma$.*
(ii) *The length of c_γ is $(1/n_\gamma)\lambda_\Gamma(\gamma)$ where n_γ is an integer $\geqslant 1$ and $\lambda_\Gamma(\gamma)$ is given by the following formula:*

$$\lambda_\Gamma(\gamma)^2 = \langle\ell(\gamma), \ell(\gamma)\rangle = \left(\sum(\log|\alpha(\gamma)|)^2\right), \tag{23}$$

where the summation is over all roots of G *with respect to* T *and* log *denotes the natural logarithm.*

Thus,

$$\mathbf{Q}\cdot \mathrm{L}(\mathfrak{X}_\Gamma) = \mathbf{Q}\cdot\{\lambda_\Gamma(\gamma) \mid \gamma \in \Gamma \textit{ semi-simple}\},$$

where $\lambda_\Gamma(\gamma)$ is given by (23).

Proof. — (i) Any closed geodesic c in $\mathfrak{X}_\Gamma$ is obtained as the image under π of a geodesic $\tilde{c}$ in $\mathfrak{X}$. Fix a point $\mathcal{K}z \in \tilde{c}$. It is known that $\tilde{c}$ admits a parametrization of the form

$$\widetilde{\varphi}(t) = \mathcal{K}\exp(t\mathrm{X})z$$

for some $\mathrm{X} \in \mathfrak{p}$ (cf. [17], Theorem 3.3(iii) in Chap. IV). After replacing X by a suitable positive-real multiple, we can assume that $\pi(\widetilde{\varphi}(0)) = \pi(\widetilde{\varphi}(1))$, and $d_{\widetilde{\varphi}(0)}\pi(\widetilde{\varphi}'(0)) = d_{\widetilde{\varphi}(1)}\pi(\widetilde{\varphi}'(1))$. Then, in particular, $\widetilde{\varphi}(1) = \widetilde{\varphi}(0)\gamma$ for some $\gamma \in \Gamma$. Since the map

$$\mathcal{K}\times\mathfrak{p} \to \mathcal{G}, \qquad (\kappa, \mathrm{Y}) \mapsto \kappa\exp(\mathrm{Y}),$$

is a diffeomorphism, the element $z\gamma z^{-1}$ can be uniquely written in the form $z\gamma z^{-1} = \kappa\exp(\mathrm{Y})$. Then $\widetilde{\varphi}(1) = \widetilde{\varphi}(0)\gamma$ yields $\mathrm{X} = \mathrm{Y}$, i.e.,

$$z\gamma z^{-1} = \kappa \exp(\mathrm{X}). \tag{24}$$

Furthermore, the curves in $\mathfrak{X}$ with the parametrizations

$$\widetilde{\varphi}_1(t) = \widetilde{\varphi}(t)\cdot\gamma \quad \text{and} \quad \widetilde{\varphi}_2(t) = \widetilde{\varphi}(t+1)$$

are both geodesics in $\mathfrak{X}$ such that

$$\widetilde{\varphi}_1(0) = \widetilde{\varphi}(0)\cdot\gamma = \widetilde{\varphi}(1) = \widetilde{\varphi}_2(0) =: p.$$

Since $\pi(\widetilde{\varphi}_1(t)) = \pi(\widetilde{\varphi}(t))$, we have

$$d_p\pi(\widetilde{\varphi}_1'(0)) = d_{\widetilde{\varphi}(0)}\pi(\widetilde{\varphi}'(0)) = d_{\widetilde{\varphi}(1)}\pi(\widetilde{\varphi}'(1)) = d_p\pi(\widetilde{\varphi}_2'(0)).$$

Thus, $\widetilde{\varphi}_1'(0) = \widetilde{\varphi}_2'(0)$, hence by the uniqueness of a geodesic through a given point in a given direction, we get $\widetilde{\varphi}_1(t) = \widetilde{\varphi}_2(t)$ for all t. Combining the definitions of $\widetilde{\varphi}$, $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ with (24), we now obtain that

$$\mathcal{K}\exp(t\mathrm{X})\kappa = \mathcal{K}\exp(t(\mathrm{Ad}\,\kappa^{-1}(\mathrm{X}))) = \mathcal{K}\exp(t\mathrm{X}),$$

which implies that κ commutes with $\exp(t\mathrm{X})$ for all t. Since the elements κ and $\exp(\mathrm{X})$ are semi-simple, we conclude that $\gamma = z^{-1}(\kappa\exp(\mathrm{X}))z$ is semi-simple. Moreover, κ and $\exp(\mathrm{X})$ are contained in a maximal $\mathbf{R}$-torus T_0 of G which is invariant under the Cartan involution. Let $\mathrm{T} = z^{-1}\mathrm{T}_0 z$. Then $\mathrm{T}(\mathbf{R}) = z^{-1}\mathrm{T}_0(\mathbf{R})z$ contains γ, and $\gamma_e = z^{-1}\kappa z$ and $\gamma_h = z^{-1}\exp(\mathrm{X})z$ in the notations introduced prior to the statement of the proposition. It is now obvious that c coincides with the geodesic c_γ. As we already explained, its length is $(1/n_\gamma)\langle \mathrm{X},\mathrm{X}\rangle^{1/2}$, where n_γ is the integer $\geqslant 1$ such that $1/n_\gamma$ is the smallest positive period of $\varphi(t) = \pi(\tilde{\varphi}(t))$.

(ii) We need to show that $\lambda_\Gamma(\gamma) := \langle \mathrm{X},\mathrm{X}\rangle^{1/2}$ $(= \langle \ell(\gamma),\ell(\gamma)\rangle^{1/2})$ is given by the equation (23). Since the Killing form is invariant under the adjoint action of $\mathcal{G}$ on $\mathfrak{g}$, we have $\langle \mathrm{X},\mathrm{X}\rangle = \langle \mathrm{X}',\mathrm{X}'\rangle$, where $\mathrm{X}' = \mathrm{Ad}z^{-1}(\mathrm{X})$ so that $\gamma_h = \exp(\mathrm{X}')$. In a suitable basis of $\mathfrak{g}$, $\mathrm{Ad}\,\gamma_h$ is represented by a diagonal matrix whose diagonal entries are 1 (repeated $\dim\mathrm{T}$ times) and $\alpha(\gamma_h)$ for all $\alpha\in\Phi(\mathrm{G},\mathrm{T})$; notice that all these numbers are real and positive. In the same basis, $\mathrm{ad}\,\mathrm{X}'$ is represented by a diagonal matrix with the diagonal entries 0 (repeated $\dim\mathrm{T}$ times) and $d\alpha(\mathrm{X}')$ for all $\alpha\in\Phi(\mathrm{G},\mathrm{T})$. For every α we clearly have

$$|\alpha(\gamma)| = |\alpha(\gamma_h)| = \exp(d\alpha(\mathrm{X}')).$$

So,

$$\langle \mathrm{X},\mathrm{X}\rangle = \langle \mathrm{X}',\mathrm{X}'\rangle = \sum_{\alpha\in\Phi(\mathrm{G},\mathrm{T})} (d\alpha(\mathrm{X}'))^2 = \sum_{\alpha\in\Phi(\mathrm{G},\mathrm{T})} (\log|\alpha(\gamma)|)^2,$$

and (23) follows. □

In order to relate the notion of length-commensurability with that of weak commensurability, we need to recast formula (23) in a slightly different form. As a root α of G with respect to T is a character of T, $|\alpha(\gamma)|^2$ is the value assumed by a positive character of T, and therefore,

$$\lambda_\Gamma(\gamma)^2 = \sum_{i=1}^{p} s_i(\log \chi_i(\gamma))^2, \tag{25}$$

where $\chi_1, \dots, \chi_p$ are certain positive characters of T and $s_1, \dots, s_p$ are positive rational numbers (whose denominators are divisors of 4).

We will now elaborate on (25) in the rank one case.

Lemma **8.6.** — *Assume that* $\mathrm{rk}_{\mathbf{R}}\, \mathrm{G} = 1$, *and let* Γ *be a discrete torsion-free subgroup of* $\mathcal{G} = \mathrm{G}(\mathbf{R})$. *Let* $\gamma \in \Gamma$ *be a semi-simple element* $\neq 1$, *and let* T *be a maximal* **R**-*torus containing it. Then*

1. $\mathrm{rk}_{\mathbf{R}}\, \mathrm{T} = 1$, *so the group of positive characters of* T *is cyclic with a generator, say,* χ; *and* $\chi(\gamma) \neq 1$.
2. *There exists a positive rational number t (whose denominator is a divisor of* 4*), t depending only on* G, *but not on* γ, Γ *or* T, *such that*

$$\lambda_\Gamma(\gamma) = \sqrt{t}\,|\log \chi(\gamma)|.$$

Proof. — (1): $\mathrm{rk}_{\mathbf{R}}\, \mathrm{T} = 0$ would imply that $\mathrm{T}(\mathbf{R})$ is compact, so the discreteness of $\langle\gamma\rangle$ would imply its finiteness. Since Γ is torsion-free, we would get $\gamma = 1$, a contradiction. As $(\mathrm{Ker}\,\chi)(\mathbf{R})$ is compact, we conclude that $\chi(\gamma) \neq 1$.

(2): The second assertion follows from (23) and (25) combined with the fact that any two maximal **R**-tori of G of real rank one are conjugate under an element of $\mathcal{G}$. □

Corollary **8.7.** — *Assume that* $\mathrm{rk}_{\mathbf{R}}\, \mathrm{G} = 1$. *Let* $\mathrm{K} \subset \mathbf{R}$ *be a number field, and let* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *be two* K-*forms of* $\overline{\mathrm{G}}$. *For* $i = 1, 2$, *let* Γ_i *be a discrete torsion-free* $(\mathscr{G}_i, \mathrm{K})$-*arithmetic subgroup of* $\mathcal{G}$. *If* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *have the same* K-*isomorphism classes of maximal* K-*tori, then*

$$\mathbf{Q} \cdot \lambda_{\Gamma_1}(\Gamma_1^{\mathrm{ss}}) = \mathbf{Q} \cdot \lambda_{\Gamma_2}(\Gamma_2^{\mathrm{ss}}), \tag{26}$$

and consequently, $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable.*

Proof. — We will assume (as we may) that $\mathrm{G} = \overline{\mathrm{G}}$, and then $\Gamma_i \subset \mathscr{G}_i(\mathrm{K})$, for $i = 1, 2$. So, any given $\gamma_1 \in \Gamma_1^{\mathrm{ss}} \setminus \{1\}$ is contained in a suitable maximal K-torus T_1 of $\mathscr{G}_1$. By our assumption, there exists a K-isomorphism $\varphi\colon \mathrm{T}_1 \to \mathrm{T}_2$ with a maximal K-torus T_2 of $\mathscr{G}_2$. Since Γ_2 is arithmetic, there exists an integer $m > 0$ such that $\gamma_2 := \varphi(\gamma_1)^m$ belongs to Γ_2.

Now, if $\chi^{(2)}$ is a generator of the group of positive characters of T_2, then $\chi^{(1)} = \varphi^*(\chi^{(2)})$ is a generator of the group of positive characters of T_1. We obviously have $\chi^{(1)}(\gamma_1)^m = \chi^{(2)}(\gamma_2)$, so it follows from Lemma 8.6(2) that

$$m \cdot \lambda_{\Gamma_1}(\gamma_1) = \lambda_{\Gamma_2}(\gamma_2),$$

yielding the inclusion

$$\mathbf{Q} \cdot \lambda_{\Gamma_1}(\Gamma_1^{ss}) \subset \mathbf{Q} \cdot \lambda_{\Gamma_2}(\Gamma_2^{ss}).$$

By symmetry, we get (26). The last assertion follows from (26) and Proposition 8.5. □

To deal with the higher rank case, we need the following.

Lemma **8.8.** — *Let* G_1 *and* G_2 *be two connected semi-simple real algebraic groups. For* $i = 1, 2$, *let* T_i *be a maximal* $\mathbf{R}$*-torus of* G_i, *and* $\gamma_i \in T_i(\mathbf{R})$. *Given two collections of characters* $\chi_1^{(1)}, \ldots, \chi_{d_1}^{(1)} \in X(T_1)$ *and* $\chi_1^{(2)}, \ldots, \chi_{d_2}^{(2)} \in X(T_2)$, *we set*

$$S_i = \{\log|\chi_1^{(i)}(\gamma_i)|, \ldots, \log|\chi_{d_i}^{(i)}(\gamma_i)|\}.$$

If γ_1, γ_2 *are not weakly commensurable and each of the sets (of real numbers)* S_1 *and* S_2 *is linearly independent over* $\mathbf{Q}$, *then so is their union* $S_1 \cup S_2$.

Proof. — According to the above discussion, there exist positive characters $\theta_1^{(1)}, \ldots, \theta_{d_1}^{(1)} \in X(T_1)$ and $\theta_1^{(2)}, \ldots, \theta_{d_2}^{(2)} \in X(T_2)$ such that

$$\theta_j^{(i)}(x) = |\chi_j^{(i)}(x)|^2 \quad \text{for all } x \in T_i(\mathbf{R}).$$

If the set $S_1 \cup S_2$ is linearly dependent over $\mathbf{Q}$, there exist integers $s_1, \ldots, s_{d_1}, t_1, \ldots, t_{d_2}$, not all zero, such that

$$\begin{aligned} &s_1 \log\theta_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log\theta_{d_1}^{(1)}(\gamma_1) \\ &\quad + t_1 \log\theta_1^{(2)}(\gamma_2) + \cdots + t_{d_2} \log\theta_{d_2}^{(2)}(\gamma_2) = 0. \end{aligned}$$

Consider the characters

$$\begin{aligned} &\psi_1 = s_1\theta_1^{(1)} + \cdots + s_{d_1}\theta_{d_1}^{(1)} \text{ of } T_1 \quad \text{and} \\ &\psi_2 = -(t_1\theta_1^{(2)} + \cdots + t_{d_2}\theta_{d_2}^{(2)}) \text{ of } T_2. \end{aligned}$$

Then $\psi_1(\gamma_1) = \psi_2(\gamma_2)$, and hence,

$$\psi_1(\gamma_1) = 1 = \psi_2(\gamma_2)$$

since γ_1 and γ_2 are *not* weakly commensurable. This means that

$$s_1 \log \theta_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \theta_{d_1}^{(1)}(\gamma_1)$$
$$= 0 = t_1 \log \theta_1^{(2)}(\gamma_2) + \cdots + t_{d_2} \log \theta_{d_2}^{(2)}(\gamma_2),$$

and therefore all the coefficients are zero because the sets S_1 and S_2 are linearly independent. □

Some of our results depend on the validity of Schanuel's conjecture in transcendental number theory (cf. [1]), and we recall here its statement.

8.9. Schanuel's conjecture. — *If $z_1, \ldots, z_n \in \mathbf{C}$ are linearly independent over $\mathbf{Q}$, then the transcendence degree (over $\mathbf{Q}$) of the field generated by*

$$z_1, \ldots, z_n; \qquad e^{z_1}, \ldots, e^{z_n}$$

is $\geqslant n$.

We will only use the fact that the truth of this conjecture implies that for algebraic numbers $z_1, \ldots, z_n$, (any values of) their logarithms

$$\log z_1, \ldots, \log z_n$$

are algebraically independent once they are linearly independent (over $\mathbf{Q}$).

8.10. *Notation.* — For $i = 1, 2$, let G_i be a connected semi-simple real algebraic subgroup of SL_n, $\mathcal{G}_i = G_i(\mathbf{R})$, and Γ_i be a Zariski-dense discrete torsion-free subgroup of $\mathcal{G}_i$. Let $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}_i/\Gamma_i$, where $\mathfrak{X}_i$ is the symmetric space of $\mathcal{G}_i$. Let K_{Γ_i} be the subfield of $\mathbf{R}$ generated by the traces $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ for $\gamma \in \Gamma_i$.

Proposition **8.11.** — *Suppose that the nontrivial semi-simple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are not weakly commensurable.*

(i) *If both G_1 and G_2 are of real rank 1 and there exists a number field $K \subset \mathbf{R}$ such that Γ_1 and Γ_2 can be conjugated into $SL_n(K)$, then $\theta = \lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)$ is transcendental over $\mathbf{Q}$; if $G_1 = G_2$, then θ is irrational for arbitrary Γ_1 and Γ_2.*
(ii) *If there exists a number field $K \subset \mathbf{R}$ such that Γ_1 and Γ_2 can be conjugated into $SL_n(K)$, and Schanuel's conjecture holds, then $\lambda_{\Gamma_1}(\gamma_1)$ and $\lambda_{\Gamma_2}(\gamma_2)$ are algebraically independent over $\mathbf{Q}$.*

Proof. — We fix a maximal $\mathbf{R}$-torus T_i of G_i such that $\gamma_i \in T_i(\mathbf{R})$.

(i) Using Lemma 8.6(1), we can pick a generator $\chi^{(i)}$ of the group of positive characters of T_i so that $\chi^{(i)}(\gamma_i) > 1$ for $i = 1, 2$. Then by Lemma 8.6(2) we have

$$\lambda_{\Gamma_i}(\gamma_i) = \sqrt{t_i} \log \chi^{(i)}(\gamma_i).$$

Since the elements γ_1 and γ_2 are *not* weakly commensurable, for any nonzero integers a, b, we have

$$\chi^{(1)}(\gamma_1)^a \neq \chi^{(2)}(\gamma_2)^b,$$

hence the ratio $\log\chi^{(1)}(\gamma_1)/\log\chi^{(2)}(\gamma_2)$ is irrational. If there exists a number field $K \subset \mathbf{R}$ such that Γ_1 and Γ_2 can be conjugated into $SL_n(K)$, then the numbers $\chi^{(i)}(\gamma_i)$ are algebraic, and therefore by a theorem proved independently by Gel'fond and Schneider in 1934 (cf. [2]), $\log\chi^{(1)}(\gamma_1)/\log\chi^{(2)}(\gamma_2)$, and so also θ, is transcendental over $\mathbf{Q}$. If $G_1 = G_2$, then $t_1 = t_2$, and hence $\theta = \log\chi^{(1)}(\gamma_1)/\log\chi^{(2)}(\gamma_2)$ is irrational for arbitrary Γ_1 and Γ_2.

(ii) According to (25), we have the following expressions

$$\lambda_{\Gamma_1}(\gamma_1)^2 = \sum_{i=1}^{p} s_i^{(1)}(\log\chi_i^{(1)}(\gamma_1))^2 \quad \text{and}$$

$$\lambda_{\Gamma_2}(\gamma_2)^2 = \sum_{i=1}^{p} s_i^{(2)}(\log\chi_i^{(2)}(\gamma_2))^2.$$

After renumbering the characters, we can assume that

$$a_1 := \log\chi_1^{(1)}(\gamma_1), \qquad \ldots, \qquad a_{m_1} := \log\chi_{m_1}^{(1)}(\gamma_1)$$
$$(\text{resp., } b_1 := \log\chi_1^{(2)}(\gamma_2), \ldots, b_{m_2} = \log\chi_{m_2}^{(2)}(\gamma_2))$$

for some $m_1, m_2 \leqslant p$, form a basis of the $\mathbf{Q}$-subspace of $\mathbf{R}$ spanned by $\log\chi_i^{(1)}(\gamma_1)$ (resp., $\log\chi_i^{(2)}(\gamma_2)$) for $i \leqslant p$ (notice that $m_1, m_2 \geqslant 1$ as otherwise the length of the corresponding geodesic would be zero, which is impossible). It follows from Lemma 8.8 that the set of numbers

$$\{a_1, \ldots, a_{m_1};\ b_1, \ldots, b_{m_2}\}$$

is linearly independent over $\mathbf{Q}$. Since by our assumption the subgroups Γ_1 and Γ_2 can be conjugated into $SL_n(K)$, the values $\chi_i^{(j)}(\gamma_j)$ are algebraic numbers, so it follows from Schanuel's conjecture that $a_1, \ldots, a_{m_1}; b_1, \ldots, b_{m_2}$ are algebraically independent over $\mathbf{Q}$. It remains to observe that $\lambda_{\Gamma_1}(\gamma_1)^2$ and $\lambda_{\Gamma_2}(\gamma_2)^2$ are given by nonzero homogeneous polynomials of degree two, with rational coefficients, in $a_1, \ldots, a_{m_1}$ and $b_1, \ldots, b_{m_2}$, respectively, and therefore they are algebraically independent. □

By combining Propositions 8.5 and 8.11 we obtain the following:

Theorem **8.12.** — *If Γ_1 and Γ_2 are not weakly commensurable, then, possibly after interchanging them, the following assertions hold.*

(i) *If* G_1 *and* G_2 *are of real rank* 1, *and either there exists a number field* $K \subset \mathbf{R}$ *such that both* Γ_1 *and* Γ_2 *can be conjugated into* $SL_n(K)$, *or* $G_1 = G_2$, *then there exists* $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ *such that for* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$, *the ratio* λ_1/λ_2 *is irrational.*

(ii) *If there exists a number field* $K \subset \mathbf{R}$ *such that both* Γ_1 *and* Γ_2 *can be conjugated into* $SL_n(K)$, *and Schanuel's conjecture holds, then there exists* $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ *which is algebraically independent from* any $\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$.

In either case, (under the above assumptions) $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are not length-commensurable.*

Remark **8.13.** — If G is a connected semi-simple real algebraic subgroup of SL_n of adjoint type such that $\mathcal{G} = G(\mathbf{R})$ does not contain any nontrivial compact normal subgroups, and it is not locally isomorphic to either $SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$, and Γ is an irreducible lattice in $\mathcal{G}$, then there exists a number field $K \subset \mathbf{R}$ such that Γ can be conjugated into $SL_n(K)$, cf. [36], 7.67 and 7.68.

The results in the rest of this section, and those of Sect. 10 (except Theorem 10.1), for locally symmetric spaces of rank > 1 *assume the truth of Schanuel's conjecture.*

Henceforth, we will assume that G_1 and G_2 are connected and absolutely simple. We will refer to the following situation as the *exceptional case:*

($\mathcal{E}$) One of the locally symmetric spaces, say, $\mathfrak{X}_{\Gamma_1}$, is 2-dimensional and the corresponding discrete subgroup Γ_1 cannot be conjugated into $PGL_2(K)$, for any number field $K \subset \mathbf{R}$, and the other space, $\mathfrak{X}_{\Gamma_2}$, has dimension > 2.

(In the exceptional case, $G_1 = PGL_2$, while G_2 is not, and Γ_1 cannot be conjugated into $PGL_2(K)$, for any number field $K \subset \mathbf{R}$.)

The following is an immediate consequence of Theorem 8.12 and Remark 8.13.

Corollary **8.14.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_i}$ *be a locally symmetric space of finite volume, of* $\mathcal{G}_i = G_i(\mathbf{R})$, *for* $i = 1, 2$. *Assume that we are not in the exceptional case* ($\mathcal{E}$). *If* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable, then* Γ_1 *and* Γ_2 *are weakly commensurable.*

Now Theorems 1, 2 and 7 immediately imply the following.

Theorem **8.15.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_i}$ *be a locally symmetric space of finite volume, of* $\mathcal{G}_i$, *for* $i = 1, 2$. *Assume that* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable, and we are not in the exceptional case* ($\mathcal{E}$). *Then* (i) *either* G_1 *and* G_2 *are of same Killing-Cartan type, or one of them is of type* B_n *and the other is of type* C_n, (ii) $K_{\Gamma_1} = K_{\Gamma_2}$.

We will now focus on *arithmetically defined* locally symmetric spaces (where the exceptional case does not occur). First, we note that if G_i, for $i = 1, 2$, is an absolutely almost simple algebraic group, and Γ_i is a torsion-free Zariski-dense discrete subgroup of $\mathcal{G}_i$, then the fact that Γ_1 and Γ_2 are commensurable up to an $\mathbf{R}$-isomorphism between G_1 and G_2 implies that the locally symmetric space $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable. Combining this with Corollary 8.14 and applying Theorems 4 and 5, we obtain the following.

Theorem **8.16.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathcal{G}_i = G_i(\mathbf{R})$, *for* $i = 1, 2$. *Then the set of arithmetically defined locally symmetric spaces* $\mathfrak{X}_{\Gamma_2}$ *of* $\mathcal{G}_2$, *which are length-commensurable to a given arithmetically defined locally symmetric space* $\mathfrak{X}_{\Gamma_1}$ *of* $\mathcal{G}_1$, *is a union of finitely many commensurability classes.*[12] *It in fact consists of a single commensurability class if* G_1 *and* G_2 *have the same type different from* A_n, D_{2n+1}, *with* $n > 1$, D_4 *and* E_6.

To see what this theorem means for hyperbolic spaces, we recall that the even-dimensional real hyperbolic space $\mathbf{H}^{2n}$ is the symmetric space of a group of type B_n, the odd-dimensional real hyperbolic space $\mathbf{H}^{2n-1}$—of a group of type D_n, the complex hyperbolic space $\mathbf{H}^n_{\mathbf{C}}$—of a group of type A_n, and the quaternionic hyperbolic space $\mathbf{H}^n_{\mathbf{H}}$—of a group of type C_{n+1}. All these spaces are of rank one. Using Theorem 4 and Proposition 8.11(i), we obtain the following result.

Corollary **8.17.** — *Let* M *be either the real hyperbolic space* $\mathbf{H}^{2n}$, *or* $\mathbf{H}^{4n+7}$, *or the quaternionic hyperbolic space* $\mathbf{H}^n_{\mathbf{H}}$, *for any* $n \geqslant 1$, *and let* M_1 *and* M_2 *be two arithmetic quotients of* M. *If* M_1 *and* M_2 *are not commensurable, then after a possible interchange of* M_1 *and* M_2, *there exists* $\lambda_1 \in L(M_1)$ *such that for any* $\lambda_2 \in L(M_2)$, *the ratio* λ_1/λ_2 *is transcendental over* $\mathbf{Q}$.

Remark **8.18.** — The construction described in 6.6 yields two nonisomorphic anisotropic $\mathbf{Q}$-forms $\mathscr{G}_1$ and $\mathscr{G}_2$ of the adjoint group $\overline{G}$ of the $\mathbf{R}$-group $G = SL_{2,\mathbf{H}}$, that have the same set of $\mathbf{Q}$-isomorphism classes of maximal $\mathbf{Q}$-tori. For $i = 1, 2$, fix a torsion-free $(\mathscr{G}_i, \mathbf{Q})$-arithmetic subgroup Γ_i of $\mathcal{G}$. Since $G \simeq \mathrm{Spin}(q)$, where q is a real quadratic form of signature $(5, 1)$, the corresponding symmetric space $\mathfrak{X}$ is $\mathbf{H}^5$. Then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, but not noncommensurable, compact hyperbolic 5-manifolds: the former follows from Corollary 8.7, and the latter from the fact that $\mathscr{G}_1$ and $\mathscr{G}_2$ are not isomorphic. A suitable modification of Example 6.6 enables one to construct examples of noncommensurable length-commensurable complex hyperbolic manifolds of any even dimension. These examples will be subsumed by general constructions in Section 9, which in particular, allow one to construct examples of this nature for real hyperbolic manifolds of any dimension of the form $4n + 1$, and for complex hyperbolic manifolds of any dimension, cf. 9.15.

---

[12] We note that as follows from Theorem 8.15(i), there are only finitely many possibilities for a real absolutely simple algebraic group G_2 that admits a locally symmetric space $\mathfrak{X}_{\Gamma_2}$ which is length-commensurable to a given $\mathfrak{X}_{\Gamma_1}$. So, in effect we obtain that there are only finitely many commensurability classes among all arithmetically defined locally symmetric spaces of simple real algebraic groups length-commensurable to a given space.

We now recall that given a discrete $(\mathscr{G}_i, K_i)$-arithmetic subgroup Γ_i of $\mathcal{G}_i$, the compactness of the quotient $\mathcal{G}_i/\Gamma_i$, and hence of the locally symmetric subspace $\mathfrak{X}_{\Gamma_i}$, is equivalent to $\mathscr{G}_i$ being K_i-anisotropic (cf. [26], Theorem 4.17). Combining this with Theorems 6 and 7, we obtain the following.

Theorem **8.19.** — *Let* G_1 *and* G_2 *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be length-commensurable locally symmetric spaces of* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined and that we are not in the exceptional case* $(\mathcal{E})$. *Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

Question **8.20.** — Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two arbitrary (i.e., not necessarily arithmetically defined) length-commensurable locally symmetric space of $\mathcal{G}_1$ and $\mathcal{G}_2$ as in the theorem, of finite volume. Does the compactness of one of the spaces always imply the compactness of the other? In other words, does Theorem 8.19 remain valid without any assumptions of arithmeticity? We recall that the non-compactness of $\mathfrak{X}_\Gamma$ is equivalent to the existence of nontrivial unipotent elements in Γ for *any* lattice Γ (cf. [36], 11.13 and 11.14). So, the above question can be reformulated as follows: Let Γ_i be a lattice in $\mathcal{G}_i$ for $i = 1, 2$, and assume that Γ_1 and Γ_2 are weakly commensurable. Does the existence of nontrivial unipotent elements in Γ_1 imply their existence in Γ_2? Theorems 6 and 7 provide an affirmative answer if at least one of the lattices is arithmetic. We note that the latter question is meaningful for arbitrary (finitely generated) Zariski-dense subgroups (which are not necessarily discrete or of finite covolume).

9. Construction of nonisomorphic groups with the same tori and noncommensurable length-commensurable locally symmetric spaces of type A_n, D_{2n+1} and E_6.

According to Theorem 7.5, if K is a number field and G_1 and G_2 are two K-forms of a connected absolutely simple group of type different from A_n, D_{2n+1} $(n > 1)$, D_4 and E_6, then the fact that every maximal K-torus T_1 of G_1 is K-isomorphic to some maximal K-torus T_2 of G_2, and vice versa, implies that G_1 and G_2 are K-isomorphic. The goal of this section is to describe a general construction of nonisomorphic K-forms of each of the types A_n, D_{2n+1}, where $n > 1$, and E_6, which have the "same" systems of maximal K-tori in a very strong sense (see below for the definition of groups with coherently equivalent systems of maximal K-tori). Furthermore, we show that arithmetic subgroups of the forms we construct lead to noncommensurable length-commensurable locally symmetric spaces, cf. Proposition 9.14. To avoid the excessive use of script letters, in the major part of this section (through 9.14), various forms of a given group over number fields will be denoted by ordinary (italic) letters, while we return to our standard notations in 9.14–9.16.

We begin by recalling the well-known Galois-cohomological parametrization of the conjugacy classes of maximal K-tori of a given group. Let G be a connected semi-simple simply connected algebraic group over a number field K. Fix a maximal K-torus T of G, and let $N = N_G(T)$ and $W = N/T$ denote respectively its normalizer and the corresponding Weyl group. For any field extension $\mathscr{K}/K$, we let $\theta_{\mathscr{K}}\colon H^1(\mathscr{K}, N) \to H^1(\mathscr{K}, W)$ denote the map induced by the natural homomorphism $N \to W$, and let

$$\mathscr{C}_{\mathscr{K}} := \mathrm{Ker}(H^1(\mathscr{K}, N) \longrightarrow H^1(\mathscr{K}, G)).$$

The maximal $\mathscr{K}$-tori of G correspond bijectively to the $\mathscr{K}$-rational points of the variety $\mathscr{T} = G/N$ of maximal tori of G. Furthermore, G acts on $\mathscr{T}$ by left multiplication (which corresponds to the conjugation action of G on the set of maximal tori), and the elements of the orbit set $G(\mathscr{K})\backslash\mathscr{T}(\mathscr{K})$ are in one-to-one correspondence with the $G(\mathscr{K})$-conjugacy classes of maximal $\mathscr{K}$-tori of G. The following is well-known.

Lemma **9.1.** — *There is a natural bijection $\delta_{\mathscr{K}} : \mathscr{C}_{\mathscr{K}} \to G(\mathscr{K})\backslash\mathscr{T}(\mathscr{K})$.*

We just recall the construction of $\delta_{\mathscr{K}}$. If $n : \sigma \mapsto n_\sigma$, $\sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$, is a $N(\overline{\mathscr{K}})$-valued Galois-cocycle representing an element of $\mathscr{C}_{\mathscr{K}}$, then there exists $g \in G(\overline{\mathscr{K}})$ such that $n_\sigma = g^{-1}\sigma(g)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$. Then the torus $T' = gTg^{-1}$ is defined over $\mathscr{K}$, and $\delta_{\mathscr{K}}$ carries the cohomology class of n to the $G(\mathscr{K})$-conjugacy class of T'.

We now establish a local-global principle pertaining to the description of maximal K-tori of G. To formulate it, we observe that there is an obvious map $W \longrightarrow \mathrm{Aut}\,T$, so for any $x \in H^1(\mathscr{K}, W)$, one can consider the corresponding twisted $\mathscr{K}$-torus ${}_xT$.

Theorem **9.2.** — *Fix $x \in H^1(K, W)$ and suppose that*

(i) *$x \in \theta_{K_v}(\mathscr{C}_{K_v})$ for all $v \in V^K$;*
(ii) *$Ш^2({}_xT) := \mathrm{Ker}(H^2(K, {}_xT) \longrightarrow \prod_{v\in V^K} H^2(K_v, {}_xT))$ is trivial (which holds if, for example, there exists $v_0 \in V^K$ such that ${}_xT$ is K_{v_0}-anisotropic, cf. [26], Proposition 6.12).*

Then $x \in \theta_K(\mathscr{C}_K)$.

Proof. — Applying the constructions from [39], Chap. I, §5.6, to the exact sequence

$$1 \to T \longrightarrow N \longrightarrow W \to 1,$$

we see that to any field extension $\mathscr{K}/K$, one can associate a natural cohomology class $\Delta_{\mathscr{K}}(x) \in H^2(\mathscr{K}, {}_xT)$ such that $x \in \theta_{\mathscr{K}}(H^1(\mathscr{K}, N))$ if and only if $\Delta_{\mathscr{K}}(x)$ is trivial. It follows from (i) that $\Delta_K(x) \in Ш^2({}_xT)$, which is trivial by (ii). Thus, $x = \theta_K(y)$ for some $y \in H^1(K, N)$. Furthermore, according to *loc.cit.*, §5.5, for any $\mathscr{K}/K$ there is a natural surjective map $\nu_{\mathscr{K}}\colon H^1(\mathscr{K}, {}_xT) \to \theta_{\mathscr{K}}^{-1}(x)$. For each $v \in V^K_\infty$, by (i), we can find $z_v \in \mathscr{C}_{K_v}$ such that $\theta_{K_v}(z_v) = x$, and then pick $t_v \in H^1(K_v, {}_xT)$ for which $\nu_{K_v}(t_v) = z_v$. By [26], Proposition 6.17, the diagonal map $H^1(K, {}_xT) \longrightarrow \prod_{v\in V^K_\infty} H^1(K_v, {}_xT)$ is surjective, so there is $t \in H^1(K, {}_xT)$ that maps to $(t_v)_{v \in V_\infty^K}$. Set $z = \nu_K(t)$. Then z maps onto $(z_v)_{v \in V_\infty^K}$ under the diagonal map $H^1(K, N) \longrightarrow \prod_{v \in V_\infty^K} H^1(K_v, N)$. Combining the fact that $z_v \in \mathscr{C}_{K_v}$ with the injectivity of the map $H^1(K, G) \longrightarrow \prod_{v \in V_\infty^K} H^1(K_v, G)$ ([26], Theorem 6.6), we obtain that $z \in \mathscr{C}_K$. Thus, $x = \theta_K(z) \in \theta_K(\mathscr{C}_K)$, as required. □

We now turn to the comparison of the sets of maximal K-tori of two absolutely simple simply connected K-groups G_1 and G_2. We assume that there exist maximal K-tori T_1^0 of G_1 and T_2^0 of G_2, and a $\overline{K}$-isomorphism $\varphi_0 : G_1 \to G_2$ whose restriction to T_1^0 is an isomorphism onto T_2^0 defined over K, and we fix these T_1^0, T_2^0 and φ_0 for the rest of the section. Clearly, φ_0 induces an isomorphism between $N_1 = N_{G_1}(T_1^0)$ and $N_2 = N_{G_2}(T_2^0)$, and hence an isomorphism φ_0^W between the Weyl groups $W_1 = N_1/T_1^0$ and $W_2 = N_2/T_2^0$.

Lemma **9.3.** — *The map* $\varphi_0^W : W_1 \to W_2$ *is defined over* K.

Proof. — Since $\varphi_0 | T_1^0$ is defined over K, for any $n \in N_1(\overline{K})$, $t \in T_1^0(\overline{K})$ and any $\sigma \in \mathrm{Gal}(\overline{K}/K)$, we have

$$\varphi_0(\sigma(ntn^{-1})) = \sigma(\varphi_0(ntn^{-1})),$$

which implies that

$$\varphi_0(\sigma(n))\varphi_0(\sigma(t))\varphi_0(\sigma(n))^{-1} = \sigma(\varphi_0(n))\sigma(\varphi_0(t))\sigma(\varphi_0(n))^{-1}.$$

Since $\varphi_0(\sigma(t)) = \sigma(\varphi_0(t))$, we conclude that $\sigma(\varphi_0(n)) \equiv \varphi_0(\sigma(n))$ modulo $T_2^0(\overline{K})$. This means that φ_0^W commutes with every $\sigma \in \mathrm{Gal}(\overline{K}/K)$, hence it is defined over K. □

Lemma 9.3 enables us to define, for any field extension $\mathscr{K}/K$, the induced isomorphism $H^1(\mathscr{K}, W_1) \to H^1(\mathscr{K}, W_2)$, which will also be denoted by φ_0^W. This isomorphism will play a critical role in comparing the maximal K-tori of G_1 and G_2. More precisely, for $i = 1, 2$, we let $\theta_{\mathscr{K}}^{(i)} : H^1(\mathscr{K}, N_i) \to H^1(\mathscr{K}, W_i)$ be the map induced by the canonical homomorphism $N_i \to W_i$. Furthermore, let $\mathscr{C}_{\mathscr{K}}^{(i)} = \mathrm{Ker}(H^1(\mathscr{K}, N_i) \to H^1(\mathscr{K}, G_i))$, and let $\delta_{\mathscr{K}}^{(i)} : \mathscr{C}_{\mathscr{K}}^{(i)} \to G_i(\mathscr{K}) \backslash \mathscr{T}_i(\mathscr{K})$ (where $\mathscr{T}_i$ is the variety of maximal tori of G_i) be the bijection provided by Lemma 9.1. Then the condition that G_1 and G_2 have the "same" maximal K-tori is basically equivalent to the following

$$\varphi_0^W(\theta_K^{(1)}(\mathscr{C}_K^{(1)})) = \theta_K^{(2)}(\mathscr{C}_K^{(2)}). \tag{27}$$

To give a precise interpretation of (27), we need to introduce the following definition.

Definition **9.4.** — *Let* $\mathscr{K}$ *be a field extension of* K *and let* T_1 *be a maximal* $\overline{\mathscr{K}}$*-torus of* G_1*. A* $\mathscr{K}$*-embedding* $\iota : T_1 \to G_2$ *will be called coherent (relative to* φ_0*) if there exists a* $\overline{\mathscr{K}}$*-isomorphism* $\varphi : G_1 \to G_2$ *of the form* $\varphi = \mathrm{Int}\, h \circ \varphi_0$*, with* $h \in G_2(\overline{\mathscr{K}})$*, such that* $\iota = \varphi | T_1$*. Furthermore, we say that* G_1 *and* G_2 *have coherently equivalent systems of maximal* $\mathscr{K}$*-tori if every maximal* $\mathscr{K}$*-torus* T_1 *of* G_1 *admits a coherent (relative to* φ_0*)* $\mathscr{K}$*-embedding into* G_2*, and every maximal* $\mathscr{K}$*-torus* T_2 *of* G_2 *admits a coherent (relative to* φ_0^{-1}*)* $\mathscr{K}$*-embedding into* G_1*.*

Lemma **9.5.** — *Let* T_1 *be a maximal* $\mathscr{K}$*-torus of* G_1*, and let* $x_1 \in \mathscr{C}^{(1)}_{\mathscr{K}}$ *be the cohomology class that corresponds to* T_1 *under* $\delta^{(1)}_{\mathscr{K}}$*. Then* T_1 *admits a coherent (relative to* φ_0*)* $\mathscr{K}$*-embedding into* G_2 *if and only if* $\varphi_0^W(\theta^{(1)}_{\mathscr{K}}(x_1)) \in \theta^{(2)}_{\mathscr{K}}(\mathscr{C}^{(2)}_{\mathscr{K}})$*. Thus, (27) is equivalent to the condition that* G_1 *and* G_2 *have coherently equivalent systems of maximal* K*-tori.*

Proof. — Pick $g_1 \in G_1(\overline{\mathscr{K}})$ so that $T_1 = g_1 T_1^0 g_1^{-1}$. Then x_1 is represented by the $N_1(\overline{\mathscr{K}})$-valued Galois-cocycle $\sigma \mapsto \alpha_\sigma := g_1^{-1}\sigma(g_1)$, $\sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$, and therefore, $\varphi_0^W(\theta^{(1)}_{\mathscr{K}}(x_1))$ is represented by the cocycle

$$\sigma \mapsto \beta_\sigma := \varphi_0(g_1^{-1}\sigma(g_1))T_2^0 \in W_2. \tag{28}$$

Let $\varphi\colon G_1 \to G_2$ be an isomorphism of the form $\varphi = \mathrm{Int}\, h \circ \varphi_0$, where $h \in G_2(\overline{\mathscr{K}})$. Then $T_2 := \varphi(T_1)$ can be written in the form $T_2 = g_2 T_2^0 g_2^{-1}$, where $g_2 = h\varphi_0(g_1)$. So, T_2 is defined over $\mathscr{K}$ if and only if $g_2^{-1}\sigma(g_2) \in N_2(\overline{\mathscr{K}})$ for all $\sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$, in which case the class x_2 corresponding to T_2 is represented by the $N_2(\overline{\mathscr{K}})$-valued Galois-cocycle $\sigma \mapsto g_2^{-1}\sigma(g_2)$. Then $\theta^{(2)}_{\mathscr{K}}(x_2)$ is represented by the cocycle

$$\sigma \mapsto \gamma_\sigma := g_2^{-1}\sigma(g_2)T_2^0 = \varphi_0(g_1)^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1))T_2^0 \in W_2. \tag{29}$$

Finally, notice that the condition that $\varphi|T_1$ is defined over $\mathscr{K}$ is equivalent to

$$\varphi(\sigma(g_1 t g_1^{-1})) = \sigma(\varphi(g_1 t g_1^{-1})) \quad \text{for all } t \in T^0(\overline{\mathscr{K}}) \text{ and } \sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K}). \tag{30}$$

The left- and the right-hand sides of (30) can be expanded as follows:

$$\varphi(\sigma(g_1 t g_1^{-1})) = h\varphi_0(\sigma(g_1 t g_1^{-1}))h^{-1} = h\varphi_0(\sigma(g_1))\varphi_0(\sigma(t))\varphi_0(\sigma(g_1))^{-1}h^{-1}$$

and

$$\begin{aligned}\sigma(\varphi(g_1 t g_1^{-1})) &= \sigma(h\varphi_0(g_1 t g_1^{-1})h^{-1})\\ &= \sigma(h)\sigma(\varphi_0(g_1))\sigma(\varphi_0(t))\sigma(\varphi_0(g_1))^{-1}\sigma(h)^{-1}.\end{aligned}$$

So, since $\varphi_0(\sigma(t)) = \sigma(\varphi_0(t))$, we see that (30) is equivalent to

$$\varphi_0(\sigma(g_1))^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1)) \in T_2^0 \quad \text{for all } \sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K}). \tag{31}$$

Now, suppose $\varphi|T_1$ is defined over $\mathscr{K}$, i.e., (31) holds. We claim that $\varphi_0^W(\theta^{(1)}_{\mathscr{K}}(x_1)) = \theta^{(2)}_{\mathscr{K}}(x_2) \in \theta^{(2)}_{\mathscr{K}}(\mathscr{C}^{(2)}_{\mathscr{K}})$. Indeed, combining (31) with (29) and (28), we see that

$$\gamma_\sigma = \varphi_0(g_1)^{-1}h^{-1}\sigma(h)\sigma(\varphi_0(g_1))T_2^0 = \varphi_0(g_1^{-1}\sigma(g_1))T_2^0 = \beta_\sigma,$$

as required.

Conversely, suppose $\varphi_0^{\mathrm{W}}(\theta_{\mathscr{K}}^{(1)}(x_1)) \in \theta_{\mathscr{K}}^{(2)}(\mathscr{C}_{\mathscr{K}}^{(2)})$. This means that there exists $g_2 \in \mathrm{G}_2(\overline{\mathscr{K}})$ such that

$$\beta_\sigma = g_2^{-1}\sigma(g_2)\mathrm{T}_2^0 \quad \text{for all } \sigma \in \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K}). \tag{32}$$

Set $h = g_2\varphi_0(g_1)^{-1}$ and $\varphi = \mathrm{Int}\, h \circ \varphi_0$. We need to show that $\varphi|\mathrm{T}_1$ is defined over $\mathscr{K}$, in other words, (31) holds. But this is obtained directly by combining (28) with (32). □

Combining Theorem 9.2 with Lemma 9.5, we obtain the following local-global principle for the existence of a coherent K-embedding of a K-torus as a maximal torus in a semi-simple group.

Theorem **9.6.** — *Let* G_1 *and* G_2 *be two connected semi-simple simply connected algebraic groups over a number field* K. *Assume that*

(∗) *there exist maximal* K*-tori* T_1^0 *of* G_1 *and* T_2^0 *of* G_2, *and a* $\overline{\mathrm{K}}$*-isomorphism* $\varphi_0\colon \mathrm{G}_1 \to \mathrm{G}_2$ *whose restriction to* T_1^0 *is an isomorphism onto* T_2^0 *defined over* K.

Let T_1 *be a maximal* K*-torus of* G_1 *such that* $Ш^2(\mathrm{T}_1)$ *is trivial (which automatically holds if there exists* $v_0 \in \mathrm{V}^{\mathrm{K}}$ *such that* T_1 *is* K_{v_0}*-anisotropic). If* T_1 *admits a coherent (relative to* φ_0*)* K_v*-embedding into* G_2 *for every* $v \in \mathrm{V}^{\mathrm{K}}$, *then it admits a coherent* K*-embedding into* G_2.

The following lemma explains why coherent embeddings of tori are easier to analyze if the ambient group is not of type D_{2n}.

Lemma **9.7.** — *Assume that* G_1 *and* G_2 *are absolutely simple simply connected and of type different from* D_{2n}, *and let* $\mathscr{K}/\mathrm{K}$ *be a field extension. If* T_1 *is a maximal* $\mathscr{K}$*-torus of* G_1 *and* $\varphi\colon \mathrm{G}_1 \to \mathrm{G}_2$ *is a* $\overline{\mathscr{K}}$*-isomorphism such that* $\iota := \varphi|\mathrm{T}_1$ *is defined over* $\mathscr{K}$, *then either* ι, *or* ι', *defined by* $\iota'(t) = \iota(t)^{-1}$, *is a* coherent $\mathscr{K}$*-embedding of* T_1 *into* G_2 *(in particular,* T_1 *admits such an embedding). Thus, if* G_1 *and* G_2 *are* $\mathscr{K}$*-isomorphic, then they have coherently equivalent systems of maximal* $\mathscr{K}$*-tori.*

Proof. — Obviously, $\mathrm{T}_2 := \varphi(\mathrm{T}_1)$ is defined over $\mathscr{K}$. Let Φ_2 be the root system of G_2 with respect to T_2. Since G_2 is not of type D_{2n}, the quotient $\mathrm{Aut}(\Phi_2)/\mathrm{W}(\Phi_2)$ is of order $\leqslant 2$, and in case it is of order 2, the automorphism $\alpha \mapsto -\alpha$ represents the nontrivial coset. Equivalently, $\mathrm{Aut}\,\mathrm{G}_2/\mathrm{Int}\,\mathrm{G}_2$ has order $\leqslant 2$, and in case it has order 2, there is an outer automorphism τ of G_2 defined over $\mathscr{K}$ such that $\tau(t) = t^{-1}$ for all $t \in \mathrm{T}_2$. Set $\varphi' = \tau \circ \varphi$, then $\varphi'|\mathrm{T}_1 = \iota'$. Since one of φ and φ' is of the form $\mathrm{Int}\, h \circ \varphi_0$, the lemma follows. □

Combined with Theorem 9.6, this lemma yields the following.

Corollary **9.8.** — *Let* G_1 *and* G_2 *be absolutely simple simply connected groups of type different from* D_{2n}, *and suppose that the condition* (∗) *of Theorem* 9.6 *holds. Assume in addition that* $Ш^2$ *is* *trivial for all maximal* K*-tori of* G_1 *and* G_2 *(which automatically holds if there exists a place* v_0 *of* K *such that* G_i *is* K_{v_0}*-anisotropic for* $i = 1, 2$*). If* $G_1 \simeq G_2$ *over* K_v*, for all* $v \in V^K$*, then* G_1 *and* G_2 *have coherently equivalent systems of maximal* K*-tori.*

Of course, if G_1 and G_2 are not of type A, D or E_6, then the assumption that $G_1 \simeq G_2$ over K_v for all $v \in V^K$ implies that $G_1 \simeq G_2$ over K, and our assertion becomes obvious (cf. Lemma 9.7). We will use Corollary 9.8 to show that for each of the types A_n, D_{2n+1}, or E_6, one can construct an arbitrarily large number of pairwise nonisomorphic absolutely simple simply connected K-groups of this type with coherently equivalent systems of maximal K-tori (cf. Theorem 9.12).

Henceforth (through 9.12), G_0 will denote a connected absolutely almost simple simply connected quasi-split K-group of one of the following types: A_n, D_{2n+1} $(n > 1)$, and E_6. We first describe a general construction of nonisomorphic inner twists G_1 and G_2 of G_0 which are isomorphic over K_v for all $v \in V^K$. Let L be the minimal Galois extension of K over which G_0 splits, and let V_0 be the set of $v \in V_f^K$ that split in L. We let C denote the center of G_0; clearly, C is L-isomorphic to μ_ℓ, the group of ℓ-th roots of unity, where $\ell = n + 1$ for G_0 of type A_n, $\ell = 4$ for type D_{2n+1}, and $\ell = 3$ for type E_6. Each $x \in G_0$ gives the inner automorphism $z \mapsto xzx^{-1}$ of G_0. This leads to the natural isomorphism i from the adjoint group $\overline{G}_0$ of G_0 onto the group of inner automorphisms $\mathrm{Int}\, G_0$ ($\subset \mathrm{Aut}\, G_0$). Any automorphism g of G_0 can be regarded as an automorphism of $\overline{G}_0$, and then for every $x \in \overline{G}_0$, we have $g \circ i(x) \circ g^{-1} = i(g(x))$ in $\mathrm{Aut}\, G_0$.

For a class $\zeta \in H^1(K, \overline{G}_0)$, in the sequel we will let $\sigma \mapsto \zeta_\sigma$, $\sigma \in \mathrm{Gal}(\overline{K}/K)$, denote a Galois-cocycle representing ζ.

For any $v \in V^K$, we have the following commutative diagram

$$\begin{array}{ccc} H^1(K, \overline{G}_0) & \xrightarrow{\alpha} & H^1(K, \mathrm{Aut}\, G_0) \\ \gamma_v \downarrow & & \downarrow \beta_v \\ H^1(K_v, \overline{G}_0) & \xrightarrow{\alpha_v} & H^1(K_v, \mathrm{Aut}\, G_0), \end{array}$$

in which α and α_v are induced by i. Furthermore, for any extension $\mathscr{K}/K$ there is a natural map $\rho_{\mathscr{K}} \colon H^1(\mathscr{K}, \overline{G}_0) \to H^2(\mathscr{K}, C)$. We will also need the map $\mu \colon H^2(K, C) \to \bigoplus_v H^2(K_v, C)$. We will use additive notation for $H^2(\mathscr{K}, C)$ etc.

Lemma **9.9.** — *Let* $\xi_1, \xi_2 \in H^1(K, \overline{G}_0)$.

(i) *If* $\rho_K(\xi_1) \neq \pm\rho_K(\xi_2)$*, then* $\alpha(\xi_1) \neq \alpha(\xi_2)$.
(ii) *If* $v \in V_f^K$ *and* $\rho_{K_v}(\gamma_v(\xi_1)) = \pm\rho_{K_v}(\gamma_v(\xi_2))$*, then* $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$.

Proof. — Notice that $\mathrm{Aut}\, G_0$ has the following semi-direct product decomposition

$$\mathrm{Aut}\, G_0 = \mathrm{Int}\, G_0 \rtimes \Sigma,$$

where Σ is a K-subgroup of order two, whose nontrivial element s is defined over K and acts on C as $c \mapsto c^{-1}$.

(i): Suppose $\alpha(\xi_1)=\alpha(\xi_2)$. Then there exists $g\in \operatorname{Aut}\mathrm{G}_0$ such that

$$i(\xi_{2\sigma})=g\circ i(\xi_{1\sigma})\circ\sigma(g)^{-1}\quad \text{for all } \sigma\in \operatorname{Gal}(\overline{\mathrm{K}}/\mathrm{K}).$$

If $g\in \operatorname{Int}\mathrm{G}_0$, then $\xi_1=\xi_2$, and therefore, $\rho_{\mathrm{K}}(\xi_1)=\rho_{\mathrm{K}}(\xi_2)$. Now, suppose $g\notin \operatorname{Int}\mathrm{G}_0$. Then $g=hs$, $h\in \operatorname{Int}\mathrm{G}_0$. The cohomology class ξ_2' in $\mathrm{H}^1(\mathrm{K},\overline{\mathrm{G}}_0)$ corresponding to the cocycle

$$\sigma\mapsto \xi_{2\sigma}'=s(\xi_{1\sigma}),\qquad \sigma\in \operatorname{Gal}(\overline{\mathrm{K}}/\mathrm{K}),$$

clearly equals ξ_2. As $s(c)=c^{-1}$ for $c\in \mathrm{C}$, we conclude that

$$\rho_{\mathrm{K}}(\xi_2)=\rho_{\mathrm{K}}(\xi_2')=-\rho_{\mathrm{K}}(\xi_1),$$

a contradiction.

(ii): Recall that ρ_{K_v} is a bijection for any $v\in \mathrm{V}_f^{\mathrm{K}}$ (cf. [26], Corollary of Theorem 6.20), so our claim is obvious if $\rho_{\mathrm{K}_v}(\gamma_v(\xi_1))=\rho_{\mathrm{K}_v}(\gamma_v(\xi_2))$. Suppose now that $\rho_{\mathrm{K}_v}(\gamma_v(\xi_1))=-\rho_{\mathrm{K}_v}(\gamma_v(\xi_2))$. Consider the $\overline{\mathrm{G}}_0(\overline{\mathrm{K}})$-valued Galois-cocycle $\sigma\mapsto\xi_{2\sigma}':=s(\xi_{2\sigma})$, and let ξ_2' be the associated cohomology class. Then for $\sigma\in \operatorname{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ we have

$$i(\xi_{2\sigma}')=s\circ i(\xi_{2\sigma})\circ s^{-1}=s\circ i(\xi_{2\sigma})\circ\sigma(s)^{-1},$$

so $\alpha(\xi_2')=\alpha(\xi_2)$. On the other hand,

$$\rho_{\mathrm{K}_v}(\gamma_v(\xi_2'))=-\rho_{\mathrm{K}_v}(\gamma_v(\xi_2))=\rho_{\mathrm{K}_v}(\gamma_v(\xi_1)).$$

Then $\gamma_v(\xi_2')=\gamma_v(\xi_1)$, and

$$\beta_v(\alpha(\xi_1))=\beta_v(\alpha(\xi_2'))=\beta_v(\alpha(\xi_2)).\qquad\square$$

Let $\widehat{\mathrm{C}}$ be the character group of C. Fix a generator χ of $\widehat{\mathrm{C}}(\mathrm{K})$, and let d denote its order. For each $v\in \mathrm{V}^{\mathrm{K}}$, χ induces a character

$$\chi_v\colon \mathrm{H}^2(\mathrm{K}_v,\mathrm{C})\to \mathrm{H}^2(\mathrm{K}_v,\mathrm{GL}_1)\subset \mathbf{Q}/\mathbf{Z}.$$

If $v\in \mathrm{V}_0$, then $\mathrm{H}^2(\mathrm{K}_v,\mathrm{C})\simeq \operatorname{Br}(\mathrm{K}_v)_\ell$ is cyclic of order ℓ, and one can choose a generator $b_v\in \mathrm{H}^2(\mathrm{K}_v,\mathrm{C})$ such that $\chi_v(b_v)=1/d$. Now, let V be a finite subset of V^{K} containing $\mathrm{V}_\infty^{\mathrm{K}}$, and suppose that for each $v\in \mathrm{V}$ we are given $\xi^{(v)}\in \mathrm{H}^1(\mathrm{K}_v,\overline{\mathrm{G}}_0)$. Fix an integer $t\geqslant 1$, and pick $2(t+1)$ places

$$v_0',v_0'',v_1',v_1'',\ldots,v_t',v_t''\in \mathrm{V}_0\setminus(\mathrm{V}_0\cap \mathrm{V}).$$

Let $\mathrm{V}_t=\{v_0',v_0'',v_1',v_1'',\ldots,v_t',v_t''\}$. Now pick $x_{v_0''}\in \mathrm{H}^2(\mathrm{K}_{v_0''},\mathrm{C})$ so that

$$\sum_{v\in \mathrm{V}}\chi_v(\rho_{\mathrm{K}_v}(\xi^{(v)}))+\chi_{v_0'}(b_{v_0'})+\chi_{v_0''}(x_{v_0''})=0.$$

Next, fix $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_t) \in \mathrm{E}_t := \prod_{i=1}^{t}\{\pm 1\}$, and consider $(x(\varepsilon)_v) \in \bigoplus_v \mathrm{H}^2(\mathrm{K}_v, \mathrm{C})$ with the following components:

$$x(\varepsilon)_v = \begin{cases} \rho_{\mathrm{K}_v}(\xi^{(v)}), & v \in \mathrm{V}, \\ b_{v_0'}, & v = v_0', \\ x_{v_0''}, & v = v_0'', \\ \varepsilon_j b_{v_j'}, & v = v_j',\ j \geqslant 1, \\ -\varepsilon_j b_{v_j''}, & v = v_j'',\ j \geqslant 1, \\ 0, & \text{for all other } v. \end{cases} \tag{33}$$

We obviously have $\sum_v \chi_v(x(\varepsilon)_v) = 0$, so it follows from a theorem of Poitou-Tate (cf. [39], Chap. II, §6, Theorem 3) that there exists $x(\varepsilon) \in \mathrm{H}^2(\mathrm{K}, \mathrm{C})$ such that $\mu(x(\varepsilon)) = (x(\varepsilon)_v)$. We now want to construct a maximal K-torus $\overline{\mathrm{T}}_0$ of $\overline{\mathrm{G}}_0$ (depending on V, $\xi^{(v)}$ for $v \in \mathrm{V}$, and V_t) such that for each $\varepsilon \in \mathrm{E}_t$, $x(\varepsilon)$ lifts to a class $\zeta(\varepsilon) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{T}}_0)$ whose image in $\mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{G}}_0)$ is $\xi^{(v)}$ for all $v \in \mathrm{V}$.

For every real v, $\xi^{(v)}$ is given by an element $g_v \in \overline{\mathrm{G}}_0(\overline{\mathrm{K}}_v)$ such that $g_v\bar{g}_v = 1$, where $\bar{g}_v$ denotes the conjugate of g_v under the nontrivial automorphism of $\overline{\mathrm{K}}_v/\mathrm{K}_v = \mathbf{C}/\mathbf{R}$. It follows from the uniqueness of the Jordan decomposition that the semi-simple and the unipotent components g_v^s, g_v^u of g_v also define cocycles. If $g_v^u \neq 1$, then the 1-dimensional connected unipotent subgroup U generated by g_v^u is defined over $\mathrm{K}_v = \mathbf{R}$. Using the fact that $\mathrm{H}^1(\mathrm{K}_v, \mathrm{U})$ is trivial, one sees that $\xi^{(v)}$ is the cohomology class given by g_v^s. So we can assume that g_v is semi-simple. Then g_v is contained in the connected centralizer $\mathrm{H} := \mathrm{Z}_{\overline{\mathrm{G}}_0}(g_v)^\circ$ (cf. [3], Corollary 11.12), and H is defined over K_v. Hence, g_v is contained in a maximal K_v-torus $\overline{\mathrm{T}}^{(v)}$ of H which is also a maximal torus of $\overline{\mathrm{G}}_0$. Now for each $v \in (\mathrm{V} \setminus \mathrm{V}_\infty^{\mathrm{K}}) \cup \mathrm{V}_t$, we pick a maximal K_v-torus $\overline{\mathrm{T}}^{(v)}$ of $\overline{\mathrm{G}}_0$ which is anisotropic over K_v (see [26], Theorem 6.21, or [9], §2.4). Using the weak approximation property for the variety of maximal tori of $\overline{\mathrm{G}}_0$ (cf. [26], Corollary 3 in §7.1), we can find a maximal K-torus $\overline{\mathrm{T}}_0$ of $\overline{\mathrm{G}}_0$ which is conjugate to $\overline{\mathrm{T}}^{(v)}$ under an element of $\overline{\mathrm{G}}_0(\mathrm{K}_v)$ for all $v \in \mathrm{V} \cup \mathrm{V}_t$. Let $\pi\colon \mathrm{G}_0 \to \overline{\mathrm{G}}_0$ be the natural K-isogeny, and $\mathrm{T}_0 = \pi^{-1}(\overline{\mathrm{T}}_0)$.

Lemma **9.10.** — *For every* $\varepsilon \in \mathrm{E}_t$, *there exists* $\zeta(\varepsilon) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{T}}_0)$ *which maps onto* $x(\varepsilon)$ *under the coboundary map* $\mathrm{H}^1(\mathrm{K}, \overline{\mathrm{T}}_0) \to \mathrm{H}^2(\mathrm{K}, \mathrm{C})$, *and whose image in* $\mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{G}}_0)$ *equals* $\xi^{(v)}$ *for all* $v \in \mathrm{V}$.

Proof. — For any real v, as $\overline{\mathrm{T}}_0$ is conjugate to $\overline{\mathrm{T}}^{(v)}$ under an element of $\overline{\mathrm{G}}_0(\mathrm{K}_v)$, and $\xi^{(v)}$ is given by $g_v \in \overline{\mathrm{T}}^{(v)}(\overline{\mathrm{K}}_v)$, there exists a cohomology class $\xi'^{(v)}$ in $\mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{T}}_0)$ which maps onto $\xi^{(v)}$ under the natural map $\mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{T}}_0) \to \mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{G}}_0)$. On the other hand, for every nonarchimedean $v \in \mathrm{V}$, as $\overline{\mathrm{T}}_0$ is anisotropic over K_v, the natural map $\mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{T}}_0) \to \mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{G}}_0)$ is onto (see the proof of Theorem 6.20 on p. 326 of [26]), there is a $\xi'^{(v)} \in \mathrm{H}^1(\mathrm{K}_v, \overline{\mathrm{T}}_0)$ which maps onto $\xi^{(v)}$.

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H^1(K, \overline{T}_0) & \xrightarrow{\delta_1} & H^2(K, C) & \xrightarrow{\delta_2} & H^2(K, T_0) \\ \eta_1 \downarrow & & \eta_2 \downarrow & & \eta_3 \downarrow \\ \bigoplus_v H^1(K_v, \overline{T_0}) & \xrightarrow{\Delta_1} & \bigoplus_v H^2(K_v, C) & \xrightarrow{\Delta_2} & \bigoplus_v H^2(K, T_0) \end{array}$$

(notice that η_2 actually coincides with μ). First, we will show that $x(\varepsilon) \in \mathrm{Im}\,\delta_1 = \mathrm{Ker}\,\delta_2$. Observe that

$$(\mathbf{34}) \qquad x(\varepsilon)_v \in \mathrm{Im}(H^1(K_v, \overline{T}_0) \to H^2(K_v, C))$$

for all v. This is obvious if $v \notin V \cup V_t$. For any real v, this follows from the fact that $x(\varepsilon)_v = \rho_{K_v}(\xi^{(v)})$, and $\xi^{(v)}$ is the image of $\xi'^{(v)} \in H^1(K_v, \overline{T}_0)$. For a nonarchimedean $v \in V \cup V_t$, by our construction T_0 is K_v-anisotropic, and it follows from the Nakayama-Tate Theorem (cf. [26], Theorem 6.2) that $H^2(K_v, T_0)$ is trivial. So the map $H^1(K_v, \overline{T}_0) \to H^2(K_v, C)$ is surjective, and (34) is automatic. Thus, $\eta_2(x(\varepsilon)) = (x(\varepsilon)_v) \in \mathrm{Im}\ \Delta_1$, so

$$\Delta_2(\eta_2(x(\varepsilon))) = \eta_3(\delta_2(x(\varepsilon))) = 0.$$

Since T_0 is anisotropic at every $v \in V_t$, we have that $Ш^2(T_0) = \mathrm{Ker}\,\eta_3$ is trivial, and hence $\delta_2(x(\varepsilon)) = 0$, as required. Fix $\zeta'(\varepsilon) \in H^1(K, \overline{T}_0)$ such that $\delta_1(\zeta'(\varepsilon)) = x(\varepsilon)$.

For an extension $\mathscr{K}/K$, we consider the natural homomorphism

$$\lambda_{\mathscr{K}} \colon H^1(\mathscr{K}, T_0) \to H^1(\mathscr{K}, \overline{T}_0),$$

and for $v \in V^K$, we let $\zeta'(\varepsilon)^{(v)}$ denote the image of $\zeta'(\varepsilon)$ under the restriction map $H^1(K, \overline{T}_0) \to H^1(K_v, \overline{T}_0)$. For each $v \in V$, the cohomology classes $\zeta'(\varepsilon)^{(v)}$ and $\xi'^{(v)}$ have the same image in $H^2(K_v, C)$, so there exists $\theta(\varepsilon)_v \in H^1(K_v, T_0)$ such that

$$\xi'^{(v)} = \lambda_{K_v}(\theta(\varepsilon)_v) \cdot \zeta'(\varepsilon)^{(v)}.$$

By ([26], Proposition 6.17), the map $H^1(K, T_0) \to \prod_{v \in V_\infty^K} H^1(K_v, T_0)$ is surjective. Pick $\theta(\varepsilon) \in H^1(K, T_0)$ which maps onto $(\theta(\varepsilon)_v)_{v \in V_\infty^K}$, and set $\zeta(\varepsilon) = \lambda_K(\theta(\varepsilon)) \cdot \zeta'(\varepsilon)$. Let $\zeta(\varepsilon)^{(v)}$ be the image of $\zeta(\varepsilon)$ under the map $H^1(K, \overline{T}_0) \to H^1(K_v, \overline{T}_0)$. Then $\delta_1(\zeta(\varepsilon)) = \delta_1(\zeta'(\varepsilon)) = x(\varepsilon)$ and $\zeta(\varepsilon)^{(v)} = \xi'^{(v)}$ for all $v \in V_\infty^K$. Finally, to show that the image of $\zeta(\varepsilon)^{(v)}$ in $H^1(K_v, \overline{G}_0)$ coincides with $\xi^{(v)}$ for nonarchimedean $v \in V$, we observe that these elements have the same image under ρ_{K_v}, which is a bijection for all $v \in V_f^K$ (Corollary in §6.4 of [26]). □

Let $\zeta(\varepsilon)$ be as in the preceding lemma, and $\xi(\varepsilon)$ be the image of $\zeta(\varepsilon)$ under the natural map $H^1(K, \overline{T}_0) \to H^1(K, \overline{G}_0)$. Then $\rho_K(\xi(\varepsilon)) = x(\varepsilon)$ and $\gamma_v(\xi(\varepsilon)) = \xi^{(v)}$ for all $v \in V$. Fix two distinct $\varepsilon_1, \varepsilon_2 \in E_t$, and let $\xi_j = \xi(\varepsilon_j)$. Since each b_v has order $\ell > 2$, it follows from (33) that $\mu(\rho_{\mathrm{K}}(\xi_1)) \neq \pm\mu(\rho_{\mathrm{K}}(\xi_2))$, hence $\rho_{\mathrm{K}}(\xi_1) \neq \pm\rho_{\mathrm{K}}(\xi_2)$, so according to Lemma 9.9(i), $\alpha(\xi_1) \neq \alpha(\xi_2)$. On the other hand, we have

$$\rho_{\mathrm{K}_v}(\gamma_v(\xi_1)) = 0 = \rho_{\mathrm{K}}(\gamma_v(\xi_2)) \quad \text{for any } v \in \mathrm{V}^{\mathrm{K}} \setminus (\mathrm{V} \cup \mathrm{V}_0),$$

$$\rho_{\mathrm{K}_v}(\gamma_v(\xi_1)) = \pm\rho_{\mathrm{K}_v}(\gamma_v(\xi_2)) \quad \text{for any } v \in \mathrm{V}_0,$$

and

$$\gamma_v(\xi_1) = \xi^{(v)} = \gamma_v(\xi_2) \quad \text{for any } v \in \mathrm{V}.$$

Using Lemma 9.9(ii), we now see that $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$ for all $v \in \mathrm{V}^{\mathrm{K}}$. Thus, we obtain the following proposition.

Proposition **9.11.** — *The* 2^t *elements* $\xi(\varepsilon) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{G}}_0)$, $\varepsilon \in \mathrm{E}_t$, *have the following properties: the elements* $\alpha(\xi(\varepsilon)) \in \mathrm{H}^1(\mathrm{K}, \mathrm{Aut}\,\mathrm{G}_0)$ *are pairwise distinct, while for any* $v \in \mathrm{V}^{\mathrm{K}}$, *the elements* $\beta_v(\alpha(\xi(\varepsilon))) \in \mathrm{H}^1(\mathrm{K}_v, \mathrm{Aut}\,\mathrm{G}_0)$ *are all equal, and, in addition,* $\gamma_v(\xi(\varepsilon)) = \xi^{(v)}$ *for all* $v \in \mathrm{V}$.

For $\xi(\varepsilon)$ as above, we let G_ε denote the form of G_0 obtained by twisting it by a cocycle representing $\alpha(\xi(\varepsilon))$. Since the cohomology classes $\alpha(\xi(\varepsilon))$, $\varepsilon \in \mathrm{E}_t$, are pairwise distinct, the corresponding groups G_ε are pairwise nonisomorphic over K. Now, fix $\varepsilon_1, \varepsilon_2 \in \mathrm{E}_t$, and set

$$\zeta_j = \zeta(\varepsilon_j) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{T}}_0), \quad \xi_j = \xi(\varepsilon_j) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{G}}_0) \quad \text{and} \quad \mathrm{G}_j = \mathrm{G}_{\varepsilon_j}$$

for $j = 1, 2$. As ξ_j is the image of ζ_j under the natural map $\mathrm{H}^1(\mathrm{K}, \overline{\mathrm{T}}_0) \to \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{G}}_0)$, there is a $\overline{\mathrm{T}}_0(\overline{\mathrm{K}})$-valued Galois cocycle $\sigma \mapsto z_{j\sigma}$, $\sigma \in \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, representing ξ_j. Therefore, there exists a $\overline{\mathrm{K}}$-isomorphism $\varphi_j \colon \mathrm{G}_0 \to \mathrm{G}_j$ such that $\varphi_j^{-1} \circ \sigma(\varphi_j) = i(z_{j\sigma})$, for all $\sigma \in \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, where i is the natural isomorphism $\overline{\mathrm{G}}_0 \to \mathrm{Int}\,\mathrm{G}_0$. Then $\varphi_j|\mathrm{T}_0$ is defined over K, and hence, $\mathrm{T}_j^0 := \varphi_j(\mathrm{T}_0)$ is a maximal K-torus of G_j. Now $\varphi_0 := \varphi_2 \circ \varphi_1^{-1}$ is a $\overline{\mathrm{K}}$-isomorphism from G_1 onto G_2 whose restriction to T_1^0 is an isomorphism onto T_2^0 defined over K. Since $\beta_v(\alpha(\xi_1)) = \beta_v(\alpha(\xi_2))$, the groups G_1 and G_2 are K_v-isomorphic, for all $v \in \mathrm{V}^{\mathrm{K}}$. In addition, for each $j = 1, 2$, and any $v \in \mathrm{V}$, the group G_j is K_v-isomorphic to the group ${}_{\xi^{(v)}}\mathrm{G}_0$ obtained from G_0 by twisting over K_v by any cocycle representing $\alpha_v(\xi^{(v)})$. So, applying Corollary 9.8, we obtain the following.

Theorem **9.12.** — *Let* G_0 *be a simple simply connected quasi-split* K*-group of one of the following types:* A_n, D_{2n+1} $(n > 1)$, *or* E_6, *and let* $\xi(\varepsilon) \in \mathrm{H}^1(\mathrm{K}, \overline{\mathrm{G}}_0)$, $\varepsilon \in \mathrm{E}_t$, *be the cohomology classes as in Proposition 9.11, and let* G_ε *be the group obtained by twisting* G_0 *by a cocycle representing* $\xi(\varepsilon)$. *Then* G_ε, $\varepsilon \in \mathrm{E}_t$, *are pairwise nonisomorphic* K*-forms of* G_0. *Moreover, if for every* $\varepsilon \in \mathrm{E}_t$, *and every maximal* K*-torus* T *of* G_ε, *we have* $Ш^2(\mathrm{T}) = 0$ *(which is automatically the case if for some* $v \in \mathrm{V}$ *the twist* ${}_{\xi^{(v)}}\mathrm{G}_0$ *is* K_v*-anisotropic), then all the groups* G_ε *have coherently equivalent systems of maximal* K*-tori.*

Remark **9.13.** — If G is an absolutely simple simply connected inner K-form of type A_n, then the condition $Ш^2(T) = \{0\}$ is automatically satisfied for any maximal K-torus T of G. Indeed, T is of the form $T = R^{(1)}_{A/K}(GL_1)$, where A is a commutative étale $(n+1)$-dimensional K-algebra. Letting $S = R_{A/K}(GL_1)$, we have the exact sequence

$$1 \to T \longrightarrow S \longrightarrow GL_1 \to 1,$$

which in conjunction with Hilbert's Theorem 90 induces the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 \longrightarrow & H^2(K,T) & \longrightarrow & H^2(K,S) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & \bigoplus_v H^2(K_v,T) & \longrightarrow & \bigoplus_v H^2(K_v,S). \end{array}$$

Since the map $H^2(K,S) \longrightarrow \bigoplus_v H^2(K_v,S)$ is injective by the Albert-Hasse-Brauer-Noether Theorem, our assertion follows.

We now return to our standard set-up: let $G_1 = G_2 =: G$ be connected absolutely almost simple algebraic groups over a field F of characteristic zero, and let $\widetilde{\mathscr{G}}_i$, for $i = 1, 2$, be a form of G_i over a number field K. We observe that if $\widetilde{\mathscr{G}}_1$ and $\widetilde{\mathscr{G}}_2$ have coherently equivalent systems of maximal K-tori then so do the corresponding adjoint groups $\mathscr{G}_1$ and $\mathscr{G}_2$. Then for any finite set $S \subset V^K_\infty$ containing V^K_∞, any $(\mathscr{G}_i, K, S)$-arithmetic subgroups $\Gamma_i \subset G_i(F)$ are weakly commensurable (provided that they are Zariski-dense) – see the argument in Example 6.5. It turns out that in this situation arithmetic subgroups provide length-commensurable locally symmetric spaces (cf. Corollary 8.7).

Proposition **9.14.** — *Let* $G_1 = G_2 = G$ *be connected absolutely simple real algebraic groups, and let* $\mathfrak{X}_i$ *be the symmetric space of* $\mathcal{G}_i = G_i(\mathbf{R})$. *For* $i = 1, 2$, *let* Γ_i *be a torsion-free* $(\mathscr{G}_i, K)$-*arithmetic subgroup of* $\mathcal{G}_i$. *If* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *have coherently equivalent systems of maximal* K*-tori, then the locally symmetric spaces* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable.*

Proof. — We will assume (as we may) that $G = \overline{G}$, and then $\Gamma_i \subset \mathscr{G}_i(K)$ for $i = 1, 2$. Let $\gamma_1 \in \Gamma_1$ be a nontrivial semi-simple element, and let $T_1 \subset \mathscr{G}_1$ be a maximal K-torus containing it. By our assumption, there exists an isomorphism $\varphi\colon \mathscr{G}_1 \to \mathscr{G}_2$ such that the restriction $\varphi|T_1$ is defined over K, hence $T_2 := \varphi(T_1)$ is a maximal K-torus of $\mathscr{G}_2$. Since $\varphi(T_1(K) \cap \Gamma_1)$ is an arithmetic subgroup of $T_2(K)$, there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n$ belongs to Γ_2. The map $\alpha \to \varphi^*(\alpha)$ defines a bijection between the root systems $\Phi(G_2, T_2)$ and $\Phi(G_1, T_1)$. It follows that the sets of complex numbers

$$\{\alpha(\gamma_1^n) \mid \alpha \in \Phi(\mathscr{G}_1, T_1)\} \quad \text{and} \quad \{\alpha(\gamma_2) \mid \alpha \in \Phi(\mathscr{G}_2, T_2)\}$$

are identical. Using the formula (23) from Proposition 8.5(ii), we see that

$$\lambda_{\Gamma_2}(\gamma_2)/\lambda_{\Gamma_1}(\gamma_1) \in \mathbf{Q}.$$

Thus, $\mathbf{Q} \cdot \lambda_{\Gamma_1}(\Gamma_1^{ss}) \subset \mathbf{Q} \cdot \lambda_{\Gamma_2}(\Gamma_2^{ss})$. By symmetry, these sets are actually equal, and therefore $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable. □

Construction **9.15.** — We finally indicate how Theorem 9.12 can be used to construct examples of weakly commensurable cocompact arithmetic and S-arithmetic subgroups, and length-commensurable compact locally symmetric spaces, which are not commensurable. Let G be a connected absolutely simple simply connected isotropic real algebraic group of one of the following types: A_n, D_{2n+1}, where $n > 1$, or E_6, and let $\mathcal{L}$ be either $\mathbf{R}$ or $\mathbf{C}$ depending on whether or not G is an inner form over $\mathbf{R}$. Fix a real quadratic extension $K/\mathbf{Q}$, and let v'_∞, v''_∞ denote its two real places. Next, pick a quadratic extension L of K so that $L \otimes_K K_{v'_\infty} = \mathcal{L}^{2/[\mathcal{L}:\mathbf{R}]}$ and $L \otimes_K K_{v''_\infty} = \mathbf{C}$, and let $\mathscr{G}_0$ denote the nonsplit quasi-split K-group of the same type as G which splits over L. Since for the types under consideration, the $\mathbf{R}$-anisotropic form is an inner twist of the corresponding nonsplit quasi-split $\mathbf{R}$-group, there exist cohomology classes $\xi^{(v'_\infty)} \in H^1(K_{v'_\infty}, \overline{\mathscr{G}}_0)$ and $\xi^{(v''_\infty)} \in H^1(K_{v''_\infty}, \overline{\mathscr{G}}_0)$ such that the twist ${}_{\xi^{(v'_\infty)}}\mathscr{G}_0$ is $\mathbf{R}$-isomorphic to G and the twist ${}_{\xi^{(v''_\infty)}}\mathscr{G}_0$ is $\mathbf{R}$-anisotropic. Then applying the construction described in Theorem 9.12 to $V = \{v'_\infty, v''_\infty\}$ and the specified cocycles, we obtain 2^t groups $\mathscr{G}_\varepsilon$, $\varepsilon \in E_t$, which are pairwise nonisomorphic over K but have coherently equivalent systems of maximal K-tori as these groups are all anisotropic over $K_{v''_\infty}$. Besides, $\mathscr{G}_\varepsilon$ is isomorphic to G over $K_{v'_\infty} = \mathbf{R}$, for every $\varepsilon \in E_t$. Thus, torsion-free arithmetic subgroups of $\mathscr{G}_\varepsilon$ yield discrete torsion-free subgroups of $\mathcal{G} = G(\mathbf{R})$, and it follows from Proposition 9.14 that the resulting locally symmetric spaces are length-commensurable, but not commensurable. Finally, for any finite subset S of V^K containing V^K_∞, the S-arithmetic subgroups of $\mathscr{G}_\varepsilon$, $\varepsilon \in E_t$, are pairwise weakly commensurable, but not commensurable (cf. Example 6.5).

Remark **9.16.** — Most of the results of this section immediately extend to a global function field K. This applies, in particular, to Theorem 9.6, yielding a local-global principle for the existence of a coherent embedding, and Theorem 9.12, containing a construction of forms of a quasi-split group G_0 belonging to one of the types A_n, D_{2n+1} ($n > 1$) or E_6, which are not K-isomorphic, but are isomorphic over K_v for all $v \in V^K$. It should be noted, however, that the construction of nonisomorphic K-groups with coherently equivalent systems of maximal K-tori, described in 9.15, extends to global function fields only for groups of type A_n. The reason is that we ensured the triviality of $Ш^2(T)$ for all maximal tori of a group under consideration by arranging that the group is anisotropic at a certain archimedean place. Over global function fields, however, any group of type different from A_n is isotropic.

10. Isospectral locally symmetric spaces

For a compact Riemannian manifold M, let

$$L(M) = \{\lambda \in \mathbf{R},\ \text{there exists a closed geodesic on M of length } \lambda\},$$

The following theorem is known. For locally symmetric spaces of rank 1, a proof is given in [13]. However, for locally symmetric spaces of rank > 1, we have not been able to find a reference for it. For the convenience of the reader we will give below its proof which was supplied to us by Alejandro Uribe and Steve Zelditch.

Theorem **10.1.** — *Let* M_1 *and* M_2 *be two compact locally symmetric spaces with nonpositive sectional curvatures. Assume* M_1 *and* M_2 *are isospectral, in the sense that the spectra of their Laplace-Beltrami operators on functions are the same (their eigenvalues and their multiplicities). Then* $L(M_1) = L(M_2)$.

As we will explain, this theorem is a direct consequence of theorems of Duistermaat and Guillemin, [10], and of Duistermaat, Kolk and Varadarajan, [11]. (In fact, the results of the latter paper alone imply this theorem, but it is conceptually better to use the main theorem of [10] in the proof.)

The results of [11] (cf. Proposition 5.15) include that, for M a compact locally symmetric space of non-compact type,

(i) L(M) is a discrete subset of $\mathbf{R}$, and
(ii) if $\lambda \in L(M)$, the set

$$Z_\lambda := \{\bar{x} \in T^1M; \text{the geodesic through } \bar{x} \text{ is closed of length } \lambda\}$$

is a finite union of closed submanifolds (possibly of different dimensions) of the unit tangent bundle T^1M of M.

Denote by Z_λ° the union of connected components of Z_λ of maximal dimension. It turns out that, in addition to the previous theorem, for M as above

$$\text{for all } \lambda \in L(M), \dim Z_\lambda^\circ \text{ and } \mathrm{Vol}\, Z_\lambda^\circ \text{ are spectrally determined.} \tag{35}$$

Here the volume is with respect to a measure naturally induced by the geodesic flow. (Equation (5.47) of [11] is a formula for this volume.)

Let us now see how one proves Theorem 10.1 and the additional statement, (35). Proposition 5.8 of [11] establishes that each Z_λ is a clean fixed-point set of the time λ map of the geodesic flow $\phi_\lambda : T^1M \to T^1M$. (Recall that this means that at each $z \in Z_\lambda$, $\ker d(\phi_\lambda)_z = T_z Z_\lambda$.) We can therefore apply the Duistermaat-Guillemin trace formula, [10], to the square root of the Laplace-Beltrami operator on M. Specifically, pick a length λ and a Schwartz function on the real line, φ, such that its Fourier transform $\widehat{\varphi}$ is compactly supported and satisfies:

$$\widehat{\varphi}(\lambda) = 1 \quad \text{and} \quad L(M) \cap \operatorname{supp} \widehat{\varphi} = \{\lambda\}.$$

(Such a φ exists by item (i) above.) Let $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$ be the square roots of the eigenvalues of the Laplace-Meltrami operator on M, listed with their multiplicities. Then, by Theorem 4.5 of [10] one has an asymptotic expansion as $\mu \to \infty$ of the form:

$$\sum_j \varphi(\mu - \mu_j) \sim e^{i\mu\lambda} \sum_{j=0}^{\infty} c_j \mu^{d_\lambda - j}. \tag{36}$$

Here $d_\lambda = (\dim Z_\lambda^\circ - 1)/2$. A key point is that the leading coefficient, c_0, is not zero because the Maslov indices (the integers σ_j in equation (4.7) in [10]) of all closed geodesics on M are zero, by Proposition 5.15 of [11]. By equation (4.8) of [10], c_0 is equal to the volume of Z_λ° times a factor that depends only on d_λ. The expansion (36) in the present context is explicitly discussed in §5.6 of [11] (see the last formula in that section which, incidentally, contains a typo: a τ is missing in the left-hand side exponent). The dimension of Z_λ° is determined spectrally by the size in μ of the left-hand side of (36), and therefore c_0 determines the volume of Z_λ°.

Theorem 10.1 and statement (35) follow from (36), the information on c_0, and the basic fact that if $L(M) \cap \operatorname{supp} \widehat{\varphi} = \emptyset$, then the left-hand side of (36) is $O(\mu^{-\infty})$. By considering all possible test functions φ as above, one can detect the set L(M) from the eigenvalues of the Laplacian. □

For $i = 1, 2$, let G_i be a connected absolutely simple adjoint real algebraic group, $\mathcal{G}_i = G_i(\mathbf{R})$, and Γ_i be a torsion-free cocompact discrete subgroup of $\mathcal{G}_i$. Let $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}_i / \Gamma_i$, where $\mathfrak{X}_i$ is the symmetric space of $\mathcal{G}_i$. Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral. According to a result proved by Hermann Weyl, any two isospectral Riemannian manifolds are of same dimension, and have equal volume, see, for example, [14], Theorem 4.2.1. It follows that the exceptional case ($\mathcal{E}$) cannot occur. So, from Theorem 10.1 and Corollary 8.14, we obtain the following.

Theorem **10.2.** — *If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, then Γ_1 and Γ_2 are weakly commensurable.*

Now, Theorem 10.2 in conjunction with Theorem 6 implies

Theorem **10.3.** — *If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and Γ_1 is arithmetic, then so is Γ_2.*

Let us now assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and that at least one of the subgroups Γ_1 or Γ_2 is arithmetic. Then as we already pointed out, $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ have equal dimension, and by Theorem 10.3, the other discrete subgroup is also arithmetic. As Γ_1 and Γ_2 are Zariski-dense and weakly commensurable, by Theorem 1, the groups G_1 and G_2 are either of the same type, or one of them is of type B_n and the other is of type C_n with $n > 2$. Moreover, according to Theorem 6.2, we have $\operatorname{rk}_{\mathbf{R}} G_1 = \operatorname{rk}_{\mathbf{R}} G_2$. Using the classification of connected real simple Lie groups and symmetric spaces (see [17], Chap. X) we find that if G_1 and G_2 are of same Killing-Cartan type, have equal real rank, and the associated symmetric spaces have equal dimension, then $G_1 = G_2$. On the other hand, if G_1 is of type B_n and G_2 is of type C_n, then the fact that they have the same real rank and the associated symmetric spaces have equal dimension implies (cf. [17], Table V on p. 518) that both groups split over $\mathbf{R}$, and hence, $G_1 = SO(n, n+1)$ and G_2 is the adjoint group of Sp_{2n}. Sai-Kee Yeung has just shown, by comparing the traces of the heat operator, that these groups cannot give rise to compact isospectral locally symmetric spaces if $n > 2$. Combining these results with Theorem 4, we obtain the following.

Theorem **10.4.** — *Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and at least one of the subgroups Γ_1 and Γ_2 is arithmetic. Then $G_1 = G_2 := G$. Moreover, unless G is of type A_n, D_{2n+1} $(n > 1)$, D_4 or E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.*

Remark **10.5.** — It would be interesting to determine if Theorem 10.4 remains valid without any assumption of arithmeticity.

The following remark is due to Peter Sarnak.

Remark **10.6.** — Let $\mathcal{G}$ be a connected semi-simple real Lie group of adjoint type and without compact factors; $\mathfrak{X}$ be its symmetric space. If Γ is a torsion-free irreducible cocompact discrete subgroup of $\mathcal{G}$, then the set of conjugacy classes of torsion-free irreducible cocompact discrete subgroups Γ' of $\mathcal{G}$ such that $\mathfrak{X}/\Gamma'$ is isospectral to $\mathfrak{X}/\Gamma$ is finite. This follows from H.C. Wang's finiteness theorem ([36], Chap. IX) if $\mathcal{G}$ is not isomorphic to $PSL_2(\mathbf{R})$, since according to a theorem of André Weil ([36], Theorem 7.63) irreducible cocompact discrete subgroups in such a $\mathcal{G}$ are locally rigid, and $\mathfrak{X}/\Gamma$ and $\mathfrak{X}/\Gamma'$, and therefore, $\mathcal{G}/\Gamma$ and $\mathcal{G}/\Gamma'$ have equal volume. On the other hand, if $\mathcal{G}$ is isomorphic to $PSL_2(\mathbf{R})$, then the finiteness of the conjugacy classes of Γ''s is proved in §5.3 of [22].

Acknowledgements

We thank Alejandro Uribe and Steve Zelditch for the proof of Theorem 10.1. We thank Skip Garibaldi, Philippe Gille, Alex Lubotzky, Dave Morris, Peter Sarnak, Ralf Spatzier and Shing-Tung Yau for their interest in the paper and their comments, and Peter Abramenko and Robert Griess for useful discussions on Weyl groups and root systems. We are grateful to the anonymous referee for his/her comments, corrections and suggestions which have helped us to improve the paper.

Both the authors were supported by the Institute for Advanced Study, the NSF (grants DMS-0400640 and DMS-0502120), the BSF (grant # 200071), the Humboldt Foundation and SFB 701 (Bielefeld). Part of this work was done while the first-named author visited Nagoya University. He would like to thank Shigeyuki Kondo for his kind hospitality.

REFERENCES

1. J. Ax, On Schanuel's conjecture, *Ann. Math. (2)*, **93** (1971), 252–268.
2. A. Baker, *Transcendental Number Theory*, 2nd ed., Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1990.
3. A. Borel, *Linear Algebraic Groups*, 2nd ed., GTM 126, Springer, Berlin, 1991
4. A. Borel and G. Prasad, Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups, *Publ. Math. IHES*, **69** (1989), 119–171.

5. A. Borel and J. Tits, Groupes réductifs, *Publ. Math. IHES*, **27** (1965), 55–151.
6. M. V. Borovoi, Abelianization of the second nonabelian Galois cohomology, *Duke Math. J.*, **72** (1993), 217–239.
7. N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, Chap. III, 1972, Chap. IV–VI, 1968.
8. T. Chinburg, E. Hamilton, D. D. Long, and A. W. Reid, Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds, *Duke Math. J.*, **145** (2008), 25–44.
9. S. DeBacker, Parametrizing conjugacy classes of maximal unramified tori via Bruhat–Tits theory, *Michigan Math. J.*, **54** (2006), 157–178.
10. J. J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.*, **29** (1975), 39–79.
11. J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, Spectra of compact locally symmetric manifolds of negative curvature, *Invent. Math.*, **52** (1979), 27–93.
12. E. V. Erovenko and A. S. Rapinchuk, Bounded generation of S-arithmetic subgroups of isotropic orthogonal groups over number fields, *J. Number Theory*, **119** (2006), 28–48.
13. R. Gangolli, The length spectra of some compact manifolds, *J. Differ. Geom.*, **12** (1977), 403–424.
14. P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, 2nd ed., CRC Press, Boca Raton, 1995,
15. P. Gille, Type des tores maximaux des groupes semi-simples, *J. Ramanujan Math. Soc.*, **19** (2004), 213–230.
16. G. Harder, Bericht über neuere Resultate der Galoiskohomologie halbeinfacher Gruppen, *Jber. Deutsch. Math.-Verein.*, **70** (1967), 182–216.
17. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, AMS, Providence, 2001.
18. K. Kariyama, On conjugacy classes of maximal tori in classical groups, *J. Algebra*, **125** (1989), 133–149.
19. C. J. Leininger, D. B. McReynolds, W. Neumann, and A. W. Reid, Length and eigenvalue equivalence, *Int. Math. Res. Notices*, **24** (2007).
20. A. Lubotzky, B. Samuels, and U. Vishne, Division algebras and noncommensurable isospectral manifolds, *Duke Math. J.*, **135** (2006), 361–379.
21. G. A. Margulis, *Discrete Subgroups of Semi-Simple Lie Groups*, Springer, Berlin, 1991.
22. H. P. McKean, The Selberg trace formula as applied to a compact Riemann surface, *Commun. Pure Appl. Math.*, **25** (1972), 225–246.
23. D. Morris, Bounded generation of SL(n, A) (after D. Carter, G. Keller, and E. Paige), *N. Y. J. Math.*, **13** (2007), 383–421.
24. A. Yu. Ol'shanskii, Periodic quotient groups of hyperbolic groups, *Math. USSR-Sb.*, **72** (1992), 519–541.
25. R. S. Pierce, *Associative Algebras*, GTM 88, Springer, Berlin, 1982.
26. V. P. Platonov and A. S. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, New York, 1994.
27. G. Prasad, Lattices in semi-simple groups over local fields, in *Adv. in Math. Studies in Algebra and Number Theory*, pp. 285–356, Academic Press, New York, 1979.
28. G. Prasad, Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits, *Bull. Soc. Math. France*, **110** (1982), 197–202.
29. G. Prasad and M. S. Raghunathan, Cartan subgroups and lattices in semi-simple groups, *Ann. Math. (2)*, **96** (1972), 296–317.
30. G. Prasad and A. S. Rapinchuk, Computation of the metaplectic kernel, *Publ. Math. IHES*, **84** (1996), 91–187.
31. G. Prasad and A. S. Rapinchuk, Irreducible tori in semisimple groups, *Int. Math. Res. Notices*, **23** (2001), 1229–1242; Erratum: Int. Math. Res. Notices **17** (2002), 919–921.
32. G. Prasad and A. S. Rapinchuk, Existence of irreducible **R**-regular elements in Zariski-dense subgroups, *Math. Res. Lett.*, **10** (2003), 21–32.
33. G. Prasad and A. S. Rapinchuk, Zariski-dense subgroups and transcendental number theory, *Math. Res. Lett.*, **12** (2005), 239–249.
34. G. Prasad and A. S. Rapinchuk, On the existence of isotropic forms of semi-simple algebraic groups over number fields with prescribed local behavior, *Adv. Math.*, **207** (2006), 646–660.
35. G. Prasad and A. S. Rapinchuk, Local-global principles for embedding of fields with involution into simple algebras with involution, available at: arXiv:0806.0596 (to appear in Comment. Math. Helv.).
36. M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, Berlin, 1972.
37. M. S. Raghunathan, Tori in quasi-split groups, *J. Ramanujan Math. Soc.*, **19** (2004), 281–287.
38. A. W. Reid, Isospectrality and commensurability of arithmetic hyperbolic 2- and 3-manifolds, *Duke Math. J.*, **65** (1992), 215–228.
39. J.-P. Serre, *Galois Cohomology*, Springer, Berlin, 1997.

40. D. J. Saltman, *A note on quaternion algebras* (preprint).
41. T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Birkhäuser, Basel, 1998.
42. T. Sunada, Riemann coverings and isospectral manifolds, *Ann. Math. (2)*, **121** (1985), 169–186.
43. J. Tits, Algebraic and abstract simple groups, *Ann. Math. (2)*, **80** (1964), 313–329.
44. J. Tits, Classification of algebraic semisimple groups, in *Algebraic Groups and Discontinuous Groups*. Proc. Sympos. Pure Math., vol. 9, pp. 33–62, Amer. Math. Soc., Providence, 1966.
45. M.-F. Vignéras, Variétés Riemanniennes isospectrales et non isométriques, *Ann. Math. (2)*, **112** (1980), 21–32.
46. E. B. Vinberg, Rings of definition of dense subgroups of semisimple linear groups, *Math. USSR Izv.*, **5** (1971), 45–55.
47. V. E. Voskresenskii, *Algebraic Groups and Their Birational Invariants*, AMS, Providence, 1998.
48. B. Weisfeiler, Strong approximation for Zariski-dense subgroups of semi-simple algebraic groups, *Ann. Math. (2)*, **120** (1984), 271–315.

G. P.
University of Michigan,
Ann Arbor, MI 48109, USA
gprasad@umich.edu

A.S. R.
University of Virginia,
Charlottesville, VA 22904, USA
asr3x@virginia.edu

Manuscrit reçu le 24 septembre 2007
publié en ligne le 31 mars 2009.