LANDAU-GINZBURG/CALABI-YAU CORRESPONDENCE, GLOBAL MIRROR SYMMETRY AND ORLOV EQUIVALENCE

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ABSTRACT

We show that the Gromov-Witten theory of Calabi-Yau hypersurfaces matches, in genus zero and after an analytic continuation, the quantum singularity theory (FJRW theory) recently introduced by Fan, Jarvis and Ruan following a proposal of Witten. Moreover, on both sides, we highlight two remarkable integral local systems arising from the common formalism of $\widehat{\Gamma}$-integral structures applied to the derived category of the hypersurface $\{W = 0\}$ and to the category of graded matrix factorizations of W. In this setup, we prove that the analytic continuation matches Orlov equivalence between the two above categories.

CONTENTS

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Notation

$W(x_1, \dots, x_N)$	weighted homogeneous polynomial (Section 1.1)
X_W	Calabi-Yau hypersurface defined by W in $\mathbf{P}(\underline{w})$ (Section 1.1)
$\boldsymbol{\mu}_d$	the group of d-th roots of unity
H	state space ($H(W, \boldsymbol{\mu}_d)$ or $H_{CR}(X_W)$, Sections 2.1.1, 2.2.1)
H'	narrow/ambient part (Section 2.3.1)
H''	broad/primitive part (Section 2.3.1)
$\{t^i\}$	linear co-ordinates on the state space associated to a basis $\{T_i\}$ (Sections 2.3, 2.4, 3.5.2)
$\widehat{\Gamma}$	Gamma class (Section 2.4.4)
$\overline{H}$	state space of twisted theories (H_{ext} or $H_{CR}(\mathbf{P}(\underline{w}))$, Section 3.5.2)
$\mathcal{M}$	global Kähler moduli space ($\mathbf{P}(1, d) \setminus \{0, v_c\}$, Section 5.1)

1. Introduction

1.1. *Overview.* — The so-called Landau-Ginzburg/Calabi-Yau correspondence (LG/CY correspondence for short) in string theory [30, 51, 67] describes a relationship between the sigma model on a Calabi-Yau hypersurface and the Landau-Ginzburg model whose potential is the defining equation of the Calabi-Yau. In Witten's gauged linear sigma model [69], the LG/CY correspondence arises, roughly speaking, from a variation of GIT quotient.

Let $w_1, \dots, w_N$ be coprime positive integers and $x_1, \dots, x_N$ be variables of degree $w_1, \dots, w_N$. Let $W(x_1, \dots, x_N)$ be a weighted homogeneous polynomial of degree d which has an isolated critical point only at the origin. We assume (i) the Calabi-Yau condition $d = w_1 + \cdots + w_N$ and (ii) the Gorenstein condition:[1] w_j divides d for all $1 \le j \le N$. In this paper, we discuss two objects:

- The Calabi-Yau hypersurface $X_W = \{W = 0\}$ in the weighted projective space $\mathbf{P}(\underline{w}) = \mathbf{P}(w_1, \dots, w_N)$. This is quasi-smooth (i.e. smooth in the sense of stacks) by the assumption on W above.
- The Landau-Ginzburg orbifold $(\mathbf{C}^N, W, \boldsymbol{\mu}_d)$. It consists of the space $\mathbf{C}^N$ equipped with an action of $\boldsymbol{\mu}_d = \{g \in \mathbf{C}^\times \mid g^d = 1\}$, $(x_1, \dots, x_N) \mapsto (g^{w_1} x_1, \dots, g^{w_N} x_N)$ and a $\boldsymbol{\mu}_d$-invariant function $W \colon \mathbf{C}^N \to \mathbf{C}$.

These two models arise from the following GIT quotient. Consider the $\mathbf{C}^\times$-action on the vector space $\mathbf{C}^N \times \mathbf{C}$ with co-ordinates $(x_1, \dots, x_N, p)$:

$$(x_1, \dots, x_N, p) \mapsto \left(t^{w_1} x_1, \dots, t^{w_N} x_N, t^{-d} p\right), \qquad t \in \mathbf{C}^\times.$$

We endow the space $\mathbf{C}^N \times \mathbf{C}$ with the $\mathbf{C}^\times$-invariant potential $\widetilde{W}(x, p) := pW(x)$. There are two possible GIT quotients of this space: one is the quotient of $(\mathbf{C}^N \setminus \{0\}) \times \mathbf{C}$ and

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[1] This means that the ambient space $\mathbf{P}(\underline{w})$ is Gorenstein. In this case, we can take W to be the Fermat type polynomial $W = x_1^{d/w_1} + \cdots + x_N^{d/w_N}$.

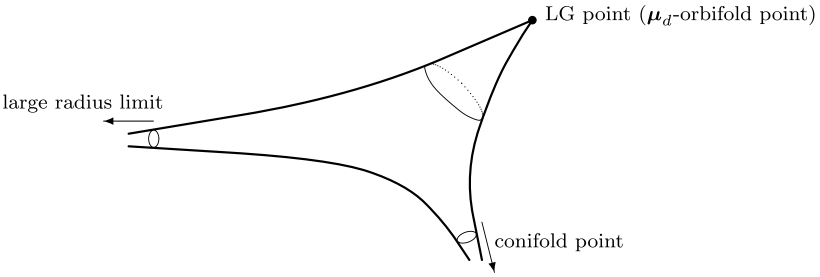

FIG. 1. — Kähler moduli space $\mathcal{M} \cong \mathbf{P}(1, d) \setminus \{\text{two points}\}$

the other is the quotient of $\mathbf{C}^N \times (\mathbf{C} \setminus \{0\})$. In the former case, we get the total space of the line bundle $\mathcal{O}(-d) \to \mathbf{P}(\underline{w})$ endowed with the function $\widetilde{W}$. This should reduce to the sigma model on the Calabi-Yau hypersurface X_W. In the latter case, we get the Landau-Ginzburg orbifold $(\mathbf{C}^N, W, \boldsymbol{\mu}_d)$.

The GIT quotient itself does not change inside a "chamber" of stability parameters, but the actual physical theory depends on a *continuous and complexified* stability parameter $r + \mathtt{i}\theta \in \mathbf{C}$. The CY theory arises for $r \to \infty$ and the LG theory for $r \to -\infty$. The stability parameter $r + \mathtt{i}\theta$ varies along a complex manifold $\mathcal{M}$ called the global Kähler moduli space. In the case at hand, it is identified with (a Zariski open subset of) the weighted projective line $\mathbf{P}(1, d)$. The local picture near the $\boldsymbol{\mu}_d$-point in $\mathbf{P}(1, d)$ corresponds to the LG model above and the $\boldsymbol{\mu}_d$-point is called LG point. The local picture near the antipodal point corresponds to the CY geometry and the antipodal point is called large radius limit point. These points are interesting asymptotically: we often work on punctured discs centered on them and refer to them as limit points (see Figure 1). There is another limit point in the Kähler moduli space where the mirror has a conifold singularity; by abuse of terminology, we refer to this limit point as a conifold point.

This paper is concerned with two aspects of topological string theory: the category of D-branes of type B (B-branes) and the closed string theory of type A (A-model). In this paper, the term "B-brane" (or "brane") has a precise mathematical meaning. On the CY side, the category of B-branes is the derived category of coherent sheaves on the Calabi-Yau hypersurface X_W. On the LG side, the category of B-branes is identified with the category $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(W)$ of graded matrix factorizations of W [37, 52, 68]. On the other hand, the mathematical A-model on the Calabi-Yau X_W is given by GW (Gromov-Witten) theory. The mathematical A-model for the Landau-Ginzburg orbifold was formulated recently by Fan-Jarvis-Ruan [26] as the intersection theory on the moduli space of W-spin curves. This is called FJRW (Fan-Jarvis-Ruan-Witten) theory. About these theories, the following LG/CY correspondences are known in mathematics:

(1) **B-brane LG/CY correspondence**: Orlov [53] constructed derived equivalences Φ_l between the categories of B-branes indexed by an integer $l \in \mathbf{Z}$:

$$\Phi_l \colon D^b(X_W) \cong \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(W).$$

(2) **A-model LG/CY correspondence**: Chiodo-Ruan [13] showed that for a quintic polynomial $W(x_1, \dots, x_5)$, GW theory of X_W is analytically continued to FJRW theory of $(\mathbf{C}^5, W, \boldsymbol{\mu}_5)$ at genus zero. Schematically, we write

$$\mathrm{GW}_{g=0}(X_W) \cong \mathrm{FJRW}_{g=0}\big(\mathbf{C}^5, W, \boldsymbol{\mu}_5\big).$$

The purpose of this paper is to extend the correspondence (2) to a general weighted homogeneous polynomial W and to describe a relationship between (1) and (2).

More precisely, "analytic continuation" in (2) means the following. In genus zero, GW theory and FJRW theory yield *quantum* D*-modules* over small neighbourhoods of the corresponding limit points; we show that these are restrictions of a certain *global* D-module over the Kähler moduli space $\mathcal{M}$. Note that the category of B-branes should be independent of the Kähler structure on X_W. Hence B-branes are "locally constant" data over the Kähler moduli space around the limit point in each theory. In fact, we associate to each B-brane a *flat section* of the quantum D-module. The flat section here is asymptotic (in the limit point) to the Chern character of the brane multiplied by the $\widehat{\Gamma}$-class. This defines a $\mathbf{Z}$-local system underlying the quantum D-module whose fibre is the numerical K-group of the category of B-branes. We call it the $\widehat{\Gamma}$*-integral structure* of the quantum D-module. This has been introduced for GW theory by Iritani [42] and Katzarkov-Kontsevich-Pantev [44]. Here the role of the $\widehat{\Gamma}$-class (see Definition 2.17) is to preserve the Euler pairing $\chi(\mathcal{E}, \mathcal{F}) := \sum_{i\in\mathbf{Z}} \dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$ in B-brane categories. Indeed, $\widehat{\Gamma}$ can be regarded as a square root of the Todd class. When X_W is a manifold, we have

$$\Big((-1)^{\frac{\deg}{2}}\widehat{\Gamma}_{X_W}\Big)\cdot\widehat{\Gamma}_{X_W} = (2\pi\mathrm{i})^{\frac{\deg}{2}}\,\mathrm{Td}_{X_W}$$

thanks to the functional equation $\Gamma(1-z)\Gamma(1+z) = \pi z/\sin(\pi z)$. Thus one can think of our flat section associated to a B-brane as a quantum version of the Mukai vector.[2] In this paper, we extend the $\widehat{\Gamma}$-integral structure to FJRW theory. Our main results are stated as follows. We refer the reader to Theorems 2.23, 2.25 for more precise statements.

Theorem **1.1.**

(i) *The ambient part quantum* D*-module of* X_W *and the narrow part quantum* D*-module of* $(\mathbf{C}^N, W, \boldsymbol{\mu}_d)$ *are analytically continued to each other,*[3] *i.e. both of them are the restrictions of a global* D*-module over the Kähler moduli space* $\mathcal{M}$.

(ii) *The analytic continuation in* (i) *matches up the* $\widehat{\Gamma}$*-integral structures on both quantum* D*-modules. Moreover, the induced isomorphism between the numerical* K*-groups of the categories of B-branes coincide with the one induced by the Orlov derived equivalence.*

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[2] In fact, it coincides with the Mukai vector for K3 surfaces.

[3] See Section 2.4.2 for the definition of the quantum D-modules. We mean by "ambient part" the cohomology classes pulled back from the ambient space, see Section 2.2.1; in FJRW side, this has a counterpart called "narrow part", see Section 2.1.1.

A prototype of our result is the work of Borisov-Horja [5], where they showed that the analytic continuation of the GKZ hypergeometric system is induced from a Fourier-Mukai transformation between the K-groups of toric Calabi-Yau orbifolds, under a suitable identification of the spaces of local solutions with the K-groups. In our case, the GKZ system is replaced with the quantum D-modules of GW/FJRW theories and the Fourier-Mukai transformation is replaced with the Orlov equivalence.

Since the global D-module over $\mathcal{M}$ have nontrivial monodromies, the analytic continuation of flat sections depends on the choice of (a homotopy type of) a path. On the other hand, the Orlov equivalence Φ_l depends on an integer $l \in \mathbf{Z}$. The recent physics paper [34] by Herbst-Hori-Page clarified (by a physical argument) the dependence of a derived equivalence on the choice of a path. We confirm the prediction of [34] that a path γ_l passing through the lth "window" corresponds to the lth Orlov equivalence Φ_l. Moreover, we check that the monodromy representation of the fundamental group of $\mathcal{M} = \mathbf{P}(1, d) \setminus \{2 \text{ points}\}$ factors through the group of autoequivalences of $\mathrm{D}^b(\mathrm{X_W})$. The following theorem refines Part (ii) of Theorem 1.1.

Theorem **1.2** *(Theorem 2.26).*

(i) *For each integer $l \in \mathbf{Z}$, there exists a path γ_l from a neighbourhood of the large radius limit point to a neighbourhood of the LG point such that the analytic continuation along γ_l^{-1} is induced by the Orlov equivalence Φ_l.*

(ii) *Let $\mathrm{N}'(\mathrm{X_W})$ be the subgroup* (**23**) *of the numerical K-group of $\mathrm{X_W}$ consisting of K-classes whose Chern characters lie in the ambient cohomology $\mathrm{H_{amb}}(\mathrm{X_W})$ and let χ be the Euler pairing. The monodromy representation of the global D-module in Theorem 1.1*

$$\rho\colon \pi_1(\mathcal{M}) \to \mathrm{Aut}\big(\mathrm{N}'(\mathrm{X_W}), \chi\big)$$

can be lifted to a group homomorphism

$$\hat{\rho}\colon \pi_1(\mathcal{M}) \to \mathrm{Auteq}\big(\mathrm{D}^b(\mathrm{X_W})\big)/[2],$$

where $[2]$ is the 2-shift functor. The homomorphism $\hat{\rho}$ sends a (clockwise) loop around the conifold point to the spherical twist by the structure sheaf.

Since the work of Seidel-Thomas [63], the monodromy group action on $\mathrm{D}^b(\mathrm{X})$ has been widely studied. Horja [38] identified the conifold monodromy of the GKZ system with the spherical twist. Aspinwall [3, Section 7.1.4] observed that the 5th power of the monodromy around the LG point corresponds to the 2-shift (for a quintic). We deduce the existence of the lift $\hat{\rho}$ from a result of Canonaco-Karp [9]. The above theorem suggests an autoequivalence group action on GW theory. However we do not know if $\hat{\rho}$ is injective. The induced homomorphism ρ is never injective when $\dim \mathrm{X_W}$ is even (since the conifold monodromy is involutive), but it is still possible that ρ is injective for an odd-dimensional Calabi-Yau $\mathrm{X_W}$.

1.2. *Mirror symmetry.* — The interaction between B-branes and the A-model above can be explained most clearly via mirror symmetry. Here we consider Hodge-theoretic mirror symmetry, Kontsevich's homological mirror symmetry [46] and their mutual relationships. See also [15] for the discussion on global mirror symmetry for finite group quotients of Calabi-Yau hypersurfaces.

The mirror of X_W is given by a certain Calabi-Yau compactification Y_v (Batyrev's mirror [4]) of the affine variety

$$Y_v^\circ := \left\{(x_1, \dots, x_N) \in \left(\mathbf{C}^\times\right)^N \,\middle|\, x_1 + \cdots + x_N = 1,\ x_1^{w_1} x_2^{w_2} \cdots x_N^{w_N} = v\right\}$$

where the parameter v is identified with an inhomogeneous co-ordinate of $\mathcal{M} = \mathbf{P}(1, d) \setminus \{2 \text{ points}\}$ such that $v = 0$ is the large radius limit and that $v = \infty$ is the LG point. The mirror Y_v may have Gorenstein terminal quotient singularities. Note that $\mathcal{M}$ now plays a role of the complex moduli of Y_v. Under mirror symmetry, the category of B-branes should be equivalent to the category of A-branes of the mirror. Mathematically, the category of A-branes is the derived Fukaya category whose objects are (twisted complexes of) graded Lagrangian submanifolds. Likewise, the A-model theory should be equivalent the B-model theory of the mirror, which is, at genus zero, the variation of Hodge structure associated to the deformation of the complex structure. We get the mirror statements of (1) and (2).

(1′) **A-brane "mirror LG/CY" correspondence:** The derived Fukaya category of Y_v is independent of $v \in \mathcal{M}$.

(2′) **B-model "mirror LG/CY" correspondence:** There exists a global variation of Hodge structure (VHS) $H^{N-2}(Y_v) = \bigoplus_{p+q=N-2} H^{p,q}(Y_v)$ over $\mathcal{M}$.

Because Y_v does not change as a symplectic manifold (or orbifold) as v varies, the Fukaya category should be independent of v (if it is defined). The B-model VHS is tautologically "analytically continued" over $\mathcal{M}$. Moreover, the category of A-branes and the B-model have a natural integration pairing. Namely, one can integrate a de Rham cohomology class on Y_v over a Lagrangian submanifold. By this pairing, an A-brane (a Lagrangian submanifold) gives rise to a dual flat section of the B-model VHS, i.e. a middle homology class in $H_{N-2}(Y_v, \mathbf{Z})$ represented by the brane. This is exactly dual to the phenomenon we described in Section 1.1.

The $\widehat{\Gamma}$-integral structure in GW theory for X_W is actually mirrored from the natural integral structure in the B-model of Y_v (see also [15, Conjecture 4.2.10]).

Theorem **1.3** *([43, Theorem 6.9]). — The ambient A-model VHS of a Calabi-Yau hypersurface* X_W *equipped with the ambient* $\widehat{\Gamma}$*-integral structure is isomorphic to the residual B-model VHS of* Y_v *equipped with the vanishing cycle integral structure near* $v = 0$ *under the mirror map* $\tau_{\mathrm{GW}} \colon \{|v| < \epsilon\} \to H^2_{\mathrm{amb}}(X_W)/\langle G \rangle$ *in Theorem 2.23.*

Here, the ambient A-model VHS is the ambient part quantum D-module in Theorem 1.1 restricted to $z = 1$ (see Remark 2.13); the ambient $\widehat{\Gamma}$-integral structure is the

$\mathbf{Z}$-local system consisting of flat sections associated to vector bundles on X_W which are restricted from the ambient space $\mathbf{P}(\underline{w})$. The residual B-model VHS is defined to be the pure part $\mathrm{Gr}^{\mathcal{W}}_{N-2} H^{N-2}(Y^\circ_v) \subset H^{N-2}(Y_v)$ of the Deligne mixed Hodge structure of the affine variety Y°_v; the vanishing cycle integral structure on it is spanned by the Poincaré duals of vanishing cycles of the function $\mathsf{x}_1 + \cdots + \mathsf{x}_N$ on the torus $\{(\mathsf{x}_1, \ldots, \mathsf{x}_N) \in (\mathbf{C}^\times)^N \mid \prod_{i=1}^N \mathsf{x}_i^{w_i} = v\}$. See [43] for the details. Because K-classes of vector bundles restricted from $\mathbf{P}(\underline{w})$ correspond, under Orlov equivalence, to the K-classes of graded Koszul matrix factorizations (Proposition 4.11), we have the following corollary (see also [15, Conjecture 4.2.11]).

Corollary **1.4.** — *The narrow A-model VHS of the Landau-Ginzburg model* $(\mathbf{C}^N, W, \boldsymbol{\mu}_d)$ *equipped with the subsystem of the* $\widehat{\Gamma}$*-integral structure spanned by* K*-classes of graded Koszul matrix factorizations is isomorphic to the residual B-model VHS of* Y_v *equipped with the vanishing cycle integral structure near* $v = \infty$ *under the mirror map* $\tau_{\mathrm{FJRW}} : \{|v|^{-1/d} < \epsilon\} \to H^2_{\mathrm{nar}}(W, \boldsymbol{\mu}_d)/\langle G\rangle$ *in Theorem 2.23.*

In particular, both the quantum D-module of X_W and of $(\mathbf{C}^N, W, \boldsymbol{\mu}_d)$ over the image of the mirror map give a polarized variation of $\mathbf{Z}$-Hodge structure.

1.3. *Plan of the paper.* — In Section 2, we introduce the $\widehat{\Gamma}$-integral structure on the quantum D-module associated to FJRW and GW theories. Then we state our main theorems in a precise way. In Section 3, we introduce twisted FJRW invariants and calculate the (twisted) I-function of FJRW theory. This provides the main ingredients of the paper. In Section 4, we calculate the analytic continuation of the I-function and show that the connection matrix matches the Orlov equivalence. In Section 5, we construct a global D-module over the Kähler moduli and prove the main theorems.

2. $\widehat{\Gamma}$-integral structure and main statements

We briefly review FJRW (Fan-Jarvis-Ruan-Witten) theory for $(W, \boldsymbol{\mu}_d)$ and GW (Gromov-Witten) theory for X_W and introduce the $\widehat{\Gamma}$-integral structure on the quantum D-modules of both theories. Then we give a precise statement of the main results.

2.1. *FJRW theory.* — The FJRW invariants "count" the number of solutions to a non-linear PDE, the so-called Witten equation. These define a cohomological field theory on the FJRW state space. In this paper, we restrict ourselves to the genus zero FJRW invariants from the "narrow part". In this case, the Witten equation has only trivial solutions and the invariants reduce to intersection numbers of tautological classes on the moduli space of d-spin curves. For the details of the full FJRW theory, we refer the reader to the original articles [25, 26].

2.1.1. *State space.* — Let $(\mathbf{C}^{\mathrm{N}}, \mathrm{W}, \boldsymbol{\mu}_d)$ be the Landau-Ginzburg orbifold in the previous Section 1.1. Let $\zeta := \exp(2\pi \mathtt{i}/d) \in \boldsymbol{\mu}_d$ denote a primitive dth root of unity. Let $(\mathbf{C}^{\mathrm{N}})_k$ denote the ζ^k-fixed subspace of $\mathbf{C}^{\mathrm{N}}$ and $\mathrm{W}_k\colon (\mathbf{C}^{\mathrm{N}})_k \to \mathbf{C}$ denote the restriction of W. We also write $\mathrm{N}_k = \dim_{\mathbf{C}}(\mathbf{C}^{\mathrm{N}})_k$. The FJRW state space $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)$ is defined to be

$$\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d) := \bigoplus_{k=0}^{d-1} \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$$

where the sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ associated to $\zeta^k \in \boldsymbol{\mu}_d$ is given by

$$\begin{aligned}\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k &:= \mathrm{H}^{\mathrm{N}_k}\big((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty}; \mathbf{C}\big)^{\boldsymbol{\mu}_d},\\ \mathrm{W}_k^{\pm\infty} &:= \big\{x \in (\mathbf{C}^{\mathrm{N}})_k\colon\ \pm\Re\big(\mathrm{W}_k(x)\big) \gg 0\big\}.\end{aligned}$$

The degree of an element $\phi \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ is defined to be

$$\deg\phi := \mathrm{N}_k + 2\sum_{i=1}^{\mathrm{N}} \langle kq_i\rangle - 2 \tag{1}$$

where $q_i := w_i/d$. This is an integer since $\sum_{i=1}^{\mathrm{N}} q_i = 1$. Let $\langle\cdot,\cdot\rangle$ denote the natural intersection pairing

$$\langle\cdot,\cdot\rangle\colon \mathrm{H}^{\mathrm{N}_k}\big((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty}; \mathbf{C}\big) \times \mathrm{H}^{\mathrm{N}_k}\big((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{-\infty}; \mathbf{C}\big) \to \mathbf{C} \tag{2}$$

and $\mathrm{I}\colon \mathbf{C}^{\mathrm{N}} \to \mathbf{C}^{\mathrm{N}}$ denote the map $(x_1, \ldots, x_{\mathrm{N}}) \mapsto (\tilde{\zeta}^{w_1}x_1, \ldots, \tilde{\zeta}^{w_{\mathrm{N}}}x_{\mathrm{N}})$ for $\tilde{\zeta} = \exp(\pi\mathtt{i}/d)$. Because $\mathrm{W}(\mathrm{I}(x)) = -\mathrm{W}(x)$, we have a map

$$\mathrm{I}^*\colon \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_{d-k} \cong \mathrm{H}^{\mathrm{N}_k}\big((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty}; \mathbf{C}\big)^{\boldsymbol{\mu}_d} \to \mathrm{H}^{\mathrm{N}_k}\big((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{-\infty}; \mathbf{C}\big)^{\boldsymbol{\mu}_d}. \tag{3}$$

We define the pairing between $\alpha_1 \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ and $\alpha_2 \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_{d-k}$ by

$$(\alpha_1, \alpha_2) := \frac{1}{d}\big\langle\alpha_1, \mathrm{I}^*\alpha_2\big\rangle. \tag{4}$$

Setting $(\alpha_1, \alpha_2) = 0$ for $\alpha_1 \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$, $\alpha_2 \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_l$ with $k + l \neq d$, we obtain a graded symmetric non-degenerate pairing $(\cdot,\cdot)$ on the state space $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)$. The pairing in this paper differs from that in [26] by the factor $1/d = 1/|\boldsymbol{\mu}_d|$. See Appendix B for this convention.

We say that a sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ is *narrow* if $(\mathbf{C}^{\mathrm{N}})_k = \{0\}$ and *broad* otherwise.[4] Each narrow sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ is one-dimensional and we denote by $\phi_{k-1} \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ the standard basis given as the identity class on $(\mathbf{C}^{\mathrm{N}})_k = \{0\}$.[5] We set

$$\mathsf{Nar} := \big\{0 \le k \le d-1\ :\ \big(\mathbf{C}^{\mathrm{N}}\big)_k = \{0\}\ \text{i.e. } kq_i \notin \mathbf{Z} \text{ for all } i\big\}$$

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[4] Fan-Jarvis-Ruan originally called these sectors "Neveu-Schwarz" and "Ramond" respectively, but they changed the names later.

[5] Note the shift of the index k by one. An element ϕ_k with $k + 1 \notin \mathsf{Nar}$ will be introduced later in Section 3 in the context of "extended theory", but notice that ϕ_k with $k + 1 \notin \mathsf{Nar}$ is *not* an element of the original FJRW state space.

and define the *narrow part* as

$$\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d) = \bigoplus_{k \in \mathsf{Nar}} \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k = \bigoplus_{k \in \mathsf{Nar}} \mathbf{C}\phi_{k-1}.$$

Note that $\deg \phi_k = 2\sum_{j=1}^{\mathrm{N}} \langle kq_j \rangle$ for $k+1 \in \mathsf{Nar}$. The degree zero element $\phi_0 \in \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_1$ plays the role of the identity in the FJRW theory. The pairing $(\cdot,\cdot)$ restricts to a non-degenerate pairing on $\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$

$$(\phi_k, \phi_l) = \frac{1}{d}\delta_{d,(k+1)+(l+1)}, \qquad k+1, l+1 \in \mathsf{Nar} \tag{5}$$

and $\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$ is orthogonal to the *broad part* $\mathrm{H}_{\mathrm{bro}}(\mathrm{W}, \boldsymbol{\mu}_d) := \bigoplus_{k \notin \mathsf{Nar}} \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$.

For a polynomial f on $\mathbf{C}^n$, we define the *Jacobi space*[6] $\Omega(f)$ by

$$\Omega(f) := \Omega^n_{\mathbf{C}^n} / \mathsf{d}f \wedge \Omega^{n-1}_{\mathbf{C}^n}, \tag{6}$$

where $\Omega^k_{\mathbf{C}^n}$ denotes the space of algebraic k-forms on $\mathbf{C}^n$. When f has an isolated critical point at the origin, we have the Grothendieck residue pairing $\mathrm{Res}_f \colon \Omega(f) \otimes \Omega(f) \to \mathbf{C}$ (see [31]):

$$\mathrm{Res}_f\big([a(y)\mathsf{d}y], [b(y)\mathsf{d}y]\big) := \mathrm{Res}\left[\frac{a(y)b(y)\mathsf{d}y}{\partial_1 f, \ldots, \partial_n f}\right],$$

where $y = (y_1, \ldots, y_n)$ is a co-ordinate system on $\mathbf{C}^n$ and $\mathsf{d}y = \mathsf{d}y_1 \wedge \cdots \wedge \mathsf{d}y_n$. The residue pairing is independent of the choice of co-ordinates. We shall consider the Jacobi space $\Omega(\mathrm{W}_k)$ associated to the homogeneous polynomial W_k on $(\mathbf{C}^{\mathrm{N}})_k$. The space $\Omega(\mathrm{W}_k)$ is graded by the usual degree: $\deg x_j = \deg \mathsf{d}x_j = w_j$.

Proposition **2.1.** — *We have a canonical isomorphism*

$$\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k \cong \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}. \tag{7}$$

Under this isomorphism, the pairing $(\cdot,\cdot)\colon \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k \times \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_{d-k} \to \mathbf{C}$ *translates into the Grothendieck residue pairing between* $\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ *and* $\Omega(\mathrm{W}_{d-k})^{\boldsymbol{\mu}_d} \cong \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$*:*

$$[\varphi] \otimes [\psi] \longmapsto (-1)^{\frac{\mathrm{N}_k(\mathrm{N}_k-1)}{2}} (2\pi \mathtt{i})^{\mathrm{N}_k} \frac{1}{d} \mathrm{Res}_{\mathrm{W}_k}\big([\varphi], (-1)^{|\psi|}[\psi]\big),$$

where $|\psi|$ *is the degree of* $\psi \in \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ *divided by* d. *Notice that the above isomorphism does not match up the grading on the FJRW state space* $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ *(which is homogeneous of degree* $\mathrm{N}_k + 2(\sum_{j=1}^{\mathrm{N}} \langle kq_j \rangle - 1)$*) and the grading on the Jacobi space* $\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$.

The isomorphism (7) is given by the Hodge decomposition (**95**). See Appendix A for the proof. This description is used in Section 4.1.1 (and in Section 2.4.4) to discuss the Chern character and Riemann-Roch for matrix factorizations.

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[6] It is isomorphic to the Jacobi ring $\mathbf{C}[y_1, \ldots, y_n]/(\partial_1 f, \ldots, \partial_n f)$, but notice that $\Omega(f)$ is not a ring.

2.1.2. *Moduli of W-curves.* — In this paper all stacks are defined over $\mathbf{C}$. By a *pointed orbicurve*, we mean a proper and connected one-dimensional Deligne-Mumford stack C which has only nodes as singularities, which is equipped with distinct marked points $\sigma_1, \dots, \sigma_n$ on the smooth locus, and which has possibly non-trivial stabilizers only at the marked points and the nodes. For a positive integer d, a pointed orbicurve C is called *d-stable* [11] if the associated pointed coarse curve $|\mathrm{C}|$ is stable and if all the stabilizers at the nodes and the marked points are isomorphic to $\boldsymbol{\mu}_d$. We always assume that every node of an orbicurve is balanced [2], i.e. formally locally near a node, the curve is isomorphic to $[\mathrm{Spec}(\mathbf{C}[x,y]/xy)/\boldsymbol{\mu}_d]$, where the action of $\boldsymbol{\mu}_d$ is given by $(x,y) \mapsto (\zeta x, \zeta^{-1} y)$. For a pointed orbicurve $(\mathrm{C}, \sigma_1, \dots, \sigma_n)$, we introduce an invertible sheaf $\omega_{\log}$ on C which may be regarded as the dualizing sheaf ω_{C} twisted at the stacky markings $\sigma_1, \dots \sigma_n$ (each of which is of degree $1/d$) or equivalently as the pullback of the dualizing sheaf $\omega_{|\mathrm{C}|}$ twisted at the coarse markings $|\sigma_1|, \dots |\sigma_n|$ (each of which is of degree one)

$$\omega_{\log} = \omega_{\mathrm{C}}(\sigma_1 + \cdots + \sigma_n) = q^*\big(\omega_{|\mathrm{C}|}\big(|\sigma_1| + \cdots + |\sigma_n|\big)\big)$$

where $q\colon \mathrm{C} \to |\mathrm{C}|$ is the natural map. In other words, $\omega_{\log}$ is the sheaf of logarithmic differential forms on C with poles only at marked points and nodes, and such that the sum of the residues at each node is zero.

A *d-spin structure* on a pointed orbicurve C is a line bundle $\mathrm{L} \to \mathrm{C}$ together with an isomorphism $\varphi\colon \mathrm{L}^{\otimes d} \cong \omega_{\log}$. Write $\mathrm{W}(x_1, \dots, x_{\mathrm{N}}) = \sum_{i=1}^{l} c_i \prod_{j=1}^{\mathrm{N}} x_j^{m_{ij}}$, where $\prod_{j=1}^{\mathrm{N}} x_j^{m_{ij}}$, $i = 1, \dots, l$ are mutually distinct monomials and $c_i \neq 0$. A W-*structure* on a pointed orbicurve C is a collection of line bundles $\mathrm{L}_1, \dots, \mathrm{L}_{\mathrm{N}}$ (corresponding to the variables $x_1, \dots, x_{\mathrm{N}}$) on C together with isomorphisms

$$\varphi_i\colon \bigotimes_{j=1}^{\mathrm{N}} \mathrm{L}_j^{\otimes m_{ij}} \cong \omega_{\log}, \qquad i = 1, \dots, l. \tag{8}$$

This generalizes the notion of a d-spin structure (see Remark 2.2). Since W is weighted homogeneous of degree d, a d-spin structure $\mathrm{L} \to \mathrm{C}$ gives rise to a W-structure by setting

$$\mathrm{L}_i = \mathrm{L}^{\otimes w_i}, \qquad i = 1, \dots, \mathrm{N}.$$

A W-structure does not necessarily arise from a d-spin structure in this way. In this paper, however, we restrict our attention to a W-structure coming from a d-spin structure.[7]

Let L be a d-spin structure on a d-stable pointed orbicurve C. The stabilizer $\boldsymbol{\mu}_d$ at a marked point σ acts on the fibre L_σ via a homomorphism $\boldsymbol{\mu}_d \to \mathbf{C}^\times$, which is of the form $t \mapsto t^k$ for a unique $0 \le k < d$. We call the rational number $\mathrm{age}_\sigma(\mathrm{L}) := k/d \in$

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[7] This is because we are interested in a *generic* weighted homogeneous polynomial W. More precisely, we can add to W a weighted homogeneous *Laurent* polynomial Z so that the group of diagonal symmetries preserving $\mathrm{W} + \mathrm{Z}$ is exactly $\boldsymbol{\mu}_d$; then every $(\mathrm{W} + \mathrm{Z})$-structure comes from a d-spin structure. This means that the group $\boldsymbol{\mu}_d$ is *admissible* in the sense of [26, Section 2.3].

$[0, 1)$ the *age* of L at σ. The generator $\zeta \in \boldsymbol{\mu}_d$ acts on the fibre of the associated W-structure $(\mathrm{L}^{\otimes w_1}, \dots, \mathrm{L}^{\otimes w_\mathrm{N}})$ at σ by $(\zeta^{kw_1}, \dots, \zeta^{kw_\mathrm{N}})$; hence in this case we regard the marked point σ as corresponding to the sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$. For $0 \le k_1, \dots, k_n \le d-1$, let $\mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$ denote the moduli stack of d-stable orbicurves C of genus zero and with n marked points $\sigma_1, \dots, \sigma_n$ endowed with a d-spin structure $\mathrm{L} \to \mathrm{C}$ such that $\mathrm{age}_{\sigma_i}(\mathrm{L}) = \langle (k_i + 1)/d \rangle$.

$$\begin{aligned}(\mathbf{9}) \qquad & \mathrm{Spin}^d_{0,n}(k_1, \dots, k_n) \\ &= \left\{ \left(\mathrm{C}; \sigma_1, \dots, \sigma_n; \mathrm{L}; \varphi \colon \mathrm{L}^{\otimes d} \cong \omega_{\log}\right) \,\middle|\, \mathrm{age}_{\sigma_i}(\mathrm{L}) = \left\langle \frac{k_i+1}{d} \right\rangle \right\} \Big/ \mathrm{isom}.\end{aligned}$$

See [13, Appendix] for a precise definition of $\mathrm{Spin}^d_{0,n}$ as a fibred category, where it is denoted by $\mathcal{R}_d$. More precisely the substack denoted by $\mathcal{R}_d(e^{2\pi \mathrm{i} \Theta_1}, \dots, e^{2\pi \mathrm{i} \Theta_n})$ in [13] corresponds to (**9**) as soon as $\Theta_j = (k_j + 1)/d$; the present choice of notation allows a more straightforward formula for the FJRW invariants, see (**12**) and the following footnote 9.

The moduli stack $\mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$ is smooth, proper and of Deligne-Mumford type [11]. It is non-empty if and only if $\chi(\mathrm{L}) = \deg \mathrm{L} - \sum_{i=1}^n \mathrm{age}_{\sigma_i}(\mathrm{L}) - 1 = (n-2)/d - \sum_{i=1}^n \langle (k_i + 1)/d \rangle - 1$ is an integer (see [1, 45, 65] for Riemann-Roch for orbicurves). When non-emtpy, it is of dimension $n - 3$. (It differs from the Deligne-Mumford space $\overline{\mathcal{M}}_{0,n}$ of stable curves only because of the stabilizers.) It is clear that the definition (**9**) extends verbatim to higher genera; we write $\mathrm{Spin}^d_{g,n}(k_1, \dots, k_n)$ for the similar moduli space at genus g. We use it only in Section 2.1.3 where we recall the general setup of [26].

Remark **2.2.** — The definition (**8**) of a W-structure originates from the Witten equation, which is a system of PDE for sections $s_i \in \mathrm{C}^\infty(\mathrm{C}, \mathrm{L}_i)$, $i = 1, \dots, \mathrm{N}$:

$$(\mathbf{10}) \qquad \overline{\partial} s_j + \overline{\partial_j \mathrm{W}(s_1, \dots, s_\mathrm{N})} = 0.$$

The equation makes sense under (**8**) and a suitable choice of a Hermitian metric on L_i (see [25]). When all the marked points correspond to the narrow sector, the zero sections are only possible solutions to the Witten equation [25, Theorem 3.3.8] and the FJRW invariant is the Euler class of the obstruction bundle over the moduli space of W-curves. We briefly review the general case.

2.1.3. *FJRW invariants.* — We review the virtual fundamental class defining the invariants of "full" FJRW theory following [26]. The general FJRW invariants are defined by an analytic moduli space of solutions to the Witten equation (**10**). We shall see in the next Section 2.1.4 that the narrow sector invariants have an algebro-geometric definition. We stress that the invariants introduced later in Section 3.1 are different from the invariants of the full theory; they are a natural and computable extension of the narrow sector invariants.

The virtual cycle[8]

$$\bigoplus_{0\leq k_1,\dots,k_n\leq d-1}\left[\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)\right]^{\mathrm{vir}}$$

of the fully developed FJRW theory is defined in all genera and lies in

$$\bigoplus_{0\leq k_1,\dots,k_n\leq d-1}\Bigg(\mathrm{H}_{2\mathrm{D}(k_1,\dots,k_n)}\big(\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n);\mathbf{C}\big) \otimes\bigotimes_{i=1}^{n}\mathrm{H}_{\mathrm{N}_{k_i}}\big((\mathbf{C}^{\mathrm{N}})_{k_i},\mathrm{W}^{+\infty}_{k_i};\mathbf{C}\big)^{\boldsymbol{\mu}_d}\Bigg),$$

where $\mathrm{D}(k_1,\dots,k_n)=(3g-3+n)+\sum_{j=1}^{\mathrm{N}}\chi(\mathrm{L}^{\otimes w_j})$ for a d-spin structure L from $\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)$. Regard the relative homology group $\mathrm{H}_{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k,\mathrm{W}^{+\infty}_k;\mathbf{C})^{\boldsymbol{\mu}_d}$ above as the dual of the sector $\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k$ of the state space. Then the virtual cycle defines a linear operator (see [26, Equation (63)]):

$$\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)^{\otimes n}\xrightarrow{[\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)]^{\mathrm{vir}}\cap}\mathrm{H}_{2\mathrm{D}(k_1,\dots,k_n)}\big(\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n);\mathbf{C}\big)$$

assigning to $\bigotimes_{i=1}^n\alpha_i$ a nonzero cycle only if α_i lies in $\mathrm{H}^{\mathrm{N}_{k_i}}((\mathbf{C}^{\mathrm{N}})_{k_i},\mathrm{W}^{+\infty}_{k_i};\mathbf{C})^{\boldsymbol{\mu}_d}$. In FJRW theory, such a cycle is pushed forward via the natural forgetful morphism

$$\mathrm{st}\colon\ \mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)\to\overline{\mathcal{M}}_{g,n}$$

forgetting the data L and φ and passing to the coarse stable curve corresponding to $(\mathrm{C};\sigma_1,\dots,\sigma_n)$. After Poincare duality and multiplication by a factor $f_g=|\mathrm{G}|^g/\deg(\mathrm{st})=d/g$, Fan, Jarvis and Ruan obtain a cohomological field theory. We show in Appendix B that the multiplication by f_g can be removed in all genera once we use the natural pairing (**4**) from Chen-Ruan cohomology. In genus zero, this simply amounts to the fact that our invariants are $1/f_0=1/d$ times the FJRW genus-zero invariants of [26]. By making explicit the cohomological field theory of [26, Definition 4.2.1] *after removing* f_g and by applying the definition of the correlators of [26, Definition 4.2.6], we obtain

$$\big\langle\tau_{b_1}(\alpha_1),\dots,\tau_{b_n}(\alpha_n)\big\rangle^{\mathrm{FJRW}}_{g,n}:=\Bigg(\left[\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)\right]^{\mathrm{vir}}\cap\prod_{i=1}^n\alpha_i\Bigg)\cap\prod_{i=1}^n\psi_i^{b_i}\tag{11}$$

for $\alpha_i\in\mathrm{H}^{\mathrm{N}_{k_i}}((\mathbf{C}^{\mathrm{N}})_{k_i},\mathrm{W}^{+\infty}_{k_i};\mathbf{C})^{\boldsymbol{\mu}_d}$ and $b_i\geq 0$. The class ψ_i is the first Chern class of the line bundle on $\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)$ whose fibre at a point $(\mathrm{C};\sigma_1,\dots,\sigma_n;\mathrm{L};\varphi)$ is the cotangent space $\mathrm{T}^*_{\sigma_i}|\mathrm{C}|$ of the coarse curve $|\mathrm{C}|$ at σ_i. Note that, since these classes are the pullback of the standard ψ classes from $\overline{\mathcal{M}}_{g,n}$, we can directly integrate on $\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)$ short-circuiting the pushforward st_* of [26, Definition 4.2.1].

---

[8] $[\mathrm{Spin}^d_{g,n}(k_1,\dots,k_n)]^{\mathrm{vir}}$ is denoted by $[\overline{\mathcal{W}}_{g,n,\boldsymbol{\mu}_d}(\mathrm{W},(k_1,\dots,k_n))]^{\mathrm{vir}}$ in [26].

2.1.4. *Narrow part FJRW invariants.* — The definition, further simplifies in genus zero and when the entries are narrow states ϕ_k with $k+1 \in \mathsf{Nar}$. Let $\pi\colon \mathcal{C} \to \mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)$ be the universal orbicurve and $\mathcal{L} \to \mathcal{C}$ be the universal d-spin structure. If all the entries are narrow states, then the following lemma shows that $\mathrm{H}^0(\mathrm{C}, \mathrm{L}^{\otimes w_j})$ vanishes all j (and for every curve C); hence $-\mathbf{R}\pi_* \bigoplus_{j=1}^{\mathrm{N}} \mathcal{L}^{\otimes w_j} = \mathrm{R}^1\pi_* \bigoplus_{j=1}^{\mathrm{N}} \mathcal{L}^{\otimes w_j}$ is a vector bundle which we refer to as the *obstruction bundle.* In these cases the virtual homology cycle is simply the Poincaré dual cycle of the top Chern class of the obstruction bundle [26, Theorem 4.1.8, Concavity].

Lemma **2.3.** — *Suppose that* $k_i + 1 \in \mathsf{Nar}$ *for all* i. *Then* $\mathrm{H}^0(\mathrm{C}, \mathrm{L}^{\otimes w_j})$ *vanishes for all* $j = 1, \dots, \mathrm{N}$ *for* $(\mathrm{C}; \sigma_1, \dots, \sigma_n; \mathrm{L}; \varphi) \in \mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)$. *In particular, the obstruction bundle* $\bigoplus_{j=1}^{\mathrm{N}} \mathrm{R}^1\pi_*(\mathcal{L}^{\otimes w_j})$ *over* $\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)$ *is locally free of rank* $2 - \mathrm{N} + \sum_{i=1}^n \sum_{j=1}^{\mathrm{N}} \langle k_i q_j \rangle$.

Proof. — Let $p\colon \mathrm{C} \to \overline{\mathrm{C}}$ denote the map forgetting the stack-theoretic structures at all the markings (but not at the nodes). Then

$$\begin{aligned} p^*p_*\big(\mathrm{L}^{\otimes w_j}\big) &= \mathrm{L}^{\otimes w_j} \otimes \mathcal{O}_{\mathrm{C}}\Big(-\sum_{i=1}^n d\,\mathrm{age}_{\sigma_i}\big(\mathcal{L}^{\otimes w_j}\big)\sigma_i\Big) \\ &= \mathrm{L}^{\otimes w_j} \otimes \mathcal{O}_{\mathrm{C}}\Big(-\sum_{i=1}^n d\big\langle (k_i+1)q_j\big\rangle \sigma_i\Big). \end{aligned}$$

Here we regard σ_i as a stacky divisor of degree $1/d$. Hence

$$\mathrm{L}' := \big(p^*p_*\big(\mathrm{L}^{\otimes w_j}\big)\big)^{\otimes(q_j^{-1})} = \omega_{\log} \otimes \mathcal{O}_{\mathrm{C}}\Big(-\sum_{i=1}^n d q_j^{-1}\big\langle (k_i+1)q_j\big\rangle \sigma_i\Big).$$

Notice that $q_j^{-1} = d/w_j$ is an integer by the Gorenstein condition and we have $q_j^{-1}\langle (k_i+1)q_j\rangle \geq 1$ since $k_i + 1 \in \mathsf{Nar}$. Hence L′ is a subsheaf of the pull-back of $\omega_{|\mathrm{C}|}$. Note that $\mathrm{H}^0(|\mathrm{C}|, \omega_{|\mathrm{C}|}) = 0$ since $|\mathrm{C}|$ is of genus zero. Hence $\mathrm{H}^0(\mathrm{C}, \mathrm{L}') = 0$ and thus $\mathrm{H}^0(\mathrm{C}, p^*p_*(\mathrm{L}^{\otimes w_j})) = 0$. This implies $\mathrm{H}^0(\mathrm{C}, \mathrm{L}^{\otimes w_j}) = 0$. Finally, by Riemann-Roch [1, 45, 65], $\mathrm{R}^1\pi_*(\mathcal{L}^{\otimes w_j})$ is locally free of rank

$$\begin{aligned} -\chi\big(\mathrm{L}^{\otimes w_j}\big) &= -1 - \deg \mathrm{L}^{\otimes w_j} + \sum_{i=1}^n \mathrm{age}_{\sigma_i}\big(\mathrm{L}^{\otimes w_j}\big) \\ &= -1 - (n-2)q_j + \sum_{i=1}^n \big\langle (k_i+1)q_j\big\rangle = -1 + 2q_j + \sum_{i=1}^n \langle q_j k_i\rangle. \end{aligned}$$

The conclusion follows. □

Remark **2.4.** — The above lemma usually fails without the Gorenstein condition. This is the main reason that we restrict ourselves to the Gorenstein case.

We can now specialize (**11**) and get the narrow part descendant FJRW invariants as

$$\left\langle \tau_{b_1}(\phi_{k_1}), \dots, \tau_{b_n}(\phi_{k_n}) \right\rangle_{0,n}^{\mathrm{FJRW}} = \int_{[\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)]} \prod_{i=1}^{n} \psi_i^{b_i} \cup \prod_{j=1}^{\mathrm{N}} c_{\mathrm{top}}\left(\mathrm{R}^1\pi_*\left(\mathcal{L}^{\otimes w_j}\right)\right), \tag{12}$$

where $\phi_{k_1}, \dots, \phi_{k_n} \in \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$ (i.e. $k_i + 1 \in \mathsf{Nar}$) and $b_1, \dots, b_n \geq 0$. We sometimes omit τ_0 from the notation, e.g. writing $\langle \phi_{k_1}, \dots, \phi_{k_n} \rangle_{0,n}^{\mathrm{FJRW}}$ for $\langle \tau_0(\phi_{k_1}), \dots, \tau_0(\phi_{k_n}) \rangle_{0,n}^{\mathrm{FJRW}}$. The FJRW invariants satisfy the homogeneity ([26, Dimension Axiom in Section 4.1])

$$\left\langle \tau_{b_1}(\alpha_1), \dots, \tau_{b_n}(\alpha_n) \right\rangle_{0,n}^{\mathrm{FJRW}} = 0 \quad \text{unless} \quad \sum_{i=1}^{n} \left(b_i + \frac{1}{2} \deg \alpha_i \right) = n + \hat{c} - 3, \tag{13}$$

where $\hat{c} := \mathrm{N} - 2$ is the dimension of the Calabi-Yau hypersurface X_{W}. Again the invariants (**12**) differ from the original definition[9] [13, 26] by the factor of $1/d$. See Appendix B.

Remark **2.5.** — Polishchuk and Vaintrob [57] recently constructed a purely algebraic cohomological field theory of singularities based on matrix factorizations. They constructed a fundamental matrix factorization on the moduli space which plays the same role as the virtual fundamental class. The role of matrix factorizations in our paper is different from theirs, but it would be interesting to study the relationships.

2.2. *GW theory.*

— GW theory for orbifolds has been developed by Chen-Ruan [10] in symplectic category and Abramovich-Graber-Vistoli [1] in algebraic category. We will work in the algebraic category. For the details, we again refer the reader to these original articles.

2.2.1. *State space.*

— The state space of orbifold GW theory is given by the *Chen-Ruan cohomology group* of the orbifold. We explain the case of the Calabi-Yau hypersurface $\mathrm{X}_{\mathrm{W}} \subset \mathbf{P}(\underline{w})$. Set

$$\begin{aligned} \mathfrak{F} &:= \{0 \leq f < 1 \mid f w_j \in \mathbf{Z} \text{ for some } 1 \leq j \leq \mathrm{N}\} \\ &= \{0 \leq f < 1 \mid fd \in \mathbf{Z},\ fd \notin \mathsf{Nar}\}. \end{aligned}$$

In the second line we used the Gorenstein condition (i.e. $w_j \mid d$). An element $f \in \mathfrak{F}$ gives rise to the stabilizer $\exp(2\pi \mathtt{i} f)$ along the substack $\mathbf{P}(\underline{w})_f \subset \mathbf{P}(\underline{w})$:

$$\mathbf{P}(\underline{w})_f := \left[\left(\left(\mathbf{C}^{\mathrm{N}} \right)_{fd} \setminus \{0\} \right) / \mathbf{C}^\times \right]$$

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[9] In [13, Section 3.1, (15)], where the case $d = 5$ was discussed, the invariant $\langle \tau_{b_1}(\phi_{k_1}), \dots, \tau_{b_n}(\phi_{k_n}) \rangle_{0,n}^{\mathrm{FJRW}}$ was defined to be $5(\prod_{i=1}^n \psi_i^{b_i} \cup c_{\mathrm{top}}(\mathrm{R}^1\pi_*(\mathcal{L}))^5) \cap [\mathcal{R}_5(e^{2\pi \mathtt{i}(k_1+1)/5}, \dots, e^{2\pi \mathtt{i}(k_n+1)/5})]$; see also [13, Section 2.3, (14)].

where recall that $(\mathbf{C}^{\mathrm{N}})_k$ is the subspace of $\mathbf{C}^{\mathrm{N}}$ fixed by $\zeta^k = \exp(2\pi \mathrm{i} k/d)$. The *inertia stacks* $\mathcal{I}\mathbf{P}(\underline{w})$, $\mathcal{I}\mathrm{X}_{\mathrm{W}}$ are defined to be

$$\mathcal{I}\mathbf{P}(\underline{w}) = \bigsqcup_{f\in\mathfrak{F}} \mathbf{P}(\underline{w})_f, \qquad \mathcal{I}\mathrm{X}_{\mathrm{W}} = \bigsqcup_{f\in\mathfrak{F}} \big(\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}}\big).$$

The Chen-Ruan cohomology $\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}})$ is defined to be the cohomology of the inertia stack:

$$\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}}) := \mathrm{H}(\mathcal{I}\mathrm{X}_{\mathrm{W}}; \mathbf{C}) = \bigoplus_{f\in\mathfrak{F}} \mathrm{H}\big(\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}}; \mathbf{C}\big).$$

The degree of $\alpha \in \mathrm{H}^k(\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}})$, as an element of $\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}})$, is defined to be

$$\deg \alpha = k + 2\sum_{j=1}^{\mathrm{N}} \langle f w_j \rangle.$$

Note that this is an integer since $\sum_{j=1}^{\mathrm{N}} f w_j = fd \in \mathbf{Z}$. Let inv: $\mathbf{P}(\underline{w})_f \cong \mathbf{P}(\underline{w})_{\langle 1-f\rangle}$ denote the natural isomorphism. For $\alpha_1 \in \mathrm{H}(\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}})$ and $\alpha_2 \in \mathrm{H}(\mathbf{P}(\underline{w})_{\langle 1-f\rangle} \cap \mathrm{X}_{\mathrm{W}})$, we define

$$(\mathbf{14}) \qquad (\alpha_1, \alpha_2) = \int_{\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}}} \alpha_1 \cup \mathrm{inv}^* \alpha_2.$$

We set $(\alpha_1, \alpha_2) = 0$ if $\alpha_1 \in \mathrm{H}(\mathbf{P}(\underline{w})_{f_1} \cap \mathrm{X}_{\mathrm{W}})$ and $\alpha_2 \in \mathrm{H}(\mathbf{P}(\underline{w})_{f_2} \cap \mathrm{X}_{\mathrm{W}})$ and $f_1 + f_2 \notin \mathbf{Z}$. Then $(\cdot,\cdot)$ defines a graded symmetric non-degenerate pairing on $\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}})$.

The *ambient part* $\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ is defined to be the image of the restriction map

$$\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}}) := \mathrm{Im}\big(i^* \colon \mathrm{H}_{\mathrm{CR}}\big(\mathbf{P}(\underline{w})\big) = \mathrm{H}\big(\mathcal{I}\mathbf{P}(\underline{w})\big) \to \mathrm{H}(\mathcal{I}\mathrm{X}_{\mathrm{W}}) = \mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}})\big).$$

Let $\mathbf{1}_f \in \mathrm{H}(\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}})$ denote the identity class on $\mathbf{P}(\underline{w})_f \cap \mathrm{X}_{\mathrm{W}}$ and $p = c_1(\mathcal{O}(1))$ denote the hyperplane class on $\mathbf{P}(\underline{w})$. The ambient part is spanned, as a $\mathbf{C}$-vector space, by $p^i \mathbf{1}_f$, $0 \le i \le \sharp\{1 \le j \le \mathrm{N} \mid f w_j \in \mathbf{Z}\} - 1, f \in \mathfrak{F}$. The pairing $(\cdot,\cdot)$ restricts to a non-degenerate pairing on $\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ and $\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ is orthogonal to the complementary *primitive part* $\mathrm{H}_{\mathrm{pri}}(\mathrm{X}_{\mathrm{W}}) := \mathrm{Ker}(i_* \colon \mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}}) \to \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w})))$.

Remark **2.6** *(Comparison of state spaces)*. — The FJRW and GW state spaces can be identified, up to Tate twist, with the *relative* Chen-Ruan cohomology (Chiodo-Nagel [12]):

$$\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d) = \mathrm{H}_{\mathrm{CR}}\big(\big[\mathbf{C}^{\mathrm{N}}/\boldsymbol{\mu}_d\big], \big[\mathrm{W}^{-1}(1)/\boldsymbol{\mu}_d\big]\big)$$
$$\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}}) = \mathrm{H}_{\mathrm{CR}}\big(\mathcal{O}_{\mathbf{P}(\underline{w})}(-d), \widetilde{\mathrm{W}}^{-1}(1)\big).$$

The first identification follows immediately from the definition. The second identification follows from the Thom isomorphism. Here $\widetilde{\mathrm{W}} \colon \mathcal{O}_{\mathbf{P}(\underline{w})}(-d) \to \mathbf{C}$ is the function in

Section 1.1. Note that the pairs $(\mathcal{O}_{\mathbf{P}(\underline{w})}(-d), \widetilde{W}^{-1}(1))$, $([\mathbf{C}^N/\boldsymbol{\mu}_d], [W^{-1}(1)/\boldsymbol{\mu}_d])$ are connected by a variation of GIT quotients. Chiodo-Ruan [14] showed that there exists a bigraded isomorphism

$$H^{p,q}(W, \boldsymbol{\mu}_d) \cong H^{p,q}_{CR}(X_W)$$

which preserves the pairings on both sides. In this paper, we will construct a graded isomorphism (preserving the pairing)

$$H^{p,p}_{nar}(W, \boldsymbol{\mu}_d) \cong H^{p,p}_{amb}(X_W)$$

which *depends* on a point of the Kähler moduli space. (Note that the narrow/ambient part has no (p, q)-Hodge component with $p \neq q$.) See Remark 2.24.

2.2.2. *GW invariants.* — For $n \geq 0$ and $\beta \in H_2(|X_W|, \mathbf{Z})$, let $(X_W)_{0,n,\beta}$ denote the moduli stack of genus zero, n-pointed, degree β stable maps to X_W (it is denoted by $\mathcal{K}_{0,n}(X_W, \beta)$ in [1]). This carries a *virtual fundamental class* $[(X_W)_{0,n,\beta}]_{vir} \in H_*((X_W)_{0,n,\beta}; \mathbf{Q})$ of degree $2(n + \hat{c} - 3)$ with $\hat{c} := N - 2 = \dim X_W$. We have the evaluation map at the ith marked point

$$ev_i \colon (X_W)_{0,n,\beta} \to \overline{\mathcal{I}}X_W$$

where $\overline{\mathcal{I}}X_W$ denotes the rigidified cyclotomic inertia stack (see [1]). Because the underlying complex analytic spaces of $\overline{\mathcal{I}}X_W$ and $\mathcal{I}X_W$ are the same, we can define the pull-back $ev_i^* \colon H_{CR}(X_W) \to H((X_W)_{0,n,\beta})$. Let ψ_i be the first Chern class of the line bundle over $(X_W)_{0,n,\beta}$ whose fibre at a stable map is the cotangent space of the coarse curve at the ith marked point. The orbifold GW invariant is defined to be

$$(\mathbf{15}) \qquad \left\langle \tau_{b_1}(\alpha_1), \ldots, \tau_{b_n}(\alpha_n) \right\rangle^{GW}_{0,n,\beta} := \int_{[(X_W)_{0,n,\beta}]_{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{b_i}.$$

Here $\alpha_1, \ldots, \alpha_n \in H_{CR}(X_W)$ and $b_1, \ldots, b_n \geq 0$. Again we sometimes omit τ_0 from the notation. The orbifold GW invariants are also homogeneous. The invariant (**15**) vanishes unless $\sum_{i=1}^{n}(b_i + \frac{1}{2}\deg \alpha_i) = n + \hat{c} - 3$.

2.3. *Quantum cohomology and quantum connection.* — We can associate the *quantum cohomology rings* to both of FJRW and GW theories. The quantum ring of FJRW theory is defined over $\mathbf{C}$, whereas the quantum ring of GW theory is defined over the *Novikov ring* $\Lambda := \mathbf{C}[[\mathrm{Eff}]]$. It is the completion of the group ring $\mathbf{C}[\mathrm{Eff}]$ of the semigroup $\mathrm{Eff} \subset H_2(|X_W|, \mathbf{Z})$ consisting of classes of effective curves. For $\beta \in \mathrm{Eff}$, we denote by Q^β the corresponding element in Λ. In Section 2.3.2 below, we see how the divisor equation reduces the ground ring to $\mathbf{C}$ (by setting $Q^\beta = 1$) for GW theory. The construction here is standard and can be applied to any (genus zero) cohomological field theories with homogeneity. See [49].

In order to describe the quantum rings of both theories in a uniform way, we use the following notation. Let K denote the ground ring. It is $\mathbf{C}$ for FJRW theory and Λ for GW theory. Let H denote the state space. It is $H(W, \boldsymbol{\mu}_d)$ or $H_{CR}(X_W) \otimes \Lambda$. Let $\{T_i\}_{i=0}^s$ be a homogeneous basis of H. We choose T_0 to be the identity class, i.e. $T_0 = \mathbf{1}_0 \in H_{CR}(X_W)$ in GW theory and $T_0 = \phi_0 \in H(W, \boldsymbol{\mu}_d)_1$ in FJRW theory. We set $g_{ij} = (T_i, T_j)$ and let (g^{ij}) denote the matrix inverse to (g_{ij}). We write[10]

$$(\mathbf{16}) \qquad \left\langle \tau_{b_1}(T_{i_1}), \dots, \tau_{b_n}(T_{i_n}) \right\rangle_{0,n} = \begin{cases} \langle \tau_{b_1}(T_{i_1}), \dots, \tau_{b_n}(T_{i_n}) \rangle_{0,n}^{\mathrm{FJRW}} & \text{for FJRW theory;} \\ \sum_{\beta \in \mathrm{Eff}} \langle \tau_{b_1}(T_{i_1}), \dots, \tau_{b_n}(T_{i_n}) \rangle_{0,n,\beta}^{\mathrm{GW}} Q^\beta & \text{for GW theory.} \end{cases}$$

Let $t^0, \dots, t^s$ denote the co-ordinates of H dual to the basis $T_0, \dots, T_s$ such that $t = \sum_{i=0}^s t^i T_i$ denotes a general point[11] on H. We regard H as a supermanifold such that t^i has the parity $|i| \equiv \deg T_i \bmod 2$ and odd co-ordinates anticommute $t^i t^j = (-1)^{|i||j|} t^j t^i$. Let $K[[t]]$ denote the tensor product of the formal power series ring in even variables and the exterior algebra in odd variables, i.e. $K[[t]] = K[[t^i : \text{even}]] \otimes_K \bigwedge_K^\bullet (\bigoplus_{|i|:\text{odd}} K t^i)$. The *quantum product* $\bullet$ is a $K[[t]]$-bilinear product on $H \otimes K[[t]]$ defined by

$$(\mathbf{17}) \qquad T_i \bullet T_j = \sum_{k,l=0}^{s} \sum_{n \ge 0} \frac{1}{n!} \langle T_i, T_j, T_k, t, \dots, t \rangle_{0,n+3} g^{kl} T_l.$$

This is super-commutative and associative by the WDVV equation. We call $(H \otimes K[[t]], \bullet)$ the quantum cohomology ring. The identity of the product $\bullet$ is given by T_0. When we set $Q = 0$ and $t = 0$ in Gromov–Witten theory, the product $\bullet$ defines the so-called *Chen-Ruan orbifold cup product* on $H_{CR}(X_W)$. This limit $Q = t = 0$ is called the *large radius limit*. On the other hand, the limit point $t = 0$ in FJRW theory is called the *Landau-Ginzburg point*.

The *quantum connection* is the set of operators ∇_i, $i = 0, \dots, s$ on $H \otimes K[[t]][z, z^{-1}]$ defined by

$$(\mathbf{18}) \qquad \nabla_i \alpha = \frac{\partial \alpha}{\partial t^i} + \frac{1}{z} T_i \bullet \alpha, \qquad \alpha \in H \otimes K[[t]]\big[z, z^{-1}\big].$$

Here z is a formal parameter. The associativity of the product $\bullet$ implies that these operators supercommute, i.e. $\nabla_i \nabla_j - (-1)^{|i||j|} \nabla_j \nabla_i = 0$. We regard the quantum cohomology

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[10] Note that the FJRW correlators are not defined for $n = 0, 1, 2$ (since the moduli spaces are empty), but the GW correlators still exist for these n because the degree β can be non-zero. We set the correlator to be zero when the subscript $(0, n)$ or $(0, n, \beta)$ is not in the stable range.

[11] We use upper indices for the co-ordinates t^i. Note that t^i does not mean the i-th power of t.

$\mathrm{H} \otimes \mathrm{K}[[t]]$ as the trivial vector bundle over the formal neighbourhood of the origin in H and ∇ as a flat connection on it with parameter z. Moreover, we can extend the connection in the z-direction because of the homogeneity in FJRW/GW theory. Define a section $\mathrm{E} \in \mathrm{H} \otimes \mathrm{K}[[t]]$ by

$$\mathrm{E} := \sum_{i=0}^{s} \left(1 - \frac{1}{2} \deg \mathrm{T}_i\right) t^i \mathrm{T}_i.$$

This corresponds to the *Euler vector field*[12] $\sum_{i=0}^{s}(1 - \frac{1}{2} \deg \mathrm{T}_i) t^i \frac{\partial}{\partial t^i}$. By abuse of notation, we also denote the vector field by E. It defines the half of the grading of variables: $\deg t^i := 2 - \deg \mathrm{T}_i$. Let Gr denote the grading operator

$$\mathsf{Gr}(\mathrm{T}_i) := \frac{\deg \mathrm{T}_i}{2} \mathrm{T}_i.$$

The connection ∇_z in the z-direction is defined to be

$$\nabla_z \alpha = \frac{\partial \alpha}{\partial z} - \frac{1}{z^2} \mathrm{E} \bullet \alpha + \frac{1}{z} \mathsf{Gr}(\alpha).$$

for $\alpha \in \mathrm{H} \otimes \mathrm{K}[[t]][z, z^{-1}]$. We have $[\nabla_z, \nabla_i] = 0$, $i = 1, \dots, s$ and the pairing $(\cdot, \cdot)$ is ∇-flat. The precise meaning of the flatness of $(\cdot, \cdot)$ is as follows. Let $(-)^* \colon \mathrm{H} \otimes \mathrm{K}[[t]][z, z^{-1}] \to \mathrm{H} \otimes \mathrm{K}[[t]][z, z^{-1}]$ denote the map $\alpha(z) \mapsto \alpha(-z)$ flipping the sign of z. By extending the pairing $(\cdot, \cdot)$ on H to $\mathrm{H} \otimes \mathrm{K}[[t]][z, z^{-1}]$ bilinearly over $\mathrm{K}[[t]][z, z^{-1}]$, we have

$$(\mathbf{19}) \qquad \frac{\partial}{\partial t^i}\big((-)^* \alpha_1, \alpha_2\big) = \big((-)^* \nabla_i \alpha_1, \alpha_2\big) + \big((-)^* \alpha_1, \nabla_i \alpha_2\big)$$

for $\alpha_1, \alpha_2 \in \mathrm{H} \otimes \mathrm{K}[[t]][z, z^{-1}]$. The flatness of the pairing follows from the Frobenius property $(\alpha_1 \bullet \alpha_2, \alpha_3) = (\alpha_1, \alpha_2 \bullet \alpha_3)$.

We have a canonical solution of the quantum connection. Define $\mathrm{L} \in \mathrm{End}(\mathrm{H}) \otimes \mathrm{K}[[t]][[z^{-1}]]$ by

$$(\mathbf{20}) \qquad \mathrm{L}(t, z)\alpha = \alpha + \sum_{i,j=1}^{s} \sum_{n \geq 0} \sum_{b \geq 0} \frac{1}{n!(-z)^{b+1}} \big\langle \tau_b(\alpha), t, \dots, t, \mathrm{T}_i \big\rangle_{0,n+2} g^{ij} \mathrm{T}_j.$$

This is an invertible endomorphism satisfying the following differential equations:

Proposition **2.7.** — *For* $\alpha \in \mathrm{H}$, *we have*

$$\nabla_i\big(\mathrm{L}(t, z) z^{-\mathsf{Gr}} \alpha\big) = 0, \qquad \nabla_z\big(\mathrm{L}(t, z) z^{-\mathsf{Gr}} \alpha\big) = 0.$$

For $\alpha_1, \alpha_2 \in \mathrm{H}$, *we have*

$$\big(\mathrm{L}(t, -z)\alpha_1, \mathrm{L}(t, z)\alpha_2\big) = (\alpha_1, \alpha_2).$$

---

[12] Since we are working in the Calabi-Yau case, the term $c_1(\mathrm{X})$ vanishes.

Proof. — These are well-known in GW theory (see e.g. [42, Proposition 2.4]) and can be proved similarly for FJRW theory. So we only sketch the outline of the proof in the case of FJRW theory. The equation $\nabla_i(L(t,z)z^{-\mathsf{Gr}}\alpha) = 0$ is a formal consequence of the Topological Recursion Relation (TRR), as shown in [54, Proposition 2] for the GW theory. The TRR in FJRW theory is proved in [26, Theorem 4.2.8]. The equation $\nabla_z(L(t,z)z^{-\mathsf{Gr}}\alpha) = 0$ follows from the homogeneity (**13**) of FJRW invariants. Since $L(t,z)\alpha_i$ is flat in the t-direction, the pairing $(L(t,-z)\alpha_1, L(t,z)\alpha_2)$ does not depend on t (see (**19**)). Therefore $(L(t,-z)\alpha_1, L(t,z)\alpha_2) = (L(0,-z)\alpha_1, L(0,z)\alpha_2) = (\alpha_1,\alpha_2)$. ☐

2.3.1. *Restriction to the narrow/ambient part.* — Recall that the state space H is decomposed as $H_{\mathrm{nar}}(W,\boldsymbol{\mu}_d) \oplus H_{\mathrm{bro}}(W,\boldsymbol{\mu}_d)$ for FJRW theory and as $H_{\mathrm{amb}}(X_W)\otimes\Lambda \oplus H_{\mathrm{pri}}(X_W)\otimes\Lambda$ for GW theory. In this section, we denote this decomposition as

$$H = H' \oplus H''$$

where H' denotes the narrow/ambient part and H'' denotes the broad/primitive part. The decomposition is orthogonal with respect to the pairing $(\cdot,\cdot)$. Moreover we have the following.

Proposition **2.8.** — *For* $\alpha_1,\dots,\alpha_n \in H'$ *and* $\gamma \in H''$, *we have*

$$\left\langle \tau_{b_1}(\alpha_1),\dots,\tau_{b_n}(\alpha_n),\tau_c(\gamma)\right\rangle_{0,n+1} = 0.$$

In particular, H' *is closed under the quantum product* $\bullet$ *when the parameter* t *is restricted to lie on* H'*; the quantum connection* ∇ *and the fundamental solution* $L(t,z)$ *preserve* H' *as far as* $t \in H'$.

Proof. — Because of the deformation invariance ([25, Theorem 1.2.5], [10, Theorem 3.4.2]) we can assume that W is of Fermat type $W = x_1^{d/w_1} + \cdots + x_N^{d/w_N}$. Then the maximal diagonal group of symmetries preserving W is given as $G_{\max} = \{(z_1,\dots,z_N) \mid z_i^{d/w_i} = 1\} \cong \boldsymbol{\mu}_{d/w_1}\times\cdots\times\boldsymbol{\mu}_{d/w_N}$. The $G_{\max}$-action on $\mathbf{C}^N$ naturally lifts to the state space H. We claim that H' is the $G_{\max}$-invariant part of H:

$$H' = H^{G_{\max}}$$

and that H'' is the sum of non-trivial irreducible $G_{\max}$-representations. The proposition follows from this claim and the $G_{\max}$-invariance of the correlator:

$$\begin{aligned}&\left\langle \tau_{b_1}(g\alpha_1),\dots\tau_{b_n}(g\alpha_n),\tau_c(g\gamma)\right\rangle_{0,n+1}\\ &\quad = \left\langle \tau_{b_1}(\alpha_1),\dots,\tau_{b_n}(\alpha_n),\tau_c(\gamma)\right\rangle_{0,n+1}, \quad g \in G_{\max}.\end{aligned}$$

First we show the claim for the FJRW state space. We use the description of $H(W,\boldsymbol{\mu}_d)_k$ as the Jacobi space $\Omega(W_k)^{\boldsymbol{\mu}_d}$ given in (**7**) If $k \in \mathsf{Nar}$, the sector $H(W,\boldsymbol{\mu}_d)_k$ is obviously $G_{\max}$-invariant. Assume that $k \notin \mathsf{Nar}$. Then each element of $H(W,\boldsymbol{\mu}_d)_k$ can be represented by a sum of monomial N_k-forms of the form $\prod_{kq_i\in\mathbf{Z}}(x_i^{a_i}dx_i)$ with $0\le a_i\le d/w_i-2$. But each summand spans a non-trivial irreducible $\mathrm{G}_{\max}$-representations and the claim follows.

Next we show the claim for the GW state space. Each twisted sector has a decomposition $\mathrm{H}(\mathbf{P}(\underline{w})_f\cap \mathrm{X}_\mathrm{W}) = \mathrm{H}_{\mathrm{amb}}(\mathbf{P}(\underline{w})_f\cap \mathrm{X}_\mathrm{W})\oplus \mathrm{H}_{\mathrm{pri}}(\mathbf{P}(\underline{w})_f\cap \mathrm{X}_\mathrm{W})$. It is obvious that $\mathrm{H}_{\mathrm{amb}}(\mathbf{P}(\underline{w})_f\cap \mathrm{X}_\mathrm{W})$ is $\mathrm{G}_{\max}$-invariant. The primitive part $\mathrm{H}_{\mathrm{pri}}(\mathbf{P}(\underline{w})_f\cap \mathrm{X}_\mathrm{W})$ is isomorphic to $\mathrm{Gr}^{\mathcal{W}}_{\mathrm{N}_k}\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}^{-1}(1))$ for $k=fd$ (see [64, p. 216]), which is $\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k$ (see Appendix A). Now the claim follows for the same reason as the previous paragraph, since $k=fd\notin \mathsf{Nar}$. ☐

Remark **2.9.** — The fact that the ambient part is closed under the quantum product (in GW theory) is shown [43, Corollary 2.5] in general for complete intersections.

2.3.2. *Divisor equation and the specialization* Q = 1.

— In orbifold GW theory, we have the following Divisor Equation [1, Theorem 8.3.1]:

$$\begin{aligned}&\left\langle \tau_{b_1}(\alpha_1),\ldots,\tau_{b_n}(\alpha_n),p\right\rangle^{\mathrm{GW}}_{0,n+1,\beta}\\ &\quad=\langle p,\beta\rangle\left\langle \tau_{b_1}(\alpha_1),\ldots,\tau_{b_n}(\alpha_n)\right\rangle^{\mathrm{GW}}_{0,n,\beta}\\ &\qquad+\sum_{i:b_i>0}\left\langle \tau_{b_1}(\alpha_1),\ldots,\tau_{b_i-1}(\alpha_i),\ldots\tau_{b_n}(\alpha_n)\right\rangle^{\mathrm{GW}}_{0,n,\beta},\end{aligned}$$

where $p=c_1(\mathcal{O}(1))$ is the hyperplane class on X_W in the untwisted sector. We choose the homogeneous basis $\{\mathrm{T}_i\}_{i=0}^s$ such that $\mathrm{T}_0=\mathbf{1}_0$ and also that $\mathrm{T}_1=p$. The divisor equation implies that

$$\mathrm{T}_i\bullet\mathrm{T}_j=\sum_{k,l=0}^{s}\sum_{n\geq0}\sum_{\beta\in\mathrm{Eff}}\left\langle \mathrm{T}_i,\mathrm{T}_j,\mathrm{T}_k,t',\ldots,t'\right\rangle^{\mathrm{GW}}_{0,n+3,\beta}e^{\langle p,\beta\rangle t^1}\mathrm{Q}^\beta g^{kl}\mathrm{T}_l,$$

where $t'=\sum_{i\neq1}t^i\mathrm{T}_i$. Therefore, the specialization $\mathrm{T}_i\bullet\mathrm{T}_j|_{\mathrm{Q}=1}$ makes sense as an element of $\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_\mathrm{W})\otimes\mathbf{C}[\![e^{t^1/w},t']\!]$. Here w is the least common multiple of $w_1,\ldots,w_\mathrm{N}$ (so that $\mathcal{O}_{\mathbf{P}(\underline{w})}(w)$ is a pull-back from the coarse moduli space $|\mathbf{P}(\underline{w})|$). Under this specialization, the limit $e^{t^1/w}=t'=0$ plays the role of the large radius limit. Similarly, the specialization $\mathrm{Q}=1$ of the fundamental solution $\mathrm{L}(t,z)$ gives

$$\begin{aligned}\mathrm{L}(t,z)\alpha|_{\mathrm{Q}=1}&=e^{-t^1p/z}\alpha+\sum_{i,j=1}^{s}\sum_{n\geq0}\sum_{b\geq0}\sum_{\beta\in\mathrm{Eff}}\frac{1}{n!(-z)^{b+1}}\\ &\qquad\times\left\langle\tau_b\left(e^{-t^1p/z}\alpha\right),t',\ldots,t',\mathrm{T}_i\right\rangle^{\mathrm{GW}}_{0,n+2,\beta}e^{\langle p,\beta\rangle t^1}g^{ij}\mathrm{T}_j.\end{aligned}$$

This is an element of $\mathrm{End}(\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_\mathrm{W}))\otimes\mathbf{C}[\![e^{t^1/w},t']\!][t^1][\![z^{-1}]\!]$ and gives a fundamental solution to the quantum connection $\nabla|_{\mathrm{Q}=1}$. In this way the divisor equation reduces the ground ring from Λ to $\mathbf{C}$.

2.4. *Quantum* D*-modules and integral structure.* — Here we define the narrow/ambient part of the quantum D-module and introduce a certain integral structure on it. In this section we entirely work over $\mathbf{C}$. Let H denote the state space over $\mathbf{C}$ and $H' \subset H$ denote the narrow/ambient part:

$$H' = \begin{cases} H_{\mathrm{nar}}(W, \boldsymbol{\mu}_d) & \text{for FJRW theory;} \\ H_{\mathrm{amb}}(X_W) & \text{for GW theory.} \end{cases} \tag{21}$$

Recall that H' is closed under the quantum product by Proposition 2.8. Let $\{T_0, T_1, \dots, T_r\}$ $(r \le s)$ be a homogeneous basis of H' such that T_0 is the identity. It is, for example, a reordering of the basis $\{\phi_{k-1} \mid k \in \mathsf{Nar}\}$ (FJRW theory) or the basis $\{p^i \mathbf{1}_f \mid i \ge 0, f \in \mathfrak{F}\}$ (GW theory). For GW theory, we choose T_1 to be the hyperplane class $p = c_1(\mathcal{O}(1))$. Let $\{t^0, \dots, t^r\}$ denote the dual co-ordinates and $t = \sum_{i=0}^r t^i T_i$ denote a general point on H'. The parity of these co-ordinates are all even. In this section, *the parameter t is restricted to lie on* H' unless otherwise stated. Also in GW theory, *we set the Novikov parameter* Q *to* 1 in the quantum product $\bullet$, the connection ∇ and the fundamental solutions $L(t, z)$ as done in Section 2.3.2.

2.4.1. *Convergence assumption.* — We assume that the quantum product $T_i \bullet T_j$, $0 \le i, j \le r$ are all convergent power series. This means

$$\begin{aligned} &T_i \bullet T_j \in H' \otimes \mathbf{C}\left\{t^0, t^1, \dots, t^r\right\} && \text{for FJRW theory;} \\ &T_i \bullet T_j \in H' \otimes \mathbf{C}\left\{t^0, e^{t^1/w}, t^2, \dots, t^r\right\} && \text{for GW theory.} \end{aligned}$$

Let $U \subset H'$ denote the domain of convergence of the product $\bullet$. For FJRW theory, U is of the form

$$\left\{\left|t^i\right| < \epsilon, \ (\forall i)\right\}$$

for some $\epsilon > 0$. For GW theory, U is of the form

$$\left\{\Re\left(t^1\right) < -M, \ \left|t^i\right| < \epsilon, \ (\forall i \ne 1)\right\}$$

for some $\epsilon, M > 0$. In practice, we do not need to assume the full convergence of the product. One can consider the quantum D-module over a submanifold of U where the product $\bullet$ is convergent. In our case, we show in Section 5.2 that the quantum product $\bullet$ is convergent on the image of the mirror map. When X_W is a manifold, we show the full convergence in Section 5.5 for both GW and FJRW theories.

Note that the convergence assumption imply that ∇ and $L(t, z)$ are analytic in $t \in U$ and $z \in \mathbf{C}^\times$.

2.4.2. *Quantum* D*-module.* — Let $U \subset H'$ be as in Section 2.4.1. The present quantum D-module is defined as a meromorphic flat connection over $U \times \mathbf{C}$. Let z denote the co-ordinate on the second factor $\mathbf{C}$ and $\pi\colon U \times \mathbf{C} \to U$ denote the projection to the first factor. Let $(-)\colon U \times \mathbf{C} \to U \times \mathbf{C}$ be the map sending (t, z) to $(t, -z)$.

Definition **2.10.** — *Let* $F = H' \times (U \times \mathbf{C}) \to U \times \mathbf{C}$ *be the trivial vector bundle with fibre* H'. *Let* ∇ *be the meromorphic flat connection (quantum connection) on* F

$$\nabla = d + \frac{1}{z}\sum_{i=0}^{r}(T_i\bullet)dt^i + \left(-\frac{1}{z}(E\bullet) + \mathsf{Gr}\right)\frac{dz}{z}$$

which can be regarded as a map

$$\nabla\colon \mathcal{O}(F) \to \mathcal{O}(F)\big(U \times \{0\}\big) \otimes \left(\pi^*\Omega^1_U \oplus \mathcal{O}_{U\times\mathbf{C}}\frac{dz}{z}\right).$$

Here $\mathcal{O}(F)(U \times \{0\})$ *denotes the sheaf of holomorphic sections of* F *with at most simple poles along* $\{z = 0\} = U \times \{0\}$. *Let* P *be an* $\mathcal{O}_{U\times\mathbf{C}}$*-bilinear pairing*

$$(-)^*\mathcal{O}(F) \otimes \mathcal{O}(F) \to z^{\hat{c}}\mathcal{O}_{U\times\mathbf{C}}$$

defined by

$$P\big((-)^*s_1, s_2\big) := (2\pi\mathtt{i}z)^{\hat{c}}\big(s_1(t,-z), s_2(t,z)\big).$$

Here $\hat{c} = \dim X_W = N - 2$ *and* $(\cdot,\cdot)$ *in the right-hand side denotes the standard pairing on the state space defined in* (**4**) *and* (**14**). *The pairing satisfies*

$$(-1)^{\hat{c}}P\big((-)^*s_1, s_2\big) = (-)^*P\big((-)^*s_2, s_1\big)$$
$$dP\big((-)^*s_1, s_2\big) = P\big((-)^*\nabla s_1, s_2\big) + P\big((-)^*s_1, \nabla s_2\big).$$

We call the tuple (F, ∇, P) the narrow part quantum D-module $\mathrm{QDM}_{\mathrm{nar}}(W, \boldsymbol{\mu}_d)$ *(in the case of FJRW theory) and* the ambient part quantum D-module $\mathrm{QDM}_{\mathrm{amb}}(X_W)$ *(in the case of GW theory).*

Remark **2.11.** — In [49] and [42, Definition 2.2], the quantum connection $\nabla_{z\partial_z}$ in the z-direction is shifted by $-\hat{c}/2$ so that it makes the ordinary pairing $P/(2\pi\mathtt{i}z)^{\hat{c}}$ flat. In this paper, we adopt the convention in [43, Definition 3.1] because it is more compatible with mirror symmetry.

Remark **2.12.** — The quantum D-module here is a (TEP($\hat{c}$)) structure in the sense of Hertling [35, Remark 2.13]. Moreover, by the given trivialization, F is extended over $U \times \mathbf{P}^1$ as a trivial vector bundle and thus gives a (trTLEP($\hat{c}$))-structure [35, Definition 5.5]. In the context of LG/CY correspondence, it is more convenient to consider the connection over $U \times \mathbf{C}$ since the extensions across $z = \infty$ do not match under analytic continuation.

Remark **2.13.** — Over the degree two subspace $\mathrm{H}'^2 \subset \mathrm{H}'$, the narrow/ambient part quantum D-module gives rise to a variation of Hodge structure, the so-called (narrow/ambient) *A-model VHS* [43, Section 6.2]. This is defined to be the restriction of $(\mathcal{F}, \nabla)$ to the subspace $(\mathrm{H}'^2 \cap \mathrm{U}) \times \{z = 1\}$ equipped with the decreasing Hodge filtration

$$\mathcal{F}_{\mathrm{A}}^{p} := \mathrm{H}'^{\leq 2(\hat{c}-p)} \otimes \mathcal{O}_{\mathrm{H}'^2 \cap \mathrm{U}}$$

and the polarization

$$\mathrm{Q}_{\mathrm{A}}(\alpha, \beta) = (2\pi\,\mathtt{i})^{\hat{c}}\big((-1)^{\deg/2}\alpha, \beta\big).$$

2.4.3. *Galois action.* — The quantum D-module has an important discrete symmetry which we call the Galois action. This symmetry is also compatible with mirror symmetry.

Proposition **2.14** *(Galois action in FJRW theory). — Let* H *be the FJRW state space and* H' *be its narrow part. Define the linear map* $\mathrm{G}\colon \mathrm{H} \to \mathrm{H}$ *by*

$$\mathrm{G}|_{\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k} = e^{-2\pi\,\mathtt{i}(k-1)/d}\,\mathrm{id}_{\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k}.$$

The map G *preserves* H'. *Without loss of generality, one can assume that the convergence domain* $\mathrm{U} \subset \mathrm{H}'$ *is preserved by* G. *The bundle map* $\mathrm{G}_{\mathrm{F}}\colon \mathrm{F} \to \mathrm{G}^*\mathrm{F}$ *defined by*

$$\begin{aligned} \mathrm{G}_{\mathrm{F}}\colon \mathrm{H}' \times (\mathrm{U} \times \mathbf{C}) &\longrightarrow \mathrm{H}' \times (\mathrm{U} \times \mathbf{C}) \\ \big(\alpha, (t, z)\big) &\longmapsto \big(e^{-2\pi\,\mathtt{i}/d}\mathrm{G}(\alpha), \big(\mathrm{G}(t), z\big)\big) \end{aligned}$$

preserves the connection ∇ *(i.e.* $\nabla_{d\mathrm{G}(v)} \circ \mathrm{G}_{\mathrm{F}} = \mathrm{G}_{\mathrm{F}} \circ \nabla_v$*) and the pairing* P. *We call it the* Galois action *of the narrow part quantum* D*-module.*

Proof. — For a d-spin structure $\mathrm{L} \to \mathrm{C}$ on a pointed orbicurve $(\mathrm{C}, \sigma_1, \dots, \sigma_n)$, we have $\deg(\mathrm{L}) - \sum_{i=1}^{n} \mathrm{age}_{\sigma_i}(\mathrm{L}) \in \mathbf{Z}$ by Riemann-Roch for orbicurves [1, 45, 65]. Thus the moduli space $\mathrm{Spin}_{0,n}^{d}(k_1, \dots, k_n)$ is empty unless $2 + \sum_{i=1}^{n} k_i \equiv 0 \bmod d$. Therefore we have

$$\big\langle \tau_{b_1}\big(\mathrm{G}(\phi_{k_1})\big), \dots, \tau_{b_n}\big(\mathrm{G}(\phi_{k_n})\big)\big\rangle_{0,n}^{\mathrm{FJRW}} = e^{2(2\pi\,\mathtt{i})/d}\big\langle \tau_{b_1}(\phi_{k_1}), \dots, \tau_{b_n}(\phi_{k_n})\big\rangle_{0,n}^{\mathrm{FJRW}}$$

for $k_1 + 1, \dots, k_n + 1 \in \mathsf{Nar}$. This fact and the formula (**5**) of the pairing show that

$$\mathrm{G}(\alpha_1) \bullet_{\mathrm{G}(t)} \mathrm{G}(\alpha_2) = \mathrm{G}(\alpha_1 \bullet_t \alpha_2)$$

for $\alpha_1, \alpha_2 \in \mathrm{H}'$. Here the subscripts of $\bullet$ denote the parameter of the quantum product. The statement follows easily from this. □

Remark **2.15.** — Since the FJRW invariants in our case are (regardless of narrow or broad) certain intersection numbers on $\mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$, the same argument shows that the Galois action preserves ∇ and P defined on the full FJRW state space H.

Proposition **2.16** *(Galois action in GW theory: [42, Proposition 2.3]). — Let* H *be the GW state space and* H' *be its ambient part. Define* $\mathrm{G}\colon \mathrm{H} \to \mathrm{H}$ *to be the affine-linear map*

$$\mathrm{G}(\alpha) = e^{2\pi \mathtt{i} f}\alpha - 2\pi \mathtt{i} p, \quad \textit{for } \alpha \in \mathrm{H}\big(\mathbf{P}(\underline{w})_f \cap \mathrm{X_W}\big),\ f \in \mathfrak{F}.$$

The map G *preserves* H'. *Without loss of generality, one can assume that the convergence domain* $\mathrm{U} \subset \mathrm{H}'$ *is preserved by* G. *The bundle map* $\mathrm{G_F}\colon \mathrm{F} \to \mathrm{G}^*\mathrm{F}$ *defined by*

$$\begin{aligned} \mathrm{H}' \times (\mathrm{U} \times \mathbf{C}) &\longrightarrow \mathrm{H}' \times (\mathrm{U} \times \mathbf{C}) \\ \big(\alpha, (t, z)\big) &\longmapsto \big(d\mathrm{G}(\alpha), \big(\mathrm{G}(t), z\big)\big) \end{aligned}$$

preserves the connection ∇ *and the pairing* P. *Here* $d\mathrm{G}$ *is the differential (linear part) of* G. *We call it the* Galois action *of the ambient part quantum* D*-module.*

Proof. — In [42, Proposition 2.3], the Galois action was defined for each orbifold line bundle. The map $\mathrm{G_F}$ here arises from $\mathcal{O}(1)$. ☐

Via the Galois action, the quantum D-module $(\mathrm{F}, \nabla, \mathrm{P})$ over $\mathrm{U} \times \mathbf{C}$ descends to a flat connection on the quotient space $(\mathrm{U}/\langle \mathrm{G} \rangle) \times \mathbf{C}$. We denote it by $(\mathrm{F}, \nabla, \mathrm{P})/\langle \mathrm{G} \rangle$.

2.4.4. *Integral structure.*

— The $\widehat{\Gamma}$-integral structure in the orbifold GW theory was introduced in [42, Section 2.4], [44, Section 3.1]. We generalize it to the case of FJRW theory for $(\mathbf{C}^{\mathrm{N}}, \mathrm{W}, \boldsymbol{\mu}_d)$.

Definition **2.17.**

(1) *In FJRW theory, the* Gamma class $\widehat{\Gamma}_{\mathrm{FJRW}} \in \mathrm{End}(\mathrm{H})$ *is defined to be*

$$\widehat{\Gamma}_{\mathrm{FJRW}} := \bigoplus_{k=0}^{d-1} \prod_{j=1}^{\mathrm{N}} \Gamma\big(1 - \langle k q_j \rangle\big)$$

where $q_j = w_j/d$ *and the kth summand acts on* $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ *by the scalar multiplication. This is similar to the Gamma class [42, Section 2.4] of the tangent bundle of the orbifold* $[\mathbf{C}^{\mathrm{N}}/\boldsymbol{\mu}_d]$.

(2) *In GW theory, the* Gamma class $\widehat{\Gamma}_{\mathrm{GW}} \in \mathrm{End}(\mathrm{H})$ *is defined to be*

$$\widehat{\Gamma}_{\mathrm{GW}} := \bigoplus_{f \in \mathfrak{F}} \frac{\prod_{i=1}^{\mathrm{N}} \Gamma(1 - \langle f w_i \rangle + w_i p)}{\Gamma(1 + dp)}$$

where $p = c_1(\mathcal{O}(1))$ *is the hyperplane class and the summand indexed by* $f \in \mathfrak{F}$ *acts on* $\mathrm{H}(\mathbf{P}(\underline{w})_f \cap \mathrm{X_W})$ *by the cup product. This is the Gamma class [42, Section 2.4] of the tangent bundle of the Calabi-Yau hypersurface* $\mathrm{X_W}$.

Remark **2.18.** — Libgober [48] observed that the Gamma class arises from periods of mirrors of Calabi-Yau hypersurfaces.

We introduce a flat section associated to a graded matrix factorization of W (see Section 4.1) or a vector bundle on $\mathrm{X_W}$. We use the Chern character map

$$\begin{aligned} &\mathrm{ch}\colon \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W}) \to \bigoplus_{k=0}^{d-1} \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d} \cong \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d) && \text{for FJRW theory;} \\ &\mathrm{ch}\colon \mathrm{D}^b(\mathrm{X_W}) \to \mathrm{H_{CR}}(\mathrm{X_W}) && \text{for GW theory.} \end{aligned}$$

The Chern character for a matrix factorization (due to Walcher [68] and Polishchuk-Vaintrob [56]) will be explained in Section 4.1.1 and we use the isomorphism $\bigoplus_{k=0}^{d-1} \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d} \cong \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)$ in Proposition 2.1. The Chern character for a orbi-vector bundle is the "stabilizer-equivariant" version which appears in the Kawasaki-Riemann-Roch formula [45, 65]. For $\mathrm{ch}\colon \mathrm{D}^b(\mathrm{X_W}) \to \mathrm{H_{CR}}(\mathrm{X_W})$, see for instance [42, Section 2.4] where it is denoted by $\widetilde{\mathrm{ch}}$.

Definition **2.19** *($\widehat{\Gamma}$-integral structure). — Let* $\deg_0\colon \mathrm{H} \to \mathrm{H}$ *be the degree operator without the shift ("bare" degree operator):*

$$\begin{aligned} \deg_0 &:= -2\,\mathrm{id}_{\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)} && \textit{for FJRW theory;} \\ \deg_0|_{\mathrm{H}^n(\mathbf{P}(\underline{w})_f \cap \mathrm{X_W})} &:= n\,\mathrm{id}_{\mathrm{H}^n(\mathbf{P}(\underline{w})_f \cap \mathrm{X_W})} && \textit{for GW theory.} \end{aligned}$$

Let $\mathrm{inv}\colon \mathrm{H} \to \mathrm{H}$ *denote the map induced from the natural isomorphisms*

$$\begin{aligned} \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k &\cong \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_{d-k} && \textit{for FJRW theory;} \\ \mathrm{H}\big(\mathbf{P}(\underline{w})_f \cap \mathrm{X_W}\big) &\cong \mathrm{H}\big(\mathbf{P}(\underline{w})_{\langle 1-f \rangle} \cap \mathrm{X_W}\big) && \textit{for GW theory.} \end{aligned}$$

Let $\mathcal{E}$ *be an object of* $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ *(in the case of FJRW theory) or an object of* $\mathrm{D}^b(\mathrm{X_W})$ *(in the case of GW theory). We define a* ∇*-flat section* $\mathfrak{s}(\mathcal{E})$ *by*

$$\mathfrak{s}(\mathcal{E})(t,z) := \frac{1}{(2\pi \mathtt{i})^{\hat{c}}} \mathrm{L}(t,z) z^{-\mathsf{Gr}} \widehat{\Gamma}\Big((2\pi \mathtt{i})^{\frac{\deg_0}{2}}\, \mathrm{inv}^* \mathrm{ch}(\mathcal{E})\Big) \tag{22}$$

where $\widehat{\Gamma}$, $\mathrm{L}(t,z)$ *and* $\mathrm{ch}(\mathcal{E})$ *are the Gamma class, the fundamental solution* (**20**) *and the Chern character in the respective theory. It is clear from the definition that* $\mathfrak{s}(\mathcal{E})$ *depends only on the numerical class of* $\mathcal{E}$.

When $\mathrm{ch}(\mathcal{E}) \in \mathrm{H}'$, $\mathfrak{s}(\mathcal{E})$ *defines a flat section of narrow/ambient part quantum* D*-module. Define the* $\mathbf{Z}$*-local system over* $\mathrm{U} \times \mathbf{C}^\times$ *by*

$$\mathrm{F}_{\mathbf{Z}} := \left\{\mathfrak{s}(\mathcal{E}) \mid \mathrm{ch}(\mathcal{E}) \in \mathrm{H}'\right\} \subset \Gamma\left(\mathrm{U} \times \mathbf{C}^\times, \mathcal{O}(\mathrm{F})\right)^\nabla$$

where $\mathcal{E}$ *ranges over objects of* $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ *or* $\mathrm{D}^b(\mathrm{X}_\mathrm{W})$. *(Note that* $\mathrm{ch}(\mathcal{E})$ *lies in* H, *but not in* H' *in general.) We call this the* $\widehat{\Gamma}$-integral structure *of the narrow/ambient part quantum* D*-module.*

Remark **2.20.** — The degree $\deg_0$ is even on the image of the Chern character map. In fact, the Chern character takes values in the (p,p)-part. See Remark 4.4.

Proposition **2.21.**

(1) *The* $\widehat{\Gamma}$*-integral structure is preserved under the Galois action, i.e.*

$$e^{-2\pi \mathtt{i}/d}\mathrm{G}\left(\mathfrak{s}(\mathcal{E})\left(\mathrm{G}^{-1}(t), z\right)\right) = \mathfrak{s}\left(\mathcal{E}(1)\right)(t,z) \qquad \textit{for FJRW theory};$$

$$d\mathrm{G}\left(\mathfrak{s}(\mathcal{E})\left(\mathrm{G}^{-1}(t), z\right)\right) = \mathfrak{s}\left(\mathcal{E} \otimes \mathcal{O}(-1)\right)(t,z) \quad \textit{for GW theory},$$

where $\mathcal{E}(1)$ *is the shift of the grading of* $\mathcal{E}$ *by* 1*. In particular,* $\mathrm{F}_{\mathbf{Z}}$ *defines an integral structure on the quotient* $(\mathrm{F}, \nabla, \mathrm{P})/\langle \mathrm{G}\rangle$.

(2) *We have*

$$\mathrm{P}\left((-)^*\mathfrak{s}(\mathcal{E}), \mathfrak{s}(\mathcal{F})\right) = (-1)^{\mathrm{N}-1}\chi(\mathcal{E},\mathcal{F}) \quad \textit{for FJRW theory};$$

$$\mathrm{P}\left((-)^*\mathfrak{s}(\mathcal{E}), \mathfrak{s}(\mathcal{F})\right) = \chi(\mathcal{E},\mathcal{F}) \qquad \textit{for GW theory},$$

where $\chi(\mathcal{E},\mathcal{F}) := \sum_{i\in\mathbf{Z}}(-1)^i \dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$ *is the Euler pairing of* $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ *or of* $\mathrm{D}^b(\mathrm{X}_\mathrm{W})$*. In particular,* P *takes values in* $\mathbf{Z}$ *on* $\mathrm{F}_{\mathbf{Z}}$.

Proof. — The proof relies on the Hirzebruch-Riemann-Roch formula in each category. We explain the case of FJRW theory. See [42, Proposition 2.10] (or [43, Definition 3.6]) for the case of GW theory. For Part (1), since the Galois action preserves ∇, it suffices to check the equality at $t = 0$ (see Remark 2.15). It follows from $\mathrm{L}(0,z) = \mathrm{id}$ and

$$e^{-2\pi \mathtt{i}/d}\mathrm{G}\left(\mathrm{inv}^*\,\mathrm{ch}(\mathcal{E})\right) = \mathrm{inv}^*\,\mathrm{ch}\left(\mathcal{E}(1)\right).$$

Next we show Part (2). Setting $\Psi(\mathcal{E}) = \widehat{\Gamma}((2\pi\mathtt{i})^{\frac{\deg_0}{2}}\,\mathrm{inv}^*\,\mathrm{ch}(\mathcal{E}))$, we have

$$\begin{aligned}
&\mathrm{P}\left((-)^*\mathfrak{s}(\mathcal{E}), \mathfrak{s}(\mathcal{F})\right)\\
&\quad = (2\pi\mathtt{i})^{-\hat{c}} z^{\hat{c}}\left((-z)^{-\mathsf{Gr}}\Psi(\mathcal{E}), z^{-\mathsf{Gr}}\Psi(\mathcal{F})\right) \quad \text{by Proposition 2.7}\\
&\quad = (2\pi\mathtt{i})^{-\hat{c}}\left((-1)^{-\mathsf{Gr}}\Psi(\mathcal{E}), \Psi(\mathcal{F})\right)\\
&\quad = (2\pi\mathtt{i})^{-\hat{c}}\sum_{k=0}^{d-1}\left(\prod_{j=1}^{\mathrm{N}}\Gamma\left(1-\langle kq_j\rangle\right)\Gamma\left(1-\left\langle (d-k)q_j\right\rangle\right)\right)(2\pi\mathtt{i})^{-2}
\end{aligned}$$

$$
\begin{aligned}
&\times (-1)^{-\frac{\mathrm{N}_k}{2}+1-\sum_j \langle kq_j\rangle} \Big(\big(\mathrm{inv}^* \operatorname{ch}(\mathcal{E})\big)_k, \big(\mathrm{inv}^* \operatorname{ch}(\mathcal{F})\big)_{d-k} \Big) \\
&= \sum_{k=0}^{d-1} (2\pi\mathtt{i})^{-\mathrm{N}_k} \Bigg(\prod_{\langle kq_j\rangle \neq 0} \frac{1}{1-e^{2\pi\mathtt{i}\langle kq_j\rangle}} \Bigg) (-1)^{\mathrm{N}-\mathrm{N}_k} (-1)^{-\frac{\mathrm{N}_k}{2}+1} \\
&\times \big(\operatorname{ch}(\mathcal{E})_{d-k}, \operatorname{ch}(\mathcal{F})_k \big)
\end{aligned}
$$

where $\operatorname{ch}(\mathcal{F})_k$ denotes the component of $\operatorname{ch}(\mathcal{F})$ in the sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ and we used the equality $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi x)$. Note that $\operatorname{ch}(\mathcal{F})_k$ vanishes if N_k is odd. By Proposition 2.1, we can write the last expression in terms of the residue pairing:

$$
\sum_{k=0}^{d-1} \Bigg(\prod_{\langle kq_j\rangle \neq 0} \frac{1}{1-e^{2\pi\mathtt{i}\langle kq_j\rangle}} \Bigg) (-1)^{\mathrm{N}-1} (-1)^{\frac{\mathrm{N}_k(\mathrm{N}_k-1)}{2}} \frac{1}{d} \operatorname{Res}_{\mathrm{W}_k} \big(\operatorname{ch}(\mathcal{E})_{d-k}, \operatorname{ch}(\mathcal{F})_k \big)
$$

where we used the fact that the degree of $\operatorname{ch}(\mathcal{F})_k$ *as an element of* $\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ is $(\mathrm{N}_k/2)d$ (see Remark 4.4; the degree here is different from the degree as an element of the FJRW state space). This equals $(-1)^{\mathrm{N}-1}\chi(\mathcal{E}, \mathcal{F})$ by Hirzebruch-Riemann-Roch Theorem 4.6. □

Remark **2.22.** — In Proposition 2.21, we do not need to assume that $\operatorname{ch}(\mathcal{E}) \in \mathrm{H}'$ or $t \in \mathrm{H}'$.

2.5. *Statements of the main results.* — In this section we summarize our main results in three theorems. Theorems 2.23, 2.25 are about analytic continuation of quantum D-modules (with integral structure) and Theorem 2.26 is about the monodromy representation and derived equivalences. These theorems are more precise versions of Theorems 1.1 and 1.2 in the introduction. The proofs take the entire paper and are completed in Section 5 (see Section 2.6).

Let v be an inhomogeneous co-ordinate of $\mathbf{P}(1,d)$ such that $v=\infty$ is the $\boldsymbol{\mu}_d$-orbifold point. Then $u = v^{-1/d}$ gives a uniformizing co-ordinate around the orbifold point (LG point). Set $\mathcal{M} := \mathbf{P}(1,d) \setminus \{v=0, v=v_{\mathrm{c}}\}$, where $v=0$ is the large radius limit point and $v = v_{\mathrm{c}} := d^{-d} \prod_{j=1}^{\mathrm{N}} w_j^{w_j}$ is the conifold point. We write $(-) \colon \mathcal{M} \times \mathbf{C}_z \to \mathcal{M} \times \mathbf{C}_z$ for the map sending (v, z) to $(v, -z)$. The following theorem is a precise version of Theorem 1.1 (without the part concerning the Orlov equivalence). The proof will be completed in Section 5.4.

Theorem **2.23.** — *There exists a locally free sheaf* $\mathcal{F}$ *over* $\mathcal{M} \times \mathbf{C}_z$ *with a meromorphic flat connection* ∇ *(with simple poles along* $z=0$*)*

$$
\nabla \colon \mathcal{F} \to \mathcal{F}\big(\mathcal{M} \times \{0\}\big) \otimes \Omega^1_{\mathcal{M}\times\mathbf{C}_z},
$$

a ∇*-flat, symmetric and non-degenerate pairing*

$$
\mathrm{P} \colon (-)^*\mathcal{F} \otimes \mathcal{F} \to z^{\hat{c}} \mathcal{O}_{\mathcal{M}\times\mathbf{C}_z}
$$

and a $\mathbf{Z}$*-local subsystem* $\mathrm{F}_{\mathbf{Z}}$ *of the same rank over* $\mathcal{M} \times \mathbf{C}_z^\times$

$$\mathrm{F}_{\mathbf{Z}} \subset (\mathcal{F}|_{\mathcal{M}\times\mathbf{C}_z^\times})^\nabla$$

such that the following holds:

(i) *For a small open neighbourhood* $\mathrm{U}_{\mathrm{FJRW}} = \{|u| < \epsilon\} \subset \mathcal{M}$ *of the LG point, we have a mirror map* $\tau_{\mathrm{FJRW}} \colon \mathrm{U}_{\mathrm{FJRW}} \to \mathrm{H}^2_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)/\langle \mathrm{G}\rangle$ *such that* $\tau_{\mathrm{FJRW}} = -u\phi_1 + \mathrm{O}(u^2)$ *and*

$$\big(\mathcal{F}, \nabla, (-1)^{\mathrm{N}-1}\mathrm{P}, \mathrm{F}_{\mathbf{Z}}\big)\big|_{\mathrm{U}_{\mathrm{FJRW}}\times\mathbf{C}_z} \cong \tau^*_{\mathrm{FJRW}}\big(\mathrm{QDM}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)/\langle \mathrm{G}\rangle\big),$$

where in the right-hand side appears the narrow part quantum D*-module (Definition 2.10) of* $(\mathbf{C}^{\mathrm{N}}, \mathrm{W}, \boldsymbol{\mu}_d)$ *equipped with the* $\widehat{\Gamma}$*-integral structure (Definition 2.19) and* G *is the Galois action (Section 2.4.3).*

(ii) *For a small open neighbourhood* $\mathrm{U}_{\mathrm{GW}} = \{|v| < \epsilon\} \subset \mathcal{M}$ *of the large radius limit point, we have a mirror map* $\tau_{\mathrm{GW}} \colon \mathrm{U}_{\mathrm{GW}} \to \mathrm{H}^2_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})/\langle \mathrm{G}\rangle$ *such that* $\tau_{\mathrm{GW}}(v) = p\log v + \mathrm{O}(v)$ *and*

$$(\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})|_{\mathrm{U}_{\mathrm{GW}}\times\mathbf{C}_z} \cong \tau^*_{\mathrm{GW}}\big(\mathrm{QDM}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})/\langle \mathrm{G}\rangle\big),$$

where in the right-hand side appears the ambient part quantum D*-module of* X_{W} *equipped with the* $\widehat{\Gamma}$*-integral structure and* G *is the Galois action.*

Remark **2.24.** — Restricting the global D-module $(\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})$ to $z = 1$, we obtain the analytic continuation between the narrow A-model VHS of $(\mathbf{C}^{\mathrm{N}}, \mathrm{W}, \boldsymbol{\mu}_d)$ and the ambient A-model VHS of X_{W} in Remark 2.13. The fibre $\mathcal{F}_{(x,1)}$ at $(x, 1) \in \mathcal{M} \times \mathbf{C}_z$ has a well-defined filtration and a polarization

$$\mathscr{F}^p(\mathcal{F}_{(x,1)}) = \big\{v \in \mathcal{F}_{(x,1)} \mid s_v(z) \text{ has a pole of order } \le \hat{c} - p \text{ at } z = 0\big\}$$
$$\mathrm{Q}(v_1, v_2) = \mathrm{P}\big(s_{v_1}(-1), s_{v_2}(1)\big), \qquad v_1, v_2 \in \mathcal{F}_{(x,1)}$$

where $s_v(z) \in \mathrm{H}^0(\mathbf{C}_z^\times, \mathcal{F}|_{\{t\}\times\mathbf{C}_z^\times})$ is a unique ∇-flat section such that $s_v(1) = v$. The filtration and the polarization coincide with those of the A-model VHS near the respective limit point. By analytic continuation, we have an isomorphism of state spaces

$$\Theta(x) \colon \big(\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d), \mathscr{F}^p_{\mathrm{A}}, \mathrm{Q}_{\mathrm{A}}\big) \cong \big(\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}}), \mathscr{F}^p_{\mathrm{A}}, \mathrm{Q}_{\mathrm{A}}\big)$$

for a point x on the universal cover $\widetilde{\mathcal{M}}$. Taking the associated graded vector space with respect to $\mathscr{F}^\bullet$, we can turn this into a graded isomorphism (preserving the polarization). Note that Θ does not map the identity to the identity because of the factor F in the asymptotics (**74**).

In the case where the Calabi-Yau hypersurface $\mathrm{X_W}$ is a smooth manifold (e.g. $\mathbf{P}(\underline{w}) = \mathbf{P}^n$ or $\mathbf{P}(1, 1, 1, 1, 2)$, $\mathbf{P}(1, 1, 1, 1, 1, 1, 1, 2, 3)$, etc.), we can use the reconstruction theorem to prove that the "big" quantum D-modules are analytically continued to each other. Here the word "big" means the quantum D-module over the full narrow/ambient sector H'. This is used in contrast with the "small" quantum D-module which is the restriction of the big one to the image of the mirror maps. The following theorem will be proved in Section 5.5.

Theorem **2.25.** — *Assume that* $\mathrm{X_W}$ *is a manifold.*

(i) *The "big" quantum product of* $(\mathbf{C}^{\mathrm{N}}, \mathrm{W}, \boldsymbol{\mu}_d)$ *on the narrow part and the "big" quantum product of* $\mathrm{X_W}$ *on the ambient part are convergent in the sense of Section 2.4.1.*

(ii) *The global* D*-module* $(\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})$ *in Theorem 2.23 can be extended to a* D*-module* $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, \mathrm{P}^{\mathrm{ext}}, \mathrm{F}^{\mathrm{ext}}_{\mathbf{Z}})$ *over a base* $\mathcal{M}_{\mathrm{ext}} \times \mathbf{C}_z$, *where* $\mathcal{M}_{\mathrm{ext}}$ *is a complex manifold of dimension* $\operatorname{rank}\mathcal{F} = \dim \mathrm{H}_{\mathrm{amb}}(\mathrm{X_W})$ *which contains a Zariski open subset* $\mathcal{M}'$ *of* $\mathcal{M}$ *as a submanifold. The extended* D*-module is identified with the "big" narrow/ambient part quantum* D*-module of FJRW/GW theory in a neighbourhood of* $\mathrm{U_{GW}}$ *or* $\mathrm{U_{FJRW}}$.

More precisely, there exists a locally free sheaf $\mathcal{F}^{\mathrm{ext}}$ *over* $\mathcal{M}_{\mathrm{ext}} \times \mathbf{C}_z$ *equipped with a meromorphic flat connection* ∇^{ext} *(with poles of order two along* $z = 0$*)*

$$\nabla^{\mathrm{ext}} \colon \mathcal{F}^{\mathrm{ext}} \to \mathcal{F}^{\mathrm{ext}}\big(\mathcal{M}_{\mathrm{ext}} \times \{0\}\big) \otimes \left(\pi^*\Omega^1_{\mathcal{M}_{\mathrm{ext}}} \oplus \mathcal{O}_{\mathcal{M}_{\mathrm{ext}}\times\mathbf{C}_z}\frac{dz}{z}\right),$$

where $\pi : \mathcal{M}_{\mathrm{ext}} \times \mathbf{C}_z \to \mathcal{M}_{\mathrm{ext}}$ *is the projection, a* ∇^{ext}*-flat, symmetric and non-degenerate pairing*

$$\mathrm{P}^{\mathrm{ext}} \colon (-)^*\mathcal{F} \otimes \mathcal{F} \to z^{\hat{c}}\mathcal{O}_{\mathcal{M}_{\mathrm{ext}}\times\mathbf{C}_z}$$

and a $\mathbf{Z}$*-local subsystem* $\mathrm{F}^{\mathrm{ext}}_{\mathbf{Z}}$ *of the same rank over* $\mathcal{M}_{\mathrm{ext}} \times \mathbf{C}^{\times}_z$

$$\mathrm{F}^{\mathrm{ext}}_{\mathbf{Z}} \subset \big(\mathcal{F}^{\mathrm{ext}}\big|_{\mathcal{M}_{\mathrm{ext}}\times\mathbf{C}^{\times}_z}\big)^{\nabla^{\mathrm{ext}}}$$

such that the following holds:

- $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, \mathrm{P}^{\mathrm{ext}}, \mathrm{F}^{\mathrm{ext}}_{\mathbf{Z}})|_{\mathcal{M}'} = (\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})|_{\mathcal{M}'}$;
- *There exist open neighbourhoods* $\mathrm{U}^{\mathrm{ext}}_{\heartsuit}$ *of* $\mathrm{U}_{\heartsuit}$ *in* $\mathcal{M}_{\mathrm{ext}}$ *and open embeddings*

$$\tau^{\mathrm{ext}}_{\heartsuit} \colon \mathrm{U}^{\mathrm{ext}}_{\heartsuit} \hookrightarrow \mathrm{H}'_{\heartsuit}/\langle \mathrm{G}\rangle, \qquad \tau^{\mathrm{ext}}_{\heartsuit}\big|_{\mathrm{U}_{\heartsuit}} = \tau_{\heartsuit}$$

for $\heartsuit = \mathrm{GW}$ *and* FJRW *such that we have isomorphisms*

$$\begin{aligned}&\big(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, (-1)^{\mathrm{N}-1}\mathrm{P}^{\mathrm{ext}}, \mathrm{F}^{\mathrm{ext}}_{\mathbf{Z}}\big)\big|_{\mathrm{U}^{\mathrm{ext}}_{\mathrm{FJRW}}}\\ &\quad\cong {\tau^{\mathrm{ext}}_{\mathrm{FJRW}}}^{*}\big(\mathrm{QDM}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)/\langle \mathrm{G}\rangle\big)\end{aligned}$$

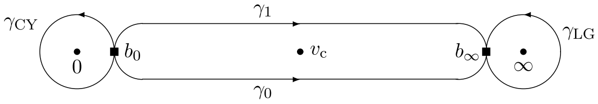


FIG. 2. — Various paths in $\mathcal{M}$

$$\left(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, \mathrm{P}^{\mathrm{ext}}, \mathrm{F}_{\mathbf{Z}}^{\mathrm{ext}}\right)\big|_{\mathrm{U}^{\mathrm{ext}}_{\mathrm{GW}}} \\ \cong \tau^{\mathrm{ext}*}_{\mathrm{GW}}\big(\mathrm{QDM}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})/\langle \mathrm{G}\rangle\big).$$

Finally we give a statement on the monodromy representation of the global quantum D-module $(\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})$. We choose base points $b_0, b_\infty \in \mathcal{M}$ near the large radius limit point and the LG point such that $b_0, b_\infty \in \mathbf{R}_{>0}$ and $0 < b_0 \ll 1 \ll b_\infty$. We choose paths $\gamma_{\mathrm{CY}}, \gamma_{\mathrm{LG}}, \gamma_0, \gamma_1$ in $\mathcal{M}$ as in Figure 2. We also define (see also Figure 3)

$$\gamma_l := \gamma_{\mathrm{LG}}^l \circ \gamma_0 \circ \gamma_{\mathrm{CY}}^l, \qquad \gamma_{\mathrm{con}} := \gamma_1^{-1} \circ \gamma_0$$

for $l \in \mathbf{Z}$. We adopt the convention that the composite $\gamma_{\mathrm{A}} \circ \gamma_{\mathrm{B}}$ means the concatenation of γ_{A} at the end of γ_{B}.

Let $\mathrm{N}(\mathrm{X}_{\mathrm{W}})$ denote the numerical K-group of X_{W}. From the definition of the $\widehat{\Gamma}$-integral structure, the fibre at b_0 of the global $\mathbf{Z}$-local system $\mathrm{F}_{\mathbf{Z}}$ is identified with

$$\mathrm{N}'(\mathrm{X}_{\mathrm{W}}) := \big\{\mathrm{E} \in \mathrm{N}(\mathrm{X}_{\mathrm{W}}) \mid \mathrm{ch}(\mathrm{E}) \in \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})\big\}. \tag{23}$$

Similarly, the fibre at b_∞ of $\mathrm{F}_{\mathbf{Z}}$ is identified with the group $\mathrm{N}'(\mathrm{W}, \boldsymbol{\mu}_d)$ of numerical classes of matrix factorizations $\mathcal{E}$ such that $\mathrm{ch}(\mathcal{E}) \in \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$. The following theorem is a detailed version of Theorem 1.2 (which also includes the part of Theorem 1.1 concerning the Orlov equivalence). The proof will be given in Section 5.6.

Theorem **2.26.** — *The* $\mathbf{Z}$*-local system* $\mathrm{F}_{\mathbf{Z}}$ *of the global* D*-module* $(\mathcal{F}, \nabla, \mathrm{P}, \mathrm{F}_{\mathbf{Z}})$ *in Theorem 2.23 induces the representation of the quiver of Figure 2 given by the assignment* $b_0 \mapsto \mathrm{N}'(\mathrm{X}_{\mathrm{W}})$, $b_\infty \mapsto \mathrm{N}'(\mathrm{W}, \boldsymbol{\mu}_d)$ *and*

$$\begin{aligned}
\gamma_{\mathrm{CY}} &\longmapsto \mathcal{O}(-1)\colon \mathrm{N}'(\mathrm{X}_{\mathrm{W}}) \to \mathrm{N}'(\mathrm{X}_{\mathrm{W}})\\
\gamma_{\mathrm{LG}} &\longmapsto (1)\colon \mathrm{N}'(\mathrm{W}, \boldsymbol{\mu}_d) \to \mathrm{N}'(\mathrm{W}, \boldsymbol{\mu}_d)\\
\gamma_l^{-1} &\longmapsto \Phi_l\colon \mathrm{N}'(\mathrm{W}, \boldsymbol{\mu}_d) \to \mathrm{N}'(\mathrm{X}_{\mathrm{W}})\\
\gamma_{\mathrm{con}}^{-1} &\longmapsto \mathrm{T}_{\mathcal{O}}\colon \mathrm{N}'(\mathrm{X}_{\mathrm{W}}) \to \mathrm{N}'(\mathrm{X}_{\mathrm{W}})
\end{aligned}$$

where $\mathcal{O}(-1)$ *denotes the tensor product by* $\mathcal{O}(-1)$, (1) *denotes the shift of the grading by* 1, Φ_l *denotes the Orlov equivalence (Section 4.2) defined for* $l \in \mathbf{Z}$ *and* $\mathrm{T}_{\mathcal{O}}$ *denotes the Seidel-Thomas spherical twist (Section 5.6) by the structure sheaf. Moreover, the monodromy representation*

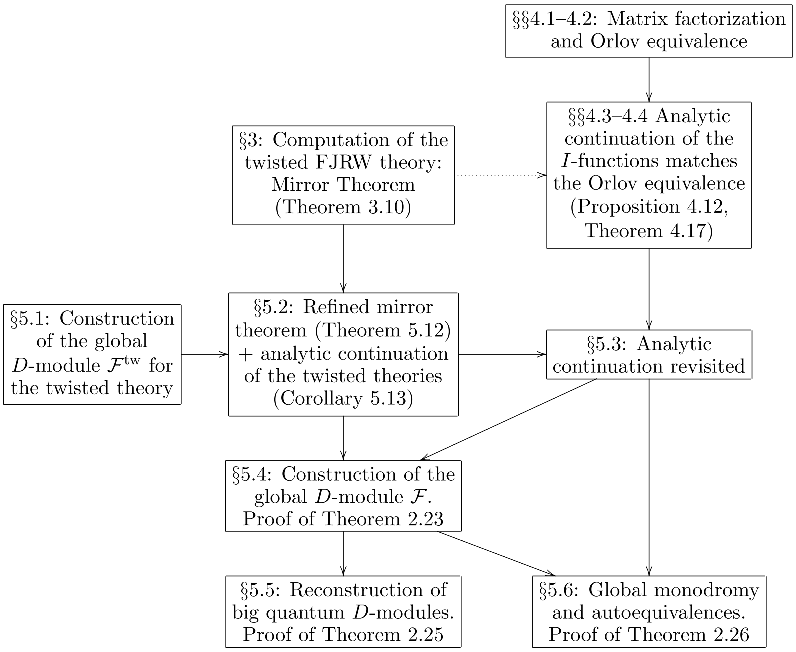
$$\rho\colon \pi_1(\mathcal{M}, b_0) \to \mathrm{Aut}\big(\mathrm{N}'(\mathrm{X}_{\mathrm{W}}), \chi\big)$$

can be lifted to a group homomorphism

$$\hat{\rho}\colon \pi_1(\mathcal{M}, b_0) \to \mathrm{Auteq}\big(\mathrm{D}^b(\mathrm{X}_\mathrm{W})\big)/[2],$$

where $[2]$ *is the* 2*-shift functor.*

2.6. *Flowchart of the proof.* — Here we show by a picture the organization of the rest of the paper. Several key results are highlighted in the chart.



The essence of the paper lies in the computation in Section 4 where the analytic continuation map **U** of the two I-functions (or more precisely the two $\mathfrak{H}$-functions) is matched up with the Orlov derived equivalence.

3. Computing FJRW theory

We compute FJRW invariants attached to narrow state space entries. In Section 3.1, we provide an extension of the definition of the invariants to a larger state space. The new invariants are zero on the extended part, but arise as the non-equivariant limit of the e_T-twisted invariants. In Sections 3.2–3.4, we calculate the twisted invariants (or more precisely the I-function) using Chiodo-Zvonkine's results [16] and Givental's symplectic formalism [29].

3.1. *Extending FJRW theory.* — Define the *extended narrow state space* (or simply the *extended state space*) to be

$$\mathrm{H}_{\mathrm{ext}} = \bigoplus_{k=1}^{d} \phi_{k-1}\mathbf{C} = \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d) \oplus \bigoplus_{k \notin \mathsf{Nar}} \phi_{k-1}\mathbf{C}. \tag{24}$$

This modified state space may be regarded as the result of the replacement of each term of the broad sector $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$, $k \notin \mathsf{Nar}$, with a one-dimensional term $\phi_{k-1}\mathbf{C}$. As we show straight away in Proposition 3.2 these new states play the role of placeholders in the theory: they only yield vanishing invariants and they allow to simplify the computation of the invariants with narrow entries.

We need to extend the grading of $\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$ to the extended state space. We set (cf. (**1**))

$$\deg \phi_{k-1} = 2\sum_{j=1}^{\mathrm{N}} \big\langle (k-1)q_j \big\rangle = 2\mathrm{N}_k + 2\sum_{j=1}^{\mathrm{N}} \langle kq_j \rangle - 2 \tag{25}$$

with $q_j := w_j/d$. The relevant moduli stack is $\mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$ defined as in (**9**), but for $k_1, \dots, k_n \in \{0, \dots, d-1\}$. Its universal curve $\pi : \mathcal{C} \to \mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$ is equipped with a universal d-spin structure $\mathcal{L}$ and a line bundle $\mathcal{M}_i = \mathcal{O}(\mathcal{D}_i)$, where $\mathcal{D}_i \subset \mathcal{C}$ denotes the divisor of the ith marking. The *extended obstruction* K*-class* is defined to be

$$-\mathbf{R}\pi_*\Big(\bigoplus_{j=1}^{\mathrm{N}} \widetilde{\mathcal{L}}^{\otimes w_j}\Big) \quad \text{for } \widetilde{\mathcal{L}} = \mathcal{L} \otimes \mathcal{M}^\vee$$

where $\mathcal{M} = \bigotimes_{i=1}^n \mathcal{M}_i$. Let $p \colon \mathcal{C} \to \overline{\mathcal{C}}$ denote the morphism forgetting the stack-theoretic structure along all the markings $\mathcal{D}_1, \dots, \mathcal{D}_n$ (but not along the nodes). Then we have

$$\mathrm{age}_{\mathcal{D}_i}(\widetilde{\mathcal{L}}) = \frac{k_i}{d}, \qquad \widetilde{\mathcal{L}}^{\otimes d} \cong p^*\overline{\omega}$$

for the relative dualizing sheaf $\overline{\omega}$ of $\overline{\pi} \colon \overline{\mathcal{C}} \to \mathrm{Spin}^d_{0,n}(k_1, \dots, k_n)$.

Proposition **3.1.** — *For any fibre* C *of* $\mathcal{C}$, *we have* $\mathrm{H}^0(\mathrm{C}, \widetilde{\mathcal{L}}^{\otimes w_j}|_{\mathrm{C}}) = 0, j = 1, \dots, \mathrm{N}$. *As a consequence,* $\mathrm{R}^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j})$ *is locally free and the extended obstruction* K*-class is represented by a vector bundle.*

Proof. — Because w_j divides d, $\widetilde{\mathcal{L}}^{\otimes w_j}$ is a root of $p^*\overline{\omega}$. On the other hand, we have $\mathrm{H}^0(\mathrm{C}, p^*\overline{\omega}) = \mathrm{H}^0(\overline{\mathrm{C}}, \overline{\omega}) = 0$ because the genus of $\overline{\mathrm{C}} = p(\mathrm{C})$ is zero. Hence $\widetilde{\mathcal{L}}^{\otimes w_j}|_{\mathrm{C}}$ does not have nonzero global sections either. □

We define the *extended FJRW invariants* to be

$$(\mathbf{26})\qquad \big\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\big\rangle_{0,n}^{\mathrm{FJRW,ext}} := \int_{[\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)]} \left(\prod_{i=1}^{n}\psi_i^{b_i}\right)\cup c_{\mathrm{top}}\left(\bigoplus_{j=1}^{\mathrm{N}} \mathrm{R}^1\pi_*\widetilde{\mathcal{L}}^{\otimes w_j}\right),$$

for $\phi_{k_1},\dots,\phi_{k_n}$ lying within the *extended state space* $\mathrm{H}_{\mathrm{ext}}$.

Proposition **3.2.** — *The above invariants vanish if one of the entries* $\phi_{k_1},\dots,\phi_{k_n}$ *does not belong to the narrow state space* $\mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)$. *Otherwise* $\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\rangle_{0,n}^{\mathrm{FJRW,ext}}$ *equals* $\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\rangle_{0,n}^{\mathrm{FJRW}}$.

Proof. — The proof parallels the argument of [13, Lemma 4.1.1]. Let us compare $\widetilde{\mathcal{L}}^{\otimes w_j}$ and $\mathcal{L}^{\otimes w_j}$ after push-forward via the morphism $p\colon \mathcal{C}\to\overline{\mathcal{C}}$ forgetting the stack-theoretic structure at the markings. We get[13]

$$(\mathbf{27})\qquad p_*\big(\widetilde{\mathcal{L}}^{\otimes w_j}\big) = p_*\big(\mathcal{L}^{\otimes w_j}\big)\otimes\mathcal{O}\Bigg(-\sum_{i:(k_i+1)w_j\in d\mathbf{Z}}\overline{\mathcal{D}}_i\Bigg),$$

where $\overline{\mathcal{D}}_i\subset\overline{\mathcal{C}}$ is the divisor supported on the ith coarse marking. Therefore, if $(k_i+1)w_j\notin d\mathbf{Z}$ for all i, we have $\mathrm{R}^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j})=\mathrm{R}^1\pi_*(\mathcal{L}^{\otimes w_j})$. This shows the second claim above.

The vanishing condition in the statement holds when $(k_i+1)w_j\in d\mathbf{Z}$ for some $1\le i\le n$ and some $1\le j\le \mathrm{N}$. This simply means that $\mathcal{L}^{\otimes w_j}|_{\mathcal{D}_i}$ is pulled back from the coarse divisor $\overline{\mathcal{D}}_i$. On the other hand $\mathcal{L}^{\otimes w_j}$ is a root of $\omega_{\log}$ and $\omega_{\log}|_{\mathcal{D}_i}\cong\mathcal{O}_{\mathcal{D}_i}$ via the residue map. Therefore, $c_1(\mathcal{L}^{\otimes w_j}|_{\mathcal{D}_i})=0$ and hence $c_1(p_*(\mathcal{L}^{\otimes w_j})|_{\overline{\mathcal{D}}_i})=0$ in the rational cohomology group.

Set $\mathcal{T}=p_*(\widetilde{\mathcal{L}}^{\otimes w_j})$. From (27), $\mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathcal{D}}_i}=p_*(\mathcal{L}^{\otimes w_j})|_{\overline{\mathcal{D}}_i}$ has vanishing first Chern class. Write the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{T}(\overline{\mathcal{D}}_i) \arrow[r] & \mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathcal{D}}_i} \arrow[r] & 0 \end{tikzcd}$$

and the induced exact sequence of vector bundles

$$(\mathbf{28})\qquad \begin{tikzcd} 0 \arrow[r] & \overline{\pi}_*\big(\mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathcal{D}}_i}\big) \arrow[r] & \mathrm{R}^1\overline{\pi}_*\mathcal{T} \arrow[r] & \mathrm{R}^1\overline{\pi}_*\mathcal{T}(\overline{\mathcal{D}}_i) \arrow[r] & 0. \end{tikzcd}$$

The vanishing $c_{\mathrm{top}}(\mathrm{R}^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j}))=c_{\mathrm{top}}(\mathrm{R}^1\overline{\pi}_*\mathcal{T})=0$ follows from $c_1(\mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathcal{D}}_i})=0$. Note that, in order to get (28), we need to show that $\mathcal{T}(\overline{\mathcal{D}}_i)$ has only trivial sections on each fibre $\overline{\mathrm{C}}$ of $\overline{\mathcal{C}}$. For $a=d/w_j$, we find that $\mathcal{T}(\overline{\mathcal{D}}_i)^{\otimes a}\cong\overline{\omega}(\overline{\mathcal{D}}_i-\sum_{l\neq i}a\langle k_l/a\rangle\overline{\mathcal{D}}_l)$ which is a subsheaf of $\overline{\omega}(\overline{\mathcal{D}}_i)$. It is easy to see that $\mathrm{H}^0(\overline{\mathrm{C}},\overline{\omega}(\overline{\mathcal{D}}_i)|_{\overline{\mathrm{C}}})=0$ for a genus zero curve $\overline{\mathrm{C}}$ by induction on the components (see [13]). Therefore $\mathrm{H}^0(\overline{\mathrm{C}},\mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathrm{C}}})=0$. □

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[13] To see this, use $p^*p_*\mathcal{E}=\mathcal{E}\otimes\mathcal{O}(-\sum_{i=1}^{n} d\,\mathrm{age}_{\mathcal{D}_i}(\mathcal{E})\mathcal{D}_i)$ for an invertible sheaf $\mathcal{E}$ on $\mathcal{C}$.

3.2. *Twisted FJRW theory and Givental's formalism.* — Let $\mathrm{K} = \mathbf{C}[[\mathbf{s}]]$ denote the completion of the polynomial ring $\mathbf{C}[s_k^{(j)} \mid 1 \leq j \leq \mathrm{N}, k \geq 0]$ with respect to the additive valuation

$$v\big(s_k^{(j)}\big) = k+1.$$

We define the ring $\mathrm{K}\{z, z^{-1}\}$ of adically convergent power series in z by

$$\mathrm{K}\big\{z, z^{-1}\big\} = \left\{ \sum_{n\in\mathbf{Z}} a_n z^n \;\middle|\; a_n \in \mathrm{K},\ v(a_n) \to \infty \text{ as } |n| \to \infty \right\}.$$

Define $\mathrm{K}\{z\}$ (resp. $\mathrm{K}\{z^{-1}\}$) to be the subring of $\mathrm{K}\{z, z^{-1}\}$ consisting of non-negative (resp. non-positive) power series in z. We introduce a symmetric non-degenerate pairing $(\cdot, \cdot)_{\mathbf{s}}$ on $\mathrm{H}_{\mathrm{ext}} \otimes \mathrm{K}$ taking values in K:

$$(\mathbf{29}) \qquad (\phi_h, \phi_k)_{\mathbf{s}} = \frac{1}{d}\bigg(\prod_{j:\langle (h+1)q_j\rangle = 0} \exp\big(-s_0^{(j)}\big) \bigg) \delta_{h+k,d-2},$$

where $\delta_{h+k,d-2}$ is 1 if $h+k \equiv d-2$ (d) and 0 otherwise. Through the entire section we adopt the convention that the index is reduced modulo d to the suitable range $\{0, \dots, d-1\}$.

Definition **3.3** *(Givental's symplectic space). — Givental's symplectic space is the space*

$$\mathcal{H} := \mathrm{H}_{\mathrm{ext}} \otimes \mathrm{K}\big\{z, z^{-1}\big\}$$

equipped with the symplectic form

$$\Omega^{\mathbf{s}}(f_1, f_2) = \mathrm{Res}_{z=0}\big(f_1(-z), f_2(z)\big)_{\mathbf{s}} dz.$$

The space $\mathcal{H}$ has a standard polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{H}_{\mathrm{ext}} \otimes \mathrm{K}\{z\}$, $\mathcal{H}_- = \mathrm{H}_{\mathrm{ext}} \otimes z^{-1}\mathrm{K}\{z^{-1}\}$ are isotropic subspaces of $\mathcal{H}$. This polarization allows us to identify $\mathcal{H}$ with the total space of the cotangent bundle of $\mathcal{H}_+$.

For the basis $\{\phi_k\}_{k=0}^{d-1}$ of $\mathrm{H}_{\mathrm{ext}}$, we write $g_{hk}^{\mathbf{s}}$ for $(\phi_h, \phi_k)_{\mathbf{s}}$ and $g_{\mathbf{s}}^{hk}$ for the coefficients of the inverse matrix. A general point of $\mathcal{H}$ can be written as $\mathbf{q} + \mathbf{p}$ with

$$(\mathbf{30}) \qquad \mathbf{q} = \sum_{b\geq 0} \sum_{k=0}^{d-1} q_b^k \phi_k z^b \in \mathcal{H}_+, \qquad \mathbf{p} = \sum_{b\geq 0} \sum_{h,k=0}^{d-1} p_{b,h} g_{\mathbf{s}}^{hk} \frac{\phi_k}{(-z)^{1+b}} \in \mathcal{H}_-.$$

Here $\{q_b^k, p_{b,k} \mid b \geq 0, 0 \leq k \leq d-1\}$ can be regarded as Darboux co-ordinates on $\mathcal{H}$. Following established practice, we denote the coordinates in Givental formalism by q_b^k, with the label b corresponding to gravitational descendants and with the label k corresponding to state space entries. We indicate explicitly where the superscript "k" index is meant instead as an exponent of a power, indeed this turns out to be useful in some very special cases.

Definition **3.4** *(Twisted FJRW theory cf. [21]). — Consider the universal characteristic class of the extended obstruction* K*-class:*

$$e(\mathbf{s}) = \exp\Bigg(\sum_{1\leq j\leq \mathrm{N}}\sum_{l=0}^{\infty} s_l^{(j)}\mathrm{ch}_l\big(\mathbf{R}\pi_*\widetilde{\mathcal{L}}^{\otimes w_j}\big)\Bigg) \in \mathrm{H}^*\big(\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n);\mathbf{C}\big)\otimes \mathrm{K} \tag{31}$$

and define the twisted FJRW invariants *as*

$$\big\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\big\rangle^{\mathbf{s}}_{0,n} = \int_{[\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)]}\Bigg(\prod_{i=1}^{n}\psi_i^{b_i}\Bigg)\cup e(\mathbf{s}). \tag{32}$$

The twisted FJRW invariants are encoded in the generating function

$$\mathbf{F}^{\mathbf{s}}_0 = \sum_{\substack{b_1,\dots,b_n\geq 0\\ 0\leq k_1,\dots,k_n\leq s}} \big\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\big\rangle^{\mathbf{s}}_{0,n}\frac{t^{k_1}_{b_1}\cdots t^{k_n}_{b_n}}{n!}. \tag{33}$$

This is a formal power series in infinitely many variables $\{t^k_b \mid 0\leq k\leq d-1, b\geq 0\}$*. Again, the superscript* k *of* t^k_b *means an index, not an exponent of a power.*

The twisted FJRW invariants here are a generalization of the extended invariants (26).

Definition **3.5** *(Givental's Lagrangian submanifold). — We relate the variables* $\{t^k_b\}$ *of* $\mathbf{F}^{\mathbf{s}}_0$ *and the co-ordinates* $\{q^k_b\}$ *on* $\mathcal{H}_+$ *by the following* dilaton shift:

$$q^k_b = -\delta^1_b\delta^k_0 + t^k_b.$$

Then $\mathbf{F}^{\mathbf{s}}_0$ *can be regarded as a function defined on a formal neighbourhood of* $-z\phi_0\in\mathcal{H}_+$*. The graph of* $d\mathbf{F}^{\mathbf{s}}_0$ *defines a Lagrangian submanifold of* $(\mathcal{H},\Omega^{\mathbf{s}})$*:*

$$\mathcal{L}^{\mathbf{s}} := \Bigg\{\mathbf{q}+\mathbf{p}\in\mathcal{H}\;\Bigg|\; p_{b,k} = \frac{\partial\mathbf{F}^{\mathbf{s}}_0}{\partial q^k_b},\; b\geq 0, 0\leq k\leq d-1\Bigg\}. \tag{34}$$

The submanifold $\mathcal{L}^{\mathbf{s}}$ *can be defined as a formal scheme over* K*. See [17, Appendix B].*

3.2.1. *The untwisted theory.* — Consider the case where $s^{(j)}_l = 0$ for all $1\leq j\leq \mathrm{N}$ and $l\geq 0$. Then $e(\mathbf{s}) = 1$ and the associated correlators give the so called *untwisted* invariants

$$\begin{aligned}(\mathbf{35})\qquad &\left\langle \tau_{b_1}(\phi_{k_1}),\dots,\tau_{b_n}(\phi_{k_n})\right\rangle^{\mathrm{un}}_{0,n}\\ &=\int_{[\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)]}\prod_{i=1}^n \psi_i^{b_i}\\ &=\begin{cases}\frac{1}{d}\frac{(\sum_{i=1}^n b_i)!}{b_1!\cdots b_n!} & \text{if } n-3=\sum_{i=1}^n b_i \text{ and } 2+\sum_{i=1}^n k_i\in d\mathbf{Z},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

One can show this by using the String Equation (since one of b_i has to be zero). The same Hodge integral over $\overline{\mathcal{M}}_{0,n}$ is given in [24, Equation (2)]; the new factor $1/d$ here comes from the fact that $\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)$ has $\boldsymbol{\mu}_d$ as the generic stabilizer and that $[\mathrm{Spin}^d_{0,n}(k_1,\dots,k_n)]=\frac{1}{d}[\overline{\mathcal{M}}_{0,n}]$. The generating function $\mathbf{F}^{\mathrm{un}}_0$ of untwisted invariants are defined similarly to (**33**). The pairing $(\cdot,\cdot)_{\mathbf{s}}$ and the symplectic form $\Omega^{\mathbf{s}}$ specialize to

$$(\phi_k,\phi_h)_{\mathrm{un}}=\frac{1}{d}\delta_{d-2,k+h},\qquad \Omega^{\mathrm{un}}(f_1,f_2)=\mathrm{Res}_{z=0}\big(f_1(-z),f_2(z)\big)_{\mathrm{un}}dz.$$

The Lagrangian submanifold $\mathcal{L}^{\mathrm{un}}\subset(\mathcal{H},\Omega^{\mathrm{un}})$ is defined as the graph of $\mathsf{d}\mathbf{F}^{\mathrm{un}}_0$ as in (**34**). (Here one should use as Darboux co-ordinates those given by the *untwisted* pairing $g^{\mathrm{un}}_{kh}=(\phi_k,\phi_h)_{\mathrm{un}}$ instead of $g^{\mathbf{s}}_{kh}$, cf. (**30**).)

Since the untwisted invariants are the usual intersection numbers on $\overline{\mathcal{M}}_{0,n}$, the generating function $\mathbf{F}^{\mathrm{un}}_0$ satisfy the well known tautological equations: String Equation (SE), Dilaton Equation (DE) and Topological Recursion Relations (TRR). Givental [29] showed that these three equations for a genus zero potential $\mathbf{F}_0$ are equivalent to the following geometric properties for the graph $\mathcal{L}$ of the differential $\mathsf{d}\mathbf{F}_0$:

- $\mathcal{L}$ is a cone in $\mathcal{H}$ with vertex at the origin $\mathbf{p}=\mathbf{q}=0$ (with the dilaton shift $q^k_a=t^k_a-\delta^1_a\delta^k_0$ understood);
- The tangent space T to $\mathcal{L}$ at any point on $\mathcal{L}$ satisfies $z\mathrm{T}=\mathcal{L}\cap\mathrm{T}$; Moreover the tangent space to $\mathcal{L}$ at any point in $z\mathrm{T}\subset\mathcal{L}$ equals T.

We refer to these properties as *Givental's geometric properties* for $\mathcal{L}$. In particular, $\mathcal{L}^{\mathrm{un}}$ satisfies Givental's geometric properties.

3.2.2. *The twisted theory.*

— The Lagrangian submanifold $\mathcal{L}^{\mathbf{s}}$ was determined by Chiodo-Zvonkine [16]. Define a linear symplectic transformation $\Delta\colon(\mathcal{H},\Omega^{\mathrm{un}})\to(\mathcal{H},\Omega^{\mathbf{s}})$ by

$$(\mathbf{36})\qquad \Delta=\bigoplus_{i=0}^{d-1}\exp\left(\sum_{j=1}^{\mathrm{N}}\sum_{l\geq 0}s^{(j)}_l\frac{\mathrm{B}_{l+1}(\langle iq_j\rangle+q_j)}{(l+1)!}z^l\right)$$

where $\mathrm{B}_n(x)$ is the Bernoulli polynomial defined by $\sum_{n=0}^{\infty}\mathrm{B}_n(x)z^n/n!=ze^{zx}/(e^z-1)$.

Theorem **3.6** *(Chiodo-Zvonkine [16]). — We have* $\mathcal{L}^{\mathbf{s}} = \Delta(\mathcal{L}^{\mathrm{un}})$.

Because Givental's geometric properties are preserved by a linear symplectic transformation, *the generating function* $\mathbf{F}_0^{\mathbf{s}}$ *of twisted FJRW invariants satisfies SE, DE and TRR.*

The adaptation of [16] to our context was explained in [13, Proposition 4.1.5]; we omit the details.

3.3. *Family of elements on the Lagrangian cone.* — The *twisted* J*-function* is a family of elements lying on $\mathcal{L}^{\mathbf{s}}$ parametrized by $t = \sum_{k=0}^{d-1} t^k \phi_k \in \mathrm{H}_{\mathrm{ext}} \otimes \mathrm{K}$:

$$\begin{aligned}(\mathbf{37})\qquad \mathrm{J}^{\mathbf{s}}(t, -z) = & -z\phi_0 + t \\ & + \sum_{n=2}^{\infty}\sum_{b=0}^{\infty}\sum_{0\le k,h\le d-1} \frac{1}{n!}\langle t, \dots, t, \tau_b(\phi_k)\rangle^{\mathbf{s}}_{0,n+1} g_{\mathbf{s}}^{kh} \frac{\phi_h}{(-z)^{b+1}}.\end{aligned}$$

Here $\mathrm{J}^{\mathbf{s}}(t, -z) \in \mathcal{H}$ is characterized as a unique point lying on $\mathcal{L}^{\mathbf{s}}$ with the property:

$$(\mathbf{38})\qquad \mathrm{J}^{\mathbf{s}}(t, -z) = -\phi_0 z + t + \mathrm{O}\big(z^{-1}\big).$$

It is known [29] that the J-function reconstructs the cone $\mathcal{L}^{\mathbf{s}}$ itself via Givental's geometric properties. Here we will find another explicit family of elements (I-function) on $\mathcal{L}^{\mathbf{s}}$.

The J-function $\mathrm{J}^{\mathrm{un}} \in \mathcal{L}^{\mathrm{un}}$ of the untwisted theory (Section 3.2.1) is the specialization of (**37**) at $\mathbf{s} = 0$. Using (**35**), we calculate

$$\mathrm{J}^{\mathrm{un}}(t, -z) = \sum_{\mathbf{k}=(k_0,\dots,k_{d-1})\in\mathbf{Z}^d_{\ge 0}} \mathrm{J}^{\mathrm{un}}_{\mathbf{k}}(t, -z), \quad \text{where}$$

$$\mathrm{J}^{\mathrm{un}}_{\mathbf{k}}(t, -z) = \frac{1}{(-z)^{|\mathbf{k}|-1}} \frac{(t^0)^{k_0}\dots(t^{d-1})^{k_{d-1}}}{k_0!\dots k_{d-1}!}\phi_{h(\mathbf{k})},$$

with $|\mathbf{k}| = \sum_{i=0}^{d-1} k_i$ and $h(\mathbf{k}) = \sum_{i=0}^{d-1} ik_i$. Here $(t^i)^{k_i}$ means the k_i-th power of the variable t^i. Introduce the *modification factor* $\mathrm{M}_{\mathbf{k}}(z)$ by

$$\mathrm{M}_{\mathbf{k}}(z) = \prod_{j=1}^{\mathrm{N}} \exp\Big(-\sum_{0\le m<\lfloor q_j h(\mathbf{k})\rfloor} \mathbf{s}^{(j)}\big(-(q_j + \langle q_j h(\mathbf{k})\rangle + m)z\big)\Big),$$

where $\mathbf{s}^{(j)}(x) = \sum_{n\ge 0} s_n^{(j)} x^n/n!$ and define the *twisted* I*-function* by

$$(\mathbf{39})\qquad \mathrm{I}^{\mathbf{s}}(t, z) = \sum_{k_0,\dots,k_{d-1}\ge 0} \mathrm{M}_{\mathbf{k}}(z)\mathrm{J}^{\mathrm{un}}_{\mathbf{k}}(t, z).$$

Using Theorem 3.6, we get the following statement.

Theorem **3.7.** — *The family* $t \mapsto \mathrm{I}^{\mathbf{s}}(t, -z)$ *of elements of* $\mathcal{H}$ *lies on* $\mathcal{L}^{\mathbf{s}}$.

Proof. — The discussion here is parallel to [17, Theorem 4.8] and [13, Theorem 4.1.6]. We give a sketch of the proof and leave the details to the reader. Introduce a function

$$G_y^{(j)}(x,z) = \sum_{m,l\geq 0} s_{l+m-1}^{(j)} \frac{B_m(y)}{m!}\frac{x^l}{l!} z^{m-1}, \quad j = 1,\dots,N$$

with $s_{-1}^{(j)} = 0$. Set $D := \sum_{k=0}^{d-1} kt^k(\partial/\partial t^k)$. Givental's geometric properties for the cone $\mathcal{L}^{\mathrm{un}}$ yield the following fact (see [17, Equation (14)] and [13, Lemma 4.1.10]):

Lemma **3.8.** — *The family* $t \mapsto \exp(-\sum_{j=1}^N G_0^{(j)}(zq_jD + zq_j, z))J^{\mathrm{un}}(t,-z)$ *lies on* $\mathcal{L}^{\mathrm{un}}$.

The conclusion of Theorem 3.7 follows from Theorem 3.6: we apply the symplectic transformation $\Delta\colon (\mathcal{H},\Omega^{\mathrm{un}}) \to (\mathcal{H},\Omega^{\mathbf{s}})$ in (**36**) to the family in Lemma 3.8. Note that we have

$$\Delta = \bigoplus_{i=0}^{d-1} \exp\left(\sum_{j=1}^N G_{\langle iq_j\rangle + q_j}^{(j)}(0,z)\right) = \bigoplus_{i=0}^{d-1} \exp\left(\sum_{j=1}^N G_0^{(j)}\big((\langle iq_j\rangle + q_j)z, z\big)\right)$$

where we used $G_y^{(j)}(x,z) = G_0^{(j)}(x+yz,z)$ in the second equality. Using the identity

$$G_0^{(j)}(x+z,z) = G_0^{(j)}(x,z) + \mathbf{s}^{(j)}(x)$$

we can easily check that

$$I^{\mathbf{s}}(t,-z) = \Delta\exp\left(-\sum_{j=1}^N G_0^{(j)}(zq_jD + zq_j, z)\right)J^{\mathrm{un}}(t,-z).$$

Theorem 3.6 and Lemma 3.8 show that $I^{\mathbf{s}}(t,-z)$ is on the cone $\mathcal{L}^{\mathbf{s}}$. □

3.4. *The twist by the equivariant Euler class.* — Let $T = (\mathbf{C}^\times)^N$ act on the extended obstruction bundle $\bigoplus_{j=1}^N R^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j})$ diagonally by scaling the fibres and trivially on the base $\mathrm{Spin}_{0,n}^d(k_1,\dots,k_n)$. Let $\lambda_1,\dots,\lambda_N \in H_T^2(\mathrm{pt})$ denote the equivariant parameters. Then the T-equivariant Euler class e_T of the extended obstruction bundle is given by

$$e_T\left(\bigoplus_{j=1}^N R^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j})\right) = \prod_{j=1}^N\sum_{l=0}^{r_j} \lambda_j^{r_j - l} c_l(R^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j}))$$

with $r_j = \mathrm{rank}(R^1\pi_*(\widetilde{\mathcal{L}}^{\otimes w_j}))$. This class can be obtained from the universal class $e(\mathbf{s})$ (**31**) by the substitution:

$$s_l^{(j)} = \begin{cases} -\log\lambda_j & l = 0; \\ (l-1)!(-\lambda_j)^{-l} & l > 0. \end{cases}$$

Note that this specialization yields $\exp(-\mathbf{s}^{(j)}(x)) = x + \lambda_j$, where $\mathbf{s}^{(j)}(x) = \sum_{n=0}^{\infty} s_n^{(j)} x^n/n!$ as before. With this choice of parameters, we obtain

- The e_{T}-*twisted pairing* as the specialization of (**29**):

$$(\phi_h, \phi_k)_{\mathrm{tw}} := \frac{1}{d}\Bigg(\prod_{j:\langle q_j(h+1)\rangle=0} \lambda_j\Bigg)\delta_{d-2,k+h};$$

- The e_{T}-*twisted FJRW invariants* $\langle \tau_{b_1}(\phi_{k_1}), \dots, \tau_{b_n}(\phi_{k_n})\rangle_{0,n}^{\mathrm{tw}}$ as the specializations of (**32**);
- The e_{T}-*twisted* J-*function* $\mathrm{J}^{\mathrm{tw}}(t, -z; \lambda)$ as the specialization of (**37**);
- The e_{T}-*twisted* I-*function* $\mathrm{I}^{\mathrm{tw}}(u, z; \lambda)$ as the specialization of $u\mathrm{I}^{\mathbf{s}}(-u\phi_1, z)$ (see (**39**)):

$$\mathrm{I}^{\mathrm{tw}}(u, z; \lambda) := z\sum_{k=1}^{\infty} u^k \frac{\prod_{j=1}^{\mathrm{N}} \prod_{0<b<q_jk,\, \langle b\rangle=\langle q_jk\rangle}(\lambda_j - bz)}{\prod_{0<b<k}(-bz)}\phi_{k-1}. \tag{40}$$

Notice that the denominator $\prod_{0<b<k}(-bz)$ here is nothing but $(k-1)!(-z)^{k-1}$, but we prefer this expression in view of our parallel treatment of I-functions in Section 4.3.1. Notice also that the non-equivariant limit $\lambda_j \to 0$ of the e_{T}-twisted FJRW invariants yield the extended invariants (**26**).

Remark **3.9.** — Note that the specialization $s_0^{(j)} = -\log\lambda_j$ does not make sense for every element in the ground ring K. For this reason, we do not try to define the e_{T}-twisted Lagrangian cone. The specializations of the I- and J-functions, however, still make sense as elements of $\mathrm{H}_{\mathrm{ext}} \otimes \mathbf{C}[z, z^{-1}][\lambda_1^{\pm}, \dots, \lambda_{\mathrm{N}}^{\pm}][[t^0, \dots, t^{d-1}]]$.

The e_{T}-twisted I-function has the following z-asymptotics

$$\mathrm{I}^{\mathrm{tw}}(u, z; \lambda) = z\mathrm{F}(u)\phi_0 + \mathrm{G}(u;\lambda) + \mathrm{O}\big(z^{-1}\big) \tag{41}$$

where $\mathrm{F} \in \mathbf{C}[[u]]$ is a scalar-valued and $\mathrm{G} \in \mathrm{H}_{\mathrm{ext}}^{\leq 2} \otimes \mathbf{C}[\lambda_1, \dots, \lambda_{\mathrm{N}}][[u]]$ is an $\mathrm{H}_{\mathrm{ext}}^{\leq 2}$-valued power series (where $\mathrm{H}_{\mathrm{ext}}^{\leq 2}$ denotes the degree ≤ 2 part of $\mathrm{H}_{\mathrm{ext}}$):

$$\mathrm{F}(u) = \sum_{k\geq 1: k\equiv 1\,(d)} u^k \frac{\prod_{j=1}^{\mathrm{N}} (kq_j - 1)_{\lceil kq_j\rceil -1}}{(k-1)!},$$

$$\begin{aligned}\mathrm{G}(u;\lambda) = &-\sum_{k\geq 2: \sum_{j=1}^{\mathrm{N}}\langle q_j(k-1)\rangle=1} u^k \frac{\prod_{j=1}^{\mathrm{N}} (kq_j - 1)_{\lceil kq_j\rceil -1}}{(k-1)!}\phi_{k-1}\\ &-\sum_{i=1}^{\mathrm{N}}\lambda_i \sum_{k\geq d+1: k\equiv 1\,(d)} u^k\Bigg(\sum_{0<a<kq_i,\, \langle a\rangle=\big\langle kq_i\big\rangle}\frac{1}{a}\Bigg)\frac{\prod_{j=1}^{\mathrm{N}} (kq_j - 1)_{\lceil kq_j\rceil -1}}{(k-1)!}\phi_0,\end{aligned} \tag{42}$$

where $(a)_n = a(a-1)\cdots(a-n+1) = \Gamma(a+1)/\Gamma(a-n+1)$ denotes the falling factorial. We define the *FJRW mirror map* to be the $\mathrm{H}^{\leq 2}_{\mathrm{ext}}$-valued function:

$$\varsigma(u;\lambda) = \frac{\mathrm{G}(u;\lambda)}{\mathrm{F}(u)} = -u\phi_1 + \mathrm{O}\big(u^2\big) \in \mathrm{H}^{\leq 2}_{\mathrm{ext}} \otimes \mathbf{C}[\lambda_1, \dots, \lambda_{\mathrm{N}}][\![u]\!].$$

We now state the mirror theorem for the e_{T}-twisted FJRW theory.

Theorem **3.10.** — *We have* $\mathrm{J}^{\mathrm{tw}}(\varsigma(u;\lambda), z; \lambda) = \mathrm{I}^{\mathrm{tw}}(u, z; \lambda)/\mathrm{F}(u)$ *for the function* $\mathrm{F}(u)$ *in* (**42**).

Proof. — Because of the problem we discussed in Remark 3.9, we use another specialization $s^{(j)}_l = \bar{s}^{(j)}_l$, where

$$\bar{s}^{(j)}_l := \begin{cases} 0 & l = 0; \\ (l-1)!(-\lambda_j)^{-l} & l \geq 0. \end{cases}$$

This specialization yields $\exp(-\bar{\mathbf{s}}^{(j)}(x)) = 1 + x/\lambda_j$. It defines a well-defined homomorphism $\mathrm{K} \to \mathbf{C}[\![\lambda^{-1}]\!] := \mathbf{C}[\![\lambda_1^{-1}, \dots, \lambda_{\mathrm{N}}^{-1}]\!]$ and the characteristic class:

$$e(\bar{\mathbf{s}}) = \left(\prod_{j=1}^{\mathrm{N}} \lambda_j^{-r_j}\right) e_{\mathrm{T}}\left(\bigoplus_{j=1}^{\mathrm{N}} \mathrm{R}^1\pi_*\big(\widetilde{\mathcal{L}}^{\otimes w_j}\big)\right).$$

The Lagrangian cone $\mathcal{L}^{\bar{\mathbf{s}}}$ can be defined as a formal scheme over $\mathbf{C}[\![\lambda^{-1}]\!]$ and $\mathrm{I}^{\bar{\mathbf{s}}}(t, -z)$ is lying on $\mathcal{L}^{\bar{\mathbf{s}}}$ by Theorem 3.7. After some computation, we find that J^{tw} and I^{tw} are related to $\mathrm{J}^{\bar{\mathbf{s}}}$ and $\mathrm{I}^{\bar{\mathbf{s}}}$ as

$$\begin{aligned} \mathrm{J}^{\mathrm{tw}}(t, z; \lambda) &= \mathrm{R}(\lambda)\mathrm{J}^{\bar{\mathbf{s}}}\big(\mathrm{R}(\lambda)^{-1}t, z; \lambda\big) \\ \mathrm{I}^{\mathrm{tw}}(u, z; \lambda) &= \mathrm{R}(\lambda)u\mathrm{I}^{\bar{\mathbf{s}}}\big(-u\mathrm{R}(\lambda)^{-1}\phi_1, z\big) \end{aligned} \tag{43}$$

where $\mathrm{R}(\lambda)\colon \mathrm{H}_{\mathrm{ext}} \to \mathrm{H}_{\mathrm{ext}}$ is defined by $\mathrm{R}(\lambda)\phi_h = (\prod_{j=1}^{\mathrm{N}} \lambda_j^{-\langle hq_j\rangle})\phi_h$. It is also easy to check that $\mathrm{I}^{\bar{\mathbf{s}}}(-u\phi_1, z)$ has the asymptotics

$$\mathrm{I}^{\bar{\mathbf{s}}}(-u\phi_1, z) = z\overline{\mathrm{F}}(u;\lambda)\phi_0 + \overline{\mathrm{G}}(u;\lambda) + \mathrm{O}\big(z^{-1}\big)$$

for a scalar valued function $\overline{\mathrm{F}} = 1 + \mathrm{O}(u)$ and an $\mathrm{H}^{\leq 2}_{\mathrm{ext}}$-valued function $\overline{\mathrm{G}}$. Here the functions F, G appearing in (**42**) are related to $\overline{\mathrm{F}}$, $\overline{\mathrm{G}}$ as

$$\mathrm{F}(u) = u\overline{\mathrm{F}}\big(\lambda^q u; \lambda\big), \qquad \mathrm{G}(u;\lambda) = u\mathrm{R}(\lambda)\overline{\mathrm{G}}\big(\lambda^q u; \lambda\big) \tag{44}$$

with $\lambda^q = \prod_{j=1}^{\mathrm{N}} \lambda_j^{q_j}$. Because $\mathcal{L}^{\bar{\mathbf{s}}}$ is a cone and $\mathrm{I}^{\bar{\mathbf{s}}}(-u\phi_1, -z)$ is on $\mathcal{L}^{\bar{\mathbf{s}}}$, we have (we regard $\mathrm{I}^{\bar{\mathbf{s}}}(-u\phi_1, -z)$ as a $\mathbf{C}[\![\lambda^{-1}]\!][\![u]\!]$-valued point on $\mathcal{L}^{\bar{\mathbf{s}}}$ and apply [17, Proposition B2]):

$$\frac{1}{\overline{\mathrm{F}}(u;\lambda)}\mathrm{I}^{\bar{\mathbf{s}}}(-u\phi_1, -z) = -z\phi_0 + \frac{\overline{\mathrm{G}}(u;\lambda)}{\overline{\mathrm{F}}(u;\lambda)} + \mathrm{O}\big(z^{-1}\big) \in \mathcal{L}^{\bar{\mathbf{s}}}.$$

By the characterization (**38**) of the J-function, this coincides with $\mathrm{J}^{\mathrm{s}}(\overline{\mathrm{G}}(u;\lambda)/\overline{\mathrm{F}}(u;\lambda), -z)$. The conclusion follows from this and the relations (**43**), (**44**). ☐

3.5. *The $e_{\mathbf{C}^\times}$-twisted quantum connection.* — Here we discuss the $e_{\mathbf{C}^\times}$-twisted quantum cohomology for both of FJRW and GW theory. We show that the non-equivariant limit $\lambda \to 0$ of the $e_{\mathbf{C}^\times}$-twisted quantum product exists and reduces to the original one (**17**) restricted to the narrow/ambient part. In the rest of the paper, we only consider the $e_{\mathbf{C}^\times}$-twisted theory as a twisted theory. Thus the word "$e_{\mathbf{C}^\times}$-twisted" is sometimes abbreviated as "twisted".

3.5.1. *A brief summary of the $e_{\mathbf{C}^\times}$-twisted theory.* — We mean by the *$e_{\mathbf{C}^\times}$-twisted FJRW theory* the e_{T}-twisted FJRW theory (Section 3.4) with the equivariant parameters $\lambda_1, \dots, \lambda_{\mathrm{N}}$ specialized to the following values:

$$\lambda_i = -q_i\lambda, \quad i = 1, \dots, \mathrm{N}.$$

The $e_{\mathbf{C}^\times}$-twisted pairing $(\phi_h, \phi_k)_{\mathrm{tw}}$ and FJRW invariants $\langle \tau_{b_1}(\phi_{k_1}), \dots, \tau_{b_n}(\phi_{k_n})\rangle_{0,n}^{\mathrm{FJRW,tw}}$ take values in $\mathbf{C}[\lambda]$. (We have put the superscript "FJRW" to distinguish them from GW invariants.)

The *$e_{\mathbf{C}^\times}$-twisted GW theory* [17, 21, 66] for $\mathbf{P}(\underline{w})$ is defined on the state space $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w})) = \mathrm{H}(\mathcal{I}\mathbf{P}(\underline{w}))$. We consider the twist by the line bundle $\mathcal{O}(d)$. Let $\mathbf{P}(\underline{w})_{0,n,\beta}$ denote the moduli stack of n-pointed stable maps to $\mathbf{P}(\underline{w})$ of genus 0 and degree β. Let $\pi\colon \mathcal{C}_{0,n,\beta} \to \mathbf{P}(\underline{w})_{0,n,\beta}$ be the universal curve and let $f\colon \mathcal{C}_{0,n,\beta} \to \mathbf{P}(\underline{w})$ be the universal stable map:

$$\begin{tikzcd}
\mathcal{C}_{0,n,\beta} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbf{P}(\underline{w}) \\
\mathbf{P}(\underline{w})_{0,n,\beta} &
\end{tikzcd}$$

It can be shown that $\mathbf{R}\pi_* f^*\mathcal{O}(d)$ is represented by a vector bundle (see [20]; this is because $\mathcal{O}(d)$ is ample and $\mathcal{O}(d)$ is pulled back from the coarse moduli $|\mathbf{P}(\underline{w})|$). The $e_{\mathbf{C}^\times}$-twisted GW invariants of $\mathbf{P}(\underline{w})$ are defined to be

$$\left\langle \tau_{b_1}(\alpha_1), \dots, \tau_{b_n}(\alpha_n)\right\rangle_{0,n,\beta}^{\mathrm{GW,tw}} = \int_{[\mathbf{P}(\underline{w})_{0,n,d}]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i)\psi_i^{b_i} \cup e_{\mathbf{C}^\times}\big(\mathbf{R}\pi_* f^*\mathcal{O}(d)\big),$$

where $\alpha_1, \dots, \alpha_n \in \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ and $\mathbf{C}^\times$ acts on $\mathrm{R}\pi_* f^*\mathcal{O}(d)$ by scaling the fibre. This is an element of $\mathrm{H}_{\mathbf{C}^\times}(\mathrm{pt}) = \mathbf{C}[\lambda]$. We endow $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ with the following twisted pairing

$$(\alpha_1, \alpha_2)_{\mathrm{tw}} = \int_{\mathcal{I}\mathbf{P}(\underline{w})} \alpha_1 \cup \alpha_2 \cup e_{\mathbf{C}^\times}\big(\mathrm{pr}^*\,\mathcal{O}(d)\big)$$

where $\mathrm{pr}\colon \mathcal{I}\mathbf{P}(\underline{w}) \to \mathbf{P}(\underline{w})$ is the natural projection.

3.5.2. *Twisted quantum product.* — We denote by $\overline{\mathrm{H}}$ the state space of the twisted theory:

$$\overline{\mathrm{H}} := \begin{cases} \mathrm{H}_{\mathrm{ext}} & \text{for FJRW theory;} \\ \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w})) & \text{for GW theory.} \end{cases}$$

Both state spaces are of dimension d. The same procedure as Section 2.3 defines the twisted quantum cohomology. The $e_{\mathbf{C}^\times}$-twisted quantum product $\bullet^{\mathrm{tw}}$ on $\overline{\mathrm{H}}$ is defined by the formula (**17**) with the correlators $\langle \cdots \rangle_{0,n}$ replaced by the $e_{\mathbf{C}^\times}$-twisted invariants and (g_{ij}) replaced by the $e_{\mathbf{C}^\times}$-twisted pairing. Because the divisor equation holds also for the twisted GW theory, we can consider the specialization $\mathrm{Q}=1$ for $\bullet^{\mathrm{tw}}$ (see Section 2.3.2). In GW theory, we shall denote by $\bullet^{\mathrm{tw}}$ the twisted quantum product with Q already specialized to 1.

Let H' denote the narrow/ambient part (**21**) of the state space. Let pr denote the natural projection

$$\mathrm{pr}\colon \overline{\mathrm{H}} \longrightarrow \mathrm{H}'.$$

Let $\mathrm{T}_0, \dots, \mathrm{T}_{d-1}$ be a homogeneous basis of $\overline{\mathrm{H}}$ such that T_0 is the identity ($\mathrm{T}_0 = \phi_0$ in FJRW theory[14] and $\mathrm{T}_0 = \mathbf{1}_0$ in GW theory). In the case of GW theory, we take $\mathrm{T}_1 = p = c_1(\mathcal{O}(1))$. Let $t^0, \dots, t^{d-1}$ denote the linear co-ordinate on $\overline{\mathrm{H}}$ dual to the basis $\mathrm{T}_0, \dots, \mathrm{T}_{d-1}$.

Proposition **3.11.** — *The $e_{\mathbf{C}^\times}$-twisted quantum products are regular at $\lambda = 0$, i.e.*

$$\mathrm{T}_i \bullet^{\mathrm{tw}} \mathrm{T}_j \in \begin{cases} \overline{\mathrm{H}} \otimes \mathbf{C}[\lambda][\![t^0, \dots, t^{d-1}]\!] & \textit{for FJRW theory;} \\ \overline{\mathrm{H}} \otimes \mathbf{C}[\lambda][\![t^0, e^{t^1/w}, t^2, \dots, t^{d-1}]\!] & \textit{for Gromow-Witten theory.} \end{cases}$$

Here w is the least common multiple of $w_1, \dots, w_n$ (see Section 2.3.2). Moreover we have

$$\lim_{\lambda \to 0} \mathrm{pr}\big(\mathrm{T}_i \bullet^{\mathrm{tw}}_t \mathrm{T}_j\big) = \mathrm{pr}(\mathrm{T}_i) \bullet_{\mathrm{pr}(t)} \mathrm{pr}(\mathrm{T}_j) \tag{45}$$

where the product in the right-hand side is the ordinary quantum product on the narrow/ambient part in Section 2.4 and the subscripts $t \in \overline{\mathrm{H}}$, $\mathrm{pt}(t) \in \mathrm{H}'$ denote the parameter of the product.

Proof. — This was proved in [43, Corollary 2.5] for GW theory, so we only discuss the case of the FJRW theory. The $e_{\mathbf{C}^\times}$-twisted FJRW quantum product can be written as

$$\phi_i \bullet^{\mathrm{tw}} \phi_j = \sum_{k=0}^{d-1} \sum_{n \geq 0} \frac{1}{n!} \langle \phi_i, \phi_j, \phi_k, t, \dots, t \rangle^{\mathrm{FJRW,tw}}_{0,n+3} \Bigg(d \prod_{j : \langle (k+1) q_j \rangle = 0} \lambda_j^{-1} \Bigg) \phi_{d-2-k}$$

---

[14] The element ϕ_0 is the identity in the twisted FJRW theory because of the string equation (see Section 3.2.2).

with $\lambda_j = -q_j\lambda$. To see that this expression is regular at $\lambda = 0$, it suffices to show that

$$\left\langle \tau_{b_1}(\phi_{k_1}), \ldots, \tau_{b_n}(\phi_{k_n}) \right\rangle_{0,n}^{\mathrm{FJRW,tw}} \in \left(\prod_{j:\langle (k_i+1)q_j \rangle = 0} \lambda_j \right) \mathbf{C}[\lambda_1, \ldots, \lambda_{\mathrm{N}}],$$

$$1 \leq \forall i \leq n.$$

This happens because $e_{\mathrm{T}}(\mathrm{R}^1\pi_*\widetilde{\mathcal{L}}^{\otimes w_j})$ is divisible by λ_j as soon as d divides $(k_i+1)w_j$. This follows from the fact that $\mathrm{R}^1\pi_*\widetilde{\mathcal{L}}^{\otimes w_j}$ contains the sub-line bundle $\overline{\pi}_*(\mathcal{T}(\overline{\mathcal{D}}_i)|_{\overline{\mathcal{D}}_i})$ whose equivariant 1st Chern class is λ_j. See (**28**).

Using Proposition 3.2 and the fact that the $e_{\mathbf{C}^\times}$-twisted invariant equals the extended invariant (**26**) in the non-equivariant limit $\lambda \to 0$, we have

$$\lim_{\lambda \to 0} \left\langle \tau_{b_1}(\phi_{k_1}), \ldots, \tau_{b_n}(\phi_{k_n}) \right\rangle_{0,n}^{\mathrm{FJRW,tw}} = \left\langle \tau_{b_1}\left(\mathrm{pr}(\phi_{k_1})\right), \ldots, \tau_{b_n}\left(\mathrm{pr}(\phi_{k_n})\right) \right\rangle_{0,n}^{\mathrm{FJRW}}. \tag{46}$$

The equality (**45**) follows easily from this. □

3.5.3. *Twisted quantum connection and the fundamental solution.* — The $e_{\mathbf{C}^\times}$*-twisted quantum connection* is defined similarly to (**18**):

$$\nabla_i^{\mathrm{tw}} = \frac{\partial}{\partial t^i} + \frac{1}{z}\mathrm{T}_i \bullet^{\mathrm{tw}}.$$

By Proposition 3.11, $e_{\mathbf{C}^\times}$-twisted quantum connection is regular at $\lambda = 0$. The non-equivariant limit is called the *e-twisted quantum connection*. In contrast with the untwisted case, the connection ∇^{tw} cannot be extended in the z-direction since the variable λ has a degree.

Let $\mathrm{L}^{\mathrm{tw}}(t, z; \lambda)$ denote the canonical fundamental solution of the connection ∇^{tw} defined by the same formula (**20**) with $\langle \cdots \rangle_{0,n}$, g_{ij} there replaced with the twisted counterparts. This satisfies (part of) the properties in Proposition 2.7:

Proposition **3.12.** — *For* $\alpha, \alpha_1, \alpha_2 \in \overline{\mathrm{H}}$, *we have*

$$\nabla_i^{\mathrm{tw}} \mathrm{L}^{\mathrm{tw}}(t, z; \lambda)\alpha = 0, \qquad \left(\mathrm{L}^{\mathrm{tw}}(t, -z; \lambda)\alpha_1, \mathrm{L}^{\mathrm{tw}}(t, z; \lambda)\alpha_2\right)_{\mathrm{tw}} = (\alpha_1, \alpha_2)_{\mathrm{tw}}.$$

In particular, the connection ∇^{tw} *is flat, i.e.* $[\nabla_i^{\mathrm{tw}}, \nabla_j^{\mathrm{tw}}] = 0$ *and that the pairing* $(\cdot, \cdot)_{\mathrm{tw}}$ *is* ∇^{tw}*-flat (see* (**19**) *for a precise meaning of the flatness of the pairing).*

Proof. — The outline of the proof is the same as Proposition 2.7. It suffices to show that the twisted theory satisfies TRR, but this follows from Givental's geometric properties (see Section 3.2.2). The same discussion as in [43, Proposition 2.1] shows the statement for the pairing. □

Remark **3.13.** — The fact that ∇^{tw} is flat implies that $\bullet^{\mathrm{tw}}$ is associative. (The commutativity of $\bullet^{\mathrm{tw}}$ is clear from the definition.)

Because L^{tw} satisfies the differential equation regular at $\lambda = 0$, it follows that L^{tw} is also regular at $\lambda = 0$ (see also [43, Proposition 2.4]).

$$\mathrm{L}^{\mathrm{tw}}(t, z; \lambda) \in \begin{cases} \mathrm{End}(\overline{\mathrm{H}}) \otimes \mathbf{C}[\lambda][\![t^0, \dots, t^{d-1}]\!][\![z^{-1}]\!] \\ \quad \text{for FJRW theory;} \\ \mathrm{End}(\overline{\mathrm{H}}) \otimes \mathbf{C}[\lambda][\![t^0, e^{t^1/w}, t^2, \dots, t^{d-1}]\!][t^1][\![z^{-1}]\!] \\ \quad \text{for GW theory.} \end{cases}$$

Let $\mathrm{L}(t, z)$ denote the fundamental solution (**20**) in the original FJRW/GW theories. (For GW theory, we specialize Q to 1.) We have the following:

Proposition **3.14.**

$$\lim_{\lambda\to 0} \mathrm{pr}\big(\mathrm{L}^{\mathrm{tw}}(t, z; \lambda)\alpha\big) = \mathrm{L}\big(\mathrm{pr}(t), z\big)\,\mathrm{pr}(\alpha).$$

Proof. — It was shown in [43, Proposition 3.24] for GW theory. For FJRW theory, the equality follows easily from (**46**). ☐

3.5.4. *Twisted J-function.* — Recall from Section 3.3 that the J-function (**37**) is a special family of elements lying on the Givental Lagrangian cone. The $e_{\mathbf{C}^\times}$-twisted J-function is defined similarly:

$$\mathrm{J}^{\mathrm{tw}}(t, z) = z\mathrm{T}_0 + t + \sum_{n\geq 0}\sum_{b\geq 0}\sum_{i,j=0}^{s} \frac{1}{n!z^{b+1}} \big\langle t, \dots, t, \tau_b(\mathrm{T}_i)\big\rangle^{\mathrm{tw}}_{0,n+1} g^{ij}_{\mathrm{tw}} \mathrm{T}_j, \tag{47}$$

where (g^{ij}_{tw}) denotes the inverse of the twisted pairing matrix $g^{\mathrm{tw}}_{ij} = (\mathrm{T}_i, \mathrm{T}_j)_{\mathrm{tw}}$. (In the case of GW theory, as in (**16**), we also take the summation over curve classes β (see [17, Equation (8)]). Then we specialize it to Q = 1 using the divisor equation.) The following relation of L^{tw} and J^{tw} is a key to understand the role of the J-function in the quantum D-module.

Proposition **3.15.** — $\mathrm{L}^{\mathrm{tw}}(t, z; \lambda)\mathrm{J}^{\mathrm{tw}}(t, z; \lambda) = z\mathrm{T}_0$.

Proof. — By Proposition 3.12, we have $\mathrm{L}^{\mathrm{tw}}(t, z; \lambda)^{-1} = \mathrm{L}^{\mathrm{tw}}(t, -z; \lambda)^*$ where $*$ denotes the adjoint with respect to the twisted pairing. Thus we have

$$\begin{aligned} \big(\mathrm{T}_i, \mathrm{L}^{\mathrm{tw}}(t, z; \lambda)^{-1} z\mathrm{T}_0\big)_{\mathrm{tw}} &= \big(\mathrm{T}_i, \mathrm{L}^{\mathrm{tw}}(t, -z; \lambda)^* z\mathrm{T}_0\big)_{\mathrm{tw}} \\ &= \big(\mathrm{L}^{\mathrm{tw}}(t, -z; \lambda)\mathrm{T}_i, \mathrm{T}_0\big)_{\mathrm{tw}} = \big(\mathrm{T}_i, \mathrm{J}^{\mathrm{tw}}(t, z; \lambda)\big)_{\mathrm{tw}}, \end{aligned}$$

where the last equality follows directly from the definitions (**20**), (**47**) of L^{tw} and J^{tw} and the string equation. The conclusion follows. ☐

4. Orlov equivalence matches Mellin-Barnes analytic continuation

4.1. *Matrix factorizations.* — Matrix factorizations were originally introduced by Eisenbud [23] for the study of maximal Cohen-Macaulay modules. Recently Kontsevich proposed that they form the category of B-branes in the Landau-Ginzburg model. References are made to [22, 34, 37, 52, 53, 56, 68]. The paper [22] contains a nice introduction to the subject.

We introduce the differential graded (dg) category of graded matrix factorizations of a degree-d weighted homogeneous polynomial $\mathrm{W} \in \mathbf{C}[x_1, \dots, x_\mathrm{N}]$ from the introduction Section 1.1. Set $\mathrm{R} := \mathbf{C}[x_1, \dots, x_\mathrm{N}]$. Notice that R is a $\mathbf{Z}_{\geq 0}$-graded ring by $\deg x_i = w_i$.

Definition **4.1** *(Graded matrix factorization [37, 68], [53, Section 3.1]).* — *A* graded matrix factorization *of* W *is a collection* $(\mathrm{E}^i, \delta_i)_{i\in\mathbf{Z}}$ *of finitely generated graded free* R*-modules* E^i *and degree-zero homomorphisms* $\delta_i \in \mathrm{Hom}_{\text{gr-R}}(\mathrm{E}^i, \mathrm{E}^{i+1})$

$$\cdots \xrightarrow{\delta_{-1}} \mathrm{E}^0 \xrightarrow{\delta_0} \mathrm{E}^1 \xrightarrow{\delta_1} \mathrm{E}^2 \xrightarrow{\delta_2} \mathrm{E}^3 \xrightarrow{\delta_3} \cdots$$

such that it is 2-periodic up to the shift of grading

$$\mathrm{E}^{i+2} = \mathrm{E}^i(d), \qquad \delta_{i+2} = \delta_i(d)$$

and that $\delta_{i+1} \circ \delta_i = \mathrm{W} \cdot \mathrm{id}_{\mathrm{E}^i} \colon \mathrm{E}^i \to \mathrm{E}^{i+2} = \mathrm{E}^i(d)$ *for all i. This is equivalent to the data* E^0, E^1, $\delta_0 \in \mathrm{Hom}_{\text{gr-R}}(\mathrm{E}^0, \mathrm{E}^1)$, $\delta_1 \in \mathrm{Hom}_{\text{gr-R}}(\mathrm{E}^1, \mathrm{E}^0(d))$ *such that* $\delta_1 \circ \delta_0 = \mathrm{W} \cdot \mathrm{id}_{\mathrm{E}^0}$ *and* $\delta_0(d) \circ \delta_1 = \mathrm{W} \cdot \mathrm{id}_{\mathrm{E}^1}$. *These data are denoted also by* $(\mathrm{E}, \delta_\mathrm{E})$, *where*

$$\mathrm{E} := \mathrm{E}^0 \oplus \mathrm{E}^1, \qquad \delta_\mathrm{E} := \begin{pmatrix} 0 & \delta_1 \\ \delta_0 & 0 \end{pmatrix} : \mathrm{E} \to \mathrm{E} \quad \textit{satisfying} \quad \delta_\mathrm{E}^2 = \mathrm{W} \cdot \mathrm{id}_\mathrm{E}\,.$$

These objects form a dg category as follows. Consider the graded matrix factorizations $\overline{\mathrm{E}} = (\mathrm{E}^i, \delta_i)_{i\in\mathbf{Z}}$ *and* $\overline{\mathrm{F}} = (\mathrm{F}^i, \delta'_i)_{i\in\mathbf{Z}}$; *the space of homomorphisms is defined to be the* $\mathbf{Z}$*-graded vector space*

$$\mathsf{Hom}^\bullet(\overline{\mathrm{E}}, \overline{\mathrm{F}}) = \left\{ (f_n)_{n\in\mathbf{Z}} \,\middle|\, f_n \in \mathrm{Hom}_{\text{gr-R}}\left(\mathrm{E}^n, \mathrm{F}^{n+\bullet}\right), f_{n+2} = f_n(d) \right\}$$

equipped with the differential

$$(\mathsf{d}f)_n = \delta'_{n+\bullet} \circ f_n - (-1)^\bullet f_{n+1} \circ \delta_n, \quad f \in \mathsf{Hom}^\bullet(\overline{\mathrm{E}}, \overline{\mathrm{F}}).$$

The homotopy category of the above dg category is denoted by $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$. *It is a triangulated category.*

Remark **4.2** *([56, Section 4.4]).* — The lower index in the notation $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ emphasizes the fact that a graded matrix factorization is automatically $\boldsymbol{\mu}_d$-equivariant. The $\boldsymbol{\mu}_d$-action on R is defined by $\zeta \cdot x_i = \zeta^{-w_i} x_i$, where $\zeta = \exp(2\pi \mathtt{i}/d) \in \boldsymbol{\mu}_d$. For a graded matrix factorization $\overline{\mathrm{E}} = (\mathrm{E}^i, \delta_i)_{i\in\mathbf{Z}}$, we define the $\boldsymbol{\mu}_d$-action on E^i by $\zeta \cdot e = \zeta^{-n} e$ for $e \in (\mathrm{E}^i)_n$. Then the R-module E^i is $\boldsymbol{\mu}_d$-linearized and δ_i is $\boldsymbol{\mu}_d$-equivariant.

We introduce a graded Koszul matrix factorization (see [8, Section 2] for the ungraded case).

Definition **4.3** *(Graded Koszul matrix factorization). — Suppose that* W *is of the form*

$$\mathrm{W} = \sum_{i=1}^{\mathrm{N}} a_i b_i \tag{48}$$

for homogeneous elements $a_i, b_i \in \mathrm{R}$ *such that* $\deg(a_i) + \deg(b_i) = d$. *Let* V *be the graded vector space* $\bigoplus_{i=1}^{\mathrm{N}} \mathbf{C} e_i$ *with* $\deg(e_i) = -\deg(a_i)$. *For* $q \in \mathbf{Z}$, *the* graded Koszul matrix factorization $\{\underline{a}, \underline{b}\}_q$ *is defined by the data*

$$\mathrm{E}^i = \bigoplus_{\substack{k=0,\dots,\mathrm{N} \\ k \equiv i\,(2)}} \mathrm{R} \otimes \Big(\bigwedge^k \mathrm{V}\Big)\Big(\frac{d(i-k)}{2} + q\Big),$$

$$\delta_i = \delta'_{\underline{a}} + \delta''_{\underline{b}} \colon \mathrm{E}^i \to \mathrm{E}^{i+1}, \qquad i \in \mathbf{Z},$$

where $\delta'_{\underline{a}}, \delta''_{\underline{b}}$ *are the Koszul differentials:*

$$\delta'_{\underline{a}} = \sum_{j=1}^{\mathrm{N}} a_j e_j \wedge, \qquad \delta''_{\underline{b}} = \sum_{j=1}^{\mathrm{N}} b_j \iota(e_j^*).$$

Observe that the grading is shifted so that the map δ_i *preserves the degree. Note also that* $\{\underline{a}, \underline{b}\}_q = \{\underline{a}, \underline{b}\}_0(q)$.

4.1.1. *Hirzebruch-Riemann-Roch Theorem.* — For graded matrix factorizations $\overline{\mathrm{E}}, \overline{\mathrm{F}}$ of W, we write $\chi(\overline{\mathrm{E}}, \overline{\mathrm{F}})$ for the Euler characteristic

$$\sum_{k \in \mathbf{Z}} (-1)^k \dim \mathrm{H}^k\big(\mathsf{Hom}^\bullet(\overline{\mathrm{E}}, \overline{\mathrm{F}}), \mathsf{d}\big).$$

This can be computed via Hirzebruch-Riemann-Roch (HRR) due to Walcher [68] and Polishchuk-Vaintrob [56] for G-equivariant matrix factorizations. To this effect we need to introduce the Chern character taking values in the orbifold Jacobi space $\bigoplus_{k=0}^{d-1} \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$, which is identified with the FJRW state space by Proposition 2.1. Let $x_{j_1}, \dots, x_{j_{\mathrm{N}_k}}$ denote the co-ordinates of the ζ^k-fixed part $(\mathbf{C}^{\mathrm{N}})_k$ where $\zeta = e^{2\pi \mathtt{i}/d}$. For a graded matrix factorization $\overline{\mathrm{E}} = (\mathrm{E}, \delta_{\mathrm{E}})$, we define [56, Theorem 3.3.3]

$$\mathrm{ch}(\overline{\mathrm{E}}) := \bigoplus_{k=0}^{d-1} \Big[\mathrm{str}_{\mathrm{R}}\big(\partial_{j_1}\delta_{\mathrm{E}} \circ \partial_{j_2}\delta_{\mathrm{E}} \circ \cdots \circ \partial_{j_{\mathrm{N}_k}}\delta_{\mathrm{E}} \circ \zeta^k\big)\big|_{(\mathbf{C}^{\mathrm{N}})_k} \mathsf{d}x_{j_1} \wedge \cdots \wedge \mathsf{d}x_{j_{\mathrm{N}_k}}\Big].$$

Here we take a free basis of $\mathrm{E} = \mathrm{E}^0 \oplus \mathrm{E}^1$ over $\mathrm{R} = \mathbf{C}[x_1, \dots, x_\mathrm{N}]$ and regard δ_E as a matrix with entries in R; the supertrace $\mathrm{str}_\mathrm{R}(f)$ of an operator $f \in \mathrm{End}_\mathrm{R}(\mathrm{E})$ is defined to be $\mathrm{tr}(f_{0,0}) - \mathrm{tr}(f_{1,1})$, where $f_{\sigma,\sigma} \colon \mathrm{E}^\sigma \to \mathrm{E}^\sigma$, $\sigma = 0, 1$ are the components of f. The right hand side are meant to be the class in $\bigoplus_{k=0}^{d-1} \Omega(\mathrm{W}_k)$ and lies in the $\boldsymbol{\mu}_d$-invariant part. This is independent of the choice of a co-ordinate ordering or the choice of a basis of E.

Remark **4.4.** — Let $\mathrm{ch}(\overline{\mathrm{E}})_k$ denote the $\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ component of $\mathrm{ch}(\overline{\mathrm{E}})$. For a graded matrix factorization $\overline{\mathrm{E}}$, one can see that $\mathrm{ch}(\overline{\mathrm{E}})_k$ vanishes if N_k is odd and $\mathrm{ch}(\overline{\mathrm{E}})_k$ is of degree $(\mathrm{N}_k/2)d$. In terms of the Hodge decomposition, the component $\mathrm{ch}(\overline{\mathrm{E}})_k$ has Hodge type $(\mathrm{N}_k/2, \mathrm{N}_k/2)$.

Example **4.5.** — For a general weighted homogeneous polynomial W, we can write $\mathrm{W} = \sum_{j=1}^{\mathrm{N}} a_j b_j$ with $a_j = q_j \partial_j \mathrm{W}$, $b_j = x_j$ $(q_j := w_j/d)$. The Chern character of the graded Koszul matrix factorization $\{\underline{a}, \underline{b}\}_q$ of W is supported on the narrow sector. In fact, by a direct calculation, we obtain

$$\mathrm{ch}\big(\{\underline{a}, \underline{b}\}_q\big) = \bigoplus_{k \in \mathsf{Nar}} \zeta^{qk} \Big(\prod_{j=1}^{\mathrm{N}} (1 - \zeta^{-w_j k}) \Big) \phi_{k-1}. \tag{49}$$

See [56, Proposition 4.3.4] where $\{\underline{a}, \underline{b}\}_0$ is denoted by k^{st}. (The case $q \neq 0$ follows from the case with $q = 0$ since q is just a shift the grading.) These Chern characters span the narrow part $\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$.

Theorem **4.6** *(Walcher [68, Section 5], Polishchuk-Vaintrob [56, Theorem 4.2.1]).* — *Let* $\overline{\mathrm{E}}, \overline{\mathrm{F}}$ *be graded matrix factorizations. The Euler characteristic* $\chi(\overline{\mathrm{E}}, \overline{\mathrm{F}})$ *is given by the formula:*

$$\sum_{k=0}^{d-1} \Big(\prod_{kw_j \notin d\mathbf{Z}} \frac{1}{1 - \zeta^{kw_j}} \Big) (-1)^{\frac{\mathrm{N}_k(\mathrm{N}_k - 1)}{2}} \frac{1}{d} \mathrm{Res}_{\mathrm{W}_k}\big(\mathrm{ch}(\overline{\mathrm{E}})_k, \mathrm{ch}(\overline{\mathrm{F}})_{d-k}\big). \tag{50}$$

Here $\mathrm{ch}(\overline{\mathrm{E}})_k$ *denotes the* $\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$*-component of* $\mathrm{ch}(\overline{\mathrm{E}})$.

Proof. — Because Polishchuk-Vaintrob considered the G-equivariant (ungraded) matrix factorizations over the ring of formal power series, we need check that the Euler characteristic does not change under the base change from the polynomial ring to the formal power series ring for *graded* matrix factorizations. Set $\widehat{\mathrm{R}} = \mathbf{C}[\![x_1, \dots, x_\mathrm{N}]\!]$. Let $(\mathrm{E}, \delta_\mathrm{E})$, $(\mathrm{F}, \delta_\mathrm{F})$ be graded matrix factorizations. Let $(\widehat{\mathrm{E}}, \hat{\delta}_\mathrm{E}) = (\mathrm{E}, \delta_\mathrm{E}) \otimes_\mathrm{R} \widehat{\mathrm{R}}$, $(\widehat{\mathrm{F}}, \hat{\delta}_\mathrm{F}) = (\mathrm{F}, \delta_\mathrm{F}) \otimes_\mathrm{R} \widehat{\mathrm{R}}$ be $\boldsymbol{\mu}_d$-equivariant matrix factorizations over $\widehat{\mathrm{R}}$ without the $\mathbf{Z}$-grading. We have the identification as $\mathbf{Z}/2$-graded complexes:

$$\mathsf{Hom}^\sigma\big((\widehat{\mathrm{E}}, \hat{\delta}_\mathrm{E}), (\widehat{\mathrm{F}}, \hat{\delta}_\mathrm{F})\big) = \widehat{\bigoplus_{j \equiv \sigma\ (2)}} \mathsf{Hom}^j\big((\mathrm{E}, \delta_\mathrm{E}), (\mathrm{F}, \delta_\mathrm{F})\big), \quad \sigma \in \mathbf{Z}/2,$$

where the completed direct sum consists of arbitrary sequences of homomorphisms bounded in the negative direction. Hence the cohomology is again the completed direct sum of the cohomology $H^j(\mathsf{Hom}^\bullet((E, \delta_E), (F, \delta_F)))$. The HRR for the left-hand side implies the finite-dimensionality and the boundedness of the cohomology of the right-hand side, and the HRR for the right-hand side as well. ☐

Remark **4.7.** — Dyckerhoff [22] identified the Hochschild homology of the category of matrix factorizations over a formal power series ring with the Jacobi space of the potential. Polishchuk-Vaintrob [56] observed that the Hochschild homology can be identified with the FJRW state space in the G-equivariant case. The Chern character naturally takes values in the Hochschild homology and the Riemann-Roch formula was derived in the categorical framework in [56].

4.2. *Orlov equivalence.* — Under the Calabi-Yau condition $d = \sum_{j=1}^{N} w_j$, Orlov [53, Theorem 2.5] constructed the equivalence of triangulated categories

$$\Phi_l\colon\ \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(W) \longrightarrow D^b(X_W) \tag{51}$$

parametrized by $l \in \mathbf{Z}$. Consider a graded matrix factorization $\overline{E} = (E^i, \delta_i)_{i\in\mathbf{Z}}$ of W

$$\cdots \to E^0 \xrightarrow{\delta_0} E^1 \xrightarrow{\delta_1} E^2 = E^0(d) \xrightarrow{\delta_2=\delta_0(d)} E^3 = E^1(d) \xrightarrow{\delta_3=\delta_1(d)} \dots$$

and set

$$S = R/(W) = \mathbf{C}[x_1, \dots, x_N]/(W).$$

By tensoring the above data $(E^i, \delta_i)_{i\in\mathbf{Z}}$ with S over R, we obtain an acyclic (see Eisenbud [23] and Buchweitz [7]) complex

$$\cdots \xrightarrow{\delta_{-1}\otimes S} \mathcal{C}^0 \xrightarrow{\delta_0\otimes S} \mathcal{C}^1 \xrightarrow{\delta_1\otimes S} \mathcal{C}^2 = \mathcal{C}^0(d) \xrightarrow{\delta_2\otimes S} \mathcal{C}^3 = \mathcal{C}^1(d) \xrightarrow{\delta_3\otimes S} \cdots.$$

By construction we can extract from $\mathcal{C}^\bullet$ a positively graded and left semiinfinite complex $L_0^\bullet$. To this effect, after expressing each $\mathcal{C}^i$ as a direct sum of S-modules of the form $S(k)$ for some $k \in \mathbf{Z}$, we mod out the S-modules of the form $S(-e)$ with $e \leq 0$. More precisely we may notice that E^0 and E^1 have the same dimension and can be expresses as

$$E^0 = \bigoplus_{1\leq h\leq r} R(-j_h), \qquad E^1 = \bigoplus_{r+1\leq h\leq 2r} R(-j_h).$$

In this way we have $\mathcal{C}^0 = \bigoplus_{1\leq h\leq r} S(-j_h)$, and $\mathcal{C}^1 = \bigoplus_{r+1\leq h\leq 2r} S(-j_h)$ and

$$\mathcal{C}^i = \bigoplus_{1\leq h-2r\langle i/2\rangle\leq r} S\big(d\lfloor i/2\rfloor - j_h\big)$$

(note that $2r\langle i/2\rangle$ equals 0 or r according to the parity of i). Then, the definition of $L_0^\bullet$ reads

$$L_0^i = \bigoplus_{\substack{1\leq h-2r\langle i/2\rangle\leq r\\ d\lfloor i/2\rfloor<j_h}} S\big(d\lfloor i/2\rfloor - j_h\big).$$

Since $\mathcal{C}^\bullet$ is acyclic, $L_0^\bullet$ is represented by a bounded complex of coherent sheaves. For simplicity, we stated the definition of the positively graded complex $L_0^\bullet$. For any $l\in\mathbf{Z}$, we can define $L_l^\bullet$ as

$$L_l^i = \bigoplus_{\substack{1\leq h-2r\langle i/2\rangle\leq r\\ d\lfloor i/2\rfloor-j_h<l}} S\big(d\lfloor i/2\rfloor - j_h\big). \tag{52}$$

This amounts to extracting from each $\mathcal{C}^i$, only the S-modules of the form $S(-e)$ with $e>l$. We have the following statement. (We stress that the equivalence of categories holds only under the CY condition $d=\sum_{j=1}^{N} w_j$, which we assume throughout the paper.)

Proposition **4.8** *(Herbst-Hori-Page [34, Section 10.6, (10.56–58)]). — The Orlov equivalence*

$$\Phi_l\colon\ \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(W) \longrightarrow D^b(X_W)$$

for $l\in\mathbf{Z}$ *assigns to* $(E,\delta_E)\in \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(W)$ *the left semiinfinite complex* (**52**)

$$\Phi_l(E,\delta_E) = L_l^\bullet \in D^b(X_W).$$

Here the graded module $S(k)$ *in* $L_l^\bullet$ *is identified with the sheaf* $\mathcal{O}(k)$ *on* X_W.

Remark **4.9.** — We point out that there are two presentations of $\Phi_l(E,\delta_E)$ in the derived category. Because the complex $\mathcal{C}^\bullet$ is acyclic, the left semiinfinite complex $L_l^\bullet$ can be equivalently represented by the (complementary) right semiinfinite complex $(L_l^c)^\bullet[1]$, where

$$\big(L_l^c\big)^i = \bigoplus_{\substack{1\leq h-2r\langle i/2\rangle\leq r\\ d\lfloor i/2\rfloor-j_h\geq l}} S\big(d\lfloor i/2\rfloor - j_h\big). \tag{53}$$

Remark **4.10** *(Herbst-Hori-Page brane transportation).* — Although we will not use this in the rest of the paper, we should mention that Orlov functors Φ_l can be constructed, for $\widetilde{R}=R[p]$ and $\widetilde{W}=pW$, by lifting the $\boldsymbol{\mu}_d$-action to a $\mathbf{C}^\times$-action and by obtaining in this way a graded and $\mathbf{C}^\times$-equivariant matrix factorization in $\mathrm{MF}^{\mathrm{gr}}_{\mathbf{C}^\times}(\widetilde{W})$. Clearly $\boldsymbol{\mu}_d$-actions are not uniquely lifted to $\mathbf{C}^\times$-actions; we need an extra datum of an integer parameter l. This point of view due to Herbst, Hori, and Page explains the presence of several Orlov functors Φ_l for $l\in\mathbf{Z}$. From $\mathrm{MF}^{\mathrm{gr}}_{\mathbf{C}^\times}(\widetilde{W})$ a natural functor leads to $D^b(X_W)$, see [34].

We apply Orlov's functor Φ_l to the graded Koszul matrix factorization $\{\underline{a}, \underline{b}\}_q$ from Example 4.5 (see also Definition 4.3).

Proposition **4.11.** — *The image via Φ_l of the graded matrix factorization $\{\underline{a}, \underline{b}\}_q$ in Example 4.5 is represented by the complex on* $\mathrm{X_W}$

$$\bigoplus_{\substack{j_1<j_2<\cdots<j_r \\ \sum_{a=1}^r w_{j_a}\leq m}} \mathcal{O}\Big(l+m-\sum_{a=1}^{r} w_{j_a}\Big)e_{j_1}\wedge\cdots\wedge e_{j_r}[r+1+2t]$$

equipped with the Koszul differential $\delta''_{\underline{b}} = \sum_{j=1}^{\mathrm{N}} x_j\iota(e_i^*)$. Here t and m denote the integer quotient and remainder of $q-l$ divided by d.

Proof. — Write $(\mathrm{E}^i, \delta_i)_{i\in\mathbf{Z}} = \{\underline{a}, \underline{b}\}_q$. Let us consider $\mathrm{E}^i \otimes_{\mathrm{R}} \mathrm{S}$

$$\bigoplus_{r\in\mathbf{Z}\mid r\equiv i\,(2)} \mathrm{S}\otimes\Big(\bigwedge^{r}\mathrm{V}\Big)\Big(\frac{d(i-r)}{2}+q\Big), \tag{54}$$

where $\mathrm{V} = \bigoplus_{j=1}^{\mathrm{N}} \mathbf{C}e_j$ with $\deg(e_j) = -\deg(a_j) = w_j - d$. Each summand is of the form

$$\begin{aligned} &\bigoplus_{j_1<\cdots<j_r} \mathrm{S}\Big(-\sum_{a=1}^{r}\deg(e_{j_a})+\frac{d(i-r)}{2}+q\Big) \\ &\quad=\bigoplus_{j_1<\cdots<j_r} \mathrm{S}\Big(-\sum_{a=1}^{r} w_{j_a}+\frac{d(i+r)}{2}+q\Big). \end{aligned} \tag{55}$$

By Proposition 4.8 and Remark 4.9 we can regard the image via Φ_l of the Koszul matrix factorization $\{\underline{a}, \underline{b}\}_q$ as the (complementary) right semiinfinite complex $(\mathrm{L}_l^c)^\bullet[1]$. The terms $\mathrm{S}(h)$ appearing in the above formula contribute to L_l^c if and only if $h\geq l$; therefore we consider the inequality

$$h = -\sum_{a=1}^{r} w_{j_a} + \frac{i+r}{2}d + q \geq l,$$

which can be rewritten as (using $q-l = td+m$)

$$m + td + \frac{i+r}{2}d \geq \sum_{a=1}^{r} w_{j_a}.$$

Since $\sum_{a=1}^r w_{j_a}$ lies in $\{0,\ldots,d\}$ by the CY condition $d = \sum_{j=1}^{\mathrm{N}} w_j$, we deduce that $h\geq l$ if and only if either we have $(i+r)/2 > -t$ (note $i+r$ is even by (**54**)) or we have $m \geq \sum_{a=1}^r w_{j_a}$ alongside with $(i+r)/2 = -t$.

Let us consider all terms of (**55**) for which $(i+r)/2 > -t$. Then, the summand of (**55**) attached to $j_1 < \cdots < j_r$ is of the form $\mathrm{S}(l+n-\sum_{a=1}^{r} w_{j_a})$ with $n \geq d$. Such summands with fixed n form an exact sequence $\mathcal{E}_n^\bullet$ on $\mathrm{X_W}$

$$(\mathbf{56}) \qquad \mathcal{E}_n^\bullet : \quad \mathcal{O}(l+n) \xleftarrow{\delta_b''} \bigoplus_j \mathcal{O}(l+n-w_j) \xleftarrow{\delta_b''} \bigoplus_{j_1<j_2} \mathcal{O}(l+n-w_{j_1}-w_{j_2})$$
$$\xleftarrow{\delta_b''} \cdots \xleftarrow{\delta_b''} \bigoplus_j \mathcal{O}\Big(l+n-\sum_{j'\neq j} w_{j'}\Big) \xleftarrow{\delta_b''} \mathcal{O}(l+n-d)$$

where we wrote $\mathcal{O}(h)$ for $\mathrm{S}(h)$ following Proposition 4.8. Therefore, all together, the sum

$$\bigoplus_{\substack{r\equiv i\,(2)\\ (i+r)/2>-t}} \mathrm{S}\otimes \Big(\bigwedge^r \mathrm{V}\Big)\Big(\frac{d(i-r)}{2}+q\Big)$$

gives an acyclic subcomplex of $(\mathrm{L}_l^c)^\bullet$. It is acyclic because it can be written as a successive extension by the acyclic complexes $\mathcal{E}_n^\bullet$ of a complex supported on arbitrarily high homological degrees. The quotient of $(\mathrm{L}_l^c)^\bullet$ by this acyclic subcomplex consists of terms of (**55**) with $(i+r)/2 = -t$ and $\sum_{a=1}^{r} w_{j_a} \leq m$. The conclusion follows. (Recall that we need to take the shift $(\mathrm{L}_l^c)^\bullet[1]$ by 1.) □

4.3. *Twisted* I*-functions and Mellin-Barnes continuation.* — We provide two parallel discussions of the twisted I-functions for GW and FJRW theories. We show that the two I-functions satisfy the same Picard-Fuchs equation under a co-ordinate change. We compute the connection matrix between the two I-functions (or more precisely the $\mathfrak{H}$-functions) using the Mellin-Barnes method of analytic continuation.

On both sides we systematically work with the $e_{\mathbf{C}^\times}$-twisted theories. On the Landau-Ginzburg side we already discussed the e_{T}-twisted FJRW theory Section 3.4 over the extended state space; its non-equivariant limit, followed by projection to the narrow state space, encodes the genus zero correlator of FJRW theory. The counterpart on the Calabi-Yau side is the $e_{\mathbf{C}^\times}$-twisted theory of $\mathbf{P}(\underline{w})$, twisted by $\mathcal{O}(d)$. It is treated and computed in genus zero in [19]; again, the non-equivariant limit, followed by the projection to the ambient part $\mathrm{H_{amb}}(\mathrm{X_W})$ of the state space yields the genus-zero correlators in GW theory. (See Section 3.5 for a review.)

4.3.1. *The $e_{\mathbf{C}^\times}$-twisted* I*-functions.* — Recall the e_{T}-twisted I-function (**40**) with N equivariant parameters $\lambda_1,\ldots,\lambda_{\mathrm{N}}$. Here, without loss of information from the point of view of non-equivariant theory, we can impose the conditions $\lambda_j = -q_j\lambda$ for all j with a single equivariant parameter λ (as in Section 3.5.1). The $e_{\mathbf{C}^\times}$-twisted I-function in FJRW theory is given by:

$$\mathrm{I}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z;\lambda) = z \sum_{k\in\mathbf{Z}_{\geq 1}} u^k \frac{\prod_{j=1}^{\mathrm{N}} \prod_{0<b<kq_j, \langle b\rangle=\langle kq_j\rangle}(-q_j\lambda - bz)}{\prod_{0<b<k, \langle b\rangle = 0}(-bz)} \phi_{k-1}.$$

Here the index $k-1$ of ϕ_{k-1} is reduced modulo d within the range $\{0,\dots,d-1\}$. This takes values in the extended state space $\mathrm{H}_{\mathrm{ext}}$ (**24**).

In GW theory, the $e_{\mathbf{C}^\times}$-twisted I-function was computed in [19]. It is given by:

$$\mathrm{I}^{\mathrm{tw}}_{\mathrm{GW}}(v,z;\lambda) = ze^{p\log v/z} \sum_{\substack{n\in\mathbf{Q}_{\geq 0} \\ \exists j,\, nw_j\in\mathbf{Z}}} v^n \frac{\prod_{0<b\leq dn, \langle b\rangle = 0}(dp+\lambda+bz)}{\prod_{j=1}^{\mathrm{N}}\prod_{0<b\leq w_j n, \langle b\rangle=\langle w_j n\rangle}(w_j p + bz)} \mathbf{1}_{\langle -n\rangle}.$$

This encodes the $e_{\mathbf{C}^\times}$-twisted GW invariants of $\mathbf{P}(\underline{w})$, twisted by the line bundle $\mathcal{O}(d)$, and takes values in $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$.

These twisted I-functions $\mathrm{I}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z)$ and $\mathrm{I}^{\mathrm{tw}}_{\mathrm{GW}}(v,z)$ are convergent respectively on the regions $\{|u| < v_{\mathrm{c}}^{-1/d}\}$ and $\{|v| < v_{\mathrm{c}}\}$, where $v_{\mathrm{c}} := d^{-d}\prod_{j=1}^{\mathrm{N}} w_j^{w_j}$, see Lemma 5.10.

4.3.2. *Picard-Fuchs equations.* — The I-function $\mathrm{I}^{\mathrm{tw}}_{\mathrm{FJRW}}$ is a solution of the Picard-Fuchs equation

$$\left[u^d \prod_{j=1}^{\mathrm{N}}\prod_{c=0}^{w_j-1}(-q_j z\mathrm{D}_u - q_j\lambda - cz) - \prod_{c=1}^{d}(-z\mathrm{D}_u + cz)\right]\mathrm{I} = 0, \tag{57}$$

for $\mathrm{D}_u = u(\partial/\partial u)$. The I-function $\mathrm{I}^{\mathrm{tw}}_{\mathrm{GW}}$ is a solution of the Picard-Fuchs equation

$$\left[\prod_{j=1}^{\mathrm{N}}\prod_{c=0}^{w_j-1}(w_j z\mathrm{D}_v - cz) - v\prod_{c=1}^{d}(dz\mathrm{D}_v + \lambda + cz)\right]\mathrm{I} = 0 \tag{58}$$

for $\mathrm{D}_v = v(\partial/\partial v)$.

Under the change of variable $u = v^{-1/d}$ and conjugation with the operator $u^{-\lambda/z} = v^{\lambda/dz}$ the two equations coincide. This happens because we have $d\mathrm{D}_v = -\mathrm{D}_u$ and $v^{-\lambda/dz}\circ(dz\mathrm{D}_v)\circ v^{\lambda/dz} = dz\mathrm{D}_v + \lambda$. In particular the limits for $\lambda\to 0$ match under $v = u^{-d}$. (We remedy the discrepancy of the equivariant Picard-Fuchs equations by introducing the unit co-ordinate t^0 (or s^0) later in Section 5.2.) The components of each of the I-functions give a basis of solutions to the Picard-Fuchs equation for generic λ (cf. Proposition 5.11, Lemma 5.15 and (**69**)).

4.3.3. *The $\mathfrak{H}$-functions.* — We introduce a constant linear transform of the I-function, the $\mathfrak{H}$-function, which is more compatible with the $\widehat{\Gamma}$-integral structure in Section 2.4.4. The relevance of such hypergeometric series in homological mirror symmetry was observed by Horja [38], Hosono [39] and Borisov-Horja [5]. The $\mathfrak{H}$-function is defined by the relation[15] (cf. (**22**)):

$$\mathrm{I}^{\mathrm{tw}}(x,z;\lambda) = z^{-\mathsf{Gr}}\widehat{\Gamma}^{\mathrm{tw}}\Big((2\pi\mathtt{i})^{\frac{\deg_0}{2}}\mathfrak{H}^{\mathrm{tw}}(x,z;\lambda)\Big). \tag{59}$$

Here the operators $\widehat{\Gamma}^{\mathrm{tw}}$, Gr, $\deg_0$ in the respective theory are defined as follows: In the twisted FJRW theory, the *twisted Gamma class* $\widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{FJRW}}$ operating on the extended state space $\mathrm{H}_{\mathrm{ext}}$ is defined to be

$$\widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{FJRW}} := \bigoplus_{k=0}^{d-1}\prod_{i=1}^{\mathrm{N}}\Gamma\big(1-\langle kq_j\rangle - q_j\xi\big), \qquad \xi=\lambda/z.$$

In the twisted GW theory, the *twisted Gamma class* $\widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{GW}}$ operating on $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ is defined to be

$$\widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{GW}} := \bigoplus_{f\in\mathfrak{F}}\frac{\prod_{i=1}^{\mathrm{N}}\Gamma(1-\langle fw_i\rangle + w_ip)}{\Gamma(1+\xi+dp)}, \qquad \xi=\lambda/z.$$

The non-equivariant limits $\lambda\to 0$ are well-defined and induce $\widehat{\Gamma}_{\mathrm{FJRW}}$ and $\widehat{\Gamma}_{\mathrm{FJRW}}$ in Definition 2.17 under the projection to the original state spaces. The grading operator Gr on $\mathrm{H}_{\mathrm{ext}}$ or on $\mathrm{H}_{\mathrm{CR}}(\mathrm{X}_{\mathrm{W}})$ is given by

$$\mathsf{Gr}(\mathrm{T}_i) = \frac{\deg \mathrm{T}_i}{2}\mathrm{T}_i$$

where "deg" denotes the degree defined in (**25**) for FJRW theory and the age-shifted degree of orbifold cohomology classes of $\mathbf{P}(\underline{w})$ for GW theory. The "bare" degree operator $\deg_0$ on $\mathrm{H}_{\mathrm{ext}}$ or on $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ is defined by (cf. Definition 2.19)

$$\begin{aligned} \deg_0(\phi_k) &= -2\phi_k && \text{for twisted FJRW theory;}\\ \deg_0\big(p^n\mathbf{1}_f\big) &= 2n\big(p^n\mathbf{1}_f\big) && \text{for twisted GW theory.}\end{aligned}$$

On the Landau-Ginzburg side, we have

---

[15] See Section 5.3, (**82**) for a precise relationship between the $\mathfrak{H}$-function and the $\widehat{\Gamma}$-integral structure.

$$
\begin{aligned}
\mathrm{I}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z;\lambda) &= z^{-\mathsf{Gr}} z \sum_{k\in\mathbf{Z}_{\geq 1}} u^k \frac{(-1)^{k-1}}{\Gamma(k)} \prod_{j=1}^{\mathrm{N}} \frac{\Gamma(\langle -q_j k\rangle - q_j\xi)}{\Gamma(1-q_j(k+\xi))}\phi_{k-1} \\
&= z^{-\mathsf{Gr}} z \sum_{k\in\mathbf{Z}_{\geq 1}} u^k \frac{(-1)^{k-1}}{\Gamma(k)} \frac{1}{\prod_{j:kq_j\in\mathbf{Z}}(-q_j\xi)} \\
&\quad \times \prod_{j=1}^{\mathrm{N}} \frac{\Gamma(1-\langle q_j k\rangle - q_j\xi)}{\Gamma(1-q_j(k+\xi))}\phi_{k-1} \\
&= z^{-\mathsf{Gr}} \widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{FJRW}} \big((2\pi \mathtt{i})^{\frac{\deg_0}{2}} \mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z;\lambda)\big),
\end{aligned}
$$

where $\xi := \lambda/z$ and

$$
\mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z;\lambda) = z \sum_{k\in\mathbf{Z}_{\geq 1}} u^k \frac{(-1)^{k-1}(2\pi\mathtt{i})}{\Gamma(k)\prod_{j:kq_j\in\mathbf{Z}}(-q_j\xi)\prod_{j=1}^{\mathrm{N}}\Gamma(1-q_j(k+\xi))}\phi_{k-1}. \tag{60}
$$

On the Calabi-Yau side, we have

$$
\begin{aligned}
\mathrm{I}^{\mathrm{tw}}_{\mathrm{GW}}(v,z;\lambda) &= z^{-\mathsf{Gr}} z e^{p\log v} \sum_{\substack{n\in\mathbf{Q}_{\geq 0} \\ \exists j,\, nw_j\in\mathbf{Z}}} v^n \frac{\Gamma(1+dp+\xi+dn)}{\Gamma(1+dp+\xi)} \\
&\quad \times \prod_{j=1}^{\mathrm{N}} \frac{\Gamma(1+w_j p - \langle -w_j n\rangle)}{\Gamma(1+w_j p + w_j n)} \mathbf{1}_{\langle -n\rangle} \\
&= z^{-\mathsf{Gr}} \widehat{\Gamma}^{\mathrm{tw}}_{\mathrm{GW}} \big((2\pi\mathtt{i})^{\frac{\deg_0}{2}} \mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}(v,z;\lambda)\big),
\end{aligned}
$$

where $\xi := \lambda/z$ and

$$
\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}(v,z;\lambda) = z e^{\frac{p}{2\pi\mathtt{i}}\log v} \sum_{\substack{n\in\mathbf{Q}_{\geq 0} \\ \exists j, nw_j\in\mathbf{Z}}} v^n \frac{\Gamma(1+d\frac{p}{2\pi\mathtt{i}}+\xi+dn)}{\prod_{j=1}^{\mathrm{N}}\Gamma(1+w_j\frac{p}{2\pi\mathtt{i}}+w_j n)} \mathbf{1}_{\langle -n\rangle}. \tag{61}
$$

4.3.4. *Mellin-Barnes analytic continuation.* — The function $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}(v,z;\lambda)$ is convergent and analytic on the region $\Re(\log v) < \log v_{\mathrm{c}}$, where $v_{\mathrm{c}} := d^{-d}\prod_{j=1}^{\mathrm{N}} w_i^{w_i}$ is the singularity of the Picard-Fuchs equation (**58**). Similarly $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}(u,z;\lambda)$ is convergent and analytic on the region $\Re(\log u) < -(\log v_{\mathrm{c}})/d$. Let $\widetilde{\mathcal{M}}^\circ$ denote the $(\log v)$-plane minus the singularities of the Picard-Fuchs equation:

$$
\widetilde{\mathcal{M}}^\circ = \mathbf{C}_{\log v} \setminus \{\log v_c + 2l\pi\mathtt{i} \mid l\in\mathbf{Z}\}. \tag{62}
$$

Under the identification $\log v = -d\log u$, we regard $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}$ as a single-valued function in the left-half of $\widetilde{\mathcal{M}}^\circ$ and $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}$ as a single-valued function on the right-half of $\widetilde{\mathcal{M}}^\circ$. Let

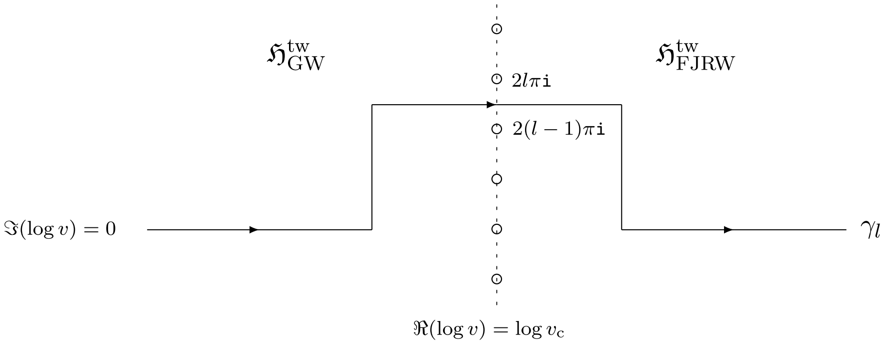

Fig. 3. — The analytic continuation path γ_l on the $(\log v)$-plane

$\gamma_l \subset \widetilde{\mathcal{M}}^\circ$ be a path from the large radius limit ($\Im(\log v)=0, \Re(\log v) \ll 0$) to the LG limit ($\Im(\log v)=0, \Re(\log v) \gg 0$) which passes through the "window" $[\log v_{\mathrm{c}} + 2(l-1)\pi\mathtt{i}, \log v_{\mathrm{c}} + 2l\pi\mathtt{i}]$. See Figure 3. We consider analytic continuation along the path γ_l.

We rewrite $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}$ by expressing the running index n as an element of $\mathfrak{F} + \mathbf{Z}_{\geq 0}$. For $f \in \mathfrak{F}$, we adopt the notation $\overline{f} = \langle 1 - f \rangle$. We get

$$\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}(v, z; \lambda) = z \sum_{f \in \mathfrak{F}} \sum_{k \in \mathbf{Z}_{\geq 0}} \frac{\Gamma(1 + \xi + d\frac{p}{2\pi\mathtt{i}} + d\overline{f} + dk)}{\prod_{j=1}^{\mathrm{N}} \Gamma(1 + w_j \frac{p}{2\pi\mathtt{i}} + w_j\overline{f} + w_j k)} v^{\frac{p}{2\pi\mathtt{i}} + \overline{f} + k} \mathbf{1}_f.$$

During the analytic continuation, we regard p as a small complex number and think of the $\mathfrak{H}$-function as a scalar valued function. At the end of the calculation, we take the Taylor expansion in p and replace p with the hyperplane class. In this way we get analytic continuation of a cohomology-valued function. We write the sum over $\mathbf{Z}_{\geq 0}$ as a sum of residues:

$$z \sum_{f \in \mathfrak{F}} \mathbf{1}_f \sum_{k \in \mathbf{Z}_{\geq 0}} \mathrm{Res}_{s=k}\, ds$$
$$\times \left(\Gamma(s)\Gamma(1-s) \frac{\Gamma(1 + \xi + d(\frac{p}{2\pi\mathtt{i}} + \overline{f} + s))}{\prod_{j=1}^{\mathrm{N}} \Gamma(1 + w_j(\frac{p}{2\pi\mathtt{i}} + \overline{f} + s))} e^{-(2l-1)\pi\mathtt{i}s} e^{(\frac{p}{2\pi\mathtt{i}} + \overline{f} + s)\log v} \right).$$

Here $l \in \mathbf{Z}$ is the index of the path γ_l. Consider the contour integrals along the path of Figure 4 of each 1-form in the above expression. The integrals are absolutely convergent (and define analytic functions of v) if $|\Im(\log v) - (2l-1)\pi| < \pi$ (see e.g. [38, Lemma 3.3]). This condition is satisfied when $\log v$ is along (the middle part of) the path γ_l. When $|v| < v_{\mathrm{c}}$, we can close the contour to the right and obtain the above sum of residues. On the other hand, if $|v| > v_{\mathrm{c}}$, we can close the contour to the left and obtain the sum of residues at $s = -m$ ($m \in \mathbf{Z}_{\geq 1}$) plus the sum of residues at

$$s = -\left(\frac{1 + \xi + k}{d} + \frac{p}{2\pi\mathtt{i}} + \overline{f} \right) \quad \text{for } k \in \mathbf{Z}_{\geq 0}.$$

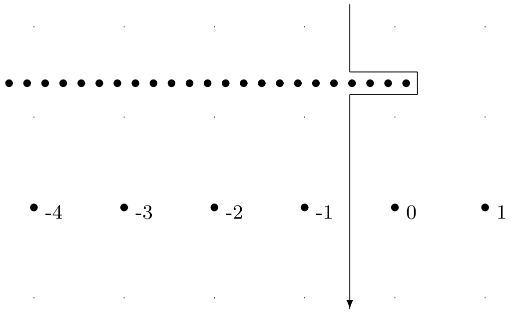

FIG. 4. — The contour of integration on the s-plane

The sum of these residues gives

$$
\begin{aligned}
(\mathbf{63}) \qquad & -z\sum_{f\in\mathfrak{F}}\mathbf{1}_f\sum_{m=1}^{\infty}\frac{\Gamma(1+\xi+d(\frac{p}{2\pi\mathtt{i}}+\overline{f}-m))}{\prod_{j=1}^{\mathrm{N}}\Gamma(1+w_j(\frac{p}{2\pi\mathtt{i}}+\overline{f}-m))}e^{(\frac{p}{2\pi\mathtt{i}}+\overline{f}-m)\log v}\\
& -z\sum_{f\in\mathfrak{F}}\mathbf{1}_f\sum_{k=0}^{\infty}\frac{\pi}{\sin(-(\frac{1+\xi+k}{d}+\frac{p}{2\pi\mathtt{i}}+\overline{f})\pi)}\frac{(-1)^k}{d\cdot k!}\\
& \times\frac{e^{(2l-1)\pi\mathtt{i}(\frac{p}{2\pi\mathtt{i}}+\overline{f}+\frac{1+\xi+k}{d})}}{\prod_{j=1}^{\mathrm{N}}\Gamma(1-q_j(1+\xi+k))}u^{1+\xi+k}.
\end{aligned}
$$

Here the overall minus sign appears because the contour closed to the left encloses each pole clockwise. We also used the co-ordinate change $\log u = -\log v/d$.

We now regard p as the hyperplane class on $\mathbf{P}(\underline{w})$. The first term of (**63**) vanishes in cohomology because the class

$$
\prod_{j:w_j\overline{f}\in\mathbf{Z}}\frac{1}{\Gamma(1+w_j\frac{p}{2\pi\mathtt{i}}+w_j(\overline{f}-m))}=\mathrm{O}\big(p^{\sharp\{j|w_j\overline{f}\in\mathbf{Z}\}}\big)
$$

is zero on the sector $\mathbf{P}(\underline{w})_f$. (Note that $\mathbf{P}(\underline{w})_f$ is of dimension $\sharp\{j\mid w_j\overline{f}\in\mathbf{Z}\}-1$.) By shifting the index k by 1 and using $\sin(x)=(e^{\mathtt{i}x}-e^{-\mathtt{i}x})/2\mathtt{i}$, we can rewrite the second term of (**63**) as

$$
z\sum_{f\in\mathfrak{F}}\mathbf{1}_f\sum_{k=1}^{\infty}\frac{1}{d}\frac{(\zeta^k e^{p+2\pi\mathtt{i}(\overline{f}+\frac{\xi}{d})})^l}{\zeta^k e^{p+2\pi\mathtt{i}(\overline{f}+\frac{\xi}{d})}-1}\cdot\frac{2\pi\mathtt{i}(-1)^{k-1}u^{\xi+k}}{(k-1)!\prod_{j=1}^{\mathrm{N}}\Gamma(1-q_j(k+\xi))}.
$$

This expression is regular at $p=0$ and can be regarded as an $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$-valued function. This is the analytic continuation of $\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}}$ along the path γ_l. Comparing this with $\mathfrak{H}_{\mathrm{FJRW}}^{\mathrm{tw}}$ (**60**), we have the following proposition:

Proposition **4.12.** — *Define a linear transformation* $\mathbf{U}_l^{\mathrm{tw}} \colon \mathrm{H}_{\mathrm{ext}} \to \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ *depending on* $l \in \mathbf{Z}$ *and the parameter* $\xi = \lambda/z$ *by*

$$\mathbf{U}_l^{\mathrm{tw}}(\phi_{k-1}) = \frac{1}{d}\sum_{f\in\mathfrak{F}} \mathbf{1}_f \frac{(\zeta^k e^{p+2\pi\mathtt{i}(\bar{f}+\frac{\xi}{d})})^l}{\zeta^k e^{p+2\pi\mathtt{i}(\bar{f}+\frac{\xi}{d})} - 1} \prod_{j:kq_j\in\mathbf{Z}} (-q_j\xi), \quad k = 1, \dots, d. \tag{64}$$

Then we have

$$u^{-\xi}\big(\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}}\big)_{\mathrm{continued}} = \mathbf{U}_l^{\mathrm{tw}}\big(\mathfrak{H}_{\mathrm{FJRW}}^{\mathrm{tw}}\big),$$

where $(\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}})_{\mathrm{continued}}$ *is the analytic continuation of* $\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}}$ *along the path* γ_l.

Remark **4.13.** — By Proposition 4.12 and (**59**), we can find the connection matrix of the twisted I-functions. We have $u^{-\xi}(\mathrm{I}_{\mathrm{GW}}^{\mathrm{tw}})_{\mathrm{continued}} = \widetilde{\mathbf{U}}_l^{\mathrm{tw}}(\mathrm{I}_{\mathrm{FJRW}}^{\mathrm{tw}})$ for the transformation

$$\widetilde{\mathbf{U}}_l^{\mathrm{tw}} = z^{-\mathsf{Gr}} \circ \widehat{\Gamma}_{\mathrm{GW}}^{\mathrm{tw}} \circ (2\pi\mathtt{i})^{\frac{\deg_0}{2}} \circ \mathbf{U}_l^{\mathrm{tw}} \circ (2\pi\mathtt{i})^{-\frac{\deg_0}{2}} \circ \big(\widehat{\Gamma}_{\mathrm{FJRW}}^{\mathrm{tw}}\big)^{-1} \circ z^{\mathsf{Gr}}.$$

The non-equivariant limit of this induces a linear transformation between the Givental symplectic vector spaces of FJRW theory and GW theory. This is the symplectic transformation computed in [13] for a quintic.

4.4. *The non-equivariant limit and Orlov equivalence.*

— Here we show that the non-equivariant limit of $\mathbf{U}_l^{\mathrm{tw}}$ exists and descends to a linear transformation between the narrow and the ambient part state spaces. We show that it matches with the numerical Orlov equivalence.

4.4.1. *The narrow-to-ambient linear transformation.*

Proposition **4.14.** — *The non-equivariant limit* $\lambda \to 0$ *of* $\mathbf{U}_l^{\mathrm{tw}}$ *exists. We have*

$$\lim_{\lambda\to 0}\big(\mathbf{U}_l^{\mathrm{tw}}(\phi_{k-1})\big) = \begin{cases} \frac{1}{d}\sum_{f\in\mathfrak{F}} \frac{(\zeta^k e^{p+2\pi\mathtt{i}\bar{f}})^l}{\zeta^k e^{p+2\pi\mathtt{i}\bar{f}}-1}\mathbf{1}_f & \text{for } k \in \mathsf{Nar}; \\ -p^{\mathrm{N}_k-1}\mathbf{1}_{\langle\frac{k}{d}\rangle}\frac{\zeta^{kl}}{d}\prod_{j:kq_j\in\mathbf{Z}}\frac{w_j}{2\pi\mathtt{i}} & \text{for } k \notin \mathsf{Nar}, \end{cases}$$

where $k = 1, \dots, d$ *and* $\mathrm{N}_k := 1 + \dim\mathbf{P}(\underline{w})_{\langle k/d\rangle} = \sharp\{j \mid kq_j \in \mathbf{Z}\}$.

Proof. — We take the Taylor expansion of the expression (**64**) in p first and check if the expansion are regular at $\xi = 0$ when evaluated in $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$.

If $k \in \mathsf{Nar}$, or equivalently $\langle k/d\rangle \notin \mathfrak{F}$, there exists no $f \in \mathfrak{F}$ such that $\zeta^k e^{2\pi\mathtt{i}\bar{f}} = 1$. Therefore (**64**) is regular at $(p, \xi) = (0, 0)$ and the conclusion follows.

If $k \notin \mathsf{Nar}$, (**64**) is not regular at $(p, \xi) = (0, 0)$. The only non-regular term in (**64**) is the one with $f = \langle k/d \rangle$ (in this case $\zeta^k e^{2\pi \mathtt{i} \overline{f}} = 1$). We compute the Taylor expansion in p of such term. By an elementary computation, we have

$$\frac{1}{e^{p + \frac{2\pi \mathtt{i} \xi}{d}} - 1} = \sum_{n=0}^{\infty} \beta_n(\xi) p^n, \qquad \beta_n(\xi) = (-1)^n \left(\frac{2\pi \mathtt{i} \xi}{d} \right)^{-n-1} + \mathrm{O}(\xi^{-n}).$$

When evaluated in the cohomology group $\mathrm{H}(\mathbf{P}(\underline{w})_f)$, this Taylor series is truncated at $n = \dim \mathbf{P}(\underline{w})_f = \mathrm{N}_k - 1$ (where we used $f = \langle k/d \rangle$). Therefore the factor $\prod_{j : kq_j \in \mathbf{Z}} (-q_j \xi)$ cancels all the negative powers of ξ in β_n. Hence $\mathbf{U}_l^{\mathrm{tw}}(\phi_{k-1})$ is regular at $\xi = 0$ and the conclusion follows. □

We have natural projections $\mathrm{H}_{\mathrm{ext}} \to \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$, $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w})) \to \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ from the state spaces of the twisted theory to the narrow/ambient part of the state spaces. We denote this projection by pr. By Proposition 4.14, $\lim_{\lambda \to 0} \mathbf{U}^{\mathrm{tw}}$ descends to these projections.

Corollary **4.15.** — *Define a linear transformation* $\mathbf{U}_l \colon \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d) \to \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ *by*

$$\mathbf{U}_l(\phi_{k-1}) = \frac{1}{d} \sum_{f \in \mathfrak{F}} \frac{(\zeta^k e^{p + 2\pi \mathtt{i} \overline{f}})^l}{\zeta^k e^{p + 2\pi \mathtt{i} \overline{f}} - 1} \mathbf{1}_f. \tag{65}$$

Then we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{ext}} \arrow[r, "\lim_{\lambda \to 0} \mathbf{U}_l^{\mathrm{tw}}"] \arrow[d, "\mathrm{pr}"'] & \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w})) \arrow[d, "\mathrm{pr}"] \\
\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d) \arrow[r, "\mathbf{U}_l"] & \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})
\end{tikzcd}$$

The operator $\mathbf{U}_l$ *gives a connection between the non-equivariant limit of* $\mathfrak{H}$*-functions, i.e.* $\mathfrak{H}_{\mathrm{GW}} = \mathbf{U}_l(\mathfrak{H}_{\mathrm{FJRW}})$ *for* $\mathfrak{H}_{\heartsuit} := \mathrm{pr}(\lim_{\lambda \to 0} \mathfrak{H}_{\heartsuit}^{\mathrm{tw}})$.

4.4.2. *The analytic continuation matches Orlov equivalences.* — Via the Chern character, the linear transformations $\mathbf{U}_l$ match the Orlov equivalences Φ_l. To show this we use the explicit expression for Orlov's equivalence for Koszul matrix factorizations (Proposition 4.11) and the equation (**49**) for the Chern character.

Lemma **4.16.** — *We have*

$$\frac{1}{d} \sum_{k=0}^{d-1} \frac{\zeta^{kj}}{\zeta^k y - 1} = \frac{y^{d\langle -j/d \rangle}}{y^d - 1},$$

where $d\langle -j/d \rangle$ *is simply* $-j$ *reduced modulo* d *within* $\{0, 1, \ldots, d-1\}$.

Proof. — Note that $(1/d)\sum_{k=0}^{d-1}\zeta^{qk}$ equals 1 if $q \in d\mathbf{Z}$ and 0 otherwise. Thus we have

$$\frac{1}{d}\sum_{k=0}^{d-1}\frac{\zeta^{kj}}{\zeta^k y-1} = -\frac{1}{d}\sum_{k=0}^{d-1}\sum_{n=0}^{\infty}(\zeta^k)^{j+n}y^n = -\sum_{n\geq 0:j+n\in d\mathbf{Z}} y^n.$$

The lemma follows. $\square$

Theorem **4.17.** — *Let* $\mathbf{U}_l\colon \mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)\to \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_{\mathrm{W}})$ *denote the map in Corollary 4.15. For a graded matrix factorization* $\mathrm{E}\in \mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ *such that* $\mathrm{ch}(\mathrm{E})\in \mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)$, *we have*

$$\mathbf{U}_l\big(\mathrm{inv}^*\,\mathrm{ch}(\mathrm{E})\big) = \mathrm{inv}^*\,\mathrm{ch}\big(\Phi_l(\mathrm{E})\big).$$

Proof. — Because Chern characters of the form $\mathrm{ch}(\{\underline{a},\underline{b}\}_q)$ in Example 4.5 span the narrow part, it suffices to show that

$$\mathbf{U}_l\Big(\mathrm{inv}^*\,\mathrm{ch}\big(\{\underline{a},\underline{b}\}_q\big)\Big) = \mathrm{inv}^*\,\mathrm{ch}\Big(\Phi_l\big(\{\underline{a},\underline{b}\}_q\big)\Big)$$

for $q\in\mathbf{Z}$ and $\underline{a}$, $\underline{b}$ in Example 4.5. Using (**65**) and (**49**), we get

$$\begin{aligned}\mathbf{U}_l\Big(\mathrm{inv}^*\,\mathrm{ch}\big(\{\underline{a},\underline{b}\}_q\big)\Big) &= \mathbf{U}_l\Big(\sum_{k\in\mathsf{Nar}}\zeta^{-qk}\big(1-\zeta^{w_1k}\big)\cdots\big(1-\zeta^{w_{\mathrm{N}}k}\big)\phi_{k-1}\Big)\\ &= \frac{1}{d}\sum_{f\in\mathfrak{F}}\sum_{k=0}^{d-1}\zeta^{-qk}\frac{(1-\zeta^{w_1k})\cdots(1-\zeta^{w_{\mathrm{N}}k})}{\zeta^k e^{p+2\pi\mathtt{i}\overline{f}}-1}\\ &\quad\times\big(\zeta^k e^{p+2\pi\mathtt{i}\overline{f}}\big)^l\mathbf{1}_f\\ &= \sum_{f\in\mathfrak{F}}\mathbf{1}_f\sum_{j_1<\cdots<j_r}y^l(-1)^r\frac{1}{d}\sum_{k=0}^{d-1}\frac{(\zeta^k)^{-q+l+w_{j_1}+\cdots+w_{j_r}}}{(\zeta^k)y-1},\end{aligned}$$

where we set $y:=e^{p+2\pi\mathtt{i}\overline{f}}$. Using Lemma 4.16, we can write the coefficient of $\mathbf{1}_f$ as

$$\frac{y^l}{1-y^d}\sum_{j_1<\cdots<j_r}(-1)^{r+1}y^{d\langle\frac{q-l}{d}-\frac{1}{d}\sum_{a=1}^r w_{j_a}\rangle}. \tag{66}$$

Let m be the remainder of $q-l$ divided by d. The sum (**66**) can be decomposed as

$$\frac{y^l}{1-y^d}\Bigg(\sum_{\substack{j_1<\cdots<j_r\\ \sum_{a=1}^r w_{j_a}\leq m}}(-1)^{r+1}y^{m-\sum_{a=1}^r w_{j_a}} + \sum_{\substack{j_1<\cdots<j_r\\ \sum_{a=1}^r w_{j_a}> m}}(-1)^{r+1}y^{m-\sum_{a=1}^r w_{j_a}+d}\Bigg).$$

This can be further rewritten as

$$\frac{y^l}{1-y^d}\left((1-y^d)\sum_{\substack{j_1<\cdots<j_r\\ \sum_{a=1}^r w_{j_a}\le m}}(-1)^{r+1}y^{m-\sum_{a=1}^r w_{j_a}}+\sum_{j_1<\cdots<j_r}(-1)^{r+1}y^{m-\sum_{a=1}^r w_{j_a}+d}\right). \tag{67}$$

The second summand equals

$$-\frac{y^{d+l+m}(1-y^{w_1})\cdots(1-y^{w_{\mathrm{N}}})}{1-y^d}.$$

This is divisible by $p^{\sharp\{j|w_jf\in\mathbf{Z}\}-1}$ and vanishes in $\mathrm{H}(\mathbf{P}(\underline{w})_f\cap \mathrm{X_W})$ for the dimensional reason (note that $\dim(\mathbf{P}(\underline{w})_f\cap\mathrm{X_W})=\sharp\{j|w_jf\in\mathbf{Z}\}-2$). Finally, the first summand of (**67**) equals the coefficient of $\mathbf{1}_f$ of $\mathrm{inv}^*\,\mathrm{ch}(\Phi_l(\{\underline{a},\underline{b}\}_q))$ by Proposition 4.11. ☐

5. Construction of global D-module

This section is devoted to the proof of the main theorems in Sections 1.1 and 2.5. We construct a global D-module over the base $\mathcal{M}=\mathbf{P}(1,d)\setminus\{2\text{ points}\}$ as an explicit GKZ-type differential system and show that the D-module is isomorphic to the quantum D-module of GW theory near $v=0$ and to the quantum D-module of FJRW theory near $v=\infty$. We use the mirror theorem in Section 3 and that of Coates-Corti-Lee-Tseng [19] (and its refinement in [43]).

5.1. *Multi-GKZ system.* — Let $v\mapsto[1,v]$ denote the inhomogeneous co-ordinate on $\mathbf{P}(1,d)$ where $v=\infty$ is the $\boldsymbol{\mu}_d$-orbifold point (LG point). Using the co-ordinate v, we set

$$\mathcal{M}:=\mathbf{P}(1,d)\setminus\{0,v_{\mathrm{c}}\},\qquad \mathcal{M}^\circ:=\mathbf{P}(1,d)\setminus\{0,v_{\mathrm{c}},\infty\}$$

where $v_{\mathrm{c}}:=d^{-d}\prod_{j=1}^{\mathrm{N}}w_j^{w_j}$ is the conifold point. Let $u:=v^{-1/d}$ denote the uniformizing co-ordinate centered at the LG point. In this section we introduce a GKZ-type (Gelf'and-Kapranov-Zelevinskii [27]) hypergeometric D-module over the base $\mathcal{M}^\circ$. The D-module here involves the parameter z which appears in the quantum D-module (see Section 2.4.2), and the equivariant parameter λ which appears in the twisted theory (see Section 3). Therefore it is defined as a sheaf over $\mathcal{M}^\circ\times\mathbf{C}_z\times\mathbf{C}_\lambda$. Let $\mathcal{R}^{\mathrm{tw}}$ denote the sheaf of algebras over $\mathcal{M}^\circ\times\mathbf{C}_z\times\mathbf{C}_\lambda$ given by the non-commutative ring of differential operators

$$\mathbf{C}\langle z,\lambda,v^{\pm},(v-v_{\mathrm{c}})^{-1},z\mathrm{D}_v\rangle$$

where $\mathrm{D}_v=v(\partial/\partial v)$. We also set

$$\mathrm{B}:=\left\{(\nu_0,\ldots,\nu_{\mathrm{N}})\in\mathbf{Z}^{\mathrm{N}+1}\;\middle|\;\nu_i+q_i\nu_0\ge 0,\ i=1,\ldots,\mathrm{N}\right\},\qquad q_i=w_i/d.$$

Definition **5.1.** — *The sheaf* $\mathcal{F}^{\mathrm{tw}}$ *over* $\mathcal{M}^{\circ} \times \mathbf{C}_z \times \mathbf{C}_\lambda$ *is defined to be the* $\mathcal{R}^{\mathrm{tw}}$*-module generated by the symbols* Δ_ν *with* $\nu \in \mathrm{B}$ *subject to the relations:*

$$\begin{aligned} \big(dz\mathrm{D}_v + \lambda + (\nu_0+1)z\big)\Delta_\nu &= \Delta_{\nu+e_0}, \\ (w_i z\mathrm{D}_v - \nu_i z)\Delta_\nu &= \Delta_{\nu+e_i}, \quad i \in \{1,\ldots,\mathrm{N}\}, \\ v\cdot\Delta_\nu &= \Delta_{\nu+(-d,w_1,\ldots,w_\mathrm{N})}. \end{aligned} \tag{68}$$

Here $\nu \in \mathrm{B}$ *and* $e_i = (0,\ldots,0,\overset{i}{1},0,\ldots,0)$, $0 \le i \le \mathrm{N}$. *This defines a GKZ-type hypergeometric differential system. In fact, it is easy to see that each generator* Δ_ν *satisfies the relation*

$$\left[v\prod_{k=1}^{d}\big(dz\mathrm{D}_v + \lambda + (\nu_0+k)z\big) - \prod_{i=1}^{\mathrm{N}}\prod_{k=0}^{w_i-1}\big(w_i z\mathrm{D}_v - (\nu_i+k)z\big)\right]\Delta_\nu = 0. \tag{69}$$

Remark **5.2.** — A multi-generated hypergeometric system similar to $\mathcal{F}^{\mathrm{tw}}$ above appeared in the recent work of Borisov-Horja [6] (also will appear in Coates-Corti-Iritani-Tseng [18]). The $\mathcal{R}^{\mathrm{tw}}$-submodule $\mathcal{R}^{\mathrm{tw}}\Delta_0$ of $\mathcal{F}^{\mathrm{tw}}$ generated by Δ_0 coincides with $\mathcal{F}^{\mathrm{tw}}$ at the generic point (Lemma 5.15), but not everywhere (for instance along $z = \lambda = 0$). A closely related multi-generation phenomena of quantum cohomology was observed by Guest-Sakai [33] for a Fano hypersurface in $\mathbf{P}(\underline{w})$. It was shown in [43] that the quantum D-module of a toric Calabi-Yau hypersurface can be described by a multi-GKZ system.

Remark **5.3.** — Givental's mirror [28] (adapted to a Calabi-Yau hypersurface $\mathrm{X_W}$ in the weighted projective space $\mathbf{P}(\underline{w})$) gives a solution to the above differential system. Let $\mathsf{x}_0,\ldots,\mathsf{x}_\mathrm{N}$ be mirror $\mathbf{C}^\times$-variables subject to the relation

$$\mathsf{x}_0^{-d}\mathsf{x}_1^{w_1}\cdots\mathsf{x}_\mathrm{N}^{w_\mathrm{N}} = v.$$

The mirror potential W_λ is defined by

$$\mathsf{W}_\lambda = \mathsf{x}_1 + \cdots + \mathsf{x}_\mathrm{N} - \mathsf{x}_0 + \lambda\log\mathsf{x}_0.$$

Then the integrals

$$\mathcal{I}_\nu(v) = \int \mathsf{x}_0^{\nu_0}\mathsf{x}_1^{\nu_1}\cdots\mathsf{x}_\mathrm{N}^{\nu_\mathrm{N}}e^{\mathsf{W}_\lambda/z}\frac{\mathsf{d}\mathsf{x}_0\wedge\mathsf{d}\log\mathsf{x}_1\wedge\cdots\wedge\mathsf{d}\log\mathsf{x}_\mathrm{N}}{\mathsf{d}\log v}, \quad \nu\in\mathrm{B}$$

satisfy the same differential relations as Δ_ν's do. The integration cycle is contained in the torus $\{(\mathsf{x}_0,\ldots,\mathsf{x}_\mathrm{N}) \in (\mathbf{C}^\times)^{\mathrm{N}+1} \mid \mathsf{x}_0^{-d}\mathsf{x}_1\cdots\mathsf{x}_\mathrm{N} = v\}$ and possibly noncompact, but here we do not try to justify the integral itself. The differential relations among $\mathcal{I}_\nu(v)$ follow from a formal computation of integration by parts.

Lemma **5.4.** — *Set* $\nu(l) := (l, -\lfloor q_1 l\rfloor, \dots, -\lfloor q_N l\rfloor) \in B$. *The sheaf* $\mathcal{F}^{\mathrm{tw}}$ *is generated by* $\triangle_{\nu(l)}$, $l = 0, \dots, d-1$ *as an* $\mathcal{R}^{\mathrm{tw}}$*-module.*

Proof. — For $\nu = (\nu_0, \dots, \nu_N) \in B$, set $l = d\langle \nu_0/d\rangle$. Observe that

$$\nu = \nu(l) + \sum_{i=1}^{N} \big(\nu_i + \lfloor q_i \nu_0\rfloor\big) e_i - \left\lfloor \frac{\nu_0}{d} \right\rfloor (-d, w_1, \dots, w_N), \qquad \nu_i + \lfloor q_i \nu_0 \rfloor \geq 0.$$

The conclusion follows from this and the defining relations (**68**) of $\mathcal{F}^{\mathrm{tw}}$. □

The sheaf $\mathcal{F}^{\mathrm{tw}}$ is a $2\mathbf{Z}_{\geq 0}$-graded $\mathcal{R}^{\mathrm{tw}}$-module with respect to the grading

$$\deg v = 0, \qquad \deg z = \deg \lambda = \deg(z\mathrm{D}_v) = 2,$$
$$\deg \triangle_\nu = 2(\nu_0 + \cdots + \nu_N).$$

(Strictly speaking, the module of global sections of $\mathcal{F}^{\mathrm{tw}}$ is graded, but we abuse the language since we are working over the affine base.)

Lemma **5.5.** — *Set* $\delta(l) := \frac{1}{2}\deg \triangle_{\nu(l)}$. *We have*

(i) $\delta(l+1) \leq \delta(l) + 1$, $\delta(l+d) = \delta(l)$.
(ii) $0 \leq \delta(l) \leq N - 1$. *We have* $\delta(l) = N-1$ *if and only if* $l \equiv -1 \bmod d$.

Proof. — We have $\delta(l) = l - \sum_{i=1}^{N} \lfloor q_i l\rfloor$. Part (i) follows from this formula. Part (ii) follows from $\delta(l) = \sum_{i=1}^{N}\langle q_i l\rangle \leq \sum_{i=1}^{N}(1-q_i) = N-1$. The equality holds iff $l \equiv -1 \bmod d/w_i$ for all i, i.e. $l \equiv -1 \bmod d$. □

Lemma **5.6.** — *The following relations hold in* $\mathcal{F}^{\mathrm{tw}}$*:*

(i) *For* $0 \leq l < d-1$, $m = \min\{l \leq l' \leq d-1 \mid \delta(l') = \delta(l)+1\}$ *exists and we have*

$$z\mathrm{D}_v \cdot \triangle_{\nu(l)} \in d^{l-m}\left(\prod_{i=1}^{N} w_i^{\lfloor q_i m\rfloor - \lfloor q_i l\rfloor}\right)\triangle_{\nu(m)} + (z, \lambda)\mathcal{F}^{\mathrm{tw}}.$$

(ii) $z\mathrm{D}_v \cdot \triangle_{\nu(d-1)} \in (z,\lambda)\mathcal{F}^{\mathrm{tw}}$.

Proof. — The existence of m follows from Lemma 5.5. We have by (**68**)

$$\big(dz\mathrm{D}_v + \lambda + (l+1)z\big)\triangle_{\nu(l)} = \prod_{i=1}^{N} \prod_{\lfloor q_i l\rfloor < k \leq \lfloor q_i(l+1)\rfloor} (w_i z\mathrm{D}_v + kz)\triangle_{\nu(l+1)}.$$

Hence

$$z\mathrm{D}_v \cdot \triangle_{\nu(l)} \in d^{-1}\left(\prod_{i=1}^{N} w_i^{\lfloor q_i(l+1)\rfloor - \lfloor q_i l\rfloor}\right)(z\mathrm{D}_v)^{\delta(l)-\delta(l+1)+1}\triangle_{\nu(l+1)} + (z,\lambda)\mathcal{F}^{\mathrm{tw}}.$$

If $\delta(l+1) \leq \delta(l)$, we can apply this formula recursively for $z\mathrm{D}_v \cdot \Delta_{\nu(l+1)}$ in the right-hand side. In general, if $\delta(l') \leq \delta(l)$ for all l' with $l < l' < m'$, we have

$$z\mathrm{D}_v \cdot \Delta_{\nu(l)} \in d^{-(m'-l)}\left(\prod_{i=1}^{\mathrm{N}} w_i^{\lfloor q_i m' \rfloor - \lfloor q_i l \rfloor}\right)(z\mathrm{D}_v)^{\delta(l)-\delta(m')+1}\Delta_{\nu(m')} + (z,\lambda)\mathcal{F}^{\mathrm{tw}}.$$

Taking m' to be m, we have (i). When $l = d-1$, we can take m' to be $l+d = 2d-1$. Then we have

$$z\mathrm{D}_v \cdot \Delta_{\nu(d-1)} - v_c z\mathrm{D}_v \cdot \Delta_{\nu(2d-1)} \in (z,\lambda)\mathcal{F}^{\mathrm{tw}}.$$

Part (ii) follows because $\Delta_{\nu(2d-1)} = v^{-1}\Delta_{\nu(d-1)}$ and $(1 - v_c/v)$ is invertible. ☐

Theorem **5.7.** — *The sheaf* $\mathcal{F}^{\mathrm{tw}}$ *is a free* $\mathcal{O}_{\mathcal{M}^\circ \times \mathbf{C}_z \times \mathbf{C}_\lambda}$*-module of rank d with the basis* $\Delta_{\nu(l)}$, $l = 0, \ldots, d-1$.

Proof. — Let $\mathcal{F}^{\mathrm{tw}\prime}$ be the $\mathcal{O}_{\mathcal{M}^\circ \times \mathbf{C}_z \times \mathbf{C}_\lambda}$-submodule of $\mathcal{F}^{\mathrm{tw}}$ generated by $\Delta_{\nu(l)}$, $l = 0, \ldots, d-1$. First we see that $\mathcal{F}^{\mathrm{tw}\prime} = \mathcal{F}^{\mathrm{tw}}$. We proceed by induction on the degree. The degree zero part $(\mathcal{F}^{\mathrm{tw}})_0$ is generated by $\Delta_0 = \Delta_{\nu(0)}$. Hence $(\mathcal{F}^{\mathrm{tw}})_0 \subset \mathcal{F}^{\mathrm{tw}\prime}$. Assume by induction that $(\mathcal{F}^{\mathrm{tw}})_{\leq 2k} \subset \mathcal{F}^{\mathrm{tw}\prime}$ for some $k \geq 0$. We shall show $(\mathcal{F}^{\mathrm{tw}})_{\leq 2(k+1)} \subset \mathcal{F}^{\mathrm{tw}\prime}$. By Lemma 5.4, it suffices to show that $z\mathrm{D}_v \cdot \Delta_{\nu(l)} \in \mathcal{F}^{\mathrm{tw}\prime}$ for $0 \leq l \leq d-1$ with $\delta(l) = k$. This follows from Lemma 5.6 and the induction hypothesis. Therefore $\mathcal{F}^{\mathrm{tw}} = \mathcal{F}^{\mathrm{tw}\prime}$.

As we will see in Proposition 5.11 below, $\mathcal{F}^{\mathrm{tw}}$ has d independent solutions. This shows that the generic rank (the rank at the generic point) of $\mathcal{F}^{\mathrm{tw}}$ equals d. By the previous paragraph, $\mathcal{F}^{\mathrm{tw}}$ is generated by $\Delta_{\nu(l)}$, $l = 0, \ldots, d-1$. Suppose we have a relation $\sum_{l=0}^{d-1} f_l(v, \lambda, z)\Delta_{\nu(l)} = 0$ with $f_l \in \mathcal{O}_{\mathcal{M}^\circ \times \mathbf{C}_\lambda \times \mathbf{C}_z}$. Then f_l should vanish at the generic point. Therefore $f_l = 0$. The conclusion follows. ☐

5.2. *Refined mirror theorem.* — We construct a basis of hypergeometric solutions of the GKZ system $\mathcal{F}^{\mathrm{tw}}$. Then we relate it to the fundamental solution L^{tw} of the $e_{\mathbf{C}^\times}$-twisted quantum connection (see Section 3.5) in Theorem 5.12. This shows the analytic continuation of *twisted* quantum connections. (In this section "twisted" always means "$e_{\mathbf{C}^\times}$-twisted".)

First we will "thicken" $\mathcal{F}^{\mathrm{tw}}$ by adding a new co-ordinate t^0. Let $\widetilde{\mathcal{M}}^\circ \to \mathcal{M}^\circ$ be the minimal abelian cover of $\mathcal{M}^\circ$ such that $\log v$ is single-valued (see (**62**)). We set

$$\widehat{\mathcal{M}} = \mathbf{C}_{t^0} \times \widetilde{\mathcal{M}}^\circ,$$

where $\mathbf{C}_{t^0}$ denotes the complex plane with co-ordinate t^0. Define another co-ordinate $s^0 \colon \widehat{\mathcal{M}} \times \mathbf{C}_\lambda \to \mathbf{C}$ by

$$s^0 = t^0 - \frac{1}{d}\lambda \log v.$$

We shall use $(t^0, v; \lambda)$ and $(s^0, u; \lambda)$ as two co-ordinate systems on $\widehat{\mathcal{M}} \times \mathbf{C}_\lambda$; $(t^0, v; \lambda)$ is a chart for GW theory and $(s^0, u; \lambda)$ is for FJRW theory. The co-ordinates t^0 and s^0 correspond to the identity direction of the state space via the mirror map we consider below. Let $\widehat{\mathcal{R}}^{\mathrm{tw}}$ be the following sheaf of algebras over $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$:

$$\widehat{\mathcal{R}}^{\mathrm{tw}} = \mathcal{O}_{\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda} \left\langle z\mathrm{D}_v, z\frac{\partial}{\partial t^0} \right\rangle.$$

Here we use the analytic structure sheaf. Note that we have

$$\mathrm{D}_u = -d\mathrm{D}_v - \lambda \frac{\partial}{\partial t^0}, \qquad \frac{\partial}{\partial s^0} = \frac{\partial}{\partial t^0} \tag{\textbf{70}}$$

under the co-ordinate change $(t^0, v) \mapsto (s^0, u)$. Let $\mathrm{pr}\colon \widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda \to \widetilde{\mathcal{M}}^\circ \times \mathbf{C}_z \times \mathbf{C}_\lambda \to \mathcal{M}^\circ \times \mathbf{C}_z \times \mathbf{C}_\lambda$ denote the natural projection. The pull back $\widehat{\mathcal{F}}^{\mathrm{tw}} := \mathrm{pr}^* \mathcal{F}^{\mathrm{tw}}$ has the structure of an $\widehat{\mathcal{R}}^{\mathrm{tw}}$-module by

$$z\frac{\partial}{\partial t^0} \cdot \triangle_\nu = \triangle_\nu, \qquad \nu \in \mathrm{B}.$$

By a *solution* of the $\widehat{\mathcal{R}}^{\mathrm{tw}}$-module $\widehat{\mathcal{F}}^{\mathrm{tw}}$, we mean an $\widehat{\mathcal{R}}^{\mathrm{tw}}$-module homomorphism $\varphi\colon \widehat{\mathcal{F}}^{\mathrm{tw}}|_{\mathrm{V}} \to \mathcal{O}_{\mathrm{V}}$ for an open subset $\mathrm{V} \subset \widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$. We construct a vector-valued solution with values in $\overline{\mathrm{H}}$ such that all of its components form a basis of solutions.

Definition **5.8.** — *The* generalized twisted I-functions $\mathrm{I}^{\mathrm{tw},\nu}$, $\nu \in \mathrm{B}$ *are defined as follows (the relevant convergence will be shown in Lemma 5.10 below):*

(i) *In the FJRW side:*

$$\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}\big(s^0, u, z; \lambda\big) = ze^{s^0/z} \sum_{k=\nu_0+1}^{\infty} u^k \frac{\prod_{j=1}^{\mathrm{N}} \prod_{0<b<kq_j+\nu_j, \langle b\rangle = \langle kq_j\rangle} (-q_j\lambda - bz)}{\prod_{0<b<k-\nu_0, b\in\mathbf{Z}} (-bz)} \phi_{k-1}$$

where we use the convention of reducing the index $k-1$ of ϕ_{k-1} modulo d. This is an $\mathrm{H}_{\mathrm{ext}}$*-valued power series convergent on the region $\{|u| < v_{\mathrm{c}}^{-1/d}\} \times \mathbf{C}_z^\times \times \mathbf{C}_\lambda$ in $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$. Note that, if $k \geq \nu_0 + 1$, then $kq_j + \nu_j \geq q_j + (q_j\nu_0 + \nu_j) \geq q_j$.*

(ii) *In the GW side (cf. [43, Definition 4.5]):*

$$\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}\big(t^0, v, z; \lambda\big) = ze^{(t^0 + p\log v)/z} \sum_{n\in\mathbf{Q}:\langle n\rangle\in\mathfrak{F}} v^n \prod_{b=1}^{dn+\nu_0} (dp + \lambda + bz)$$
$$\times \prod_{i=1}^{\mathrm{N}} \frac{\prod_{b\leq 0, \langle b\rangle = \langle w_i n\rangle} (w_i p + bz)}{\prod_{b\leq w_i n - \nu_i, \langle b\rangle = \langle w_i n\rangle} (w_i p + bz)} \mathbf{1}_{\langle -n\rangle}.$$

This is an $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$*-valued power series convergent on the region* $\{|v| < v_{\mathrm{c}}\} \times \mathbf{C}_z^\times \times \mathbf{C}_\lambda$ *in* $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$. *Note that the term*

$$\prod_{i=1}^{\mathrm{N}} \frac{\prod_{b\leq 0,\langle b\rangle=\langle w_i n\rangle}(w_i p + bz)}{\prod_{b\leq w_i n-\nu_i,\langle b\rangle=\langle w_i n\rangle}(w_i p + bz)} \mathbf{1}_{\langle -n\rangle}$$

vanishes if $w_i n - \nu_i < 0$ *for all* i *such that* $w_i n \in \mathbf{Z}$ *(because the factor* $\prod_{\langle w_i n\rangle=0}(w_i p)$ *in the numerator vanishes on* $\mathbf{P}(\underline{w})_{\langle -n\rangle}$ *for dimensional reason). Thus one can assume that there exists* i *such that* $w_i n \in \mathbf{Z}$ *and* $w_i n - \nu_i \geq 0$. *In this case we have* $q_i(dn + \nu_0) \geq q_i(dn + \nu_0) + \nu_i - w_i n = q_i \nu_0 + \nu_i \geq 0$. *Hence one can assume* $dn + \nu_0 \geq 0$ *in the summation.*

Remark **5.9.** — For $\nu = 0$, $\mathrm{I}^{\mathrm{tw},0}_{\mathrm{FJRW}}(0, u, z)$ and $\mathrm{I}^{\mathrm{tw},0}_{\mathrm{GW}}(0, v, z)$ coincide with the original twisted I-functions in Section 4.3.1. Also note that the generalized I-function $\mathrm{I}^{\mathrm{tw},\nu}$ is homogeneous of degree $2 + \deg \triangle_\nu = 2(1 + \nu_0 + \cdots + \nu_{\mathrm{N}})$ with respect to the degree $\deg s^0 = \deg t^0 = \deg z = \deg \lambda = 2$, $\deg u = \deg v = 0$ and the grading on $\overline{\mathrm{H}}$.

Lemma **5.10.** — *The function* $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}(s^0, u, z; \lambda)$ *is convergent on the region* $\{|u| < v_{\mathrm{c}}^{-1/d}\} \times \mathbf{C}_z^\times \times \mathbf{C}_\lambda$ *in* $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$; $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}(t^0, v, z; \lambda)$ *is convergent on the region* $\{|v| < v_{\mathrm{c}}\} \times \mathbf{C}_z^\times \times \mathbf{C}_\lambda$ *in* $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$.

Proof. — Write $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}(s^0, u, z; \lambda) = z e^{s^0/z} \sum_{k=\nu_0+1}^{\infty} u^k \square_k(z, \lambda)$ with $\square_k(z, \lambda)$ an element of $\mathrm{H}_{\mathrm{ext}}$. Fix a norm $\|\cdot\|$ on $\mathrm{H}_{\mathrm{ext}}$ such that $\|\phi_k\| = 1$. Then we have

$$\frac{\|\square_{k+d}(z,\lambda)\|}{\|\square_k(z,\lambda)\|} = \frac{\prod_{j=1}^{\mathrm{N}} \prod_{a=0}^{w_j-1} |q_j\lambda + (kq_j + \nu_j + a)z|}{\prod_{a=0}^{d-1} |(k - \nu_0 + a)z|}.$$

This converges to $\prod_{j=1}^{\mathrm{N}} q_j^{w_j} = v_{\mathrm{c}}$ as $k \to \infty$ for a fixed $(z, \lambda) \in \mathbf{C}^\times \times \mathbf{C}$. This implies that the convergence radius of $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}$ as a power series of u is $v_{\mathrm{c}}^{-1/d}$. Similarly, write $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}(t^0, v, z; \lambda) = z e^{(t^0 + p \log v)/z} \sum_{n\in\mathbf{Q}:\langle n\rangle\in\mathfrak{F}} \Diamond_n(z, \lambda)$ with $\Diamond_n(z, \lambda)$ an element of $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$. Fix a norm $\|\cdot\|$ on $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$. We have

$$\frac{\|\Diamond_{n+1}(z,\lambda)\|}{\|\Diamond_n(z,\lambda)\|} \leq \left\| \frac{\prod_{a=1}^{d}(dp + \lambda + (dn + \nu_0 + a)z)}{\prod_{i=1}^{\mathrm{N}} \prod_{a=1}^{w_i}(w_i p + (w_i n - \nu_i + a)z)} \right\|$$

where in the right-hand side $\|\cdot\|$ means the operator norm. The right-hand side converges to $d^d / \prod_{i=1}^{\mathrm{N}} w_i^{w_i} = v_{\mathrm{c}}^{-1}$ as $n \to \infty$ for a fixed $(z, \lambda) \in \mathbf{C}^\times \times \mathbf{C}$. Hence the convergence radius of $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}(t^0, v, z)$ as a power series of v is no less than v_{c}. □

Proposition **5.11.** — *For each* $\varphi \in \mathrm{Hom}(\overline{\mathrm{H}}, \mathbf{C})$, *the map*

$$\mathrm{I}^\varphi : \widehat{\mathcal{F}}^{\mathrm{tw}} \longrightarrow \mathcal{O}, \qquad \triangle_\nu \longmapsto z^{-1}\varphi\big(\mathrm{I}^{\mathrm{tw},\nu}\big), \quad \nu \in \mathrm{B}$$

defines a solution to the $\widehat{\mathcal{R}}^{\mathrm{tw}}$*-module* $\widehat{\mathcal{F}}^{\mathrm{tw}}$*, i.e. a homomorphism of* $\widehat{\mathcal{R}}^{\mathrm{tw}}$*-modules. Moreover, for a* $\mathbf{C}$*-basis* $\varphi_1, \dots, \varphi_d$ *of* $\mathrm{Hom}(\overline{\mathrm{H}}, \mathbf{C})$*, the corresponding solutions* $\mathrm{I}^{\varphi_1}, \dots, \mathrm{I}^{\varphi_d}$ *are linearly independent. (In fact, they form a basis of solutions by Theorem 5.7.)*

Proof. — For the former statement, it suffices to check that $\mathrm{I}^{\mathrm{tw},\nu} = \mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}$ or $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}$ satisfies the following differential equations (cf. (**68**); note also the co-ordinate change (**70**)):

$$\begin{aligned}&\big(-z\mathrm{D}_u + (\nu_0+1)z\big)\mathrm{I}^{\mathrm{tw},\nu} = \mathrm{I}^{\mathrm{tw},\nu+e_0},\\ &(-q_i z\mathrm{D}_u - q_i\lambda - \nu_i z)\mathrm{I}^{\mathrm{tw},\nu} = \mathrm{I}^{\mathrm{tw},\nu+e_i}, \qquad i = 1,\dots,\mathrm{N},\\ &v\cdot \mathrm{I}^{\mathrm{tw},\nu} = \mathrm{I}^{\mathrm{tw},\nu+(-d,w_1,\dots,w_\mathrm{N})}, \qquad z\frac{\partial}{\partial s^0}\mathrm{I}^{\mathrm{tw},\nu} = \mathrm{I}^{\mathrm{tw},\nu}.\end{aligned} \tag{71}$$

They follow from a straightforward computation. Let $\nu(l)$ be as in Lemma 5.4. For the FJRW I-functions, we have

$$z^{-1}\mathrm{I}^{\mathrm{tw},\nu(l)}_{\mathrm{FJRW}} \sim e^{s^0/z}u^{l+1}\big(\phi_l + \mathrm{O}(u)\big), \qquad l = 0,\dots,d-1. \tag{72}$$

Since the leading terms span $\mathrm{H}_{\mathrm{ext}}$, it follows that the solutions $\mathrm{I}^{\varphi_1}_{\mathrm{FJRW}}, \dots, \mathrm{I}^{\varphi_d}_{\mathrm{FJRW}}$ are linearly independent. For the GW I-functions, we have if $\langle l/d\rangle \in \mathfrak{F}$,

$$z^{-1}\mathrm{I}^{\mathrm{tw},\nu(l)}_{\mathrm{GW}} \sim e^{(t^0+p\log v)/z}v^{-l/d}\big(\mathbf{1}_{\langle \frac{l}{d}\rangle} + \mathrm{O}(v^{1/d})\big).$$

Thus

$$\left(z\mathrm{D}_v + \frac{l}{d}z\right)^i z^{-1}\mathrm{I}^{\mathrm{tw},\nu(l)}_{\mathrm{GW}} \sim e^{(t^0+p\log v)/z}v^{-l/d}\big(p^i\mathbf{1}_{\langle \frac{l}{d}\rangle} + \mathrm{O}(v^{1/d})\big). \tag{73}$$

These leading terms span $\mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$. Hence $\mathrm{I}^{\varphi_1}_{\mathrm{GW}}, \dots, \mathrm{I}^{\varphi_d}_{\mathrm{GW}}$ are linearly independent. □

The twisted I-function $\mathrm{I}^{\mathrm{tw},0}$ in each theory has the z^{-1}-asymptotics of the form (cf. (**41**)):

$$\mathrm{I}^{\mathrm{tw},0} = z\mathrm{F}\cdot \mathrm{T}_0 + \mathrm{G} + \mathrm{O}(z^{-1}) \tag{74}$$

where F and G are functions on a domain in $\widehat{\mathcal{M}} \times \mathbf{C}_\lambda$ on which $\mathrm{I}^{\mathrm{tw},0}$ converges; F takes values in $\mathbf{C}$ and G takes values in the degree ≤ 2 part $\overline{\mathrm{H}}^{\leq 2} = \overline{\mathrm{H}}^0 \oplus \overline{\mathrm{H}}^2$. More precisely, in the FJRW side, we have (cf. (**42**)):

$$\begin{aligned}\mathrm{F} &= \mathrm{F}_{\mathrm{FJRW}}(u) = \sum_{k\geq 1: k\equiv 1\,(d)} u^k \frac{\prod_{j=1}^{\mathrm{N}}(kq_j-1)_{\lceil kq_j\rceil - 1}}{(k-1)!}\\ \mathrm{G} &= \mathrm{G}_{\mathrm{FJRW}}(s^0, u; \lambda)\end{aligned}$$

$$
\begin{aligned}
&= s^0 \mathrm{F}_{\mathrm{FJRW}}(u)\phi_0 - \sum_{k\geq 2:\sum_{j=1}^{\mathrm{N}}\langle q_j(k-1)\rangle=1} u^k \frac{\prod_{j=1}^{\mathrm{N}}(kq_j-1)_{\lceil kq_j\rceil -1}}{(k-1)!}\phi_{k-1}\\
&\quad + \lambda \sum_{k\geq d+1:k\equiv 1(d)} u^k \left(\sum_{i=1}^{\mathrm{N}} \sum_{0<a<kq_i,\langle a\rangle=\langle kq_i\rangle} \frac{q_i}{a}\right)\frac{\prod_{j=1}^{\mathrm{N}}(kq_j-1)_{\lceil kq_j\rceil -1}}{(k-1)!}\phi_0
\end{aligned}
$$

and in the GW side, we have:

$$
\begin{aligned}
\mathrm{F} &= \mathrm{F}_{\mathrm{GW}}(v) = \sum_{n=0}^{\infty} v^n \frac{(dn)!}{\prod_{j=1}^{\mathrm{N}}(w_j n)!}\\
\mathrm{G} &= \mathrm{G}_{\mathrm{GW}}\big(t^0, v; \lambda\big)\\
&= t^0 \mathrm{F}_{\mathrm{GW}}(v)\mathbf{1} + \sum_{n\in\mathbf{Q}_{>0}:\langle n\rangle\in\mathfrak{F},\sum_{j=1}^{\mathrm{N}}\langle -w_j n\rangle=1} v^n \frac{(dn)!}{\prod_{j=1}^{\mathrm{N}}(w_j n)_{\lceil w_j n\rceil}}\mathbf{1}_{\langle -n\rangle}\\
&\quad + \left[\mathrm{F}_{\mathrm{GW}}(v)\log v + \sum_{n=1}^{\infty} v^n \left(\sum_{a=1}^{dn}\frac{d}{a} - \sum_{i=1}^{\mathrm{N}}\sum_{a=1}^{w_i n}\frac{w_i}{a}\right)\frac{(dn)!}{\prod_{j=1}^{\mathrm{N}}(w_j n)!}\right] p\mathbf{1}\\
&\quad + \lambda \sum_{n=1}^{\infty} v^n \left(\sum_{a=1}^{dn}\frac{1}{a}\right)\frac{(dn)!}{\prod_{j=1}^{\mathrm{N}}(w_j n)!}\mathbf{1}
\end{aligned}
$$

where $(a)_n = a(a-1)\cdots(a-n+1) = \Gamma(a+1)/\Gamma(a-n+1)$ denotes the falling factorial. We define the mirror map ς to be the $\overline{\mathrm{H}}^{\leq 2}$-valued function:

$$
\varsigma = \frac{\mathrm{G}}{\mathrm{F}}. \tag{75}
$$

The FJRW mirror map $\varsigma_{\mathrm{FJRW}}$ is defined over $\{|u| < v_{\mathrm{c}}^{-1/d}\} \times \mathbf{C}_\lambda$ and the GW mirror map ς_{GW} is defined over $\{|v| < v_{\mathrm{c}}\} \times \mathbf{C}_\lambda$. The following mirror theorem gives a refinement of Theorem 3.10 and [17, 19], namely, the special case $\nu = 0$ corresponds to the original mirror theorem (see (**78**) in the proof). A similar refinement was given in [43, Theorem 4.6] for GW theory of complete intersections in toric orbifolds.

Theorem **5.12.** — *In both FJRW and GW theories, there exist* $\overline{\mathrm{H}}$*-valued complex analytic functions* $\Upsilon^{\mathrm{tw},\nu}$*,* $\nu \in \mathrm{B}$ *defined on an open set* $\widehat{\mathrm{U}} \times \mathbf{C}_z \times \mathbf{C}_\lambda \subset \widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$ *such that*

$$
\mathrm{L}^{\mathrm{tw}}\big(\varsigma(x;\lambda), z; \lambda\big)\mathrm{I}^{\mathrm{tw},\nu}(x, z; \lambda) = z\Upsilon^{\mathrm{tw},\nu}(x, z; \lambda). \tag{76}
$$

Here ς *denotes the mirror map* (**75**) *in each theory. The fundamental solution* $\mathrm{L}(\varsigma(x;\lambda), z; \lambda)$ *is also analytic over* $\widehat{\mathrm{U}} \times \mathbf{C}_z^\times \times \mathbf{C}_\lambda$*. The open subset* $\widehat{\mathrm{U}} \subset \widehat{\mathcal{M}}$ *is of the form* $\{|u| < \epsilon\}$ *for FJRW theory and is of the form* $\{|v| < \epsilon\}$ *for GW theory. For* $\nu = 0$*, we have* $\Upsilon^{\mathrm{tw},0} = \mathrm{F}\cdot \mathrm{T}_0$ *where* F *is the function appearing in* (**74**)*.*

Proof. — First we discuss the case of FJRW theory. By Theorem 3.10, we have

$$\mathrm{F}(u)\mathrm{J}^{\mathrm{tw}}\big(\varsigma\big(s^0, u; \lambda\big), z; \lambda\big) = \mathrm{I}^{\mathrm{tw},0}\big(s^0, u, z; \lambda\big) \quad \text{for } \; s^0 = 0 \tag{77}$$

where the subscripts "FJRW" are omitted. We have $\varsigma(s^0, u; \lambda) = s^0\mathrm{T}_0 + \varsigma(0, u; \lambda)$ and $\mathrm{I}^{\mathrm{tw},0}(s^0, z; \lambda) = e^{s^0/z}\mathrm{I}^{\mathrm{tw},0}(0, z; \lambda)$. By the string equation for the twisted invariants (see Section 3.2.2), we have

$$\mathrm{J}^{\mathrm{tw}}\big(s^0\mathrm{T}_0 + \varsigma(0, u; \lambda), z; \lambda\big) = e^{s^0/z}\mathrm{J}^{\mathrm{tw}}\big(\varsigma(0, u; \lambda), z; \lambda\big).$$

Hence (**77**) holds for arbitrary s^0. Therefore, by Proposition 3.15, we have

$$\mathrm{L}^{\mathrm{tw}}\big(\varsigma\big(s^0, u; \lambda\big), z; \lambda\big)\mathrm{I}^{\mathrm{tw},0}\big(s^0, u, z; \lambda\big) = z\mathrm{F}(u)\mathrm{T}_0. \tag{78}$$

This shows that one can take $\Upsilon^{\mathrm{tw},0}(s^0, u, z; \lambda) = \mathrm{F}(u)\mathrm{T}_0$. The other $\Upsilon^{\mathrm{tw},\nu}$'s are obtained from this by differentiation. To see this, we use the fact that the generalized I-functions satisfy (**71**) and that we have by Proposition 3.12

$$\big(\varsigma^*\nabla^{\mathrm{tw}}_{z\mathrm{D}_u}\big) \circ \mathrm{L}^{\mathrm{tw}}\big(\varsigma\big(s^0, u; \lambda\big), z; \lambda\big) = \mathrm{L}^{\mathrm{tw}}\big(\varsigma\big(s^0, u; \lambda\big), z; \lambda\big) \circ z\mathrm{D}_u,$$

where $\varsigma^*\nabla^{\mathrm{tw}}_{z\mathrm{D}_u} = z\mathrm{D}_u + (\mathrm{D}_u\varsigma(s^0, u; \lambda))\bullet^{\mathrm{tw}}$. For example, one obtains $z\Upsilon^{\mathrm{tw},e_i}$ as

$$\begin{aligned}
&\mathrm{L}^{\mathrm{tw}}\big(\varsigma\big(s^0, u; \lambda\big), z; \lambda\big)\mathrm{I}^{\mathrm{tw},e_i}\big(s^0, u, z; \lambda\big)\\
&\quad = \mathrm{L}^{\mathrm{tw}}\big(\varsigma\big(t^0, u; \lambda\big), z; \lambda\big)(-q_iz\mathrm{D}_u - q_i\lambda) \cdot \mathrm{I}^{\mathrm{tw},0}\big(s^0, u, z; \lambda\big)\\
&\quad = \big(-q_i\varsigma^*\nabla^{\mathrm{tw}}_{z\mathrm{D}_u} - q_i\lambda\big)\big(z\mathrm{F}(u)\mathrm{T}_0\big).
\end{aligned}$$

To obtain $\Upsilon^{\mathrm{tw},\nu}$ for a general ν, we use the following differential operator:

$$\mathrm{P}_\nu(z\mathrm{D}_u) = v^k \prod_{b=1}^{\nu_0+kd}(-z\mathrm{D}_u + bz) \cdot \prod_{i=1}^{\mathrm{N}} \frac{\prod_{b=-\infty}^{\nu_i-kw_i-1}(-q_iz\mathrm{D}_u - q_i\lambda - bz)}{\prod_{b=-\infty}^{-1}(-q_iz\mathrm{D}_u - q_i\lambda - bz)}$$

where k is an integer such that $\nu_0 + kd \geq 0$. When $\nu_i - kw_i < 0$ for some i, we expand the factor $(-q_iz\mathrm{D}_u - q_i\lambda - bz)^{-1}$ in the λ^{-1}-series

$$\sum_{n=0}^{\infty}(-q_i\lambda)^{-n-1}(bz + q_iz\mathrm{D}_u)^n.$$

Then we have $\mathrm{P}_\nu(z\mathrm{D}_u)\mathrm{I}^{\mathrm{tw},0} = \mathrm{I}^{\mathrm{tw},\nu}$. By applying $\mathrm{P}_\nu(\varsigma^*\nabla_{z\mathrm{D}_u})$ to (**78**), one obtains (**76**) with $\Upsilon^{\mathrm{tw},\nu}(s^0, u; \lambda) = \mathrm{P}_\nu(\varsigma^*\nabla_{z\mathrm{D}_u})\mathrm{F}(u)\mathrm{T}_0$. Note that this expression makes sense as an element of $\overline{\mathrm{H}} \otimes \mathbf{C}[z]((\lambda^{-1}))[\![s^0, u]\!]$. This is because $\varsigma^*\nabla_{z\mathrm{D}_u} = z\mathrm{D}_u + (\mathrm{D}_u\varsigma)\bullet^{\mathrm{tw}} = z\mathrm{D}_u + \mathrm{O}(u)$ (note that $\varsigma(s^0, u) = s^0\phi_0 - u\phi_1 + \mathrm{O}(u^2)$). A posteriori, we know that $\Upsilon^{\mathrm{tw},\nu}$ belongs to $\overline{\mathrm{H}} \otimes \mathbf{C}[z, \lambda][\![s^0, u]\!]$ by (**76**) since L^{tw}, I^{tw} and ς are regular at $\lambda = 0$.

Next we show the analyticity of $\Upsilon^{\mathrm{tw},\nu}$ and L^{tw}. It suffices to show the analyticity of L^{tw}. By (**76**) we have

$$
\begin{aligned}
z^{-1}\begin{bmatrix} | & & | \\ u^{-1}\mathrm{I}^{\mathrm{tw},\nu(0)} & \dots & u^{-d}\mathrm{I}^{\mathrm{tw},\nu(d-1)} \\ | & & | \end{bmatrix} \\
= \left(\mathrm{L}^{\mathrm{tw}}\right)^{-1}\begin{bmatrix} | & & | \\ u^{-1}\Upsilon^{\mathrm{tw},\nu(0)} & \dots & u^{-d}\Upsilon^{\mathrm{tw},\nu(d-1)} \\ | & & | \end{bmatrix}
\end{aligned} \tag{79}
$$

where $\nu(l)$ is as in Lemma 5.4. Using the basis $\phi_0, \dots, \phi_{d-1}$ of $\overline{\mathrm{H}}$, one can view this as an equality of (d, d) matrices. Write $\gamma(z)$ for the left-hand side. It is invertible near $u = 0$ because of the asymptotics (**72**). Hence $\mathrm{S}^1 \ni z \mapsto \gamma(z)$ defines an element of the loop group $\mathrm{LGL}_d = \mathrm{C}^\infty(\mathrm{S}^1, \mathrm{GL}_d)$. As observed by Coates-Givental [21] and Guest [32], we can regard (**79**) as a Birkhoff factorization of $\gamma(z)$ because $(\mathrm{L}^{\mathrm{tw}})^{-1} = \mathrm{id} + \mathrm{O}(z^{-1})$ and $\Upsilon^{\mathrm{tw},\nu}$ is regular at $z = 0$. Birkhoff's theorem [58] says that the multiplication map $\mathrm{L}^-_1\mathrm{GL}_d \times \mathrm{L}^+\mathrm{GL}_d \to \mathrm{LGL}_d$ is an isomorphism onto an open and dense subset called the "big cell". Here $\mathrm{L}^-_1\mathrm{GL}_d$ is the subgroup consisting of the boundary values of holomorphic maps $\gamma_- \colon \{z \in \mathbf{C} \cup \{\infty\} \mid |z| < 1\} \to \mathrm{GL}_d$ satisfying $\gamma_-(\infty) = \mathrm{id}$ and $\mathrm{L}^+\mathrm{GL}_d$ is the subgroup consisting of the boundary values of holomorphic maps $\gamma_+ \colon \{z \in \mathbf{C} \mid |z| > 1\} \to \mathrm{GL}_d$. The asymptotics (**72**) ensures that $\gamma(z)$ is in the big cell for $|u| < \epsilon$ and $|\lambda| \leq 1$ for sufficiently small $\epsilon > 0$. Hence its Birkhoff factor $\gamma_-(z) = \mathrm{L}^{\mathrm{tw}}(\varsigma(s^0, u; \lambda), z; \lambda)^{-1}$ is analytic on the region $\{|u| < \epsilon, |z| > 1, |\lambda| < 1\}$. The homogeneity of $(\mathrm{L}^{\mathrm{tw}})^{-1}$ implies that it is in fact analytic on $\{|u| < \epsilon, z \in \mathbf{C}^\times, \lambda \in \mathbf{C}\}$.

For GW theory, the theorem follows from the proof of [43, Theorem 4.6]. When $\langle l/d \rangle \in \mathfrak{F}$, the function $\mathrm{I}^{\mathrm{tw},\nu(l)}_{\mathrm{GW}}$ coincides with $zv^{-l/d}\mathbf{I}^{\langle l/d\rangle}_\lambda|_{\mathrm{Q}=1}$ there and we can take $\Upsilon^{\mathrm{tw},\nu(l)} = v^{l/d}\widetilde{\Upsilon}_{\langle l/d\rangle}|_{\mathrm{Q}=1}$ in the notation of *loc. cit.* We can get the other $\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}$ from $\mathrm{I}^{\mathrm{tw},\nu(l)}_{\mathrm{GW}}$, $\langle l/d \rangle \in \mathfrak{F}$ by applying differential operators in $\widehat{\mathcal{R}}^{\mathrm{tw}}$, so the other $\Upsilon^{\mathrm{tw},\nu}$ as well. ☐

Let $\widehat{\mathrm{U}}_{\mathrm{FJRW}} = \{|u| < \epsilon\}$, $\widehat{\mathrm{U}}_{\mathrm{GW}} = \{|v| < \epsilon\}$ be sufficiently small open subsets of $\widehat{\mathcal{M}}$ as in Theorem 5.12. The following corollary gives a twisted version of Theorem 2.23.

Corollary **5.13.** — *Via the $\widehat{\mathcal{R}}^{\mathrm{tw}}$-module $\widehat{\mathcal{F}}^{\mathrm{tw}}$ over $\widehat{\mathcal{M}}$, the $e_{\mathbf{C}^\times}$-twisted quantum connections ∇^{tw} of FJRW theory and of GW theory are analytically continued to each other. More precisely, we have a local trivialization of $\widehat{\mathcal{F}}^{\mathrm{tw}}$ over $\widehat{\mathrm{U}}_\heartsuit$*

$$
\begin{aligned}
\mathrm{Mir}_\heartsuit \colon \widehat{\mathcal{F}}^{\mathrm{tw}}|_{\widehat{\mathrm{U}}_\heartsuit \times \mathbf{C}_z \times \mathbf{C}_\lambda} &\cong \overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}_{\widehat{\mathrm{U}}_\heartsuit \times \mathbf{C}_z \times \mathbf{C}_\lambda}, \qquad \heartsuit = \mathrm{FJRW} \text{ or } \mathrm{GW} \\
\Delta_\nu &\longmapsto \Upsilon^{\mathrm{tw},\nu}_\heartsuit
\end{aligned}
$$

such that, under the trivialization, the action of $\widehat{\mathcal{R}}^{\mathrm{tw}}$ is given by the $e_{\mathbf{C}^\times}$-twisted quantum connection

$$zD \longmapsto \varsigma_{\heartsuit}^{*}\nabla^{\mathrm{tw},\heartsuit}_{zD}, \qquad D \textit{ is a vector field on } \widehat{U}_{\heartsuit},$$

where $\varsigma_{\heartsuit}\colon \widehat{U}_{\heartsuit} \times \mathbf{C}_{\lambda} \to \overline{H}^{\leq 2}_{\heartsuit}$ *is the mirror map* (**75**) *in the respective theory.*

Proof. — We omit the subscript "FJRW" or "GW" throughout the proof. By Proposition 5.11, the generalized twisted I-functions define an $\overline{H}$-valued solution:

$$(\mathbf{80}) \qquad \widehat{\mathcal{F}}^{\mathrm{tw}}|_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}} \to \overline{H} \times \mathcal{O}_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}}, \qquad \Delta_{\nu} \mapsto z^{-1}I^{\mathrm{tw},\nu},$$

which is an isomorphism (see the asymptotics (**72**), (**73**)). On the other hand, the twisted quantum connection $\varsigma^{*}\nabla^{\mathrm{tw}}$ also has an $\overline{H}$-valued solution (Proposition 3.12)

$$L^{\mathrm{tw}}\big(\varsigma(\cdot\,;\lambda), z; \lambda\big)^{-1}\colon \big(\overline{H} \otimes \mathcal{O}_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}}, \varsigma^{*}\nabla^{\mathrm{tw}}\big) \to \overline{H} \otimes \mathcal{O}_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}},$$

which sends $\Upsilon^{\mathrm{tw},\nu}$ to $z^{-1}I^{\mathrm{tw},\nu}$ by Theorem 5.12. This is also an isomorphism. Therefore we have an isomorphism $\mathrm{Mir}\colon \widehat{\mathcal{F}}^{\mathrm{tw}}|_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}} \cong \overline{H} \otimes \mathcal{O}_{\widehat{U}\times\mathbf{C}_{z}^{\times}\times\mathbf{C}_{\lambda}}$ such that $\mathrm{Mir}(\Delta_{\nu}) = \Upsilon^{\mathrm{tw},\nu}$. It extends across $z = 0$ as $\Upsilon^{\mathrm{tw},\nu}$ is regular at $z = 0$. Now it suffices to show that $\Upsilon^{\mathrm{tw},\nu}$, $\nu \in B$ generate $\overline{H}$ along $z = 0$. In the case of FJRW theory, this follows from the fact that the factor $[u^{-1}\Upsilon^{\mathrm{tw},\nu(0)}, \dots, u^{-d}\Upsilon^{\mathrm{tw},\nu(d-1)}]$ in the Birkhoff factorization (**79**) is invertible at $z = 0$. The discussion is similar for GW theory. □

Remark **5.14.** — Mann-Mignon [50, Theorem 1.2] described explicitly the twisted quantum D-module (with $\lambda = 0$) for a smooth nef complete intersection in a toric manifold.

5.3. *Analytic continuation* $\mathbf{U}^{\mathrm{tw}}$ *revisited.*

Lemma **5.15.** — *The submodule* $\mathcal{R}^{\mathrm{tw}}\Delta_0$ *of* $\mathcal{F}^{\mathrm{tw}}$ *coincides with* $\mathcal{F}^{\mathrm{tw}}$ *at the generic point on* $\mathcal{M}^{\circ} \times \mathbf{C}_{z} \times \mathbf{C}_{\lambda}$.

Proof. — In view of the isomorphism (**80**), it suffices to show that $\prod_{b=1}^{l}(zD_u - bz)z^{-1}I^{\mathrm{tw},0}_{\mathrm{FJRW}}$, $l = 0, \dots, d-1$ form a basis of $\overline{H} = H_{\mathrm{ext}}$ for generic (z, λ) and sufficiently small $|u|$. This follows from the asymptotics near $u = 0$:

$$\begin{aligned}&\prod_{b=1}^{l}(zD_u - bz)z^{-1}I^{\mathrm{tw},0}_{\mathrm{FJRW}}\\ &\qquad \sim e^{s^0/z}u^{l+1}(-1)^{l}\Bigg(\prod_{j=1}^{N}\prod_{\substack{0<b<(l+1)q_j\\ \langle b\rangle=\langle(l+1)q_j\rangle}}(-q_j\lambda - bz)\Bigg)\phi_l + O\big(u^{l+2}\big).\end{aligned}$$

□

We calculated in Proposition 4.12 a linear transformation $\mathbf{U}^{\mathrm{tw}}_{l}$ (**64**)

$$\mathbf{U}^{\mathrm{tw}}_{l}\colon H_{\mathrm{ext}} \otimes \mathcal{O}_{\Delta_{\xi}} \to H_{\mathrm{CR}}\big(\mathbf{P}(\underline{w})\big) \otimes \mathcal{O}_{\Delta_{\xi}}$$

from analytic continuation of the "$\mathfrak{H}$-functions", where $\Delta_\xi = \{|\xi| < \varepsilon\}$ denotes a sufficiently small disc in the $\xi := (\lambda/z)$-plane. We give an interpretation of $\mathbf{U}_l^{\mathrm{tw}}$ as analytic continuation of flat sections of the global D-module $\widehat{\mathcal{F}}^{\mathrm{tw}}$. Using the trivialization $\mathrm{Mir}_\heartsuit$ in Corollary 5.13, we define a flat section $\mathfrak{f}_\heartsuit^{\mathrm{tw}}(\alpha)$ of $\widehat{\mathcal{F}}^{\mathrm{tw}}$ over $\widehat{\mathrm{U}}_\heartsuit \times \{(z,\lambda) \in \mathbf{C}^\times \times \mathbf{C} \mid |\xi| = |\lambda/z| < \varepsilon\}$ parametrized by $\alpha \in \overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}(\Delta_\xi)$:

$$\mathfrak{f}_\heartsuit^{\mathrm{tw}}(\alpha)(x,z;\lambda) := \mathrm{L}_\heartsuit^{\mathrm{tw}}\big(\varsigma_\heartsuit(x;\lambda),z;\lambda\big) z^{-\mathsf{Gr}}\widehat{\Gamma}_\heartsuit^{\mathrm{tw}}\big((2\pi\mathtt{i})^{\frac{\deg_0}{2}}\alpha\big), \quad \alpha \in \overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}(\Delta_\xi) \tag{81}$$

where $\mathrm{L}_\heartsuit^{\mathrm{tw}}$, $\widehat{\Gamma}_\heartsuit^{\mathrm{tw}}$ are the fundamental solution and the twisted Gamma class (Section 4.3.3) in each theory and $\heartsuit = \mathrm{GW}$ or FJRW. We extend the $\mathfrak{H}$-functions in the s^0- or t^0-direction as follows:

$$\mathfrak{H}_{\mathrm{FJRW}}^{\mathrm{tw}}\big((s^0,u),z;\lambda\big) = e^{s^0/z}\mathfrak{H}_{\mathrm{FJRW}}^{\mathrm{tw}}(u,z;\lambda)$$
$$\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}}\big((t^0,v),z;\lambda\big) = e^{t^0/z}\mathfrak{H}_{\mathrm{GW}}^{\mathrm{tw}}(v,z;\lambda)$$

so that we have $\mathrm{I}_\heartsuit^{\mathrm{tw}} = z^{-\mathsf{Gr}}\widehat{\Gamma}_\heartsuit^{\mathrm{tw}}((2\pi\mathtt{i})^{\frac{\deg_0}{2}}\mathfrak{H}_\heartsuit^{\mathrm{tw}})$ (cf. (**59**)). By this relation and Theorem 5.12, we have

$$\mathrm{Mir}_\heartsuit(z\Delta_0) = z\Upsilon_\heartsuit^{\mathrm{tw},0}(x,z;\lambda) = \mathfrak{f}_\heartsuit^{\mathrm{tw}}\big(\mathfrak{H}_\heartsuit^{\mathrm{tw}}(x,z;\lambda)\big)(x,z;\lambda). \tag{82}$$

Namely *the $\mathfrak{H}$-function represents the section $z\Delta^0$ in the flat frame* $\mathfrak{f}_\heartsuit^{\mathrm{tw}}$. The relationships between $\Upsilon_\heartsuit^{\mathrm{tw},0}$, $z\Delta_0$, $\mathrm{I}_\heartsuit^{\mathrm{tw}}$, $\mathfrak{H}_\heartsuit^{\mathrm{tw}}$ are given in the following diagram:

$$\begin{tikzcd}
\mathcal{F}^{\mathrm{tw}}|_{\mathcal{B}_\heartsuit} \arrow[r, "\mathrm{Mir}_\heartsuit"] & (\overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}_{\mathcal{B}_\heartsuit}, \varsigma_\heartsuit^*\nabla^{\mathrm{tw}}) & (\overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}_{\mathcal{B}_\heartsuit}, \mathsf{d}) \arrow[l, "\varsigma_\heartsuit^*\mathrm{L}_\heartsuit^{\mathrm{tw}}"'] & (\overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}_{\mathcal{B}_\heartsuit}, \mathsf{d}) \arrow[l, "z^{-\mathsf{Gr}}\widehat{\Gamma}_\heartsuit^{\mathrm{tw}}(2\pi\mathtt{i})^{\frac{\deg_0}{2}}"'] \arrow[ll, bend left, "\mathfrak{f}_\heartsuit^{\mathrm{tw}}"] \\
z\Delta_0 \arrow[r, mapsto] & z\Upsilon_\heartsuit^{\mathrm{tw},0} & \mathrm{I}_\heartsuit^{\mathrm{tw}} \arrow[l, mapsto] & \mathfrak{H}_\heartsuit^{\mathrm{tw}} \arrow[l, mapsto]
\end{tikzcd}$$

where $\mathcal{B}_\heartsuit = \widehat{\mathrm{U}}_\heartsuit \times \{(z,\lambda) \in \mathbf{C}^\times \times \mathbf{C} \mid |\lambda/z| < \epsilon\}$ and d stands for the trivial connection (all the connections here are only defined in the $\widehat{\mathrm{U}}_\heartsuit$-direction).

Recall the path γ_l in $\widetilde{\mathcal{M}}^\circ$ (Section 4.3.4, Figure 3) defined for each integer l. It can be lifted to a path $\hat{\gamma}_l$ in $\widehat{\mathcal{M}}$ starting from the GW base point $\log v \ll 0$, $t^0 = 0$ and ending at the FJRW base point $\log u \ll 0$, $s^0 = 0$. The homotopy type of the lift $\hat{\gamma}_l$ is unambiguous. For convenience, we take the following lift $\hat{\gamma}_l$:

$$\begin{aligned}
&\big(\log v \ll 0,\, t^0 = 0\big) \\
&\quad \rightsquigarrow \big(\log u \ll 0,\, t^0 = 0\big) = \big(\log u \ll 0,\, s^0 = \lambda \log u\big) && (\text{moving along } \gamma_l) \\
&\quad \rightsquigarrow \big(\log u \ll 0,\, s^0 = 0\big) && (\text{shifting } s_0).
\end{aligned}$$

Because the shift of s^0 has an effect of the multiplication by the factor $u^{\lambda/z} = u^{\xi}$ on the $\mathfrak{H}$-function $\mathfrak{H}_{\mathrm{FJRW}}$, we have from Proposition 4.12 that

$$(\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}})_{\mathrm{continued}} = \mathbf{U}^{\mathrm{tw}}_{l}(\mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}) \tag{83}$$

where the left-hand side now denotes the analytic continuation of $\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}}$ along $\hat{\gamma}_l$.

Proposition **5.16.** — *Along the path $\hat{\gamma}_l^{-1}$, the FJRW flat section $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{FJRW}}(\alpha)$, $\alpha \in \mathrm{H}_{\mathrm{ext}}$ is analytically continued to the GW flat section $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{GW}}(\mathbf{U}^{\mathrm{tw}}_{l}\alpha)$. Here the values of z and λ are fixed during the analytic continuation and chosen so that $|\xi| = |\lambda/z|$ is sufficiently small.*

Proof. — Note that $\widehat{\Gamma}^{\mathrm{tw}}_{\heartsuit}$ is invertible for sufficiently small $|\xi|$. Therefore $\{\mathfrak{f}^{\mathrm{tw}}_{\heartsuit}(\mathrm{T}_i)\}_{i=0}^{d-1}$ forms a basis of flat sections for sufficiently small $\xi = \lambda/z$. Hence for a fixed such (z, λ), there exists an invertible linear transformation $\mathbf{V}_l \colon \mathrm{H}_{\mathrm{ext}} \to \mathrm{H}_{\mathrm{CR}}(\mathbf{P}(\underline{w}))$ such that $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{FJRW}}(\mathrm{T}_i)$ is analytically continued to $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{GW}}(\mathbf{V}_l\mathrm{T}_i)$ along $\hat{\gamma}_l^{-1}$. Because $\mathfrak{f}^{\mathrm{tw}}_{\heartsuit}(\mathfrak{H}^{\mathrm{tw}}_{\heartsuit}) = \mathrm{Mir}_{\heartsuit}(z\Delta_0)$ (**82**) and $z\Delta_0$ is a global section of $\widehat{\mathcal{F}}^{\mathrm{tw}}$, we have

$$(\mathfrak{H}^{\mathrm{tw}}_{\mathrm{GW}})_{\mathrm{continued}} = \mathbf{V}_l(\mathfrak{H}^{\mathrm{tw}}_{\mathrm{FJRW}}).$$

Because $z\Delta_0$ is a generator of $\widehat{\mathcal{F}}^{\mathrm{tw}}$ at the generic point (Lemma 5.15), this relation uniquely determines $\mathbf{V}_l$ for a generic (z, λ). By (**83**), we know that $\mathbf{V}_l = \mathbf{U}^{\mathrm{tw}}_{l}$. ☐

5.4. *The non-equivariant limit and its reduction.* — Here we prove Theorem 2.23. By taking the non-equivariant limit $\lambda = 0$ in Corollary 5.13, we obtain analytic continuation between e-twisted quantum connections. (Recall that e stands for the non-equivariant Euler class.) We shall show that it reduces to analytic continuation between ambient and narrow part quantum D-modules. This reduction was described more explicitly in terms of the Picard-Fuchs ideal in a recent paper of Mann-Mignon [50, Theorem 1.2] for the quantum cohomology of a smooth nef complete intersection in a toric manifold.

(Step 0). — Note that $\widetilde{\mathcal{M}}^{\circ} \times \mathbf{C}_z$ is contained in $\widehat{\mathcal{M}} \times \mathbf{C}_z \times \mathbf{C}_\lambda$ as the locus $\{\lambda = t^0 = 0\} = \{\lambda = s^0 = 0\}$. We consider the restriction

$$\widetilde{\mathcal{G}} := \widehat{\mathcal{F}}^{\mathrm{tw}}|_{\lambda = t^0 = 0}$$

of $\widehat{\mathcal{F}}^{\mathrm{tw}}$ to $\widetilde{\mathcal{M}}^{\circ} \times \mathbf{C}_z$. This is also identified with the pull-back of

$$\mathcal{G} := \mathcal{F}^{\mathrm{tw}}|_{\lambda = 0}$$

by $\widetilde{\mathcal{M}}^{\circ} \times \mathbf{C}_z \to \mathcal{M}^{\circ} \times \mathbf{C}_z$. Let $\widetilde{\mathrm{U}}_{\heartsuit}$ denote the open subset of $\widetilde{\mathcal{M}}^{\circ}$ given by $\widetilde{\mathrm{U}}_{\mathrm{GW}} = \{|v| < \epsilon\}$ or $\widetilde{\mathrm{U}}_{\mathrm{FJRW}} = \{|u| < \epsilon\}$ where ϵ is the same as in Corollary 5.13. Over $\widetilde{\mathrm{U}}_{\heartsuit} \times \mathbf{C}_z$, $\widetilde{\mathcal{G}}$ is identified with the e-twisted quantum connection ∇^{tw} on $\overline{\mathrm{H}}_{\heartsuit} \times (\widetilde{\mathrm{U}}_{\heartsuit} \times \mathbf{C}_z) \to \widetilde{\mathrm{U}}_{\heartsuit} \times \mathbf{C}_z$

by Corollary 5.13. By Proposition 3.11, under the natural projection $\mathrm{pr}\colon \overline{\mathrm{H}} \twoheadrightarrow \mathrm{H}'$, the e-twisted quantum connection projects to the quantum connection of the respective theory:

$$\begin{tikzcd}
\widetilde{\mathcal{G}}|_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z} \arrow[r, "\cong"] & (\overline{\mathrm{H}}_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, \varsigma_\heartsuit^* \nabla^{\mathrm{tw}}) \arrow[d, "\mathrm{pr}"] \\
& (\mathrm{H}'_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, (\mathrm{pr} \circ \varsigma_\heartsuit)^* \nabla),
\end{tikzcd} \tag{84}$$

where $\varsigma_\heartsuit \colon \widetilde{\mathrm{U}}_\heartsuit \to \overline{\mathrm{H}}^2_\heartsuit$ denotes the mirror map (**75**) restricted to $\lambda = t^0 = 0$. Here the meromorphic flat connection ∇ on $\widetilde{\mathcal{G}}$ (or $\mathcal{G}$) is given by the action of $z\mathrm{D}_v \in \mathcal{R}^{\mathrm{tw}}|_{\lambda=0}$, i.e. we define $\nabla_{\mathrm{D}_v} := z^{-1}$(the action of $z\mathrm{D}_v$) on $\widetilde{\mathcal{G}}$ (or $\mathcal{G}$).

(Step 1). — Let $\mathrm{U}_\heartsuit \subset \mathcal{M}^\circ$ be the image of $\widetilde{\mathrm{U}}_\heartsuit$ under the projection $\widetilde{\mathcal{M}}^\circ \to \mathcal{M}^\circ$. We show that the diagram (**84**) descends to the quotient $\widetilde{\mathrm{U}}_\heartsuit \twoheadrightarrow \mathrm{U}_\heartsuit$. First notice that the Galois symmetry in Propositions 2.14, 2.16 extends to the twisted theory. The map $\mathrm{G}\colon \mathrm{H}' \to \mathrm{H}'$ there is extended to $\overline{\mathrm{H}}$ as

$$\mathrm{G}(\phi_k) = e^{-2\pi \mathtt{i} k/d} \phi_k \qquad \text{for FJRW theory;}$$
$$\mathrm{G}(\mathbf{1}_f) = e^{2\pi \mathtt{i} f} \mathbf{1}_f - 2\pi \mathtt{i} p \qquad \text{for GW theory.}$$

Then the conclusions of Propositions 2.14, 2.16 hold for this G (except that we do not have the connection in the z-direction in the twisted theory). The proof is similar. This shows that the fundamental solution L^{tw} in the twisted theory (see Proposition 3.12) has the following symmetry:

$$e^{-2\pi \mathtt{i}/d} \mathrm{G} \circ \mathrm{L}^{\mathrm{tw}}_{\mathrm{FJRW}}\big(\mathrm{G}^{-1}(t), z; \lambda\big) = \mathrm{L}^{\mathrm{tw}}_{\mathrm{FJRW}}(t, z; \lambda) \circ e^{-2\pi \mathtt{i}/d} \mathrm{G}$$
$$\mathsf{d}\mathrm{G} \circ \mathrm{L}^{\mathrm{tw}}_{\mathrm{GW}}\big(\mathrm{G}^{-1}(t), z; \lambda\big) = \mathrm{L}^{\mathrm{tw}}_{\mathrm{GW}}(t, z; \lambda) \circ e^{-2\pi \mathtt{i} p/z} \mathsf{d}\mathrm{G}.$$

On the other hand, the deck transformation of $\widetilde{\mathrm{U}}_\heartsuit \twoheadrightarrow \mathrm{U}_\heartsuit$ acts on $\mathrm{I}^{\mathrm{tw},\nu}$ as

$$e^{-2\pi \mathtt{i}/d} \mathrm{G}\Big(\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}\big(\log u + (2\pi \mathtt{i}/d), z\big)\Big|_{s^0=\lambda=0}\Big) = \mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{FJRW}}(\log u, z)\Big|_{s^0=\lambda=0}$$
$$e^{-2\pi \mathtt{i} p/z} \mathsf{d}\mathrm{G}\Big(\mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}(\log v + 2\pi \mathtt{i}, z)\Big|_{t^0=\lambda=0}\Big) = \mathrm{I}^{\mathrm{tw},\nu}_{\mathrm{GW}}(\log v, z)\Big|_{t^0=\lambda=0}.$$

Hence the mirror maps (with $t^0 = \lambda = 0$) satisfy

$$\begin{aligned} \mathrm{G}\big(\varsigma_{\mathrm{FJRW}}\big(\log u + (2\pi \mathtt{i}/d)\big)\big) &= \varsigma_{\mathrm{FJRW}}(\log u), \\ \mathrm{G}\big(\varsigma_{\mathrm{GW}}(\log v + 2\pi \mathtt{i})\big) &= \varsigma_{\mathrm{GW}}(\log v). \end{aligned} \tag{85}$$

This shows that the deck transformation of $\widetilde{\mathrm{U}}_\heartsuit$ is conjugate to the Galois action on $\overline{\mathrm{H}}^2$ via the mirror maps. By the relation (**76**) and the above calculations, we find that (again over the locus $\lambda = t^0 = 0$)

$$\begin{aligned} e^{-2\pi\mathtt{i}/d}\mathrm{G}\big(\Upsilon^{\mathrm{tw},\nu}_{\mathrm{FJRW}}(\log u + (2\pi\mathtt{i})/d, z)\big) &= \Upsilon^{\mathrm{tw},\nu}_{\mathrm{FJRW}}(\log u, z) \\ \mathsf{d}\mathrm{G}\big(\Upsilon^{\mathrm{tw},\nu}_{\mathrm{GW}}(\log v + 2\pi\mathtt{i}, z)\big) &= \Upsilon^{\mathrm{tw},\nu}_{\mathrm{GW}}(\log v, z). \end{aligned}$$

This shows that the induced Galois symmetry on the sheaf $(\overline{\mathrm{H}} \times \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, \varsigma_\heartsuit^*\nabla^{\mathrm{tw}})$ is compatible with the deck transformation on $\widetilde{\mathcal{G}}|_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}$ because the deck-transformation-invariant section $\Delta_\nu \in \widetilde{\mathcal{G}}$ corresponds to $\Upsilon^{\mathrm{tw},\nu}$. Moreover the projection $\mathrm{pr}\colon \overline{\mathrm{H}} \to \mathrm{H}'$ is compatible with the Galois action, so the diagram (**84**) descends to

$$\begin{tikzcd} \mathcal{G}|_{\mathrm{U}_\heartsuit \times \mathbf{C}_z} \arrow[r, "\cong"] & \big(\overline{\mathrm{H}}_\heartsuit \times \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, \varsigma_\heartsuit^*\nabla^{\mathrm{tw}}\big)/\langle \mathrm{G}\rangle \arrow[d, "\mathrm{pr}"] \\ & \big(\mathrm{H}'_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, (\mathrm{pr}\circ\varsigma_\heartsuit)^*\nabla\big)/\langle \mathrm{G}\rangle. \end{tikzcd} \tag{86}$$

Notice that the bundle in the second line is the pull-back of the quantum D-module $(\mathrm{F}, \nabla)/\langle \mathrm{G}\rangle$ in Definition 2.10 by the mirror map

$$\tau_\heartsuit := [\mathrm{pr}\circ\varsigma_\heartsuit]\colon \mathrm{U}_\heartsuit \to \mathrm{H}'^{2}_\heartsuit/\langle \mathrm{G}\rangle.$$

In the diagram (**86**), we do not consider the flat connection ∇_z in the z-direction and the pairing P. However, we can introduce ∇_z for $\mathcal{G}$ and make the diagram compatible with ∇_z as follows. Recall that (the module of global sections of) $\mathcal{F}^{\mathrm{tw}}$ is $2\mathbf{Z}$-graded by $\deg u = \deg v = 0$, $\deg \Delta_\nu = 2\sum_{i=0}^{\mathrm{N}} \nu_i$, $\deg z = \deg \lambda = 2$. Thus $\mathcal{G} = \mathcal{F}^{\mathrm{tw}}|_{\lambda=0}$ is also graded. The grading defines the meromorphic flat connection ∇_z on $\mathcal{G}$ (with logarithmic poles along $z = 0$) as

$$\nabla_z \Delta_\nu = \frac{1}{z}\frac{\deg \Delta_\nu}{2}\Delta_\nu.$$

Because all the morphisms in the diagram (**86**) preserve the grading and the Euler vector field vanishes on the image of the mirror map $\mathrm{pr}\circ\varsigma_\heartsuit$, the projection $\mathrm{pr}\colon \mathcal{G}|_{\mathrm{U}_\heartsuit \times \mathbf{C}_z} \to (\tau_\heartsuit)^*(\mathrm{F}, \nabla)/\langle \mathrm{G}\rangle$ induced from the diagram (**86**) preserves the connection ∇_z as well.

(Step 2). — The diagram (**86**) defines for each $(x, z) \in \mathrm{U}_\heartsuit \times \mathbf{C}_z$ a projection $\mathcal{G}_{(x,z)} \twoheadrightarrow \mathrm{H}'$, i.e. an element of the Grassmannian $Gr(\mathcal{G}_{(x,z)})$. The kernel of the projection is flat for ∇ (including the z-direction). We show that this section of the Grassmannian bundle $Gr(\mathcal{G})$ extends globally over $\mathcal{M}^\circ \times \mathbf{C}_z$.

Recall the flat section $\mathfrak{f}^{\mathrm{tw}}_\heartsuit(\alpha)$ of the twisted theory in (**81**). When restricted to the locus $\lambda = t^0 = 0$, this defines a flat section of $\mathcal{G}$. On the other hand, we can define a flat section of the quantum D-module $(\mathrm{H}'_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z}, (\mathrm{pr}\circ\varsigma_\heartsuit)^*\nabla)$ by an analogous formula:

$$\mathfrak{f}_\heartsuit(\alpha) = \mathrm{L}_\heartsuit\big(\mathrm{pr}\circ\varsigma_\heartsuit(x), z\big) z^{-\mathsf{Gr}}\widehat{\Gamma}_\heartsuit\big((2\pi\mathtt{i})^{\frac{\deg_0}{2}}\alpha\big), \quad \alpha \in \mathrm{H}'_\heartsuit, \tag{87}$$

where $\mathrm{L}_{\heartsuit}(t,z)$ and $\widehat{\Gamma}_{\heartsuit}$ are the fundamental solution and the Gamma class in the respective theory (as appear in Definition 2.19). By Proposition 3.14 and the definitions of $\mathfrak{f}^{\mathrm{tw}}_{\heartsuit}$ and $\mathfrak{f}_{\heartsuit}$, we have

$$\mathrm{pr}\big(\mathfrak{f}^{\mathrm{tw}}_{\heartsuit}(\alpha)\big|_{\lambda=t^0=0}\big) = \mathfrak{f}_{\heartsuit}\big(\mathrm{pr}(\alpha)\big) \tag{88}$$

for $\alpha \in \overline{\mathrm{H}}$.

Lemma **5.17.** — *The section of $Gr(\mathcal{G})$ over $(\mathrm{U}_{\mathrm{GW}} \cup \mathrm{U}_{\mathrm{FJRW}}) \times \mathbf{C}_z$ given by the diagram* (**86**) *extends to* $((\mathrm{U}_{\mathrm{GW}} \cup \mathrm{U}_{\mathrm{FJRW}}) \times \mathbf{C}_z) \cup (\mathcal{M}^{\circ} \times \mathbf{C}_z^{\times})$.

Proof. — By the flat connection ∇ on $\mathcal{G}$, the section of $Gr(\mathcal{G})$ over $\mathrm{U}_{\mathrm{GW}} \times \mathbf{C}_z^{\times}$ can be extended along any path in $\mathcal{M}^{\circ} \times \mathbf{C}_z^{\times}$. We see that the given section of $Gr(\mathcal{G})|_{\mathrm{U}_{\mathrm{GW}} \times \mathbf{C}_z^{\times}}$ is analytically continued to the given section of $Gr(\mathcal{G})|_{\mathrm{U}_{\mathrm{FJRW}} \times \mathbf{C}_z^{\times}}$ along the path γ_l in Section 4.3.4. By considering the $\lambda = 0$ limit in Proposition 5.16, we know that $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{FJRW}}(\alpha)|_{\lambda=t^0=0}$ is analytically continued to $\mathfrak{f}^{\mathrm{tw}}_{\mathrm{GW}}(\mathbf{U}^{\mathrm{tw}}_l \alpha)|_{\lambda=t^0=0}$ along γ_l^{-1}, for $\alpha \in \overline{\mathrm{H}}_{\mathrm{FJRW}} = \mathrm{H}_{\mathrm{ext}}$. By (**88**), the projections of these flat sections by pr are $\mathfrak{f}_{\mathrm{FJRW}}(\mathrm{pr}(\alpha))$ and $\mathfrak{f}_{\mathrm{GW}}(\mathrm{pr}(\lim_{\lambda\to 0} \mathbf{U}^{\mathrm{tw}}_l \alpha))$. Diagrammatically:

(**89**)
$$\begin{tikzcd}
\mathfrak{f}^{\mathrm{tw}}_{\mathrm{FJRW}}(\alpha)|_{\lambda=t^0=0} \arrow[r, "\text{analytic continuation}", "\text{along } \gamma_l^{-1}"'] \arrow[d, "\mathrm{pr}"'] & \mathfrak{f}^{\mathrm{tw}}_{\mathrm{GW}}\big(\mathbf{U}^{\mathrm{tw}}_l \alpha\big)\big|_{\lambda=t^0=0} \arrow[d, "\mathrm{pr}"'] \\
\mathfrak{f}_{\mathrm{FJRW}}\big(\mathrm{pr}(\alpha)\big) & \mathfrak{f}_{\mathrm{GW}}\big(\mathbf{U}_l\, \mathrm{pr}(\alpha)\big)
\end{tikzcd}$$

Here we used the fact (Corollary 4.15) that there exists a unique operator $\mathbf{U}_l \colon \mathrm{H}'_{\mathrm{FJRW}} \to \mathrm{H}'_{\mathrm{GW}}$ such that $\mathrm{pr} \circ (\lim_{\lambda\to 0} \mathbf{U}^{\mathrm{tw}}_l) = \mathbf{U}_l \circ \mathrm{pr}$. The existence of such an operator shows that the sections pr of $Gr(\mathcal{G})|_{\mathrm{U}_{\mathrm{GW}} \times \mathbf{C}_z^{\times}}$ and $Gr(\mathcal{G})|_{\mathrm{U}_{\mathrm{FJRW}} \times \mathbf{C}_z^{\times}}$ coincide under analytic continuation along γ_l. Because this holds for all the paths γ_l with $l \in \mathbf{Z}$, the conclusion follows. ☐

Lemma **5.18.** — *The section of $Gr(\mathcal{G})$ in the previous lemma extends to* $\mathcal{M}^{\circ} \times \mathbf{C}_z$.

Proof. — The section of $Gr(\mathcal{G})$ here is flat for ∇ on $\mathcal{G}$. Therefore, the corresponding element of $Gr(\mathcal{G}_{(x,z)})$ at $(x,z) \in \mathcal{M}^{\circ} \times \mathbf{C}_z^{\times}$ can be represented by a matrix independent of z when we write it in terms of the homogeneous basis $z^{-\deg \Delta_{\nu(l)}/2} \Delta_{\nu(l)}$, $l = 0, \dots, d-1$ of $\mathcal{G}_{(x,z)}$. Therefore, via the basis $\Delta_{\nu(l)}$, $\nu = 0, \dots, d-1$, the section $\{x\} \times \mathbf{C}_z^{\times} \to Gr(\mathcal{G}|_{\{x\} \times \mathbf{C}_z^{\times}})$ can be represented by an algebraic map $\mathbf{C}_z^{\times} \to Gr(\mathbf{C}^d)$, which extends across $z = 0$ by the completeness of $Gr(\mathbf{C}^d)$. This proves the lemma. ☐

(Step 3). — The previous step shows that there exists a projection $\mathcal{G} \twoheadrightarrow \mathcal{F}$ to a locally free sheaf $\mathcal{F}$ over $\mathcal{M}^{\circ} \times \mathbf{C}_z$. The sheaf $\mathcal{F}$ is equipped with a meromorphic flat connection with simple poles along $z = 0$.

$$\nabla \colon \mathcal{F} \to \mathcal{F}\big(\mathcal{M}^{\circ} \times \{0\}\big) \otimes \Omega^1_{\mathcal{M}^{\circ} \times \mathbf{C}_z}.$$

Also $\mathcal{F}$ is isomorphic to the pulled-back quantum D-module $(\mathrm{H}'_{\heartsuit}\otimes\mathcal{O}_{\widetilde{\mathrm{U}}_{\heartsuit}\times\mathbf{C}_z}, (\mathrm{pr}\circ\varsigma_{\heartsuit})^*\nabla)/\langle\mathrm{G}\rangle$ over the open subset $\mathrm{U}_{\heartsuit}\times\mathbf{C}_z$. In particular, $\mathcal{F}$ extends across the orbifold point $u=0$ as an orbi-sheaf with flat connection (i.e. $\boldsymbol{\mu}_d$-equivariant flat bundle on a d-fold cover). We denote this extension over $\mathcal{M}\times\mathbf{C}_z$ by the same symbol $\mathcal{F}$.

We claim that there is a global $\mathbf{Z}$-local subsystem $\mathrm{F}_{\mathbf{Z}}$ of $(\mathcal{F}|_{\mathcal{M}\times\mathbf{C}_z^\times},\nabla)$ such that it coincides with the $\widehat{\Gamma}$-integral structure over $\mathrm{U}_{\heartsuit}\times\mathbf{C}_z^\times$. By (**89**), the flat section $\mathfrak{f}_{\mathrm{FJRW}}(\alpha)$, $\alpha\in\mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)$ is analytically continued to $\mathfrak{f}_{\mathrm{GW}}(\mathbf{U}_l\alpha)$ along the path γ_l^{-1}. Note that $\mathfrak{f}(\alpha)$ (**87**) is related to the flat section $\mathfrak{s}(\mathcal{E})$ (**22**) defining the $\widehat{\Gamma}$-integral structure by

$$\mathfrak{s}(\mathcal{E})=\frac{1}{(2\pi\mathtt{i})^{\hat{c}}}\mathfrak{f}\big(\mathrm{inv}^*\operatorname{ch}(\mathcal{E})\big)(x,z)$$

where $\mathcal{E}$ is an object of $\mathrm{D}^b(\mathrm{X_W})$ or $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ such that $\operatorname{ch}(\mathcal{E})\in\mathrm{H}'$. Therefore by Theorem 4.17 we know that

$$\mathfrak{s}_{\mathrm{FJRW}}(\mathcal{E})\text{ is analytically continued to }\mathfrak{s}_{\mathrm{GW}}\big(\Phi_l(\mathcal{E})\big)\text{ along }\gamma_l^{-1}\tag{\textbf{90}}$$

for $\mathcal{E}\in\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ with $\operatorname{ch}(\mathcal{E})\in\mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)$. This shows the existence of a global $\mathbf{Z}$-local system and that the analytic continuation along γ_l^{-1} corresponds to the Orlov equivalence Φ_l.

Finally we show that $\mathcal{F}$ admits a global ∇-flat pairing

$$\mathrm{P}\colon(-)^*\mathcal{F}\otimes\mathcal{F}\to z^{\hat{c}}\mathcal{O}_{\mathcal{M}\times\mathbf{C}_z},\qquad \hat{c}=\mathrm{N}-2,$$

which coincides with the pairings P_{GW}, $(-1)^{\mathrm{N}-1}\mathrm{P}_{\mathrm{FJRW}}$ of the quantum D-modules. In order to see that the global pairing exists over $\mathcal{M}\times\mathbf{C}^\times$, in view of (**90**), it suffices to check that

$$(-1)^{\mathrm{N}-1}\mathrm{P}_{\mathrm{FJRW}}\big((-)^*\mathfrak{s}_{\mathrm{FJRW}}(\mathcal{E}_1),\mathfrak{s}_{\mathrm{FJRW}}(\mathcal{E}_2)\big)=\mathrm{P}_{\mathrm{GW}}\big((-)^*\mathfrak{s}_{\mathrm{GW}}(\Phi_l\mathcal{E}_1),\mathfrak{s}_{\mathrm{GW}}(\Phi_l\mathcal{E}_2)\big)\tag{\textbf{91}}$$

for $\mathcal{E}_1,\mathcal{E}_2\in\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W},\boldsymbol{\mu}_d)$ such that $\operatorname{ch}(\mathcal{E}_i)\in\mathrm{H}_{\mathrm{nar}}(\mathrm{W},\boldsymbol{\mu}_d)$. Recall that the pairing between the flat sections $\mathfrak{s}(\mathcal{E})$ coincides with the Euler form up to sign (Proposition 2.21). Because the categorical equivalence preserves the Euler pairing $\chi(\mathcal{E},\mathcal{F})=\chi(\Phi_l\mathcal{E},\Phi_l\mathcal{F})$, (**91**) follows. The global pairing P over $\mathcal{M}\times\mathbf{C}_z^\times$ extends across $z=0$ (with zeros of order $\hat{c}$) by Hartog's principle because it already extends over $\mathrm{U}_{\heartsuit}\times\mathbf{C}_z$. The non-degeneracy of $\mathrm{P}/(2\pi\mathtt{i}z)^{\hat{c}}$ along $z=0$ holds for the same reason.

Now the proof of Theorem 2.23 is complete.

Remark **5.19.** — We described the global D-module $\mathcal{F}$ as a quotient of $\mathcal{G}=\mathcal{F}^{\mathrm{tw}}|_{\lambda=0}$. In [43, Theorem 6.13], with the aid of mirror symmetry, it was described as a *submodule* of another multi-GKZ system. We can translate this result in our setting as follows. Define the shift map $\mathrm{S}\colon\mathcal{M}^\circ\times\mathbf{C}_z\times\mathbf{C}_\lambda\to\mathcal{M}^\circ\times\mathbf{C}_z\times\mathbf{C}_\lambda$ by $\mathrm{S}(x,z,\lambda)=(x,z,\lambda-z)$. Then the map

$$\sigma:\mathcal{F}^{\mathrm{tw}}\to\mathrm{S}^*\mathcal{F}^{\mathrm{tw}},\qquad \Delta_\nu\longmapsto\mathrm{S}^*\Delta_{\nu+e_0}$$

is a morphism of $\mathcal{R}^{\mathrm{tw}}$-modules by the relations (**68**). This is an isomorphism at the generic point because both $\mathcal{R}^{\mathrm{tw}}\Delta_0$ and $\mathcal{R}^{\mathrm{tw}}\Delta_{e_0}$ equal $\mathcal{F}^{\mathrm{tw}}$ at the generic point (see Lemma 5.15; the proof there applies also to $\mathcal{R}^{\mathrm{tw}}\Delta_{e_0}$). However, σ is not an isomorphism over $\lambda = 0$ and we have $\mathcal{F} = \mathrm{Im}(\sigma|_{\lambda=0} \colon \mathcal{F}^{\mathrm{tw}}|_{\lambda=0} \to (\mathrm{S}^*\mathcal{F}^{\mathrm{tw}})|_{\lambda=0})$. See also [50].

5.5. *Reconstruction of the big quantum* D*-module.* — Here we prove Theorem 2.25. When $\mathrm{X_W}$ is a manifold, the orbifold cohomology consists only of untwisted sectors. In particular $\mathrm{H_{amb}}(\mathrm{X_W})$ is spanned $\mathbf{1}, p\mathbf{1}, \dots, p^{\dim \mathrm{X_W}}\mathbf{1}$. This allows us to use the reconstruction theorem [36, 41, 47, 59, 60] to obtain the big quantum cohomology from the small one.

More specifically, we apply the reconstruction theorem of a (TE) structure by Hertling-Manin [36, Theorem 2.5] to the global D-module $(\mathcal{F}, \nabla)$ over $\mathcal{M}$ (which is itself a (TE) structure). For this, one has to check the injectivity condition (IC) and the generation condition (GC) for $(\mathcal{F}, \nabla)$. More concretely, (IC) means

$$z\mathrm{D}_v\Delta_0|_{z=0} \neq 0,$$

and (GC) means

$$\left\{(z\mathrm{D}_v)^n\Delta_0 \;\middle|\; n \geq 0\right\} \text{ generates } \mathcal{F}|_{z=0} \text{ over } \mathcal{O}_{\mathcal{M}}.$$

We claim that $(z\mathrm{D}_v)^n\Delta_0$, $n = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ is a basis of $\mathcal{F}|_{z=0}$ over the open subsets $\mathrm{U_{FJRW}}$ and $\mathrm{U_{GW}}$. (Here $\mathrm{U_{FJRW}}$ does not contain $u = 0$.) We work over the cyclic cover $\widetilde{\mathrm{U}}_\heartsuit \subset \widetilde{\mathcal{M}}^\circ$ of $\mathrm{U}_\heartsuit$. First observe that we have D-module isomorphisms (cf. (**80**)):

$$\begin{tikzcd}
(\mathcal{F}, \nabla)|_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z^\times} \arrow[r, "\mathrm{Mir}_\heartsuit"] & \left(\mathrm{H}'_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z^\times}, (\mathrm{pr} \circ \varsigma_\heartsuit)^*\nabla\right) \arrow[r, "\mathrm{L}(\mathrm{pr} \circ \varsigma(x), z)^{-1}"] & \left(\mathrm{H}'_\heartsuit \otimes \mathcal{O}_{\widetilde{\mathrm{U}}_\heartsuit \times \mathbf{C}_z^\times}, d\right) \\
\Delta_\nu \arrow[r, mapsto] & \Upsilon^\nu := \mathrm{pr}\left(\Upsilon^{\mathrm{tw},\nu}\right) \arrow[r, mapsto] & z^{-1}\mathrm{I}^\nu_\heartsuit := z^{-1}\,\mathrm{pr}\left(\mathrm{I}^{\mathrm{tw},\nu}_\heartsuit|_{\lambda=t^0=0}\right).
\end{tikzcd}$$

Here the first map is induced from the mirror isomorphism in Corollary 5.13 (see also (**84**)) and the second map is given by the inverse of the fundamental solution $\mathrm{L}(t, z)$ in each theory. The relation $\mathrm{L}(\mathrm{pr} \circ \varsigma(x), z)^{-1}\Upsilon^\nu(x, z) = z^{-1}\mathrm{I}^\nu(x, z)$ follows from (**76**) and Proposition 3.14. Similarly to (**79**), the two maps $\mathrm{Mir}_\heartsuit$, L^{-1} can be viewed as the Birkhoff factors of the composition since $\mathrm{Mir}_\heartsuit$ extends regularly to $z = 0$ and L^{-1} extends regularly to $z = \infty$. We want to check that $(z\mathrm{D}_v)^i\Delta_0$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ form a basis. Under the above map, these sections map to

$$(z\mathrm{D}_v)^i z^{-1}\mathrm{I}^0_{\mathrm{GW}} = e^{p\log v/z}\left(p^i\mathbf{1} + \mathrm{O}\left(v^{1/d}\right)\right)$$

over $\widetilde{\mathrm{U}}_{\mathrm{GW}}$. From these asymptotics, we know that the matrix with the column vectors $(z\mathrm{D}_v)^i z^{-1}\mathrm{I}^0_{\mathrm{GW}}$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ is Birkhoff factorizable (i.e. in the "big cell") for sufficiently small $|v|$; this means that $(z\mathrm{D}_v)^i\Delta_0$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ is a basis of $\mathcal{F}|_{z=0}$ over $\widetilde{\mathrm{U}}_{\mathrm{GW}}$.

Over $\widetilde{\mathrm{U}}_{\mathrm{FJRW}}$, the calculation is a little more involved. Instead of $(z\mathrm{D}_v)^i$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$, we consider the differential operator P_i, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ defined inductively by

$$\mathrm{P}_0 = u^{-1}, \qquad \mathrm{P}_i := u^{-\operatorname{ord}_i}(z\partial_u)\mathrm{P}_{i-1}$$

where $\operatorname{ord}_i \in \mathbf{N}$ is determined by $(z\partial_u)\mathrm{P}_{i-1}\mathrm{I}^0_{\mathrm{FJRW}} = \mathrm{O}(u^{\operatorname{ord}_i})$. It suffices to show that $\mathrm{P}_i\triangle_0$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ is a basis of $\mathcal{F}|_{z=0}$ since $\{\mathrm{P}_i\triangle_0\}$ and $\{(z\mathrm{D}_v)^i\triangle_0\}$ are related by an invertible matrix along $z = 0$. We have

$$\mathrm{P}_i z^{-1}\mathrm{I}^0_{\mathrm{FJRW}} = c_i\frac{\phi_{k_i-1}}{z^{l_i-i}} + \mathrm{O}(u)$$

where k_i is the $(i+1)$-th smallest element of the set $\mathsf{Nar} \subset \{1, \dots, d-1\}$, $c_i \neq 0$ and $l_i := \deg(\phi_{k_i-1})/2$. It is not difficult to show that $l_i = i$ when X_{W} is a manifold. Therefore the matrix having the column vectors $\mathrm{P}_i z^{-1}\mathrm{I}^0_{\mathrm{FJRW}}$, $i = 0, \dots, \operatorname{rank}\mathcal{F} - 1$ is Birkhoff factorizable for small $|u|$. The claim now follows also over $\widetilde{\mathrm{U}}_{\mathrm{FJRW}}$.

Because (IC) and (GC) are open conditions, they hold in a Zariski open subset $\mathcal{M}'$ of $\mathcal{M}$ containing U_{GW} and $\mathrm{U}_{\mathrm{FJRW}}$. At each point $x \in \mathcal{M}'$, we have a universal unfolding [36, Definition 2.3] of $(\mathcal{F}, \nabla)|_{(\mathcal{M},x)\times\mathbf{C}_z}$ over the analytic germ $(\mathcal{M}, x) \times (\mathbf{C}^{\operatorname{rank}\mathcal{F}-1}, 0) \times \mathbf{C}_z$. By the universality, they will patch together to form a global (TE) structure $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}})$ over $\mathcal{M}_{\mathrm{ext}} \supset \mathcal{M}'$. By [36, Lemma 3.2], the pairing P over $\mathcal{M} \times \mathbf{C}_z$ extends to $\mathcal{M}_{\mathrm{ext}} \times \mathbf{C}_z$ and we have a (TEP) structure $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, \mathrm{P}^{\mathrm{ext}})$. The extension of the $\mathbf{Z}$-local system $\mathrm{F}_{\mathbf{Z}}$ is automatic.

Next we show that $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}}, \mathrm{P}^{\mathrm{ext}})$ coincides with the "big" quantum D-module over a neighbourhood of $\mathrm{U}_{\mathrm{FJRW}}$ or U_{GW}. We review the reconstruction of the big FJRW quantum cohomology. Over $\mathrm{U}_{\mathrm{FJRW}}$, we already identified $(\mathcal{F}, \nabla, \mathrm{P})$ with the quantum D-module over the image of the mirror map $\tau = \mathrm{pr} \circ \varsigma$. We take a basis $\{\mathrm{T}_i\}_{i=0}^r$ of H' such that $\mathrm{T}_0 = \phi_0$, $\mathrm{T}_1 = \phi_1$ and write the big quantum product as $\mathrm{T}_i \bullet \mathrm{T}_j = \sum_{k=0}^r \mathrm{C}_{ij}^k(t)\mathrm{T}_k$, where $t = (t^0, \dots, t^r)$ is the co-ordinates of $\mathrm{H}' = \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$ dual to $\{\mathrm{T}_i\}_{i=0}^r$. Using the frame $\{\mathrm{T}_i\}_{i=0}^r$, one can write the connection ∇ of $\mathcal{F}|_{\mathrm{U}_{\mathrm{FJRW}}}$ as

$$\nabla_u = \frac{\partial}{\partial u} + \frac{1}{z}\sum_{i=0}^r \frac{\partial\tau^i(u)}{\partial u}\left(\mathrm{C}_{i\alpha}^\beta\left(\tau(u)\right)\right)_{\alpha,\beta}.$$

Here $\tau(u) = \sum_{i=0}^r \tau^i(u)\mathrm{T}_i$ denotes the mirror map. The structure constants $\mathrm{C}_{ij}^k(t)$ are a priori formal power series in t, but we know from the mirror theorem that the above connection ∇_u is convergent. Because $\tau(u) = -u\phi_1 + \mathrm{O}(u^2)$, we can use $(u, t^0, t^2, \dots, t^r) \mapsto \tau(u) + \sum_{j\neq 1} t^j\mathrm{T}_j$ as a co-ordinate patch of H' near the origin. We want to reconstruct the connection operators

$$\nabla_u^{\mathrm{ext}} = \frac{\partial}{\partial u} + \frac{1}{z}\mathrm{A}(u, t),$$

$$\nabla_i^{\mathrm{ext}} = \frac{\partial}{\partial t^i} + \frac{1}{z}\left(\mathrm{C}_{i\alpha}^{\beta}\left(\tau(u) + \sum_{j\neq 1} t^j \mathrm{T}_j\right)\right)_{\alpha,\beta}, \qquad i \neq 1$$

satisfying $\nabla_u^{\mathrm{ext}}|_{t=0} = \nabla_u$, $[\nabla_i^{\mathrm{ext}}, \nabla_u^{\mathrm{ext}}] = [\nabla_i^{\mathrm{ext}}, \nabla_j^{\mathrm{ext}}] = 0$ and $\nabla_i \mathrm{T}_0 = \mathrm{T}_i$. Following the method of [36, Lemma 2.9], [41, Section 4.4], one can solve for such $\mathrm{C}_{i\alpha}^{\beta}(\tau(u) + \sum_{j\neq 1} t^j \mathrm{T}_j)$ uniquely as a power series in t. This is because $\mathrm{T}_0 = \phi_0$ is asymptotic to $u^{-1}\Delta_0$ as $u \to 0$, so is also a cyclic vector of the action of $[z\nabla_u]|_{z=0}$ for a sufficiently small $u \neq 0$. This reconstruction can be done either over the formal Laurent series ring $\mathbf{C}((u))$ or for a fixed small $u \neq 0$. In the former case, we recover the big quantum product as a formal power series in (u, t); in the latter case, we get $\mathrm{C}_{i\alpha}^{\beta}(\tau(u) + \sum_{j\neq 1} t^j \mathrm{T}_j)$ as a convergent power series of t ([36, Lemma 2.9]). Therefore $\mathrm{C}_{i\alpha}^{\beta}(\tau(u) + \sum_{j\neq 1} t^j \mathrm{T}_j)$ is a formal power series in t whose coefficients are analytic functions on $\{u \in \mathbf{C} \mid |u| < \epsilon\}$. Moreover for each u with $0 < |u| < \epsilon$, it is convergent as a power series in t. By [40, Lemma 6.5], such a function is holomorphic in a neighbourhood of $(u, t) = (0, 0)$. This shows the convergence of the big quantum product and that $(\mathcal{F}^{\mathrm{ext}}, \nabla^{\mathrm{ext}})$ is isomorphic to the big quantum D-module in a neighbourhood of $\mathrm{U}_{\mathrm{FJRW}}$. The discussion on the GW side is similar and omitted.

5.6. *Monodromy and autoequivalences.* — Here we prove Theorem 2.26. We study the relationships between monodromy of the global quantum D-module $\mathcal{F}$ and category equivalences.

An object E of $\mathrm{D}^b(\mathrm{X_W})$ is said to be *spherical* [63, Definition 1.1] if $\mathrm{Hom}^n(\mathrm{E}, \mathrm{E}) = \mathrm{Hom}(\mathrm{E}, \mathrm{E}[n])$ is isomorphic to the cohomology of a sphere, i.e.

$$\mathrm{Hom}^n(\mathrm{E}, \mathrm{E}) = \begin{cases} \mathbf{C} & n = 0 \text{ or } \dim \mathrm{X_W} \\ 0 & \text{otherwise.} \end{cases}$$

Seidel-Thomas [63] introduced a functor $\mathrm{T_E} \colon \mathrm{D}^b(\mathrm{X_W}) \to \mathrm{D}^b(\mathrm{X_W})$, called *spherical twist*, for a spherical object E. This gives an auto-equivalence with the following property:

$$\mathrm{T_E}(\mathrm{F}) \cong \mathrm{Cone}\big(\mathrm{Hom}^\bullet(\mathrm{E}, \mathrm{F}) \otimes \mathrm{E} \to \mathrm{F}\big).$$

Example **5.20.** — A line bundle $\mathcal{O}(i)$ on $\mathrm{X_W}$ is a spherical object (since $\mathrm{X_W}$ is Calabi-Yau).

By Proposition 2.21 and (**85**), the monodromy of flat sections $\mathfrak{s}(\mathcal{E})$ around the paths γ_{CY}, γ_{LG} (Figure 2) comes from the autoequivalences $\mathcal{O}(-1)$, (1) of $\mathrm{D}^b(\mathrm{X_W})$ and $\mathrm{MF}^{\mathrm{gr}}_{\boldsymbol{\mu}_d}(\mathrm{W})$ respectively. We already saw in (**90**) that the analytic continuation along γ_l^{-1} (Figure 3) is induced by the Orlov equivalence Φ_l. Thus the monodromy along $\gamma_{\mathrm{con}}^{-1}$ corresponds to the composition $\Phi_0 \circ \Phi_1^{-1}$. The following proposition shows that the monodromy around $\gamma_{\mathrm{con}}^{-1}$ comes from the spherical twist $\mathrm{T}_{\mathcal{O}}$.

Proposition **5.21.** — *For* $\mathrm{E} \in \mathrm{D}^b(\mathrm{X}_\mathrm{W})$ *such that* $\mathrm{ch}(\mathrm{E}) \in \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_\mathrm{W})$, *we have* $[\Phi_l \Phi_{l+1}^{-1}(\mathrm{E})] = [\mathrm{T}_{\mathcal{O}(l)}(\mathrm{E})]$ *in the numerical* K*-group.*

Proof. — Let $\{\underline{a}, \underline{b}\}_q$ be the graded Koszul matrix factorization in Example 4.5. Recall that $\mathrm{ch}(\{\underline{a}, \underline{b}\}_q)$, $q \in \mathbf{Z}$ span $\mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d)$. Hence by Theorem 4.17, $\mathrm{ch}(\Phi_l(\{\underline{a}, \underline{b}\}_q))$, $q \in \mathbf{Z}$ also span $\mathrm{H}_{\mathrm{amb}}(\mathrm{X}_\mathrm{W})$ since $\mathbf{U}_l \colon \mathrm{H}_{\mathrm{nar}}(\mathrm{W}, \boldsymbol{\mu}_d) \cong \mathrm{H}_{\mathrm{amb}}(\mathrm{X}_\mathrm{W})$. Therefore, it suffices to check that $[\mathrm{T}_{\mathcal{O}(l)} \Phi_{l+1}(\{\underline{a}, \underline{b}\}_q)] = [\Phi_l(\{\underline{a}, \underline{b}\}_q)]$ in the K-group. By Proposition 4.11, we have

$$\left[\Phi_{l+1}\big(\{\underline{a}, \underline{b}\}_q\big)\right] = \sum_{\substack{j_1 < \cdots < j_r \\ \sum_{a=1}^r w_{j_a} \le m'}} (-1)^{r+1} \left[\mathcal{O}\left(l + 1 + m' - \sum_{a=1}^r w_{j_a}\right)\right]$$

$$\left[\Phi_{l}\big(\{\underline{a}, \underline{b}\}_q\big)\right] = \sum_{\substack{j_1 < \cdots < j_r \\ \sum_{a=1}^r w_{j_a} \le m}} (-1)^{r+1} \left[\mathcal{O}\left(l + m - \sum_{a=1}^r w_{j_a}\right)\right]$$

where m (resp. m') is the remainder of $q - l$ (resp. $q - l - 1$) divided by d. Because $[\mathrm{T}_{\mathcal{O}(l)}\mathrm{E}] = [\mathrm{E}] - \chi(\mathrm{E}(-l))[\mathcal{O}(l)]$, we have

$$\begin{aligned} [\mathrm{T}_{\mathcal{O}(l)} \Phi_{l+1}\big(\{\underline{a}, \underline{b}\}_q\big)] = & \sum_{\substack{j_1 < \cdots < j_r \\ \sum_{a=1}^r w_{j_a} \le m'}} (-1)^{r+1} \left[\mathcal{O}\left(l + 1 + m' - \sum_{a=1}^r w_{j_a}\right)\right] \\ & + \sum_{\substack{j_1 < \cdots < j_r \\ \sum_{a=1}^r w_{j_a} \le m'}} (-1)^{r} \chi\left(\mathcal{O}\left(1 + m' - \sum_{a=1}^r w_{j_a}\right)\right)[\mathcal{O}(l)]. \end{aligned} \tag{92}$$

Here we use the following fact: For $1 \le i \le d$, we have

$$\begin{aligned} \chi\big(\mathcal{O}(i)\big) &= \dim \mathrm{H}^0\big(\mathrm{X}_\mathrm{W}, \mathcal{O}(i)\big) \\ &= \begin{cases} \sharp\{k_1 \le \cdots \le k_s \mid s \ge 0, \sum_{b=1}^s w_{k_b} = i\} & \text{if } 1 \le i \le d-1 \\ \sharp\{k_1 \le \cdots \le k_s \mid s \ge 0, \sum_{b=1}^s w_{k_b} = i\} - 1 & \text{if } i = d. \end{cases} \end{aligned}$$

Therefore the second term of the right-hand side of (**92**) gives

$$[\mathcal{O}(l)]\Bigg(-\delta_{m', d-1} + \sum_{\substack{j_1 < \cdots < j_r,\ k_1 \le \cdots \le k_s \\ \sum_{a=1}^r w_{j_a} + \sum_{b=1}^s w_{k_b} = m'+1,\ \sum_{a=1}^r w_{j_a} \le m'}} (-1)^r\Bigg). \tag{93}$$

We claim that for $m' \ge 0$

$$\sum_{\substack{j_1 < \cdots < j_r,\ k_1 \le \cdots \le k_s \\ \sum_{a=1}^r w_{j_a} + \sum_{b=1}^s w_{k_b} = m'+1}} (-1)^r = 0.$$

The claim follows from the comparison of the coefficient of $t^{m'+1}$ in the following equality:

$$\begin{aligned} 1 &= \frac{(1-t^{w_1})(1-t^{w_2})\cdots(1-t^{w_{\mathrm{N}}})}{(1-t^{w_1})(1-t^{w_2})\cdots(1-t^{w_{\mathrm{N}}})} \\ &= \sum_{p,q\geq 0}\Bigg(\sum_{\substack{j_1<\cdots<j_r \\ \sum_{a=1}^r w_{j_a}=p}} (-1)^r t^p\Bigg)\Bigg(\sum_{\substack{k_1\leq\cdots\leq k_s \\ \sum_{b=1}^s w_{k_b}=q}} t^q\Bigg). \end{aligned}$$

By the above claim, (**93**) can be rewritten as

$$[\mathcal{O}(l)]\Bigg(-\delta_{m',d-1} - \sum_{\substack{j_1<\cdots<j_r \\ \sum_{a=1}^r w_{j_a}=m'+1}} (-1)^r\Bigg).$$

This gives the second term of the right-hand side of (**92**).

First consider the case where $m' < d-1$. In this case, by the above calculation, $[\mathrm{T}_{\mathcal{O}(l)}\Phi_{l+1}(\{\underline{a},\underline{b}\}_q)]$ equals $[\Phi_l(\{\underline{a},\underline{b}\}_q)]$ because $m = m'+1$. Next consider the case where $m' = d-1$. In this case, we have $m=0$ and

$$\big[\mathrm{T}_{\mathcal{O}(l)}\Phi_{l+1}\big(\{\underline{a},\underline{b}\}_q\big)\big] = \sum_{j_1<\cdots<j_r} (-1)^{r+1}\Bigg[\mathcal{O}\Bigg(l+d-\sum_{a=1}^{r} w_{j_a}\Bigg)\Bigg] - [\mathcal{O}(l)].$$

We know from the Koszul complex $\mathcal{E}_d$ (**56**) that the first term in the right-hand side vanishes. Because $m=0$ we have $[\Phi_l(\{\underline{a},\underline{b}\}_q)] = -[\mathcal{O}(l)]$. The conclusion follows. ☐

Remark **5.22.** — We should have an isomorphism of functors $\mathrm{T}_{\mathcal{O}(l)} \cong \Phi_l \circ \Phi_{l+1}^{-1}$, but this does not seem to be proved in the literature. E. Segal [62, Theorem 3.13] showed a similar (object-wise) relationship in the category of B-branes on the LG model $(\mathrm{K}_{\mathbf{P}(\underline{w})}, \widetilde{\mathrm{W}})$ (which should be equivalent to $\mathrm{D}^b(\mathrm{X}_{\mathrm{W}})$).

We speculate that the relations in the fundamental groupoid of $\mathcal{M}$

$$\begin{aligned} &\gamma_{l+1} = \gamma_{\mathrm{LG}} \circ \gamma_l \circ \gamma_{\mathrm{CY}}, \\ &\gamma_{\mathrm{con}} = \gamma_1^{-1} \circ \gamma_0, \\ &\gamma_{\mathrm{LG}}^d = \mathrm{id} \end{aligned}$$

should be lifted to category equivalences as

$$\begin{aligned} &\Phi_{l+1}^{-1} \cong (1) \circ \Phi_l^{-1} \circ \mathcal{O}(-1), \\ &\mathrm{T}_{\mathcal{O}}^{-1} \cong \Phi_1 \circ \Phi_0^{-1}, \\ &(d) \cong [2]. \end{aligned}$$

The second relation is conjectural (see Remark 5.22) but the other two are easy to show. Note that the identity in the fundamental groupoid is lifted to the 2-shift [2] in the third relation. This is the reason why we have to mod out by [2] in the statement of Theorem 2.26.

Finally we check the last statement in Theorem 2.26. The fundamental group of $\mathcal{M}$ is generated by γ_{CY}, γ_{con} and is defined by the relation

$$(\gamma_{\mathrm{CY}} \circ \gamma_{\mathrm{con}})^d = \mathrm{id}\,.$$

We define the lift $\hat{\rho}\colon \pi_1(\mathcal{M}, b_0) \to \mathrm{Auteq}(\mathrm{D}^b(\mathrm{X}))/[2]$ by sending γ_{CY} to $\mathcal{O}(-1)$ and γ_{con} to $\mathrm{T}_{\mathcal{O}}^{-1}$ as we speculated above. It suffices to check the relation:

$$\left(\mathrm{O}(-1) \circ \mathrm{T}_{\mathcal{O}}^{-1}\right)^d \cong [2].$$

This was proved by Canonaco-Karp [9]. The proof of Theorem 2.26 (hence of Theorem 1.2) is now complete.

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Appendix A: Proof of Proposition 2.1

When $\mathrm{N}_k = 0$, both sides of (**7**) are one-dimensional and their pairings match. When $\mathrm{N}_k = 1$, both sides are zero. Assume that $\mathrm{N}_k \geq 2$. The relative cohomology exact sequence identifies $\mathrm{H}^{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty})$ with $\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{+\infty}) \cong \mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$. Therefore

$$\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k \cong \mathrm{H}^{\mathrm{N}_k-1}\left(\mathrm{W}_k^{-1}(1)\right)^{\boldsymbol{\mu}_d}.$$

We use the following result of Steenbrink:

Theorem **A.1** *([64, Theorem 1]). — The Deligne weight filtration $\mathcal{W}_\bullet$ on $\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$ is of the form*

$$0=\mathcal{W}_{\mathrm{N}_k-2}\subset\mathcal{W}_{\mathrm{N}_k-1}\subset\mathcal{W}_{\mathrm{N}_k}=\mathrm{H}^{\mathrm{N}_k-1}\big(\mathrm{W}_k^{-1}(1)\big).$$

Take a set $\{\varphi_1,\dots,\varphi_{\mathrm{L}}\}\subset\Omega^{\mathrm{N}_k}_{(\mathbf{C}^{\mathrm{N}})_k}$ of homogeneous N_k-forms which gives a basis of $\Omega(\mathrm{W}_k)$. Let $|i|$ denote the degree of φ_i divided by d. Define $\eta_i\in\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$ by

$$\eta_i:=c_i\,\mathrm{Res}_{\mathrm{W}_k(x)=1}\left(\frac{\varphi_i}{(\mathrm{W}_k(x)-1)^{\lceil|i|\rceil}}\right)\tag{94}$$

with $c_i=\Gamma(1-\langle-|i|\rangle)(\lceil|i|\rceil-1)!$. Then the set $\{\eta_i\mid\mathrm{N}_k-1-p<|i|<\mathrm{N}_k-p\}$ gives a basis of $\mathrm{Gr}^p_{\mathcal{F}}(\mathcal{W}_{\mathrm{N}_k-1})$; the set $\{\eta_i\mid|i|=\mathrm{N}_k-p\}$ gives a basis of $\mathrm{Gr}^p_{\mathcal{F}}(\mathcal{W}_{\mathrm{N}_k}/\mathcal{W}_{\mathrm{N}_k-1})$.

There is a typo in the statement of [64, Theorem 1] about the index of the Hodge filtration and we corrected it above. The prefactor c_i is not important in the above statement, but is chosen for our later purpose. Since $\{\eta_i\mid|i|\in\mathbf{Z}\}$ gives a basis of the $\boldsymbol{\mu}_d$-invariant part of $\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$, by the theorem, the $\boldsymbol{\mu}_d$-invariant part splits the weight filtration:

$$\mathcal{W}_{\mathrm{N}_k}/\mathcal{W}_{\mathrm{N}_k-1}\cong\mathrm{H}^{\mathrm{N}_k-1}\big(\mathrm{W}_k^{-1}(1)\big)^{\boldsymbol{\mu}_d}.$$

Therefore the sector $\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k\cong\mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))^{\boldsymbol{\mu}_d}$ has a pure Hodge structure of weight N_k. Moreover the theorem gives an isomorphism

$$\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}\cong\mathrm{Gr}^\bullet_{\mathcal{F}}\,\mathrm{H}^{\mathrm{N}_k-1}\big(\mathrm{W}_k^{-1}(1)\big)^{\boldsymbol{\mu}_d},\quad[\varphi_i]\longmapsto[\eta_i]$$

independent of the choice of representatives φ_i. The isomorphism (**7**) is defined by the Hodge decomposition which splits the above isomorphism:

$$\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k\cong\mathrm{H}^{\mathrm{N}_k-1}\big(\mathrm{W}_k^{-1}(1)\big)^{\boldsymbol{\mu}_d}=\bigoplus_{p=0}^{\mathrm{N}_k}\mathcal{F}^p\cap\overline{\mathcal{F}}^{\mathrm{N}_k-p}\cong\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}.\tag{95}$$

Next we study the pairing on the FJRW state space. The form $e^{-\mathrm{W}_k}\varphi_i$ defines a cohomology class in $\mathrm{H}^{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k,\mathrm{W}_k^{+\infty})$ via the integration over non-compact Lefschetz thimbles $\Gamma\in\mathrm{H}_{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k,\mathrm{W}_k^{+\infty})$ of W_k:

$$\Gamma\longmapsto\int_\Gamma e^{-\mathrm{W}_k}\varphi_i.$$

The following lemma shows that the set $\{e^{-\mathrm{W}_k}\varphi_i\mid|i|\in\mathbf{Z},\ |i|\leq\mathrm{N}_k-p\}$ of relative cohomology classes forms a basis of $\mathcal{F}^p\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k$. It also shows that $[\varphi_i]\in\Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ corresponds to an element of the form $[(\varphi_i+\sum_{|j|<|i|}a_{ji}\varphi_j)e^{-\mathrm{W}_k}]\in\mathrm{H}(\mathrm{W},\boldsymbol{\mu}_d)_k$ under (**95**).

Lemma **A.2.** — *Under the isomorphism* $\mathrm{H}^{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty}) \cong \mathrm{H}^{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$, *the class represented by* $e^{-\mathrm{W}_k}\varphi_i$ *corresponds to the class* η_i *in* (**94**).

Proof. — Let Γ be a Lefschetz thimble of W_k in $\mathrm{H}_{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty})$ and $\mathrm{C} \in \mathrm{H}_{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(t))$ be the corresponding cycle. (Note that $\mathrm{H}_{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, \mathrm{W}_k^{+\infty}) \cong \mathrm{H}_{\mathrm{N}_k-1}(\mathrm{W}_k^{-1}(1))$.) The image of Γ under W_k is assumed to be the positive real line. Then we have

$$\int_\Gamma e^{-\mathrm{W}_k}\varphi_i = \int_0^\infty e^{-t}\mathrm{P}(t)\mathsf{d}t. \tag{96}$$

Here we set

$$\mathrm{P}(t) := \int_{\Gamma\cap\{\mathrm{W}_k(x)=t\}} \frac{\varphi_i}{d\mathrm{W}_k} = \frac{1}{2\pi\mathtt{i}}\int_{\mathrm{T}} \frac{\varphi_i}{\mathrm{W}_k(x)-t} \tag{97}$$

where T is a circle bundle over $\Gamma \cap \{\mathrm{W}_k(x) = t\}$. Using the homogeneity, one can deduce from the co-ordinate change $x_i \mapsto t^{-w_i/d}x_i$ that

$$\mathrm{P}(t) = t^{|i|-1}\mathrm{P}(1).$$

Therefore by (**96**),

$$\int_\Gamma e^{-\mathrm{W}_k}\varphi_i = \Gamma\big(|i|\big)\mathrm{P}(1) = \Gamma\big(1-\langle -|i|\rangle\big)\mathrm{P}^{(\lceil |i|\rceil -1)}(1). \tag{98}$$

By differentiating (**97**) and setting $t = 1$, we find

$$\Gamma\big(1-\langle -|i|\rangle\big)\mathrm{P}^{(\lceil |i|\rceil -1)}(1) = \int_{\mathrm{C}} \eta_i. \tag{99}$$

The lemma follows from (**98**) and (**99**). □

Consider the tame deformation $\mathrm{W}_{k,s}$ of W_k:

$$\mathrm{W}_{k,s}(x) = \mathrm{W}_k(x) + \sum_{i\in \mathrm{F}_k} s_i x_i,$$

where $\mathrm{F}_k := \{1 \le j \le \mathrm{N} \mid \zeta^{kw_j} = 1\}$ is the index set of co-ordinates on $(\mathbf{C}^{\mathrm{N}})_k$. For generic values of s, $\mathrm{W}_{k,s}$ has only non-degenerate critical points (i.e. it is a Morse function). Let $z \in \mathbf{C}^\times$ and $\langle\cdot,\cdot\rangle\colon \mathrm{H}^{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, (\mathrm{W}_{k,s}/z)^{+\infty}) \times \mathrm{H}^{\mathrm{N}_k}((\mathbf{C}^{\mathrm{N}})_k, (\mathrm{W}_{k,s}/z)^{-\infty}) \to \mathbf{C}$ denote the intersection pairing (cf. (**2**)). Set

$$\mathrm{G}_{ij}(s,z) := \big\langle \big[e^{-\mathrm{W}_{k,s}/z}\varphi_i\big], \big[e^{\mathrm{W}_{k,s}/z}\varphi_j\big]\big\rangle.$$

This is a presentation of K. Saito's higher residue pairing [61] by Pham [55]. The invariance of the pairing under the co-ordinate change $x_j \mapsto \lambda^{w_j/d}x_j$ shows the following.

Lemma **A.3.** — *With respect to the degree* $\deg s_i := 1 - (w_i/d)$ *and* $\deg z := 1$, *the function* $G_{ij}(s, z)$ *is homogeneous of degree* $|i| + |j|$.

Lemma **A.4.** — *The function* $G_{ij}(s, z)$ *is regular at* $z = 0$. *Moreover*

$$G_{ij}(s,z) = (-1)^{\frac{N_k(N_k-1)}{2}} (2\pi \mathtt{i} z)^{N_k} \big(\mathrm{Res}_{W_{k,s}}\big([\varphi_k], [\varphi_l]\big) + O(z)\big).$$

Proof. — This is remarked in [55, 2ème Partie, Section 4.3, Remarque], but we include a proof for the convenience of the reader. Suppose that s is generic so that $x \mapsto \Re(W_{k,s}(x)/z)$ is a Morse function. Let $\Gamma_1^+, \dots, \Gamma_L^+$ (resp. $\Gamma_1^-, \dots, \Gamma_L^-$) denote the Lefschetz thimbles emanating from the critical points $\sigma_1, \dots, \sigma_L$ of $\Re(W_{k,s}/z)$ given by the upward (resp. downward) gradient flow. Choose an orientation of $\Gamma_i^\pm$ such that $\Gamma_a^+ \cdot \Gamma_b^- = \delta_{ab}$. We have

$$G_{ij}(s,z) = \sum_{a=1}^{L} \left(\int_{\Gamma_a^+} e^{-W_{k,s}/z} \varphi_i \right) \cdot \left(\int_{\Gamma_a^-} e^{W_{k,s}/z} \varphi_j \right).$$

For a fixed argument of z, we have the stationary phase expansion as $z \to 0$.

$$\int_{\Gamma_a^+} e^{-W_{k,s}/z} \varphi_i \sim \pm \frac{(2\pi z)^{N_k/2}}{\sqrt{\mathrm{Hess}\, W_{k,s}(\sigma_a)}} \big(f_i(\sigma_a) + O(z)\big).$$

Here we set $\varphi_i = f_i(x) \bigwedge_{j\in F_k} \mathrm{d}x_j$, $\mathrm{Hess}\, W_{k,s}(\sigma_a) := \det((\partial_{x_i}\partial_{x_j} W_{k,s})_{i,j\in F_k})$ is the Hessian of $W_{k,s}$ at σ_a and ± is the sign depending on the orientation of Γ_a^+. Therefore

$$G_{ij}(s,z) \sim (-1)^{\frac{N_k(N_k-1)}{2}} (2\pi \mathtt{i} z)^{N_k} \left(\sum_{a=1}^{N} \frac{f_i(\sigma_a) f_j(\sigma_a)}{\mathrm{Hess}\, W_{k,s}(\sigma_a)} + O(z) \right)$$

where the lowest order term in the right-hand side equals the Grothendieck residue. The sign $(-1)^{\frac{N_k(N_k-1)}{2}}$ comes from a local computation[16] of the orientation. Since this holds for an arbitrarily fixed argument of z, and $G_{ij}(s, z)$ is holomorphic in $z \in \mathbf{C}^\times$, the conclusion follows for a generic s. By analytic continuation, the same holds for all s. ☐

By Lemma A.3 and Lemma A.4, we have

$$G_{ij}(0,z) = \begin{cases} 0 & \text{if } |i| + |j| < N_k \\ (-1)^{\frac{N_k(N_k-1)}{2}} (2\pi \mathtt{i} z)^{N_k} \mathrm{Res}_{W_k}([\varphi_i], [\varphi_j]) & \text{if } |i| + |j| = N_k. \end{cases} \tag{100}$$

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[16] This comes from $\bigwedge_{j=1}^{N_k} \mathrm{d}u_j \wedge \bigwedge_{j=1}^{N_k} \mathrm{d}v_j = (-1)^{\frac{N_k(N_k-1)}{2}} \bigwedge_{j=1}^{N_k} (\mathrm{d}u_j \wedge \mathrm{d}v_j)$ where $\{u_j + \sqrt{-1}v_j \mid j = 1, \dots, N_k\}$ is a local co-ordinate system centered at a critical point.

This shows the Hodge-Riemann bilinear relation:

$$(\mathbf{101}) \qquad \left(\mathcal{F}^p \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k, \mathcal{F}^q \mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_{d-k}\right) = 0 \quad \text{if } p+q > \mathrm{N}_k.$$

For i, j such that $|i|, |j| \in \mathbf{Z}$, we take lifts

$$\left[e^{-\mathrm{W}_k}\hat{\varphi}_i\right] \in \mathcal{F}^p \cap \overline{\mathcal{F}}^{\mathrm{N}_k - p}, \qquad \left[e^{-\mathrm{W}_k}\hat{\varphi}_j\right] \in \mathcal{F}^q \cap \overline{\mathcal{F}}^{\mathrm{N}_k - q}$$

which correspond to $[\varphi_i], [\varphi_j] \in \Omega(\mathrm{W}_k)^{\boldsymbol{\mu}_d}$ under the isomorphism (**95**). When $p+q > \mathrm{N}_k$, the pairing $([e^{-\mathrm{W}_k}\hat{\varphi}_i], [e^{-\mathrm{W}_k}\hat{\varphi}_j]) = 0$ vanishes by (**101**). When $p+q < \mathrm{N}_k$, the pairing again vanishes because of the Hodge-Riemann bilinear relation (**101**) for $\overline{\mathcal{F}}$. When $p+q = \mathrm{N}_k$, we have

$$\begin{aligned}
&\left(\left[e^{-\mathrm{W}_k}\hat{\varphi}_i\right], \left[e^{-\mathrm{W}_k}\hat{\varphi}_j\right]\right) \\
&\quad = \left(\left[e^{-\mathrm{W}_k}\varphi_i\right], \left[e^{-\mathrm{W}_k}\varphi_j\right]\right) \quad \text{by } (\mathbf{101}) \\
&\quad = \frac{1}{d}\left\langle\left[e^{-\mathrm{W}_k}\varphi_i\right], (-1)^{|j|}\left[e^{\mathrm{W}_k}\varphi_j\right]\right\rangle \quad \text{by } (\mathbf{4}) \\
&\quad = (-1)^{\frac{\mathrm{N}_k(\mathrm{N}_k-1)}{2}}(2\pi \mathtt{i})^{\mathrm{N}_k}\frac{1}{d}\mathrm{Res}_{\mathrm{W}_k}\left([\varphi_i], (-1)^{|j|}[\varphi_j]\right) \quad \text{by } (\mathbf{100}).
\end{aligned}$$

The factor $(-1)^{|j|}$ comes from the map I^* in (**3**). The proof of Proposition 2.1 is complete.

Remark **A.5.** — In the proof we observed that $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ has a Hodge structure of weight N_k. In order to make the weight compatible with the FJRW grading, we consider the Tate twist by $\sum_{i=1}^{\mathrm{N}} \langle kq_i \rangle - 1$ so that $\mathrm{H}(\mathrm{W}, \boldsymbol{\mu}_d)_k$ is of weight $\mathrm{N}_k + 2(\sum_{i=1}^{\mathrm{N}} \langle kq_i \rangle - 1)$.

Appendix B: Compatibility with FJRW setup

In [26] a factor $f_g = |\mathrm{G}|^g / \deg(\mathrm{st})$ multiplies all genus-g n-pointed invariants as well as all the homomorphisms

$$\begin{aligned}
&\Lambda_{g,n}^{\mathrm{W},\mathrm{G}} \colon \mathrm{H}(\mathrm{W}, \mathrm{G})^{\otimes n} \to \mathrm{H}^*(\overline{\mathcal{M}}_{g,n}; \mathbf{Q}), \\
&\qquad (\alpha_1, \ldots, \alpha_n) \mapsto f_g \mathrm{st}_*\left(\left[\overline{\mathcal{W}}_{g,n,\mathrm{G}}\left(\mathrm{W}, (k_1, \ldots k_n)\right)\right]^{\mathrm{vir}} \cap \prod_{i=1}^n \alpha_i\right)
\end{aligned}$$

defining a cohomological field theory via Poincaré duality. These operators embody all the relevant invariants via the definition

$$\left\langle \tau_{b_1}(\alpha_1), \ldots, \tau_{b_n}(\alpha_n)\right\rangle_{g,n}^{\mathrm{W},\mathrm{G}} = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}^{\mathrm{W},\mathrm{G}}(\alpha_1, \ldots, \alpha_n) \prod_{i=1}^n \psi_i^{b_i}.$$

The cycle $[\overline{\mathcal{W}}_{g,n,\mathrm{G}}(\mathrm{W},(k_1,\dots k_n))]^{\mathrm{vir}} \cap \prod_{i=1}^{n}\alpha_i$ is a cycle in the moduli space of (W, G)-curves; in this paper G equals $\boldsymbol{\mu}_d$ and, in genus zero and for narrow state space entries, we may regard this as the top Chern class of the obstruction bundle. The degree of st is simply the degree of the map forgetting the (W, G)-structure and retaining only the underlying coarse stable curve; for $(\mathrm{W},\mathrm{G}) = (\mathrm{W},\boldsymbol{\mu}_d)$ the morphism st is the natural forgetful map $\mathrm{Spin}^{d}_{g,n}(k_1,\dots,k_n) \to \overline{\mathcal{M}}_{g,n}$. We have $\deg(\mathrm{st}) = |\mathrm{G}|^{2g-1}$ in general; therefore, f_g equals $1/|\mathrm{G}|^{g-1}$, and the setup of [26] is consistent with that of Witten's original tentative treatment [70] of quantum singularity theory. In genus zero and for $\mathrm{G} = \boldsymbol{\mu}_d$, this amounts to an overall factor d appearing also in [14, (14)].

We point out that all these different factors f_g can be removed once we take into account that the pairing used (**4**) comes from orbifold Chen-Ruan cohomology (in its relative version) and acquires an overall factor $1/|\mathrm{G}|$ equal to the degree of BG over Spec **C** (we recall that the pairing of [26] maps the pair (ϕ_k,ϕ_l) to $\delta_{d-1,k+l}$ without any factor). In particular, removing the factor $f_0 = d$ in the definition of the genus-zero invariants does not change the quantum product: in the definition (**17**) of $\mathrm{T}_i \bullet \mathrm{T}_j$, the factor f_0 is absorbed into $g^{k,l} = d\delta_{d-2,k+l}$. Furthermore, removing the factors f_g from the cohomological field theory homomorphisms $\Lambda^{\mathrm{W,G}}_{g,n}$ does not affect the composition axioms [26, (62), (64)]. Let $g = g_1 + g_2$; let $n = n_1 + n_2$; and let $\rho_{\mathrm{tree}}\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ be the gluing morphism. Then the forms

$$\widetilde{\Lambda}^{\mathrm{W,G}}_{g,n}(\alpha_1,\dots,\alpha_n) = \mathrm{st}_*\left(\left[\overline{\mathcal{W}}_{g,n,\boldsymbol{\mu}_d}\big(\mathrm{W},(k_1,\dots k_n)\big)\right]^{\mathrm{vir}} \cap \prod_{i=1}^{n}\alpha_i\right) = \frac{\Lambda^{\mathrm{W,G}}_{g,n}}{f_g}$$

satisfy the composition property stated in [26, (62)]

$$\begin{aligned}&\rho^*_{\mathrm{tree}}\widetilde{\Lambda}^{\mathrm{W,G}}_{g,n}(\alpha_1,\alpha_2,\dots,\alpha_n)\\&\quad= \sum_{\mu,\nu} g^{\mu,\nu}\widetilde{\Lambda}^{\mathrm{W,G}}_{g_1,n_1+1}(\alpha_{i_1},\dots,\alpha_{i_{n_1}},\mu) \otimes \widetilde{\Lambda}^{\mathrm{W,G}}_{g_2,n_2+1}(\alpha_{i_{n_1+1}},\dots,\alpha_{i_{n_1+n_2}},\nu),\end{aligned}$$

for all $\alpha_i \in \mathrm{H}(\mathrm{W},\mathrm{G})$, for μ and ν running through a basis of $\mathrm{H}(\mathrm{W},\mathrm{G})$, and for $g^{\mu,\nu}$ denoting the inverse of the pairing $(\ ,\ )$ with respect to the chosen basis. This happens because, by rescaling the pairing, we have multiplied $g^{\mu,\nu}$ by $|\mathrm{G}|$; the canceled factors on the two sides of the above identity match

$$f_g = \frac{1}{|\mathrm{G}|^{g-1}} = \frac{1}{|\mathrm{G}|}\frac{1}{|\mathrm{G}|^{g_1-1}}\frac{1}{|\mathrm{G}|^{g_2-1}} = \frac{1}{|\mathrm{G}|}f_{g_1}f_{g_2}.$$

The same happens for the gluing morphism $\rho_{\mathrm{loop}}\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$. We have

$$\rho^*_{\mathrm{loop}}\widetilde{\Lambda}^{\mathrm{W,G}}_{g,n}(\alpha_1,\alpha_2,\dots,\alpha_n) = \sum_{\mu,\nu} g^{\mu,\nu}\widetilde{\Lambda}^{\mathrm{W,G}}_{g-1,n+2}(\alpha_1,\dots,\alpha_n,\mu,\nu).$$

Taking again into account that, in this paper, the matrix $(g^{\mu,\nu})$ has been multiplied by an overall factor $|G|$, the cancellation of factors from the analogue identity [26, (64)] yields the same quantity on both sides

$$f_g = \frac{1}{|G|}\frac{1}{|G|^{g-2}} = \frac{1}{|G|}f_{g-1}.$$

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