A CATEGORY OF KERNELS FOR EQUIVARIANT FACTORIZATIONS AND ITS IMPLICATIONS FOR HODGE THEORY

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ABSTRACT

We provide a factorization model for the continuous internal Hom, in the homotopy category of k-linear dg-categories, between dg-categories of equivariant factorizations. This motivates a notion, similar to that of Kuznetsov, which we call the extended Hochschild cohomology algebra of the category of equivariant factorizations. In some cases of geometric interest, extended Hochschild cohomology contains Hochschild cohomology as a subalgebra and Hochschild homology as a homogeneous component. We use our factorization model for the internal Hom to calculate the extended Hochschild cohomology for equivariant factorizations on affine space.

Combining the computation of extended Hochschild cohomology with the Hochschild-Kostant-Rosenberg isomorphism and a theorem of Orlov recovers and extends Griffiths' classical description of the primitive cohomology of a smooth, complex projective hypersurface in terms of homogeneous pieces of the Jacobian algebra. In the process, the primitive cohomology is identified with the fixed subspace of the cohomological endomorphism associated to an interesting endofunctor of the bounded derived category of coherent sheaves on the hypersurface. We also demonstrate how to understand the whole Jacobian algebra as morphisms between kernels of endofunctors of the derived category.

Finally, we present a bootstrap method for producing algebraic cycles in categories of equivariant factorizations. As proof of concept, we show how this reproves the Hodge conjecture for all self-products of a particular K3 surface closely related to the Fermat cubic fourfold.

1. Introduction

The subject of matrix factorizations has, in recent years, found itself at the crossroads between commutative algebra, homological algebra, theoretical physics, and algebraic geometry. One of the deepest manifestations of this junction is D. Orlov's σ-model/Landau-Ginzburg correspondence [Orl09] which intimately links projective varieties to equivariant factorization categories. With Orlov's work as inspiration, this paper provides a thorough investigation of equivariant factorizations in broad generality. The central technical result is a factorization model for B. Töen's internal Hom dg-category [Toë07] between these dg-categories. The novelty lies in the range of applications, including those to classical problems in algebraic geometry and Hodge theory.

In this article, we will examine some of the more immediate consequences of the main result, such as some special cases of the Hodge conjecture and a new proof of Griffith's classical result [Gri69] relating the Dolbeault cohomology of a complex projective hypersurface to the Jacobian algebra of its defining polynomial. In the sequel to this article [BFK13], we will construct categorical coverings, calculate Rouquier dimension, investigate Orlov spectra, and connect our work to Homological Mirror Symmetry, all as applications of the central theorem presented here. Now, before we delve into detailed statements, let us try to provide some context for the results.

Perhaps the simplest class of singular rings is that of hypersurface rings, i.e. rings which are the quotient of a regular ring by a single element (also called hypersurface

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singularities). In the foundational paper, [Eis80], D. Eisenbud introduced matrix factorizations and demonstrated their precise relationship with maximal Cohen-Macaulay (MCM) modules over a hypersurface singularity. Building on Eisenbud's description, R.-O. Buchweitz introduced the proper categorical framework in [Buc86]. Buchweitz showed that the homotopy category of matrix factorizations, the stable category of MCM modules over the associated hypersurface ring, and the stable derived category of the associated hypersurface ring are all equivalent descriptions of the same triangulated category.

Outside of commutative algebra, interest in matrix factorizations grew due to intimate connections with physics; physical theories with potentials, called Landau-Ginzburg models, are ubiquitous. Building on the large body of work on Landau-Ginzburg models without boundary, (see, for example, C. Vafa's computation, [Vaf91], of the closed string topological sector as the Jacobian algebra of the potential), M. Kontsevich proposed matrix factorizations as the appropriate category of D-branes for the topological B-model in the presence of a potential [KL03a, Section 7.1].

In physics, A. Kapustin and Y. Li confirmed Kontsevich's prediction and gave a mathematically conjectural description of the Chern character map and the pairing on Hochschild homology for the category of matrix factorizations, [KL03a, KL03b].

In mathematics, several foundational papers by Orlov soon followed: [Orl04, Orl06, Orl09]. In particular, Orlov gave a global model for the stable bounded derived category of a Noetherian scheme possessing enough locally-free sheaves. He called this the category of singularities. Orlov also proved that the category of B-branes for an LG-model is equivalent to the coproduct of the categories of singularities of the fibers, and, to reiterate, the main inspiration for this work was the tight relationship he provided between the bounded derived categories of coherent sheaves on a projective hypersurface and the equivariant factorization category of affine space together with the defining polynomial.

In another early development, signaling the fertility of the marriage of physical inspiration to matrix factorizations, M. Khovanov and L. Rozansky categorified the HOMFLY polynomial using matrix factorizations, [KR08a, KR08b]. In the process, Khovanov and Rozansky also introduced several important ideas to the study of matrix factorizations. Central to their work is a construction which associates functors between categories of matrix factorizations to matrix factorizations of the difference potential. A strong, and precise, analogy exists between Khovanov and Rozansky's construction and the calculus of kernels of integral transforms between derived categories of coherent sheaves on algebraic varieties. Through this analogy, factorizations of the difference potential can be viewed as categorified correspondences for factorization categories.

Numerous further articles have elucidated the connection between factorization categories and Hodge theory. In [KKP08], the third author, Kontsevich, and T. Pantev give explicit constructions describing the Hodge theory associated to the category of matrix factorizations. For the case of an isolated local hypersurface singularity, T. Dyckerhoff proved, in [Dyc11], that the category of kernels introduced in [KR08a] is the correct one from the perspective of [Toë07]. More precisely, the dg-category of kernels from [KR08a] and [Dyc11] is quasi-equivalent to the internal homomorphism dg-category in the homotopy category of dg-categories. Using this result, Dyckerhoff rigorously established Kapustin and Li's description of the Hochschild homology of the dg-category of matrix factorizations. D. Murfet gave a mathematical derivation of the Kapustin-Li pairing [Mur09] which subsequently was expanded in [DM12]. In addition, E. Segal gave a description of the Kapustin-Li package in [Seg09].

Following this lead, several groups of authors extended Dyckerhoff's results. For a finite group, G, A. Polishchuk and A. Vaintrob gave a description of the Chern character, the bulk-boundary map, and proved an analog of Hirzebruch-Riemann-Roch in the case of the G-equivariant category of singularities of a local isolated hypersurface ring [PV12]. Orlov defined a category of matrix factorizations for a non-affine scheme with a global regular function and proved it is equivalent to the category of singularities of the associated hypersurface in the case when the ambient scheme is regular [Orl12]. K. Lin and D. Pomerleano also tackled non-affine matrix factorizations [LP11]. Contemporaneously, A. Preygel, using genuinely new ideas rooted in derived algebraic geometry, handled matrix factorizations on derived schemes, [Pre11].

Extending Dyckerhoff's results from the case of a local hypersurface to a global hypersurface, i.e. using a section of a line bundle instead of a global regular function, was also vigorously pursued. The first such results were obtained by A. Căldăraru and J. Tu in [CT10]. Căldăraru and Tu defined a curved A_∞-algebra associated to a hypersurface in projective space and computed the Borel-Moore homology of the curved algebra. Furthermore, in [Tu10] Tu clarified the relationship between Borel-Moore homology and Hochschild homology. In [PV10], Polishchuk and Vaintrob gave a definition of a category of matrix factorizations on a stack satisfying appropriate conditions and proved that their category of matrix factorizations coincided with the category of singularities of the underlying hypersurface. In [Pos09], L. Positselski, using his work on co- and contra-derived categories of curved dg-modules over a curved dg-algebra, defined an enlargement of the category of matrix factorizations in the case of a section of line bundle. He also defined in [Pos11], a relative singularity category for an embedding of Y in X and proved that the relative singularity category of the hypersurface defined by a section of a line bundle coincides with his category of factorizations even if the ambient scheme is not regular.

Continuing in this direction, this paper completely handles the case of a global hypersurface. Moreover, it also allows for an action of an affine algebraic group. Thus, in particular, it handles factorizations on any smooth algebraic stack with enough locally-free sheaves [Tot04]. The first main result of our paper provides an internal description of the functor category between categories of equivariant factorizations i.e. as another category of equivariant factorizations. To state it appropriately, let us recall some work of Töen, with the simplifying assumption that k is a field.

In [Toë07], Töen studies the structure of the localization of the category of dg-categories over a field, dg-cat$_k$, by the class of quasi-equivalences. Töen calls this localization, the homotopy category of dg-categories, and denotes it as Ho(dg-cat$_k$). For two dg-categories, C and D, Töen then defines a dg-category, denoted $\mathbf{R}\mathrm{Hom}(\mathsf{C}, \mathsf{D})$, which is the internal Hom dg-category in Ho(dg-cat$_k$). Töen defines $\mathbf{R}\mathrm{Hom}_c(\mathsf{C}, \mathsf{D})$ to be the full dg-subcategory of $\mathbf{R}\mathrm{Hom}(\mathsf{C}, \mathsf{D})$ whose objects induce coproduct-preserving functors between the homotopy categories. He calls such functors continuous.

The category, $\mathbf{R}\mathrm{Hom}_c(\mathsf{C}, \mathsf{D})$, lies at the heart of Töen's derived Morita result of [Toë07]. Indeed, it seems to be a robust and general prescription for picking out the "geometrically correct" functor category for familiar dg/triangulated categories. Let us give attention to an important example: derived categories of sheaves on varieties, X and Y.

An object, $\mathcal{K} \in \mathrm{D}(\mathrm{Qcoh}\, \mathrm{X} \times \mathrm{Y})$, gives a coproduct-preserving, exact functor,

$$\mathbf{R}q_*\big(\mathcal{K} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathbf{L}p^*\bullet\big) : \mathrm{D}(\mathrm{Qcoh}\, \mathrm{X}) \to \mathrm{D}(\mathrm{Qcoh}\, \mathrm{Y}),$$

where $p : \mathrm{X} \times \mathrm{Y} \to \mathrm{X}$ and $q : \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$ are the projections. However, it is well-known that the category of exact, coproduct-preserving functors from $\mathrm{D}(\mathrm{Qcoh}\, \mathrm{X})$ to $\mathrm{D}(\mathrm{Qcoh}\, \mathrm{Y})$ is *not* equivalent to $\mathrm{D}(\mathrm{Qcoh}\, \mathrm{X} \times \mathrm{Y})$, see [CS10] for an example. Passage from the category of chain complexes to triangulated categories is too brutal, we need to remember a bit more information. In [Toë07], Toën proves that, in Ho(dg-cat$_k$), there is an isomorphism,

$$\mathbf{R}\mathrm{Hom}_c\big(\mathsf{Inj}(\mathrm{X}), \mathsf{Inj}(\mathrm{Y})\big) \cong \mathsf{Inj}(\mathrm{X} \times \mathrm{Y})$$

where $\mathsf{Inj}(\mathrm{Z})$ is a particular dg-enhancement of $\mathrm{D}(\mathrm{Qcoh}\, \mathrm{Z})$. Similar work for varieties and other higher objects was carried out in [BFN10].

Hence, the failure of a Morita-type result for derived categories is remedied by lifting to dg-categories and working in Ho(dg-cat$_k$). This makes $\mathbf{R}\mathrm{Hom}_c$ the correct functor category to study. However, in general, if two dg-categories, $\mathcal{C}$ and $\mathcal{D}$, come from some geometric framework, such as derived categories of sheaves, it is not clear a priori from Töen's definition of the internal Hom how $\mathbf{R}\mathrm{Hom}_c(\mathcal{C}, \mathcal{D})$ reflects the underlying geometry. One must identify $\mathbf{R}\mathrm{Hom}_c(\mathcal{C}, \mathcal{D})$ geometrically. This is the first goal of the paper.

Let us define our dg-categories of matrix factorizations. Let k be an algebraically closed field of characteristic zero and let G and H be affine algebraic groups. Let X and Y be smooth varieties. Assume that G acts on X and H acts on Y. Let $\mathcal{L}$ be an invertible G-equivariant sheaf on X and let $w \in \mathrm{H}^0(\mathrm{X}, \mathcal{L})^{\mathrm{G}}$. Similarly, let $\mathcal{L}'$ be an invertible H-equivariant sheaf on X and let $v \in \mathrm{H}^0(\mathrm{Y}, \mathcal{L}')^{\mathrm{H}}$. Let $\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w)$ and $\mathsf{Inj}(\mathrm{Y}, \mathrm{H}, v)$ be the dg-categories of equivariant factorizations with injective components. Let $\mathrm{U}(\mathcal{L})$ be the geometric vector bundle corresponding to $\mathcal{L}$ with the zero section removed and denote the regular function induced by w on $\mathrm{U}(\mathcal{L})$ by f_w. Similarly, let $\mathrm{U}(\mathcal{L}')$ be the geometric vector bundle corresponding to $\mathcal{L}'$ with the zero section removed and denote the regular function induced by v on $\mathrm{U}(\mathcal{L}')$ by f_v. Equip $\mathrm{U}(\mathcal{L}) \times \mathrm{U}(\mathcal{L}')$ with the natural $\mathrm{G} \times \mathrm{H}$-action and allow $\mathbf{G}_m$ to scale the fibers of $\mathrm{U}(\mathcal{L}) \times \mathrm{U}(\mathcal{L}')$ diagonally. Let

$$(-f_w) \boxplus f_v := -f_w \otimes_k 1 + 1 \otimes_k f_v.$$

The following is the main result of Section 5.

Theorem **1.1.** — *In the homotopy category of k-linear dg-categories, there is an equivalence,*

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_c\big(\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w), \mathsf{Inj}(\mathrm{Y}, \mathrm{H}, v)\big)\\ &\quad\cong \mathsf{Inj}\big(\mathrm{U}(\mathcal{L}) \times \mathrm{U}\big(\mathcal{L}'\big), \mathrm{G} \times \mathrm{H} \times \mathbf{G}_m, (-f_w) \boxplus f_v\big).\end{aligned}$$

This result follows work in the ungraded case by Dyckerhoff, [Dyc11]. Our methods in proving Theorem 1.1 are in line with [LP11] as we rely on generation statements for singularity categories and use Positselski's absolute derived category, [Pos09, Pos11] as the model for our "large" triangulated category whose compact objects are (up to summands) coherent factorizations.

In contemporaneous and independent work, [PV11], Polishchuk and Vaintrob prove Theorem 1.1 in the case X and Y are affine, G and H are finite extensions of $\mathbf{G}_m$, and both w and v have an isolated critical locus. Polishchuk and Vaintrob also give a computation of the Hochschild homology of the category of equivariant matrix factorizations in this case. Despite the overlap in these foundational results, their inspiration and focus are ultimately distinct from the work here. They provide a purely algebraic version of FJRW-theory [FJR07] by way of matrix factorizations. The authors find this to be a beautiful illustration of the range and magnitude of this subject of study.

One significant advantage of a geometric description of the internal Hom category is greater computational power. As defined by Töen, Hochschild cohomology of a dg-category is the cohomology of the dg-algebra of endomorphisms of the identity, viewed as an object of the internal Hom dg-category in $\mathrm{Ho}(\text{dg-cat}_k)$. In the setting of G-equivariant factorizations, there is a natural extension, which we call extended Hochschild cohomology. For a dg-category, C, we denote its homotopy category by $[\mathsf{C}]$. Let $\widehat{\mathrm{G}}$ be the group of characters of G. The extended Hochschild cohomology is defined as

$$\mathrm{HH}_e^{(\chi,t)}(\mathrm{X}, \mathrm{G}, w) := \bigoplus_{\chi \in \widehat{\mathrm{G}}, t \in \mathbf{Z}} \mathrm{Hom}_{[\mathbf{R}\mathrm{Hom}_c(\mathsf{Inj}(\mathrm{X},\mathrm{G},w),\mathsf{Inj}(\mathrm{X},\mathrm{G},w))]}\big(\mathrm{Id}, (\chi)[t]\big).$$

Under certain assumptions on X, G, and w, the Hochschild homology of is a homogeneous component of $\mathrm{HH}_e^{\bullet}(\mathrm{X}, \mathrm{G}, w)$.

We use Theorem 1.1 to compute the extended Hochschild cohomology of $(\mathrm{X}, \mathrm{G}, w)$ when X is affine, G is a finite extension of $\mathbf{G}_m$, and w is semi-homogeneous regular function of non-torsion degree. The computation is along the lines of [PV12].

Theorem **1.2.** — *Let* G *act linearly on* $\mathbf{A}^n$ *and let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^G$. *Assume that the kernel of* χ, K_χ, *is finite and* $\chi : G \to \mathbf{G}_m$ *is surjective. Assume that the singular locus of the zero set,* $Z_{(-w)\boxplus w}$, *is contained in the product of the zero sets,* $Z_w \times Z_w$.

Then,

$$\begin{aligned}
&\mathrm{HH}_e^{(\rho,t)}(\mathbf{A}^n, G, w)\\
&\cong \Bigg(\bigoplus_{\substack{g \in K_\chi, l \geq 0 \\ t - \mathrm{rk}\, W_g = 2u}} H^{2l}(dw_g)\big(\rho - \kappa_g + (u-l)\chi\big)\\
&\quad \oplus \bigoplus_{\substack{g \in K_\chi, l \geq 0 \\ t - \mathrm{rk}\, W_g = 2u+1}} H^{2l+1}(dw_g)\big(\rho - \kappa_g + (u-l)\chi\big)\Bigg)^G
\end{aligned}$$

where $H^\bullet(dw_g)$ *denotes the Koszul cohomology of the Jacobian ideal of* $w_g := w|_{(\mathbf{A}^n)^g}$, W_g *is the conormal sheaf of* $(\mathbf{A}^n)^g$, *and* κ_g *is the character of* G *corresponding to* $\Lambda^{\mathrm{rk}\, W_g} W_g$.

If, additionally, we assume the support of the Jacobian ideal (dw) *is* $\{0\}$, *then we have*

$$\begin{aligned}
&\mathrm{HH}_e^{(\rho,t)}(\mathbf{A}^n, G, w)\\
&\cong \Bigg(\bigoplus_{\substack{g \in K_\chi \\ t - \mathrm{rk}\, W_g = 2u}} \mathrm{Jac}(w_g)(\rho - \kappa_g + u\chi)\\
&\quad \oplus \bigoplus_{\substack{g \in K_\chi \\ t - \mathrm{rk}\, W_g = 2u+1}} \mathrm{Jac}(w_g)(\rho - \kappa_g + u\chi)\Bigg)^G
\end{aligned}$$

where $\mathrm{Jac}(w)$ *denotes the Jacobian algebra of* w.

After building these foundations, we apply our results to Hodge theory. The primary observation is that Orlov's relationship between graded categories of singularities and derived categories of coherent sheaves [Orl09] has some very interesting geometric consequences when combined with Theorem 1.1.

Let C be a saturated dg-category over k. The Hochschild homology of C, $\mathrm{HH}_*(\mathsf{C})$, is an invariant that plays an important role in the noncommutative Hodge theory of C, [KKP08]. When X is a smooth proper algebraic variety over k, one can let $\mathsf{C} = \mathsf{Inj}_{\mathrm{coh}}(X)$ be the dg-category of bounded below complexes of injective $\mathcal{O}_X$-modules with bounded and coherent cohomology. There is a Hochschild-Kostant-Rosenberg isomorphism, see [HKR62, Swa96, Kon03]

$$\phi_{\mathrm{HKR}} : \mathrm{HH}_i\big(\mathsf{Inj}_{\mathrm{coh}}(X)\big) =: \mathrm{HH}_i(X) \to \bigoplus_{q-p=i} H^p\big(X, \Omega_X^q\big).$$

The HKR isomorphism allows one to study questions of Hodge theory by means of category theory. In Section 6.1, we combine Orlov's theorem, the HKR isomorphism, and the computations of Theorem 1.2 to reproduce a classic result of Griffiths [Gri69] describing the primitive cohomology of a projective hypersurface.

Theorem **1.3.** — *Let* Z *be a smooth, complex projective hypersurface defined by a homogeneous polynomial* $w \in \mathbf{C}[x_1, \dots, x_n]$ *of degree* d*. For each* $0 \leq p \leq n/2 - 1$*, Orlov's theorem and the HKR isomorphism induce an isomorphism,*

$$\mathrm{H}^{p,n-2-p}_{\mathrm{prim}}(Z) \cong \mathrm{Jac}(w)_{d(n-1-p)-n}.$$

In the process, we show that the primitive cohomology of Z is exactly the fixed locus of the action of the endofunctor

$$\begin{aligned}\{1\} := \mathrm{L}_{\mathcal{O}_Z} \circ \mathrm{T}_{\mathcal{O}_Z(1)} : \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) &\to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) \\ \mathcal{E} &\mapsto \mathrm{Cone}\big(\oplus_{i\in\mathbf{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z)}\big(\mathcal{O}_Z, \mathcal{E}(i)[i]\big) \\ &\qquad \otimes_k \mathcal{O}_Z[-i] \xrightarrow{ev} \mathcal{E}(1)\big)\end{aligned}$$

on Hochschild homology, $\mathrm{HH}_\bullet(Z)$. Furthermore, when Z is Calabi-Yau, for the kernel, $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z \times Z)$, of $\{1\}$, we have an injective homomorphism of graded rings,

$$\mathrm{Jac}(w) \to \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z\times Z)}\big(\Delta_* \mathcal{O}_Z, \mathcal{K}^{*i}\big)$$

whose appropriate graded pieces are the isomorphisms of Theorem 1.3, at least after application of the HKR isomorphism. Thus, we have a categorical realization of Griffiths' fundamental result that sees the entire Jacobian algebra.

Following this categorical path further, we study algebraic cycles by understanding the image of the Chern character map in Hochschild homology. In Section 6.2, we prove a result that allows one to bootstrap, via group homomorphisms, the Hodge conjecture for categories of equivariant matrix factorizations. We give one application of this procedure to the Hodge conjecture for varieties: we apply the results of Orlov in [Orl09] and work of Kuznetsov [Kuz09, Kuz10] to reprove the Hodge conjecture for n-fold products a certain K3 surface associated to a Fermat cubic fourfold. This case of the Hodge conjecture was originally handled in [RM08]. We thank P. Stellari for pointing out the reference, [RM08].

2. Background on equivariant sheaves

For the entirety of this paper, k will denote an algebraically-closed field of characteristic zero.

In this section, we recall some facts about quasi-coherent equivariant sheaves on separated, schemes/algebraic spaces of finite type following [MFK94]. A nice reference for basic facts, with a full set of details, is [Blu07, Chapter 3]. The results here will be used in later sections. Let X be a separated scheme of finite type over k and G be an affine algebraic group over k acting on X. Denote by $m : G \times G \to G$, $i : G \to G$, and $e : \operatorname{Spec} k \to G$, the group action, the inversion and the identity, respectively. Let $\sigma : G \times X \to X$ denote the G-action and $\pi : G \times X \to X$ the projection onto X.

Definition **2.1.** — *A quasi-coherent* **G-equivariant sheaf** *on* X *is a quasi-coherent sheaf,* $\mathcal{F}$*, on* X *together with an isomorphism,* $\theta : \sigma^*\mathcal{F} \to \pi^*\mathcal{F}$*, satisfying,*

$$\big((1_G \times \sigma) \circ (\tau \times 1_X)\big)^*\theta \circ (1_G \times \pi)^*\theta = (m \times 1_X)^*\theta,$$

on $G \times G \times X$ *where* $\tau : G \times G \times X \to G \times G \times X$ *switches the two factors of* G*, and,*

$$s^*\theta = 1_{\mathcal{F}},$$

where $s : X \to G \times X$ *is induced by* e*. If* $\mathcal{F}$ *is a coherent, respectively locally-free, sheaf on* X*, then we say the equivariant sheaf,* $(\mathcal{F}, \theta)$*, is coherent, respectively locally-free.*

The isomorphism, θ*, is called the* **equivariant structure***. We often refer to a quasi-coherent* G*-equivariant sheaf simply as* $\mathcal{E}$*. If the context is ambiguous, we denote the equivariant structure of* $\mathcal{E}$ *by* $\theta^{\mathcal{E}}$*.*

Remark **2.2.** — For each closed point $g \in G$, we get an automorphism

$$\sigma_g := \sigma(g, \bullet) : X \to X.$$

These satisfy $\sigma_{g_1} \circ \sigma_{g_2} = \sigma_{g_1 g_2}$. If $\mathcal{E}$ is a quasi-coherent G-equivariant sheaf, then θ gives isomorphisms

$$\theta_g := \theta|_{\{g\} \times X} : \sigma_g^*\mathcal{E} \to \mathcal{E}.$$

for each $g \in G$ with $\theta_{g_2 g_1} = \theta_{g_1} \circ \sigma_{g_1}^*\theta_{g_2}$. Checking a subsheaf $\mathcal{F}$ of $\mathcal{E}$ inherits the equivariant structure, i.e. $\theta(\sigma^*\mathcal{F}) \subseteq \pi^*\mathcal{F}$, boils down to checking that it is preserved by each θ_g.

Definition **2.3.** — *Let* $\operatorname{Qcoh}_G X$ *be the Abelian category of quasi-coherent* G*-equivariant sheaves on* X*. Analogously, we let* $\operatorname{coh}_G X$ *be the Abelian category of coherent* G*-equivariant sheaves.*

Definition **2.4.** — *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be quasi-coherent* G*-equivariant sheaves on* X*. The* **tensor product** *of* $\mathcal{E}$ *and* $\mathcal{F}$ *is the quasi-coherent sheaf* $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ *together with the equivariant structure,* $\theta^{\mathcal{E}} \otimes_{\mathcal{O}_{G\times X}} \theta^{\mathcal{F}}$*.*

The **sheaf of homomorphisms** *from* $\mathcal{E}$ *to* $\mathcal{F}$ *is the quasi-coherent sheaf* $\mathcal{H}om_X(\mathcal{E}, \mathcal{F})$ *together with the equivariant structure* $\theta^{\mathcal{F}} \circ (\bullet) \circ (\theta^{\mathcal{E}})^{-1}$*.*

Definition **2.5.** — *Let* X *and* Y *be separated, finite-type schemes equipped with actions,* σ_{X} *and* σ_{Y}*, of* G *and projections* $\pi_{\mathrm{X}}, \pi_{\mathrm{Y}}$*. A morphism of schemes,* $f : \mathrm{X} \to \mathrm{Y}$*, is* **G-equivariant** *if the diagram*

$$\begin{tikzcd} \mathrm{G} \times \mathrm{X} \arrow[r, "1 \times f"] \arrow[d, "\sigma_{\mathrm{X}}"'] & \mathrm{G} \times \mathrm{Y} \arrow[d, "\sigma_{\mathrm{Y}}"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

commutes. Given such an f*, we get an adjoint pair of functors,*

$$\begin{aligned} f^* : \operatorname{Qcoh}_{\mathrm{G}} \mathrm{Y} &\to \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X} \\ (\mathcal{F}, \theta) &\mapsto \big(f^*\mathcal{F}, (1 \times f)^*\theta\big), \\ f_* : \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X} &\to \operatorname{Qcoh}_{\mathrm{G}} \mathrm{Y} \\ (\mathcal{F}, \theta) &\mapsto \big(f_*\mathcal{F}, (1 \times f)_*\theta\big). \end{aligned}$$

Remark **2.6.** — The definition of f_* and f^* are sensible (as interpreted through natural isomorphisms) as $\sigma_{\mathrm{X}}, \pi_{\mathrm{X}}$ are flat and the squares

$$\begin{tikzcd} \mathrm{G} \times \mathrm{X} \arrow[r, "1 \times f"] \arrow[d, "\sigma_{\mathrm{X}}"'] & \mathrm{G} \times \mathrm{Y} \arrow[d, "\sigma_{\mathrm{Y}}"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{G} \times \mathrm{X} \arrow[r, "1 \times f"] \arrow[d, "\pi_{\mathrm{X}}"'] & \mathrm{G} \times \mathrm{Y} \arrow[d, "\pi_{\mathrm{Y}}"] \\ \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \end{tikzcd}$$

are Cartesian.

Definition **2.7.** — *Given an affine algebraic group,* G*, we let*

$$\widehat{\mathrm{G}} := \operatorname{Hom}_{\text{alg grp}}(\mathrm{G}, \mathbf{G}_m).$$

The finitely-generated Abelian group, $\widehat{\mathrm{G}}$*, is called the* **group of characters** *of* G*. As* $\widehat{\mathrm{G}}$ *is Abelian, we shall use additive notation for group structure on* $\widehat{\mathrm{G}}$*.*

For a character, $\chi \in \widehat{\mathrm{G}}$*, we let* K_χ *denote the kernel of* χ*. We also get an auto-equivalence*

$$\begin{aligned} (\chi) : \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X} &\to \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X} \\ \mathcal{E} &\mapsto \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} p^*\mathcal{L}_\chi \end{aligned}$$

where $p : \mathrm{X} \to \operatorname{Spec} k$ *is the structure map and* $\mathcal{L}_\chi$ *is the object of* $\operatorname{Qcoh}_{\mathrm{G}}(\operatorname{Spec} k)$ *corresponding to* χ*.*

Lemma **2.8.** — *Let* G *act on* X *and* Y*. Assume we have an equivariant morphism,* $f : X \to Y$*. For* $\mathcal{E} \in \operatorname{Qcoh}_G Y$ *locally-free and* $\mathcal{F} \in \operatorname{Qcoh}_G X$*, there is a natural isomorphism*

$$f_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}).$$

Proof. — This follows from the usual projection formula applied both to $\mathcal{E}$ and θ. □

We will also need a more general pull-back functor.

Definition **2.9.** — *Let* H *and* G *be affine algebraic groups and let* X *and* Y *be separated schemes of finite type equipped with actions,* $\sigma_{H,X} : H \times X \to X$ *and* $\sigma_{G,Y} : G \times Y \to Y$*. Let* $\psi : H \to G$ *be a homomorphism of algebraic groups. A* **ψ-equivariant morphism***, or a* **morphism equivariant with respect to ψ***, is a morphism of schemes,* $f : X \to Y$*, such that diagram*

$$\begin{tikzcd} H \times X \arrow[r, "\psi \times f"] \arrow[d, "\sigma_{H,X}"'] & G \times Y \arrow[d, "\sigma_{G,Y}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. Given a ψ*-equivariant morphism,* f*, we can define the pull-back functor,*

$$\begin{aligned} f^* : \operatorname{Qcoh}_G Y &\to \operatorname{Qcoh}_H X \\ (\mathcal{F}, \theta) &\mapsto \big(f^*\mathcal{F}, (\psi \times f)^*\theta\big). \end{aligned}$$

In the case that $X = Y$*, we denote this functor by* Res_ψ*. If, in addition,* $\psi : H \to G$ *is a closed subgroup, the pull-back is called the* **restriction functor** *and denoted by* Res^G_H*.*

Remark **2.10.** — While there is a bit of notational conflict here, we will always try to eliminate this confusion with exposition.

Definition **2.11.** — *Let* G *and* H *be affine algebraic groups,* X *and* Y *separated schemes of finite type equipped with actions* $G \times X \to X$ *and* $H \times Y \to Y$*. Let* $\pi_1 : X \times Y \to X$ *and* $\pi_2 : X \times Y \to Y$ *be the two projections. The projection,* π_1*, is equivariant with respect to the projection* $G \times H \to G$ *while* π_2 *is equivariant with respect to the projection* $G \times H \to H$*. Let* $\mathcal{E} \in \operatorname{Qcoh}_G X$ *and* $\mathcal{F} \in \operatorname{Qcoh}_H Y$*. The* **exterior product** *of* $\mathcal{E}$ *and* $\mathcal{F}$ *is the quasi-coherent* $G \times H$*-equivariant sheaf*

$$\mathcal{E} \boxtimes \mathcal{F} := \pi_1^*\mathcal{E} \otimes_{\mathcal{O}_{X\times Y}} \pi_2^*\mathcal{F}.$$

Let H be a closed subgroup of G and let $\sigma : H \times X \to X$ be an action of G on X. The product, $G \times X$, carries an action of H defined by

$$\tau : H \times G \times X \to G \times X$$

$$(h, g, x) \mapsto \big(m\big(g, i(h)\big), \sigma(h, x)\big).$$

Lemma **2.12.** — *The fppf quotient of* $G \times X$ *by* H *exists as a separated algebraic space of finite type over* k. *It is denoted by* $G \overset{H}{\times} X$.

Proof. — By Artin's Theorem, see [Ana73, Theorem 3.1.1], $G \overset{H}{\times} X$ exists as a separated algebraic space of finite type. □

Let $\iota : X \to G \overset{H}{\times} X$ be the inclusion, $x \mapsto (e, x)$. This is equivariant with respect to the inclusion of H in G.

Lemma **2.13.** — *The pull-back functor,* $\iota^* : \mathrm{Qcoh}_G(G \overset{H}{\times} X) \to \mathrm{Qcoh}_H X$, *is an equivalence. Moreover, it induces an equivalence between the subcategories of coherent equivariant sheaves and an equivalence between the subcategories of locally-free equivariant sheaves.*

Proof. — This is an immediate consequence of faithfully-flat descent, see [Tho97, Lemma 1.3]. □

Definition **2.14.** — *Let* H *be a closed subgroup of* G *and assume we have an action,* $\sigma : G \times X \to X$. *The action,* σ, *descends to a* G*-equivariant morphism,* $\alpha : G \overset{H}{\times} X \to X$. *The* ***induction functor***,

$$\mathrm{Ind}_H^G : \mathrm{Qcoh}_H X \to \mathrm{Qcoh}_G X$$

is defined to be the composition, $\alpha_* \circ (\iota^*)^{-1}$.

Lemma **2.15.** — *Let* H *be a closed subgroup of* G *and assume we have an action,* $\sigma : G \times X \to X$. *The functor,* Ind_H^G, *is right adjoint to the restriction,* Res_H^G, *and*

$$\mathrm{Res}_H^G \cong \iota^* \circ \alpha^*.$$

Proof. — Note that the identity map on X can be factored as

$$X \xrightarrow{\iota} G \overset{H}{\times} X \xrightarrow{\alpha} X.$$

Thus, $\mathrm{Res}_H^G = \iota^* \circ \alpha^*$ which is left adjoint to $\alpha_* \circ (\iota^*)^{-1}$. □

Lemma **2.16.** — *Let* H *be a closed subgroup of* G *and let* X *be a separated scheme of finite type equipped with an action,* $\sigma : G \times X \to X$. *Let* $p : G/H \times X \to X$ *be the projection onto* X.

(a) *The* H*-crossed product,* $G \overset{H}{\times} X$, *is a scheme,* G*-equivariantly isomorphic to* $G/H \times X$, *with the diagonal* G*-action.*

(b) *The functor,* $\mathrm{Res}^{\mathrm{G}}_{\mathrm{H}}$, *is exact.*
(c) *For* $\mathcal{E} \in \mathrm{Qcoh}_{\mathrm{H}} \mathrm{X}$ *and* $\mathcal{F} \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$ *locally-free, there is the following projection formula, i.e. a natural isomorphism,*

$$\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}\big(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} \mathcal{F}\big) \cong \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}.$$

(d) *There is a natural isomorphism*

$$\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \circ \mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} \cong p_* p^*$$

of functors.
(e) *If we, additionally, assume that* G/H *is affine, then* $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}$ *is exact. In particular, if* H *is normal, then* $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}$ *is exact.*

Proof. — For (a), as we are over k, the quotient of G by H, as a fppf sheaf, exists as a quasi-projective scheme. By [Wat79, Theorem 16.1], one can find a G-representation, V, with a subspace, W, whose stabilizer is exactly H. Let $n = \dim \mathrm{W}$. Passing to the Grassmannian, $\mathrm{G}(n, \mathrm{V})$, H is exhibited as the stabilizer of a closed point and by [DeGa70, III, §3, Proposition 5.2] is representable by scheme with a locally-closed embedding into $\mathrm{G}(n, \mathrm{V})$. Now, the H-crossed product, $\mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}$, is G-equivariantly isomorphic to the product, $\mathrm{G}/\mathrm{H} \times \mathrm{X}$, with the diagonal G-action, via the isomorphism

$$\begin{aligned} \Phi : \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} &\to \mathrm{G}/\mathrm{H} \times \mathrm{X} \\ (g, x) &\mapsto (g\mathrm{H}, gx). \end{aligned}$$

For $\alpha : \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} \to \mathrm{X}$, we have $\alpha = p \circ \Phi$.

For (b), recall that $\mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} \cong \iota^* \circ \alpha^*$. Then,

$$\mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} \cong \iota^* \circ \Phi^* \circ p^*.$$

Both ι^* and Φ^* are equivalences so both are exact while p^* is exact as G/H is flat over k.

For (c), let $\mathcal{E} \in \mathrm{Qcoh}_{\mathrm{H}} \mathrm{X}$ and $\mathcal{F} \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$ with $\mathcal{F}$ locally-free. Since ι^* is an equivalence, we can write $\mathcal{E} = \iota^* \mathcal{E}'$ for $\mathcal{E}' \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}$,

$$\begin{aligned} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}\big(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} \mathcal{F}\big) &\cong \alpha_* \big(\iota^*\big)^{-1} \big(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \iota^* \alpha^* \mathcal{F}\big) \\ &\cong \alpha_* \big(\iota^*\big)^{-1} \big(\iota^* \mathcal{E}' \otimes_{\mathcal{O}_{\mathrm{X}}} \iota^* \alpha^* \mathcal{F}\big) \\ &\cong \alpha_* \big(\mathcal{E}' \otimes_{\mathcal{O}_{\mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}}} \alpha^* \mathcal{F}\big) \\ &\cong \alpha_* \mathcal{E}' \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F} \\ &\cong \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F} \end{aligned}$$

where we used the projection formula for α and the fact the ι^* is a monoidal functor.

For (d), we have isomorphisms

$$\begin{aligned}\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G &\cong \alpha_* \circ \left(\iota^*\right)^{-1} \circ \iota^* \circ \alpha^* \\ &\cong \alpha_* \circ \alpha^* \\ &\cong p_* \circ \Phi_* \circ \Phi^* \circ p^* \\ &\cong p_* \circ p^*.\end{aligned}$$

We used the fact that $\Phi_* \cong (\Phi^*)^{-1}$ as Φ is an isomorphism.

For (e), the map p is affine so p_* is exact. Consequently, $\mathrm{Ind}_H^G \cong \alpha_* \circ (\iota^*)^{-1} \cong p_* \circ \Phi_* \circ (\iota^*)^{-1}$ is a composition of exact functors. If H is normal, then G/H is an affine algebraic group, [Wat79, Theorem 16.3]. □

Remark **2.17.** — Notice that when H is not normal we may only consider G/H as a scheme with an action of G and not as an affine algebraic group. Furthermore, G/H possesses a transitive G-action and, since the base field has characteristic zero, is generically smooth. Consequently, G/H is a smooth variety.

Lemma **2.18.** — *Let* H *be a closed normal subgroup of* G. *Assume that* G/H *is Abelian. Then, there is a natural isomorphism*

$$\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G \mathcal{E} \cong \bigoplus_{\chi \in \widehat{G/H}} \mathcal{E}(\chi)$$

where we view χ *as a character of* G *via the homomorphism* $G \to G/H$.

Proof. — By Lemma 2.16, we have an isomorphism

$$\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G \cong p_* p^*$$

where $p : G/H \times X \to X$ is the projection. Thus,

$$\mathrm{Ind}_H^G \circ \mathrm{Res}_H^G \mathcal{E} \cong p_* p^* \mathcal{E} \cong \Gamma(G/H, \mathcal{O}_{G/H}) \otimes_k \mathcal{E}.$$

Since G/H is Abelian, $\Gamma(G/H, \mathcal{O}_{G/H}) \cong k[\widehat{G/H}]$ and

$$\Gamma(G/H, \mathcal{O}_{G/H}) \otimes_k \mathcal{E} \cong \bigoplus_{\chi \in \widehat{G/H}} \mathcal{E}(\chi).$$

□

Lemma **2.19.** — *Let* $\psi : G \to H$ *be a flat homomorphism of affine algebraic groups. Let* G *act on the algebraic varieties* Z *and* X *and* H *act on the algebraic varieties* Y *and* W. *Assume we have* *a Cartesian square*

$$\begin{tikzcd} Z \arrow[r, "u'"] \arrow[d, "v'"'] & Y \arrow[d, "v"] \\ X \arrow[r, "u"] & W \end{tikzcd}$$

where u' and u are ψ-equivariant while v' is G-equivariant and v is H-equivariant. Assume that u is flat. Then, we have a natural isomorphism of functors

$$u^* \circ v_* \cong v'_* \circ u'^* : \mathrm{Qcoh}_{\mathrm{H}} \mathrm{Y} \to \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}.$$

Proof. — For a H-equivariant quasi-coherent sheaf, $(\mathcal{E}, \theta)$, we have $u^* v_* \mathcal{E} \cong v'_* u'^* \mathcal{E}$ via flat base change. We also have a Cartesian diagram

$$\begin{tikzcd} \mathrm{G} \times \mathrm{Z} \arrow[r, "\psi \times u'"] \arrow[d, "1_{\mathrm{G}} \times v'"'] & \mathrm{H} \times \mathrm{Y} \arrow[d, "1_{\mathrm{H}} \times v"] \\ \mathrm{G} \times \mathrm{X} \arrow[r, "\psi \times u"] & \mathrm{H} \times \mathrm{W} \end{tikzcd}$$

and $\psi \times u$ is flat. So $(\psi \times u)^*(1 \times v)_* \cong (1 \times v')_*(\psi \times u)^*$ via flat base change, again. Using this fact on θ, we get an equivariant isomorphism between $u^* v_* \mathcal{E}$ and $v'_* u'^* \mathcal{E}$. $\square$

Definition **2.20.** — *Let X be a separated scheme of finite type over k. Let $\sigma : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ act on X and N be a closed normal subgroup of G such that $\sigma|_{\mathrm{N}\times\mathrm{X}} : \mathrm{N} \times \mathrm{X} \to \mathrm{X}$ is the trivial action. Consider a quasi-coherent G-equivariant sheaf $(\mathcal{E}, \theta)$ and the restriction of θ to N*

$$\theta|_{\mathrm{N}\times\mathrm{X}} : \sigma|_{\mathrm{N}\times\mathrm{X}}^* \mathcal{E} \cong \pi^* \mathcal{E} \to \pi^* \mathcal{E}.$$

Via adjunction, we have a morphism,

$$\mathcal{E} \xrightarrow{u_{\mathcal{E}}} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E} \xrightarrow{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k \mathcal{E}} - \pi_* \theta|_{\mathrm{N}\times\mathrm{X}}} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E}$$

where $u : \mathrm{Id} \to \pi_* \pi^*$ is the unit of adjunction. Let $\mathcal{E}^{\mathrm{N}}$ be the kernel of this total morphism. Then, θ preserves $\mathcal{E}^{\mathrm{N}}$ and the pair $(\mathcal{E}^{\mathrm{N}}, \theta|_{\sigma^* \mathcal{E}^{\mathrm{N}}})$ is a G-equivariant sheaf that naturally descends to a quasi-coherent G/N-equivariant sheaf on X. Denote the functor by

$$(\bullet)^{\mathrm{N}} : \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X} \to \mathrm{Qcoh}_{\mathrm{G}/\mathrm{N}} \mathrm{X}.$$

We shall often, interchangeably, view $\mathcal{E}^{\mathrm{N}}$ as a G-equivariant sheaf or as a G/N-equivariant sheaf without additional notational adornment.

Remark **2.21.** — The local sections of the sheaf $\mathcal{F}^{\mathrm{N}}$ over an open subset $\mathrm{U} \subseteq \mathrm{X}$ are

$$\mathcal{F}^{\mathrm{N}}(\mathrm{U}) = \left\{ f \in \mathcal{F}(\mathrm{U}) \mid \theta_n^{\mathcal{F}}(f) = f, \forall n \in \mathrm{N} \right\}.$$

In fact, this description can be taken as a definition of $\mathcal{F}^{\mathrm{N}}$.

Lemma **2.22.** — *The functor* $(\bullet)^{\mathrm{N}}$ *is right adjoint to* Res_π *for the quotient homomorphism,* $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{N}$.

Proof. — Let $\phi : \mathrm{Res}_\pi \mathcal{E} \to \mathcal{F}$ be a G-equivariant morphism. Since $\theta_n^{\mathrm{Res}_\pi \mathcal{E}} = \theta_{\pi(n)}^{\mathcal{E}} = 1_{\mathcal{E}}$, N acts trivially on Res_π. As ϕ is G-equivariant, we have

$$\theta_n^{\mathcal{F}} \circ \phi = \phi \circ \theta_n^{\mathrm{Res}_\pi \mathcal{E}} = \phi$$

for all $n \in \mathrm{N}$, and the image of $\mathrm{Res}_\pi \mathcal{E}$ under ϕ must lie in $\mathcal{F}^{\mathrm{N}}$. So any G-equivariant morphism from $\mathrm{Res}_\pi \mathcal{E}$ factors through $\mathcal{F}^{\mathrm{N}}$ uniquely. Of course, any G-equivariant morphism, $\mathrm{Res}_\pi \mathcal{E} \to \mathcal{F}^{\mathrm{N}}$, induces a G-equivariant morphism, $\mathrm{Res}_\pi \mathcal{E} \to \mathcal{F}$, via composition with the inclusion, $\mathcal{F}^{\mathrm{N}} \to \mathcal{F}$. Hence, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}}(\mathrm{Res}_\pi \mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}}\left(\mathrm{Res}_\pi \mathcal{E}, \mathcal{F}^{\mathrm{N}}\right).$$

As both $\mathrm{Res}_\pi \mathcal{E}$ and $\mathcal{F}^{\mathrm{N}}$ are N-invariant, any G-equivariant morphism, $\mathrm{Res}_\pi \mathcal{E} \to \mathcal{F}^{\mathrm{N}}$, uniquely descends to a G/N-equivariant morphism. So,

$$\mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}}\left(\mathrm{Res}_\pi \mathcal{E}, \mathcal{F}^{\mathrm{N}}\right) \cong \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}/\mathrm{N}} \mathrm{X}}\left(\mathcal{E}, \mathcal{F}^{\mathrm{N}}\right). \qquad \square$$

Lemma **2.23.** — *For any* $\mathcal{F}_1 \in \mathrm{Qcoh}_{\mathrm{G}/\mathrm{N}} \mathrm{X}$ *and* $\mathcal{F}_2 \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$, *there is a natural isomorphism of* G*-equivariant sheaves*

$$(\mathrm{Res}_\pi \mathcal{F}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_2)^{\mathrm{N}} \cong \mathcal{F}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_2^{\mathrm{N}}.$$

Proof. — Since $\mathrm{Res}_\pi \mathcal{F}_1$ is completely N-invariant, we have an isomorphism

$$\theta_n^{\mathrm{Res}_\pi \mathcal{F}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_2} := \theta_n^{\mathrm{Res}_\pi \mathcal{F}_1} \otimes_{\mathcal{O}_{\mathrm{X}}} \theta_n^{\mathcal{F}_2} \cong 1_{\mathcal{F}_1} \otimes \theta_n^{\mathcal{F}_2}$$

for all $n \in \mathrm{N}$. Thus, $\theta_n^{\mathrm{Res}_\pi \mathcal{F}_1 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_2}$ is the identity on a local section $f_1 \otimes f_2$ if and only if $\theta_n^{\mathcal{F}_2}$ is the identity on f_2. The result follows from Remark 2.21. $\square$

Corollary **2.24.** — *Let* N *be a closed normal subgroup of an affine algebraic subgroup* G. *Assume that* G *acts on* X *and* G/N *acts on* Y. *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a morphism equivariant with respect to the quotient homomorphism* $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{N}$. *We have the pullback* $f^* : \mathrm{Qcoh}_{\mathrm{G}/\mathrm{N}} \mathrm{Y} \to \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$. *Consider* Y *with the induced* G *action to have the pushforward* $f_* : \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X} \to \mathrm{Qcoh}_{\mathrm{G}} \mathrm{Y}$. *The composition,* $(f_*)^{\mathrm{N}}$, *is right adjoint to* f^*.

Proof. — The functor f^* is the composition of $f^* : \mathrm{Qcoh}_{\mathrm{G}} \mathrm{Y} \to \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$ and Res_{π}. As we have adjunctions, $f^* \dashv f_*$ and $\mathrm{Res}_{\mathrm{G}}^{\mathrm{N}} \dashv (\bullet)^{\mathrm{N}}$, the latter by Lemma 2.22, we get the desired statement. □

Lemma **2.25.** — *Let* G *act on* X *and* Y. *Let* N *be a closed normal subgroup which acts trivially on* X *and* Y *and let* $f : \mathrm{X} \to \mathrm{Y}$ *be a* G*-equivariant morphism. For any* $\mathcal{E} \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}$*, there is a natural isomorphism*

$$(f_*\mathcal{E})^{\mathrm{N}} \cong f_*\mathcal{E}^{\mathrm{N}}.$$

Proof. — By definition, $(f_*\mathcal{E})^{\mathrm{N}}$ is the kernel of the composition

$$f_*\mathcal{E} \to \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f_*\mathcal{E} \overset{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k f_*\mathcal{E}} - \pi_{\mathrm{Y}*}(1_{\mathrm{G}} \times f)_*\theta|_{\mathrm{N}\times\mathrm{X}}}{\to} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f_*\mathcal{E}$$

where $\pi_{\mathrm{Y}} : \mathrm{G} \times \mathrm{Y} \to \mathrm{Y}$ is the projection. The above is

$$f_*\big(\mathcal{E} \to \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E} \overset{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k \mathcal{E}} - \pi_{\mathrm{X}*}\theta|_{\mathrm{N}\times\mathrm{X}}}{\to} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E}\big)$$

where $\pi_{\mathrm{X}} : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ is the projection. This is the definition of $f_*\mathcal{E}^{\mathrm{N}}$. □

Lemma **2.26.** — *Let* N *be a closed normal subgroup of* G. *Let* G *act on* X *and* Y *with* N *acting trivially on both* X *and* Y. *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a flat* G*-equivariant morphism. For each* $\mathcal{E} \in \mathrm{Qcoh}_{\mathrm{G}} \mathrm{Y}$*, there is a natural isomorphism of* G*-equivariant sheaves*

$$f^*\mathcal{E}^{\mathrm{N}} \cong \big(f^*\mathcal{E}\big)^{\mathrm{N}}.$$

Proof. — By definition, $(f^*\mathcal{E})^{\mathrm{N}}$ is the kernel of the composition

$$f^*\mathcal{E} \to \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f^*\mathcal{E} \overset{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k f^*\mathcal{E}} - \pi_{\mathrm{X}*}(1_{\mathrm{G}} \times f)^*\theta|_{\mathrm{N}\times\mathrm{X}}}{\to} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f^*\mathcal{E}$$

where $\pi_{\mathrm{X}} : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ is the projection. Therefore, by flat base change this is equal to the kernel of the composition

$$f^*\mathcal{E} \to \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f^*\mathcal{E} \overset{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k f^*\mathcal{E}} - f^*\pi_{\mathrm{Y}*}\theta|_{\mathrm{N}\times\mathrm{X}}}{\to} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k f^*\mathcal{E}$$

where $\pi_{\mathrm{Y}} : \mathrm{G} \times \mathrm{Y} \to \mathrm{Y}$ is the projection.

Since f is flat, this is isomorphic to f^* applied to the kernel of the composition

$$\mathcal{E} \to \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E} \overset{1_{\Gamma(\mathrm{N},\mathcal{O}_{\mathrm{N}})\otimes_k \mathcal{E}} - \pi_{\mathrm{Y}*}\theta|_{\mathrm{N}\times\mathrm{X}}}{\to} \Gamma(\mathrm{N}, \mathcal{O}_{\mathrm{N}}) \otimes_k \mathcal{E}.$$

This kernel is the definition of $\mathcal{E}^{\mathrm{N}}$. □

Definition **2.27.** — *Let $f : X \to Y$ be a morphism of separated schemes of finite type. We say that X possesses an f-**ample family of line bundles** if there is a set of invertible sheaves, $\mathcal{L}_\alpha$, $\alpha \in A$, such that for any quasi-coherent sheaf, $\mathcal{E}$, the natural morphism*

$$\bigoplus_{\alpha \in A} \mathcal{L}_\alpha \otimes_{\mathcal{O}_X} f^* f_* (\mathcal{L}_\alpha^\vee \otimes_{\mathcal{O}_X} \mathcal{E}) \to \mathcal{E}$$

*is an epimorphism. If $f : X \to \operatorname{Spec} k$ is the structure morphism, we shall simply refer to the set $\mathcal{L}_\alpha$, $\alpha \in A$ as an **ample family**. When X possess an ample family it is called **divisorial**. If X and Y possess an action of G, f is G-equivariant, and each $\mathcal{L}_\alpha$ admits an equivariant structure, then we will say that the f-ample family is equivariant.*

Remark **2.28.** — This is one of the multitude of equivalent definitions of an f-ample family [Ill71, Proposition 2.2.3].

Let us recall the following result of Thomason.

Theorem **2.29.** — *Let X be a normal scheme of finite type acted on by an affine algebraic group G. Assume that X is divisorial. Then, X possesses an equivariant ample family. In particular, for any coherent G-equivariant sheaf, $\mathcal{E}$, there exists a locally-free G-equivariant sheaf of finite rank, $\mathcal{V}$, and an epimorphism, $\mathcal{V} \to \mathcal{E}$.*

Proof. — The conclusion is true replacing G by the connected component of the identity, G_0, by [Tho97, Lemma 2.10]. Applying [Tho97, Lemma 2.14] shows it is also true for G. □

Remark **2.30.** — In what follows, we often assume that a scheme is divisorial and implicitly use the theorem above to obtain an equivariant ample family.

We finish the section by recalling a simple fact about the global dimensions of categories of equivariant sheaves. Let G be an affine algebraic group and let X be a separated scheme of finite type.

Definition **2.31.** — *Recall that the **global dimension** of an Abelian category, $\mathcal{A}$, is the maximal n such that $\operatorname{Ext}^n_{\mathcal{A}}(A, B)$ is nonzero for some pair of objects, A and B, of $\mathcal{A}$. Let $\operatorname{gldim} \mathcal{A}$ denote the global dimension of $\mathcal{A}$.*

*Let $\mathcal{A}$ be an Abelian category and let A be an object. The **projective dimension** of A is*

$$\operatorname{pdim} A := \min\{s \mid \operatorname{Ext}^s_{\mathcal{A}}(A, \bullet) = 0\}.$$

*It is defined to be infinite if no such s exists. The object, A, is said to have **locally-finite projective dimension** if for each $A' \in \mathcal{A}$, there exists an s_0 such that*

$$\operatorname{Ext}^s_{\mathcal{A}}(A, A') = 0$$

for all $s \geq s_0$.

Note that the global dimension of $\mathcal{A}$ is

$$\sup_{\mathrm{A}} \operatorname{pdim} \mathrm{A}.$$

Lemma **2.32.** — *Let $\mathcal{E}$ be a quasi-coherent* G*-equivariant sheaf. If $\mathcal{E}$ has locally-finite projective dimension as an object of* $\operatorname{Qcoh} \mathrm{X}$*, then it has locally-finite projective dimension as an object of* $\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}$. *Moreover, we have the following inequalities,*

$$\operatorname{pdim}(\mathcal{E}, \theta) \leq \operatorname{pdim} \mathcal{E} + \operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k$$

$$\operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X} \leq \operatorname{gldim} \operatorname{Qcoh} \mathrm{X} + \operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k.$$

In particular, if X *is smooth, then* $\operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}$ *is finite.*

Proof. — Since, by definition,

$$\operatorname{Hom}_{\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}}(\mathrm{E}, \mathrm{F}) = \operatorname{Hom}_{\operatorname{Qcoh} \mathrm{X}}(\mathrm{E}, \mathrm{F})^{\mathrm{G}},$$

there is a spectral sequence

$$\mathrm{E}_2^{p,q} : \operatorname{Ext}^p_{\operatorname{Qcoh} \mathrm{X}}(\mathrm{E}, \mathrm{F})^{\mathbf{R}^q \mathrm{G}} \Rightarrow \operatorname{Ext}^{p+q}_{\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}}(\mathrm{E}, \mathrm{F}).$$

Let

$$p_0 := \sup\big\{p \mid \operatorname{Ext}^p_{\operatorname{Qcoh} \mathrm{X}}(\mathrm{E}, \mathrm{F}) \neq 0\big\}.$$

As $\operatorname{Ext}^q_{\operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k}(k, \mathrm{M}) = \mathrm{M}^{\mathbf{R}^q \mathrm{G}}$, we see that $\operatorname{Ext}^r_{\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}}(\mathrm{E}, \mathrm{F})$ vanishes for

$$r > \operatorname{gldim} \operatorname{Qcoh} \mathrm{X} + \operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k \geq p_0 + \operatorname{gldim} \operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k.$$

This gives the stated inequality.

Choose a closed embedding of $\mathrm{G} \subset \mathrm{GL}_n$. Then, $\mathrm{M}^{\mathrm{G}} \cong (\operatorname{Ind}_{\mathrm{G}}^{\mathrm{GL}_n} \mathrm{M})^{\mathrm{GL}_n}$ and the functor of GL_n-invariants is exact. Thus, $\mathrm{M}^{\mathbf{R}^q \mathrm{G}} = 0$ for $q > \dim \mathrm{GL}_n / \mathrm{G}$ as $\operatorname{Ind}_{\mathrm{G}}^{\mathrm{GL}_n}$ is the composition of $(\iota^*)^{-1}$ and the pushforward of $\mathrm{GL}_n / \mathrm{G} \to \operatorname{Spec} k$. Since

$$\operatorname{Ext}^s_{\operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k}(\mathrm{V}, \mathrm{W}) \cong \operatorname{Ext}^s_{\operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k}\big(k, \operatorname{Hom}_k(\mathrm{V}, \mathrm{W})\big) \cong \operatorname{Hom}_k(\mathrm{V}, \mathrm{W})^{\mathbf{R}^q \mathrm{G}}$$

the global dimension of $\operatorname{Qcoh}_{\mathrm{G}} \operatorname{Spec} k$ is finite.

Thus, if $\mathcal{E}$ has locally-finite projective dimension as an object of $\operatorname{Qcoh} \mathrm{X}$, then it has locally-finite projective dimension as an object of $\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}$.

If X is smooth, it is well-known that

$$\operatorname{gldim} \operatorname{Qcoh} \mathrm{X} = \dim \mathrm{X}. \qquad \square$$

Remark **2.33.** — In general, the global dimension of $\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X}$ can be strictly smaller than the global dimension of $\operatorname{Qcoh} \mathrm{X}$. Indeed, $\operatorname{Qcoh}_{\mathrm{G}} \mathrm{G}$, with the left action of G on itself, is equivalent to $\operatorname{Qcoh} \operatorname{Spec} k$ and, therefore, must have global dimension zero. We thank Kuznetsov for pointing this out.

3. Equivariant factorizations

Let G be an affine algebraic group and let X be a smooth variety equipped with an action $\sigma : G \times X \to X$. Let $w \in \Gamma(X, \mathcal{L})^G$ be a G-invariant section of an invertible equivariant sheaf, $\mathcal{L}$.

Definition **3.1.** — *The* **dg-category of factorizations** *of* w*, is denoted by* $\mathsf{Fact}(X, G, w)$. *The objects of* $\mathsf{Fact}(X, G, w)$ *are pairs,*

$$\mathcal{E}_{-1} \xrightarrow{\phi_0^{\mathcal{E}}} \mathcal{E}_0 \xrightarrow{\phi_{-1}^{\mathcal{E}}} \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}$$

of morphisms in $\mathrm{Qcoh}_G X$*, satisfying*

$$\phi_{-1}^{\mathcal{E}} \circ \phi_0^{\mathcal{E}} = w$$
$$\big(\phi_0^{\mathcal{E}} \otimes \mathcal{L}\big) \circ \phi_{-1}^{\mathcal{E}} = w.$$

We denote such an object by $(\mathcal{E}_{-1}, \mathcal{E}_0, \phi_{-1}^{\mathcal{E}}, \phi_0^{\mathcal{E}})$ *or simply by* $\mathcal{E}$ *when there is no confusion. The morphism complex between two objects,* $\mathcal{E}$ *and* $\mathcal{F}$*, as a graded vector space, can be described as follows. For* $n = 2l$*, we have*

$$\begin{aligned}&\mathrm{Hom}^n_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})\\ &\quad = \mathrm{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_{-1}, \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^l\big) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_0, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^l\big)\end{aligned}$$

and for $n = 2l + 1$*, we have*

$$\begin{aligned}&\mathrm{Hom}^n_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})\\ &\quad = \mathrm{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_0, \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{l+1}\big) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_{-1}, \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^l\big).\end{aligned}$$

The differential applied to $(f_{-1}, f_0) \in \mathrm{Hom}^n_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})$

$$= \begin{cases} ((f_0 \circ \phi_0^{\mathcal{E}} - (\phi_0^{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{L}^l) \circ f_{-1}, \\ \ (f_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}) \circ \phi_{-1}^{\mathcal{E}} - (\phi_{-1}^{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{L}^l) \circ f_0) & \text{if } n = 2l \\ ((f_0 \circ \phi_0^{\mathcal{E}} + (\phi_{-1}^{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{L}^l) \circ f_{-1}, \\ \ (f_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}) \circ \phi_{-1}^{\mathcal{E}} + (\phi_0^{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{L}^{l+1}) \circ f_0) & \text{if } n = 2l + 1. \end{cases}$$

Given an additive subcategory of $\mathrm{Qcoh}_G X$, we can form a corresponding dg-subcategory of $\mathsf{Fact}(X, G, w)$ by requiring the components, $\mathcal{E}_{-1}$ and $\mathcal{E}_0$, to be objects from that additive subcategory.

Definition **3.2.** — *Denote by* $\mathsf{fact}(X, G, w)$, $\mathsf{Vect}(X, G, w)$, $\mathsf{vect}(X, G, w)$, *and* $\mathsf{Inj}(X, G, w)$, *respectively, the full dg-subcategory of* $\mathsf{Fact}(X, G, w)$ *whose components, respectively, are coherent, locally-free, locally-free of finite rank, and injective as quasi-coherent* G*-equivariant sheaves.*

Remark **3.3.** — Categories of projective factorizations only prove useful when X is affine and G is reductive. Then, any locally-free G-equivariant sheaf of finite rank is projective.

Definition **3.4.** — *The* **shift**, *denoted by* $[1]$, *sends a factorization,* $\mathcal{E}$, *to the factorization,*

$$\mathcal{E}[1] := \left(\mathcal{E}_0, \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}, -\phi_0^{\mathcal{E}}, -\phi_{-1}^{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{L}\right).$$

Lemma **3.5.** — *We have an equality*

$$\mathrm{Hom}^n_{\mathsf{Fact}(X,G,w)}(\mathcal{E},\mathcal{F}) = \mathrm{Hom}^0_{\mathsf{Fact}(X,G,w)}\left(\mathcal{E},\mathcal{F}[n]\right).$$

Proof. — This is a straightforward check and is suppressed. □

One can pass to an associated Abelian category. It has the same objects as $\mathsf{Fact}(X, G, w)$, but morphisms between $\mathcal{E}$ and $\mathcal{F}$ are closed degree-zero morphisms in $\mathrm{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E},\mathcal{F})$. Denote this category by $Z^0\mathsf{Fact}(X, G, w)$. The category, $Z^0\mathsf{Fact}(X, G, w)$, with component-wise kernels and cokernels is an Abelian category.

Definition **3.6.** — *Given a complex of objects from* $Z^0\mathsf{Fact}(X, G, w)$,

$$\cdots \to \mathcal{E}^b \xrightarrow{f^b} \mathcal{E}^{b+1} \xrightarrow{f^{b+1}} \cdots \xrightarrow{f^{t-1}} \mathcal{E}^t \to \cdots,$$

the **totalization**, $\mathcal{T}$, *is the factorization*

$$\mathcal{T}_{-1} := \bigoplus_{i=2l} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-l} \oplus \bigoplus_{i=2l-1} \mathcal{E}^i_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-l}$$

$$\mathcal{T}_0 := \bigoplus_{i=2l} \mathcal{E}^i_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-l} \oplus \bigoplus_{i=2l+1} \mathcal{E}^i_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}^{-l}$$

$$\phi_0^{\mathcal{T}} := \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 \\ \ddots & -\phi_{-1}^{\mathcal{E}^{-1}} & 0 & 0 & 0 \\ 0 & f_0^{-1} & \phi_0^{\mathcal{E}^0} & 0 & 0 \\ 0 & 0 & f_{-1}^0 & -\phi_{-1}^{\mathcal{E}^1} \otimes \mathcal{L}^{-1} & 0 \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

$$\phi_{-1}^{\mathcal{T}} := \begin{pmatrix} \ddots & 0 & 0 & 0 & 0 \\ \ddots & -\phi_0^{\mathcal{E}^{-1}} \otimes \mathcal{L} & 0 & 0 & 0 \\ 0 & f_{-1}^{-1} \otimes \mathcal{L} & \phi_{-1}^{\mathcal{E}^0} & 0 & 0 \\ 0 & 0 & f_0^0 & -\phi_0^{\mathcal{E}^1} & 0 \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

For any closed morphism of cohomological degree zero, $f : \mathcal{E} \to \mathcal{F}$, *in* $\mathsf{Fact}(X, G, w)$, *we can form the cone factorization,* $C(f)$, *as the totalization of the complex*

$$\mathcal{E} \xrightarrow{f} \mathcal{F}$$

where $\mathcal{F}$ *is in degree zero.*

Proposition **3.7.** — *The homotopy category,* $[\mathsf{Fact}(X, G, w)]$, *is a triangulated category.*

Proof. — The translation is [1] and the class of triangles is given by sequences of morphisms

$$\mathcal{E} \xrightarrow{f} \mathcal{F} \to C(f) \to \mathcal{E}[1].$$

The proof now runs completely analogously to proving that the homotopy category of chain complexes of an Abelian category is triangulated. It is therefore suppressed. □

Definition **3.8.** — *(Positselski) Let* $\mathsf{Acyc}(X, G, w)$ *denote the full subcategory of objects of* $\mathsf{Fact}(X, G, w)$ *consisting of totalizations of bounded exact complexes from* $Z^0\mathsf{Fact}(X, G, w)$. *Objects of* $\mathsf{Acyc}(X, G, w)$ *are called* **acyclic**. *Similarly, let* $\mathsf{acyc}(X, G, w)$ *denote the subcategory of totalizations of bounded exact complexes of coherent factorizations.*

We will also need the analogs for factorizations with locally-free components. The full subcategory of objects of $\mathsf{Vect}(X, G, w)$ *consisting of totalizations of bounded exact complexes from* $Z^0\mathsf{Vect}(X, G, w)$ *is denoted by* $\mathsf{AcycVect}(X, G, w)$. *Similarly, let* $\mathsf{acycvect}(X, G, w)$ *denote the subcategory of totalizations of bounded exact complexes of coherent locally-free factorizations.*

Definition **3.9.** — *(Positselski) The* **absolute derived category** *of* $[\mathsf{Fact}(X, G, w)]$ *is the Verdier quotient of* $[\mathsf{Fact}(X, G, w)]$ *by* $[\mathsf{Acyc}(X, G, w)]$,

$$D^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big] := \big[\mathsf{Fact}(X, G, w)\big]/\big[\mathsf{Acyc}(X, G, w)\big].$$

The **absolute derived category** *of* $[\mathsf{fact}(X, G, w)]$ *is the Verdier quotient of* $[\mathsf{fact}(X, G, w)]$ *by* $[\mathsf{acyc}(X, G, w)]$,

$$D^{\mathrm{abs}}\big[\mathsf{fact}(X, G, w)\big] := \big[\mathsf{fact}(X, G, w)\big]/\big[\mathsf{acyc}(X, G, w)\big].$$

The **absolute derived category** *of* $[\mathsf{Vect}(X, G, w)]$ *is the Verdier quotient of* $[\mathsf{Vect}(X, G, w)]$ *by* $[\mathsf{AcycVect}(X, G, w)]$

$$D^{\mathrm{abs}}\big[\mathsf{Vect}(X, G, w)\big] := \big[\mathsf{Vect}(X, G, w)\big]/\big[\mathsf{AcycVect}(X, G, w)\big].$$

The **absolute derived category** *of* $[\mathsf{vect}(X, G, w)]$ *is the Verdier quotient of* $[\mathsf{vect}(X, G, w)]$ *by* $[\mathsf{acycvect}(X, G, w)]$,

$$D^{\mathrm{abs}}\big[\mathsf{vect}(X, G, w)\big] := \big[\mathsf{vect}(X, G, w)\big]/\big[\mathsf{acycvect}(X, G, w)\big].$$

We say that two factorizations are **quasi-isomorphic** *if they are isomorphic in the appropriate absolute derived category.*

We will also use versions of these categories with support conditions. Let Z be a closed G-invariant subset of X and set $U := X \setminus Z$. Let $j : U \to X$ be the inclusion.

Definition **3.10.** — *The category,* $D^{abs}_Z[\mathsf{Fact}(X, G, w)]$, *is the kernel of the functor,*

$$j^* : D^{abs}_Z\big[\mathsf{Fact}(X, G, w)\big] \to D^{abs}_Z\big[\mathsf{Fact}(U, G, w|_U)\big].$$

Define $D^{abs}_Z[\mathsf{fact}(X, G, w)]$, $D^{abs}_Z[\mathsf{Vect}(X, G, w)]$, $D^{abs}_Z[\mathsf{vect}(X, G, w)]$ *analogously.*

Let us recall some useful facts, due essentially to Positselski, about $D^{abs}[\mathsf{Fact}(X, G, w)]$.

Proposition **3.11.** — *Factorizations with injective components are right orthogonal to acyclic complexes in* $[\mathsf{Fact}(X, G, w)]$. *Moreover, the composition,*

$$\big[\mathsf{Inj}(X, G, w)\big] \to \big[\mathsf{Fact}(X, G, w)\big] \to D^{abs}\big[\mathsf{Fact}(X, G, w)\big]$$

is an equivalence.

Proof. — This is a version of [Pos09, Theorem 3.6] of Positselski. In this generality, it is a special case of [BDFIK12, Corollary 3.3]. □

Definition **3.12.** — *We let* $\mathsf{Inj}_{coh}(X, G, w)$ *be the full dg-category of* $\mathsf{Fact}(X, G, w)$ *consisting of factorizations that have injective components and that are quasi-isomorphic to a factorization with coherent components.*

Corollary **3.13.** — *The composition,*

$$\big[\mathsf{Inj}_{coh}(X, G, w)\big] \to \big[\mathsf{Fact}(X, G, w)\big] \to D^{abs}\big[\mathsf{fact}(X, G, w)\big]$$

is an equivalence.

Proof. — This is an immediate corollary of Proposition 3.11. □

Proposition **3.14.** — *The natural functor,*

$$D^{abs}\big[\mathsf{Vect}(X, G, w)\big] \to D^{abs}\big[\mathsf{Fact}(X, G, w)\big],$$

is an equivalence as is the natural functor,

$$D^{abs}\big[\mathsf{vect}(X, G, w)\big] \to D^{abs}\big[\mathsf{fact}(X, G, w)\big].$$

Moreover, if X *is affine and* G *is reductive, factorizations with locally-free components are left orthogonal to acyclic complexes in* $[\mathsf{Fact}(X, G, w)]$ *and the compositions*

$$[\mathsf{Vect}(X, G, w)] \to [\mathsf{Fact}(X, G, w)] \to D^{\mathrm{abs}}[\mathsf{Fact}(X, G, w)]$$
$$[\mathsf{vect}(X, G, w)] \to [\mathsf{fact}(X, G, w)] \to D^{\mathrm{abs}}[\mathsf{fact}(X, G, w)]$$

are equivalences.

Proof. — We first check that any factorization is quasi-isomorphic to a locally-free factorization. Moreover, if the original factorization is coherent, then the locally-free factorization can be chosen to have finite rank. The argument is contained in the proof of [Pos09, Theorem 3.6]. Let $\mathcal{E}$ be a factorization. By Theorem 2.29, we can find locally-free G-equivariant sheaves, $\mathcal{V}_{-1}$ and $\mathcal{V}_0$ and epimorphisms

$$\mathcal{V}_{-1} \xrightarrow{f_{-1}} \mathcal{E}_{-1}$$
$$\mathcal{V}_0 \xrightarrow{f_0} \mathcal{E}_0.$$

Form the factorization, $G^+(\mathcal{V})$,

$$\mathcal{V}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \oplus \mathcal{V}_{-1} \xrightarrow{\begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix}} \mathcal{V}_{-1} \oplus \mathcal{V}_0 \xrightarrow{\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}} \mathcal{V}_0 \oplus \mathcal{V}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}.$$

The maps,

$$\mathcal{V}_0 \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \oplus \mathcal{V}_{-1} \xrightarrow{(0\ f_{-1})} \mathcal{E}_{-1}$$
$$\mathcal{V}_{-1} \oplus \mathcal{V}_0 \xrightarrow{(0\ f_0)} \mathcal{E}_0,$$

give an epimorphism in $Z^0\mathsf{Fact}(X, G, w)$. Thus, for any factorization, there exists a factorization with locally-free components mapping epimorphically onto it. We can construct an exact complex of objects of $Z^0\mathsf{Fact}(X, G, w)$

$$\cdots \to \mathcal{V}^s \to \cdots \to \mathcal{V}^1 \to \mathcal{E} \to 0$$

where each $\mathcal{V}^j$ is a factorization with locally-free components. Let $\mathcal{K}^s$ be the kernel of $\mathcal{V}^s \to \mathcal{V}^{s-1}$ for $s > \dim X$. Since X is smooth, the components of $\mathcal{K}^s$ are locally-free. Thus, we have an exact sequence

$$0 \to \mathcal{K}^s \to \mathcal{V}^s \to \cdots \to \mathcal{V}^1 \to \mathcal{E} \to 0.$$

In $D^{\mathrm{abs}}[\mathsf{fact}(X, G, w)]$, we have an isomorphism

$$\mathcal{T} \to \mathcal{E}$$

where $\mathcal{T}$ is the totalization of $\mathcal{K}^s \to \mathcal{V}^s \to \cdots \to \mathcal{V}^1$. The factorization, $\mathcal{T}$, has locally-free components.

Thus, the natural functors,

$$\mathrm{D}^{\mathrm{abs}}\big[\mathsf{Vect}(\mathrm{X}, \mathrm{G}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\big]$$
$$\mathrm{D}^{\mathrm{abs}}\big[\mathsf{vect}(\mathrm{X}, \mathrm{G}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)\big],$$

are essentially surjective. We next check fully-faithfulness.

For fully-faithfulness, it suffices to show that given a short exact sequence

$$0 \to \mathcal{E}^3 \to \mathcal{E}^2 \to \mathcal{E}^1 \to 0 \tag{3.1}$$

there exists a factorization, $\mathcal{S} \in \mathsf{AcycVect}(\mathrm{X}, \mathrm{G}, w)$, that is isomorphic to the totalization, $\mathcal{T}$, of (3.1) in $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$. Moreover, if $\mathcal{E}_i$ are all coherent, then $\mathcal{S}$ can be taken to have finite rank.

Using what we have already proven, we can find a locally-free factorization $\mathcal{V}_1^1$ and an epimorphism

$$\mathcal{V}_1^1 \to \mathcal{E}^1.$$

Next choose a locally-free factorization $\mathcal{V}_1^2$ and an epimorphism onto the fiber product

$$\mathcal{V}_1^2 \to \mathcal{E}^2 \times_{\mathcal{E}^1} \mathcal{V}_1^1.$$

Let $\mathcal{V}_1^3$ be the kernel of the map $\mathcal{V}_1^2 \to \mathcal{E}^2 \times_{\mathcal{E}^1} \mathcal{V}_1^1 \to \mathcal{V}_1^1$. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}^3 \arrow[r] & \mathcal{E}^2 \arrow[r] & \mathcal{E}^1 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{V}_1^3 \arrow[r] \arrow[u] & \mathcal{V}_1^2 \arrow[r] \arrow[u] & \mathcal{V}_1^1 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with the vertical morphisms being epimorphisms. Replacing (3.1) the kernels of the vertical morphisms, repeating the argument, and iterating, we get an exact sequence of short exact sequences

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{E}^3 \arrow[r] \arrow[u] & \mathcal{E}^2 \arrow[r] \arrow[u] & \mathcal{E}^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{V}_1^3 \arrow[r] \arrow[u] & \mathcal{V}_1^2 \arrow[r] \arrow[u] & \mathcal{V}_1^1 \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & \mathcal{V}_s^3 \arrow[r] \arrow[u] & \mathcal{V}_s^2 \arrow[r] \arrow[u] & \mathcal{V}_s^1 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where each $\mathcal{V}_j^i$ is locally-free, and of finite rank if each $\mathcal{E}^i$ is coherent. The long exact sequence of short exact sequences gives rise to a long exact sequence of the totalizations of these short exact sequences

$$0 \to \mathcal{T}_s \to \mathcal{T}_{s-1} \to \cdots \to \mathcal{T}_1 \to \mathcal{T} \to 0.$$

Each $\mathcal{T}_j$ lies in $\mathsf{AcycVect}(X, G, w)$, or in $\mathsf{acycvect}(X, G, w)$ if each $\mathcal{E}^i$ is coherent. Thus, the totalization of $\mathcal{T}_s \to \mathcal{T}_{s-1} \to \cdots \to \mathcal{T}_1$ lies in $\mathsf{AcycVect}(X, G, w)$, or in $\mathsf{acycvect}(X, G, w)$ if each $\mathcal{E}^i$ is coherent, and is isomorphic to $\mathcal{T}$ in $D^{abs}[\mathsf{Fact}(X, G, w)]$.

If we assume that X is affine and G is reductive, then any G-equivariant locally-free sheaf is projective. The result in this case is a version of [Pos09, Theorem 3.6] of Positselski. For this generality, we argue as follows. By [BDFIK12, Lemma 2.14], factorizations with projective components are left orthogonal to acyclic factorizations. Thus, the compositions

$$\big[\mathsf{Vect}(X, G, w)\big] \to \big[\mathsf{Fact}(X, G, w)\big] \to D^{abs}\big[\mathsf{Fact}(X, G, w)\big]$$
$$\big[\mathsf{vect}(X, G, w)\big] \to \big[\mathsf{fact}(X, G, w)\big] \to D^{abs}\big[\mathsf{fact}(X, G, w)\big]$$

are fully-faithful. As we have already seen they are essentially surjective, they must both be equivalences. □

For a definition of a compactly-generated triangulated category and compact generators, refer to Section 4.

Proposition **3.15.** — *The triangulated category,* $D^{abs}[\mathsf{Fact}(X, G, w)]$, *is compactly-generated. The objects of* $D^{abs}[\mathsf{fact}(X, G, w)]$ *are a set of compact generators.*

Proof. — The proof of this fact is a repetition of the argument of [Pos09, Theorem 3.11.2] using the fact that any quasi-coherent G-equivariant sheaf on X, hence any factorization, is a union of its coherent subsheaves [Tho97, Lemma 1.4]. More precisely, one can use Lemma 4.7 (which is a consequence of Thomason's result) and follow Positselski's argument verbatim. □

Remark **3.16.** — It is a subtle problem to determine whether or not all compact objects of $D^{abs}[\mathsf{Fact}(X, G, w)]$ are isomorphic to objects of $D^{abs}[\mathsf{fact}(X, G, w)]$. By Proposition 3.15 and [Nee92, Theorem 2.1], every compact object is a summand of an object of $D^{abs}[\mathsf{fact}(X, G, w)]$ under a splitting in $D^{abs}[\mathsf{Fact}(X, G, w)]$. However, those summands may not be representable by coherent factorizations. See [Orl11] for an investigation of the relationship with completions of X.

To handle the possible idempotent incompleteness of our factorizations categories, we make the following definitions.

Definition **3.17.** — *Let* $\overline{\mathsf{Inj}}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)$ *be the full dg-subcategory of* $\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations which are compact in* $[\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w)] \cong \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$.

Let $\overline{\mathsf{vect}}(\mathrm{X}, \mathrm{G}, w)$ *be the full dg-subcategory of* $\mathsf{Vect}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations which are compact in* $\mathrm{D}^{\mathrm{abs}}[\mathsf{Vect}(\mathrm{X}, \mathrm{G}, w)] \cong \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$.

Let $\overline{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)]}$ *denote the idempotent-completion of* $\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)]$. *Note that by Proposition 3.11, we have*

$$\left[\overline{\mathsf{Inj}}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)\right] \cong \overline{\mathrm{D}^{\mathrm{abs}}\left[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)\right]}.$$

If X *is affine and* G *is reductive, by Proposition 3.14, we have*

$$\left[\overline{\mathsf{vect}}(\mathrm{X}, \mathrm{G}, w)\right] \cong \overline{\mathrm{D}^{\mathrm{abs}}\left[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)\right]}.$$

From a complex on the zero locus of w, one can form a factorization.

Definition **3.18.** — *Let* Y *be the zero locus of* w *in* X. *Denote by* $\mathsf{Qcoh}_{\mathrm{G}}\mathrm{Y}$ *the dg-category of chain complexes of quasi-coherent* G*-equivariant sheaves on* Y.

We have the dg-functor, see [Pos11, Section 3.7],

$$\begin{aligned}
\Upsilon : \mathsf{Qcoh}_{\mathrm{G}}\mathrm{Y} &\to \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \\
\mathcal{C} &\mapsto \Bigg(\bigoplus_{l \in \mathbf{Z}} i_* \mathcal{C}^{2l-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}^{-l}, \bigoplus_{l \in \mathbf{Z}} i_* \mathcal{C}^{2l} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}^{-l}, \\
&\qquad \bigoplus_{l \in \mathbf{Z}} i_* d_{\mathcal{C}}^{2l-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}^{-l}, \bigoplus_{l \in \mathbf{Z}} i_* d_{\mathcal{C}}^{2l} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}^{-l}\Bigg),
\end{aligned}$$

In the case that $\mathcal{C}$ *is a coherent* G*-equivariant sheaf and the context allows, we will denote* $\Upsilon\mathcal{C}$ *simply by* $\mathcal{C}$

Note that $\Upsilon\mathcal{C}$ is the totalization of the chain complex

$$\cdots \to \Upsilon\mathcal{C}^b \to \cdots \to \Upsilon\mathcal{C}^t \to \cdots.$$

It is clear that Υ takes bounded acyclic chain complexes in $\mathsf{Qcoh}_{\mathrm{G}}\mathrm{Y}$ to acyclic chain complexes on $[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$. Thus, Υ descends to a functor

$$\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{Y}) \to \mathrm{D}^{\mathrm{abs}}\left[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\right].$$

Moreover, Υ takes bounded complexes of coherent sheaves to coherent factorizations so it induces a functor

$$\Upsilon : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{Y}) \to \mathrm{D}^{\mathrm{abs}}\left[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)\right].$$

Now, we give an explicit construction, due essentially to Eisenbud [Eis80, Section 7], of a factorization associated to certain invariant closed subschemes in the zero locus of w. Consider an equivariant morphism

$$\mathcal{E} \xrightarrow{s} \mathcal{O}_X$$

with $\mathcal{E}$ locally-free of finite rank. We notationally identify s with the corresponding global section of $\mathcal{E}^\vee$. Further, assume there exists an equivariant morphism $t : \mathcal{O}_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ making the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "s"] \arrow[d, "w"'] & & \mathcal{O}_X \arrow[d, "w"] \arrow[dll, "t"'] \\ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L} \arrow[rr, "s \otimes_{\mathcal{O}_X} \mathcal{L}"] & & \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} \end{tikzcd}$$

commute.

Definition **3.19.** — *The* **Koszul factorization** *associated to the data* $(\mathcal{E}, s, t)$ *is defined as*

$$\mathcal{K}_{-1}(s,t) := \bigoplus_{l \geq 0} \left(\Lambda^{2l+1}\mathcal{E}\right) \otimes_{\mathcal{O}_X} \mathcal{L}^l$$

$$\mathcal{K}_0(s,t) := \bigoplus_{l \geq 0} \left(\Lambda^{2l}\mathcal{E}\right) \otimes_{\mathcal{O}_X} \mathcal{L}^l$$

$$\phi_0^{\mathcal{K}}, \phi_{-1}^{\mathcal{K}} := \bullet \lrcorner\, s + \bullet \wedge t.$$

Proposition **3.20.** — *Assume that* $(\mathcal{E}, s, t)$ *as above exist. Let* $\mathcal{O}_{Z_s}$ *be the cokernel of* s. *If* $\operatorname{rank} \mathcal{E} = \operatorname{codim} Z_s$, *then* $\mathcal{K}(s,t)$ *is quasi-isomorphic to the factorization,* $\Upsilon \mathcal{O}_{Z_s}$.

Let $\mathcal{O}_{Z_{t^\vee}}$ *be the cokernel of* $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \xrightarrow{t^\vee} \mathcal{O}_X$. *If* $\operatorname{rank} \mathcal{E} = \operatorname{codim} Z_{t^\vee}$, *then* $\mathcal{K}(s,t)$ *is quasi-isomorphic to the factorization* $\Upsilon(\mathcal{O}_{Z_{t^\vee}} \otimes_{\mathcal{O}_X} \Lambda^{\operatorname{rk} \mathcal{E}} \mathcal{E}[-\operatorname{rk} \mathcal{E}])$.

Proof. — Each is an application of [BDFIK12, Lemma 3.6], see also [Bec12, Section 3.2]. □

Lemma **3.21.** — *We have an isomorphism of factorizations,*

$$\mathcal{K}(s,t)^\vee \cong \mathcal{K}\left(t^\vee, s^\vee\right).$$

Proof. — This is immediate from the definitions. □

We describe some functors associated with natural operations on factorizations, mirroring those discussed in Section 2.

Definition **3.22.** — *Let* X *be a smooth variety equipped with an action of* G. *Assume we have* $w, v \in \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}}$. *We define a dg-functor,*

$$\otimes_{\mathcal{O}_{\mathrm{X}}} : \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Fact}(\mathrm{X}, \mathrm{G}, v) \to \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w+v),$$

by setting

$$\begin{aligned}
(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F})_{-1} &:= \mathcal{E}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_0 \oplus \mathcal{E}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_{-1} \\
(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F})_{0} &:= \mathcal{E}_{0} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_0 \oplus \mathcal{E}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \\
\phi_0^{\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}} &:= \begin{pmatrix} \phi_0^{\mathcal{E}} \otimes_{\mathcal{O}_{\mathrm{X}}} 1_{\mathcal{F}_0} & 1_{\mathcal{E}_0} \otimes_{\mathcal{O}_{\mathrm{X}}} \phi_0^{\mathcal{F}} \\ -1_{\mathcal{E}_{-1}} \otimes_{\mathcal{O}_{\mathrm{X}}} \phi_{-1}^{\mathcal{F}} & \phi_{-1}^{\mathcal{E}} \otimes_{\mathcal{O}_{\mathrm{X}}} 1_{\mathcal{F}_{-1}} \end{pmatrix} \\
\phi_{-1}^{\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}} &:= \begin{pmatrix} \phi_{-1}^{\mathcal{E}} \otimes_{\mathcal{O}_{\mathrm{X}}} 1_{\mathcal{F}_0} & -1_{\mathcal{E}_{-1}} \otimes_{\mathcal{O}_{\mathrm{X}}} \phi_0^{\mathcal{F}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \\ 1_{\mathcal{E}_0} \otimes_{\mathcal{O}_{\mathrm{X}}} \phi_{-1}^{\mathcal{F}} & \phi_0^{\mathcal{E}} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \otimes_{\mathcal{O}_{\mathrm{X}}} 1_{\mathcal{F}_{-1}} \end{pmatrix}.
\end{aligned}$$

Given $\alpha : \mathcal{E} \to \mathcal{E}'[r]$ *and* $\beta : \mathcal{F} \to \mathcal{F}'[s]$, *one has*

$$\alpha \otimes_{\mathcal{O}_{\mathrm{X}}} \beta : \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{E}' \to \mathcal{F} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}'[r+s]$$

defined by

$$\alpha \otimes_{\mathcal{O}_{\mathrm{X}}} \beta = \begin{cases}
\left(\begin{pmatrix} \alpha_{-1} \otimes \beta_0 & 0 \\ 0 & \alpha_0 \otimes \beta_{-1} \end{pmatrix}, \begin{pmatrix} \alpha_0 \otimes \beta_0 & 0 \\ 0 & \alpha_{-1} \otimes \beta_{-1} \otimes \mathcal{L} \end{pmatrix} \right) \\
\quad r, s \text{ even} \\
\left(\begin{pmatrix} 0 & \alpha_0 \otimes \beta_{-1} \\ -\alpha_{-1} \otimes \beta_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\alpha_{-1} \otimes \beta_{-1} \otimes \mathcal{L} \\ \alpha_0 \otimes \beta_0 & 0 \end{pmatrix} \right) \\
\quad r \text{ even}, s \text{ odd} \\
\left(\begin{pmatrix} \alpha_{-1} \otimes \beta_0 & 0 \\ 0 & \alpha_0 \otimes \beta_{-1} \end{pmatrix}, \begin{pmatrix} \alpha_0 \otimes \beta_0 & 0 \\ 0 & \alpha_{-1} \otimes \beta_{-1} \otimes \mathcal{L} \end{pmatrix} \right) \\
\quad r \text{ odd}, s \text{ even} \\
\left(\begin{pmatrix} 0 & -\alpha_0 \otimes \beta_{-1} \\ \alpha_{-1} \otimes \beta_0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha_{-1} \otimes \beta_{-1} \otimes \mathcal{L} \\ -\alpha_0 \otimes \beta_0 & 0 \end{pmatrix} \right) \\
\quad r, s \text{ odd}.
\end{cases}$$

For a locally-free factorization, $\mathcal{V}$, *the functor,*

$$\mathcal{V} \otimes_{\mathcal{O}_{\mathrm{X}}} \bullet : \big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)\big] \to \big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w+v)\big],$$

preserves acyclic complexes and descends to a functor.

$$\mathcal{V} \otimes_{\mathcal{O}_{\mathrm{X}}} \bullet : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w+v)\big].$$

For $\mathcal{E} \in \mathsf{Fact}(X, G, w)$, *we define*

$$\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \bullet := \mathcal{V} \otimes_{\mathcal{O}_X} \bullet,$$

where $\mathcal{V}$ *is a locally-free factorization quasi-isomorphic to* $\mathcal{E}$.

Lemma **3.23.** — *The functor,*

$$\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \bullet : D^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, v)\big] \to D^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w + v)\big]$$

is well-defined, i.e. it does not depend on the choice of representative of the quasi-isomorphism class.

Proof. — By Proposition 3.14, inclusion of $\mathsf{Vect}(X, G, v)$ into $\mathsf{Fact}(X, G, v)$ induces an equivalence

$$D^{\mathrm{abs}}\big[\mathsf{Vect}(X, G, v)\big] \to D^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, v)\big].$$

We may therefore view the derived functor on the absolute derived category of locally-free factorizations,

$$\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \bullet : D^{\mathrm{abs}}\big[\mathsf{Vect}(X, G, v)\big] \to D^{\mathrm{abs}}\big[\mathsf{Vect}(X, G, w + v)\big].$$

Since tensoring with a locally-free sheaf is exact, tensoring with a locally-free factorization preserves acyclic factorizations and we have natural quasi-isomorphisms

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{W} \cong \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{W} =: \mathcal{E} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{W}.$$

when $\mathcal{W}$ is locally-free and $\mathcal{V}$ is locally-free and quasi-isomorphic to $\mathcal{E}$. □

Definition **3.24.** — *Let* X *be a smooth variety equipped with an action of* G. *Assume we have* $w \in \Gamma(X, \mathcal{L})^G$. *Let* $p : X \to \operatorname{Spec} k$ *be the structure morphism. Let* $(\mathcal{C}, d)$ *be a bounded complex of vector spaces. Let* $\mathcal{E} \in \mathsf{Fact}(X, G, w)$. *Define a factorization* $\mathcal{E} \otimes_k \mathcal{C}$ *by*

$$\mathcal{E} \otimes_k \mathcal{C} := \mathcal{E} \otimes_{\mathcal{O}_X} p^*(\Upsilon \mathcal{C}).$$

Denote the corresponding functor by

$$\mathcal{E} \otimes_k \bullet : \mathsf{Qcoh}^{\mathrm{b}}(\operatorname{Spec} k) \to \mathsf{Fact}(X, G, w).$$

This functor takes an exact chain complex to an acyclic factorization in $\mathsf{Fact}(X, G, w)$. *Thus, it descends to a functor*

$$\mathcal{E} \otimes_k \bullet : D^{\mathrm{b}}(\mathrm{Qcoh} \operatorname{Spec} k) \to D^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big].$$

Next we give a version of sheaf Hom.

Definition **3.25.** — *Let* X *be a smooth variety equipped with an action of* G. *Assume we have sections,* $w, v \in \Gamma(X, \mathcal{L})^G$. *We define a dg-functor,*

$$\mathcal{H}om_X : \mathsf{Fact}(X, G, w)^{\mathrm{op}} \otimes_k \mathsf{Fact}(X, G, v) \to \mathsf{Fact}(X, G, v - w),$$

by setting

$$\begin{aligned}
\mathcal{H}om_X(\mathcal{E}, \mathcal{F})_{-1} &:= \mathcal{H}om_X(\mathcal{E}_{-1}, \mathcal{F}_0) \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \oplus \mathcal{H}om_X(\mathcal{E}_0, \mathcal{F}_{-1}) \\
\mathcal{H}om_X(\mathcal{E}, \mathcal{F})_0 &:= \mathcal{H}om_X(\mathcal{E}_0, \mathcal{F}_0) \oplus \mathcal{H}om_X(\mathcal{E}_{-1}, \mathcal{F}_{-1}) \\
\phi_0^{\mathcal{H}om_X(\mathcal{E},\mathcal{F})} &:= \begin{pmatrix} (\bullet) \circ \phi_{-1}^{\mathcal{E}} & \phi_0^{\mathcal{F}} \circ (\bullet) \\ (\phi_{-1}^{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}) \circ (\bullet) & (\bullet) \circ \phi_0^{\mathcal{E}} \end{pmatrix} \\
\phi_{-1}^{\mathcal{H}om_X(\mathcal{E},\mathcal{F})} &:= \begin{pmatrix} -(\bullet) \circ \phi_0^{\mathcal{E}} & \phi_0^{\mathcal{F}} \circ (\bullet) \\ \phi_{-1}^{\mathcal{F}} \circ (\bullet) & -(\bullet) \circ (\phi_{-1}^{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}). \end{pmatrix}
\end{aligned}$$

Given $\alpha : \mathcal{E} \to \mathcal{E}'[r]$ *and* $\beta : \mathcal{F} \to \mathcal{F}'[s]$, *one has*

$$\mathcal{H}om_X(\alpha, \beta) : \mathcal{H}om_X\big(\mathcal{E}', \mathcal{F}\big) \to \mathcal{H}om_X\big(\mathcal{E}, \mathcal{F}'\big)[r+s]$$

defined by

$$\begin{pmatrix} \beta_0 \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l+1}) & 0 \\ 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) \end{pmatrix},$$
$$\begin{pmatrix} \beta_0 \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) & 0 \\ 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) \end{pmatrix}$$

if $r = 2l$, $s = 2j$,

$$\begin{pmatrix} 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) \\ -\beta_0 \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l+1}) & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -\beta_{-1} \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) \\ \beta_0 \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) & 0 \end{pmatrix}$$

if $r = 2l$, $s = 2j + 1$,

$$\begin{pmatrix} -\beta_0 \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) & 0 \\ 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) \end{pmatrix},$$
$$\begin{pmatrix} \beta_0 \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) & 0 \\ 0 & -\beta_{-1} \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l-1}) \end{pmatrix}$$

if $r = 2l + 1$, $s = 2j$, *and*

$$\begin{pmatrix} 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) \\ \beta_0 \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l}) & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \beta_{-1} \circ (\bullet) \circ (\alpha_0 \otimes \mathcal{L}^{-l-1}) \\ \beta_0 \circ (\bullet) \circ (\alpha_{-1} \otimes \mathcal{L}^{-l}) & 0 \end{pmatrix}$$

if $r = 2l+1, s = 2j+1$.

For a locally-free factorization, $\mathcal{V}$*, the functor,*

$$\mathcal{H}om_{\mathrm{X}}(\mathcal{V}, \bullet) : \big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)\big] \to \big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v - w)\big],$$

preserves acyclic complexes and descends to a functor.

$$\mathcal{H}om_{\mathrm{X}}(\mathcal{V}, \bullet) : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v - w)\big].$$

For $\mathcal{E} \in \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)$*, we define*

$$\mathbf{R}\mathcal{H}om_{\mathrm{X}}(\mathcal{E}, \bullet) := \mathcal{H}om_{\mathrm{X}}(\mathcal{V}, \bullet)$$

where $\mathcal{V}$ *is a locally-free factorization quasi-isomorphic to* $\mathcal{E}$*.*

Lemma **3.26.** — *The functor,*

$$\mathbf{R}\mathcal{H}om_{\mathrm{X}}(\mathcal{E}, \bullet) : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, v - w)\big]$$

is well-defined, i.e. it does not depend on the choice of representative of the quasi-isomorphism class.

Proof. — The proof is completely analogous to that of Lemma 3.23 and is therefore suppressed. □

Proposition **3.27.** — *Let* X *be a smooth variety equipped with an action of an affine algebraic group* G*. Let* $w, v \in \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}}$ *be invariant sections of an invertible equivariant sheaf,* $\mathcal{L}$*. For* $\mathcal{E} \in \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)$, $\mathcal{F} \in \mathsf{Fact}(\mathrm{X}, \mathrm{G}, v)$ *and* $\mathcal{G} \in \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w + v)$*, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w+v)}(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)}\big(\mathcal{E}, \mathcal{H}om_{\mathrm{X}}(\mathcal{F}, \mathcal{G})\big).$$

Proof. — We first check this for Hom^0. We have

$$\begin{aligned}
&\mathrm{Hom}^0_{\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w+v)}(\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}, \mathcal{G}) \\
&\quad := \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}\big((\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F})_{-1}, \mathcal{G}_{-1}\big) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}\big((\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F})_0, \mathcal{G}_0\big) \\
&\quad := \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}(\mathcal{E}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_0, \mathcal{G}_{-1}) \oplus \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}(\mathcal{E}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_{-1}, \mathcal{G}_{-1}) \\
&\qquad \oplus \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}(\mathcal{E}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_0, \mathcal{G}_0) \\
&\qquad \oplus \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}(\mathcal{E}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{F}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}, \mathcal{G}_0).
\end{aligned}$$

Applying Hom-tensor adjunction for G-equivariant sheaves, we have an isomorphism

$$\cong \mathrm{Hom}_{\mathrm{Qcoh}_{\mathrm{G}}\mathrm{X}}\big(\mathcal{E}_{-1}, \mathcal{H}om_{\mathrm{X}}(\mathcal{F}_0, \mathcal{G}_{-1})\big)$$

$$\begin{aligned}
&\oplus \operatorname{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_0, \mathcal{H}om_X(\mathcal{F}_{-1}, \mathcal{G}_{-1})\big)\\
&\oplus \operatorname{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_0, \mathcal{H}om_X(\mathcal{F}_0, \mathcal{G}_0)\big)\\
&\oplus \operatorname{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_{-1}, \mathcal{H}om_X(\mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}, \mathcal{G}_0)\big)\\
=:\ & \operatorname{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_{-1}, \mathcal{H}om_X(\mathcal{F}, \mathcal{G})_{-1}\big) \oplus \operatorname{Hom}_{\mathrm{Qcoh}_G X}\big(\mathcal{E}_0, \mathcal{H}om_X(\mathcal{F}, \mathcal{G})_0\big)\\
=:\ & \operatorname{Hom}^0_{\mathsf{Fact}(X,G,w+v)}\big(\mathcal{E}, \mathcal{H}om_X(\mathcal{F}, \mathcal{G})\big).
\end{aligned}$$

Since $\operatorname{Hom}^0(\bullet, \bullet[n]) = \operatorname{Hom}^n(\bullet, \bullet)$, this defines the natural transformation on the whole morphism space of Fact. It is straightforward to check that these maps commute with the differentials. □

Corollary **3.28.** — *We have an adjoint pair of derived functors*

$$\begin{aligned}
\bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} &: \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w+v)\big]\\
\mathbf{R}\mathcal{H}om_X(\mathcal{F}, \bullet) &: \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w+v)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big].
\end{aligned}$$

Proof. — This follows by replacing the first entry in a morphism space by a locally-free factorization and the second by an injective factorization and applying Proposition 3.27. □

Definition **3.29.** — *We will focus on a particular case of sheaf-Hom. Consider the factorization* $\Upsilon\mathcal{O}_X$ *of* $0 \in \Gamma(X, \mathcal{L})^G$. *Denote it by* $\mathcal{O}_X$. *We get functors*

$$\begin{aligned}
(\bullet)^{\vee} &:= \mathcal{H}om_X(\bullet, \mathcal{O}_X) : \mathsf{Fact}(X, G, w)^{\mathrm{op}} \to \mathsf{Fact}(X, G, -w)\\
(\bullet)^{\mathbf{L}\vee} &:= \mathbf{R}\mathcal{H}om_X(\bullet, \mathcal{O}_X) : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big]^{\mathrm{op}}\\
&\qquad \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, -w)\big].
\end{aligned}$$

Lemma **3.30.** — *The functor,*

$$(\bullet)^{\mathbf{L}\vee} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{fact}(X, G, w)\big]^{\mathrm{op}} \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{fact}(X, G, -w)\big],$$

is an equivalence.

Proof. — It is simple to check that for a locally-free factorization of finite rank, $\mathcal{F}$, we have a natural isomorphism

$$\mathcal{F} \cong \mathcal{F}^{\vee\vee}.$$

Any object of $\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(X, G, w)^{\mathrm{op}}]$ is quasi-isomorphic to a locally-free factorization of finite rank by Proposition 3.14. □

Lemma **3.31.** — *Let* $\mathcal{V} \in \mathsf{vect}(X, G, w)$*. Then, there is an isomorphism*

$$\mathcal{V}^\vee \otimes_{\mathcal{O}_X} \bullet \cong \mathcal{H}om_X(\mathcal{V}, \bullet).$$

Similarly, for $\mathcal{E} \in \mathsf{fact}(X, G, w)$*, there is an isomorphism*

$$\mathcal{E}^{\mathbf{L}\vee} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \bullet \cong \mathbf{R}\mathcal{H}om_X(\mathcal{E}, \bullet).$$

Proof. — The first isomorphism follows immediately from inspection of the definitions. The second is a quick consequence of the first. □

Assume we have two smooth varieties, X and Y, both carrying a G-action, and a morphism, $f : X \to Y$. Let $w \in \Gamma(Y, \mathcal{L})^G$. We have pull-back and pushforward functors.

Definition **3.32.**

$$\begin{aligned} f^* : \mathsf{Fact}(Y, G, w) &\to \mathsf{Fact}\big(X, G, f^*w\big) \\ \big(\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0\big) &\mapsto \big(f^*\mathcal{E}_{-1}, f^*\mathcal{E}_0, f^*\phi^{\mathcal{E}}_{-1}, f^*\phi^{\mathcal{E}}_0\big) \end{aligned}$$

and

$$\begin{aligned} f_* : \mathsf{Fact}\big(X, G, f^*w\big) &\to \mathsf{Fact}(X, G, w) \\ \big(\mathcal{F}_{-1}, \mathcal{F}_0, \phi^{\mathcal{F}}_{-1}, \phi^{\mathcal{F}}_0\big) &\mapsto \big(f_*\mathcal{F}_{-1}, f_*\mathcal{F}_0, f_*\phi^{\mathcal{F}}_{-1}, f_*\phi^{\mathcal{F}}_0\big). \end{aligned}$$

Note that by the projection formula $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}) \cong (f_*\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}$ *under which* $f_*(f^*w)$ *corresponds to* w *so this is well-defined.*

Definition **3.33.** — *For a factorization,* $\mathcal{E}$*, of* $0 \in \Gamma(X, \mathcal{L})^G$*. We let the* **unfolding** *of* $\mathcal{E}$ *be the complex* $\eth\mathcal{E} \in \mathsf{Qcoh}_G(X)$ *with*

$$(\eth\mathcal{E})_j = \begin{cases} \mathcal{E}_{-1} \otimes \mathcal{L}^l & j = 2l - 1 \\ \mathcal{E}_0 \otimes \mathcal{L}^l & j = 2l. \end{cases}$$

We shall also use a slightly different version of the pushforward. Let X *be equipped with an action of* G *and consider the structure morphism,* $p : X \to \operatorname{Spec} k$*. It is* G*-equivariant if we equip* $\operatorname{Spec} k$ *with the trivial action. Then, we have a pushforward*

$$\begin{aligned} p_* : \mathsf{Fact}(X, G, 0) &\to \mathsf{Qcoh}_G(\operatorname{Spec} k) \\ \mathcal{F} &\mapsto p_*(\eth\mathcal{F}) \end{aligned}$$

where $p_* : \mathsf{Qcoh}_G(X) \to \mathsf{Qcoh}_G(\operatorname{Spec} k)$ *is the usual pushforward of equivariant sheaves.*

Lemma **3.34.** — *Let* $\mathcal{E}, \mathcal{F} \in \mathsf{Fact}(X, G, w)$. *Then, we have an isomorphism of complexes*

$$\left(p_* \mathcal{H}om_X(\mathcal{E}, \mathcal{F})\right)^G \cong \mathrm{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F}).$$

Proof. — This is immediate from the definitions. □

Lemma **3.35.** — *Push-forward,* f_**, is right adjoint to pull-back,* f^*.

Proof. — Applying the standard adjunction between f^* and f_* for equivariant sheaves to the components of the factorization gives the statement. □

We also define their derived analogs.

Definition **3.36.** — *Define the left-derived functor of* f^* *by*

$$\begin{aligned} \mathbf{L}f^* : \mathrm{D}^{\mathrm{abs}}\left[\mathsf{Fact}(Y, G, w)\right] &\to \mathrm{D}^{\mathrm{abs}}\left[\mathsf{Fact}\left(X, G, f^*w\right)\right] \\ \mathcal{E} &\mapsto f^*\mathcal{V} \end{aligned}$$

where $\mathcal{V}$ *is a factorization with locally-free components quasi-isomorphic to* $\mathcal{E}$.

Define the right-derived functor of f_* *by*

$$\begin{aligned} \mathbf{R}f_* : \mathrm{D}^{\mathrm{abs}}\left[\mathsf{Fact}\left(X, G, f^*w\right)\right] &\to \mathrm{D}^{\mathrm{abs}}\left[\mathsf{Fact}(X, G, w)\right] \\ \mathcal{E} &\mapsto f_*\mathcal{I} \end{aligned}$$

where $\mathcal{I}$ *is a factorization with injective components quasi-isomorphic to* $\mathcal{E}$.

Lemma **3.37.** — *Both* $\mathbf{L}f^*$ *and* $\mathbf{R}f_*$ *are well-defined, i.e. they do not depend on the choices of representatives of a quasi-isomorphism class.*

Proof. — The derived push-forward is well-defined by Proposition 3.11 since $[\mathsf{Inj}(X, G, f^*w)] \cong \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X, G, f^*w)]$.

The derived pull-back functor, f^*, is well-defined by Proposition 3.14 since $\mathrm{D}^{\mathrm{abs}}[\mathsf{Vect}(X, G, w)] \cong \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X, G, w)]$ and f^* preserves acyclic complexes of locally-free sheaves. □

Lemma **3.38.** — *For each,* $\mathcal{E} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(Y, G, w)]$ *and* $\mathcal{F} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(X, G, f^*w)]$*, there is a natural isomorphism*

$$\mathbf{R}f_*\mathcal{F} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{E} \cong \mathbf{R}f_*\left(\mathcal{F} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathbf{L}f^*\mathcal{E}\right).$$

Proof. — This follows from replacing $\mathcal{E}$ by a factorization with locally-free components, $\mathcal{F}$ by a factorization with injective components, and applying the projection formula, Lemma 2.8, to the components of the factorizations. □

We also have an extension of pullback to allow for a group homomorphism.

Definition **3.39.** — *Assume we have two smooth varieties,* X *and* Y*, and two affine algebraic groups,* G *and* H*. Let* $\psi : G \to H$ *be a homomorphism and assume that* G *acts on* X *while* H *acts on* Y*. Let* $f : X \to Y$ *be a* ψ*-equivariant morphism. Let* $w \in \Gamma(Y, \mathcal{L})^H$ *so that* $f^*w \in \Gamma(X, f^*\mathcal{L})^G$*. We have a functor,*

$$\begin{aligned} f^* : \mathsf{Fact}(Y, H, w) &\to \mathsf{Fact}\big(X, G, f^*w\big) \\ \big(\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0\big) &\mapsto \big(f^*\mathcal{E}_{-1}, f^*\mathcal{E}_0, f^*\phi^{\mathcal{E}}_{-1}, f^*\phi^{\mathcal{E}}_0\big). \end{aligned}$$

The left-derived functor of f^* *is*

$$\begin{aligned} \mathbf{L}f^* : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(Y, H, w)\big] &\to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}\big(X, G, f^*w\big)\big] \\ \mathcal{E} &\mapsto f^*\mathcal{V} \end{aligned}$$

where $\mathcal{V}$ *is a factorization with locally-free components quasi-isomorphic to* $\mathcal{E}$*.*

Lemma **3.40.** — *The functor,* $\mathbf{L}f^*$*, is well-defined, i.e. it does not depend on the choice of representatives of a quasi-isomorphism class.*

Proof. — The proof is completely analogous to that of Lemma 3.37. $\square$

We also extend the restriction and induction functors.

Definition **3.41.** — *Let* X *be a smooth variety equipped with an action of an affine algebraic group,* G*. Let* $w \in \Gamma(G, \mathcal{L})^G$*. Let* $\psi : H \to G$ *be a closed subgroup of* G*.*

$$\begin{aligned} \mathrm{Res}^G_H : \mathsf{Fact}(X, G, w) &\to \mathsf{Fact}(X, H, w) \\ \big(\mathcal{E}_{-1}, \mathcal{E}_0, \phi^{\mathcal{E}}_{-1}, \phi^{\mathcal{E}}_0\big) &\mapsto \big(\mathrm{Res}^G_H \mathcal{E}_{-1}, \mathrm{Res}^G_H \mathcal{E}_0, \mathrm{Res}^G_H \phi^{\mathcal{E}}_{-1}, \mathrm{Res}^G_H \phi^{\mathcal{E}}_0\big) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ind}^G_H : \mathsf{Fact}(X, H, w) &\to \mathsf{Fact}(X, G, w) \\ \big(\mathcal{F}_{-1}, \mathcal{F}_0, \phi^{\mathcal{F}}_{-1}, \phi^{\mathcal{F}}_0\big) &\mapsto \big(\mathrm{Ind}^G_H \mathcal{F}_{-1}, \mathrm{Ind}^G_H \mathcal{F}_0, \mathrm{Ind}^G_H \phi^{\mathcal{F}}_{-1}, \mathrm{Ind}^G_H \phi^{\mathcal{F}}_0\big). \end{aligned}$$

The action on morphisms is clear.

The restriction functor, Res^G_H*, is exact so it immediately descends to*

$$\mathrm{Res}^G_H : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, H, w)\big].$$

The induction functor, Ind^G_H*, is left-exact so we have its right-derived functor,*

$$\begin{aligned} \mathbf{R}\mathrm{Ind}^G_H : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, H, w)\big] &\to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big] \\ \mathcal{E} &\mapsto f_*\mathcal{I} \end{aligned}$$

where $\mathcal{I}$ *is a factorization with injective components quasi-isomorphic to* $\mathcal{E}$*.*

Lemma **3.42.** — *The functor,* $\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}}$, *is left adjoint to the functor,* $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}$.

Proof. — This is an immediate consequence of Lemma 2.15. □

Corollary **3.43.** — *We have an adjoint pair of functors,*

$$\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{H}, w)\big]$$
$$\mathbf{R}\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{H}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\big].$$

Proof. — This is an immediate consequence of Lemma 3.42. □

Lemma **3.44.** — *For each,* $\mathcal{E} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$ *and* $\mathcal{F} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{H}, w)]$, *there is a natural isomorphism*

$$\mathbf{R}\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \mathcal{F} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{\mathrm{Y}}} \mathcal{E} \cong \mathbf{R}\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\big(\mathcal{F} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{\mathrm{X}}} \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} \mathcal{E}\big).$$

Proof. — This follows from replacing $\mathcal{F}$ by a factorization with injective components and applying the projection formula, Lemma 2.16, to the components of the factorizations. □

Finally, we extend the functor of invariants.

Definition **3.45.** — *Let* N *be a closed normal subgroup of* G. *Let* X *be a smooth variety equipped with an action of* G *on which* N *acts trivially. Let* $\mathcal{L}$ *be an invertible* G/N*-equivariant sheaf. Note that* $\mathcal{L}$ *inherits a* G*-equivariant structure. Consider a section* $w \in \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}} \cong \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}/\mathrm{N}}$. *We define*

$$(\bullet)^{\mathrm{N}} : \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{Fact}(\mathrm{X}, \mathrm{G}/\mathrm{N}, w)$$
$$\big(\mathcal{E}_{-1}, \mathcal{E}_0, \phi_{-1}^{\mathcal{E}}, \phi_0^{\mathcal{E}}\big) \mapsto \big(\mathcal{E}_{-1}^{\mathrm{N}}, \mathcal{E}_0^{\mathrm{N}}, (\phi_{-1}^{\mathcal{E}})^{\mathrm{N}}, (\phi_0^{\mathcal{E}})^{\mathrm{N}}\big).$$

The derived functor of invariants is

$$(\bullet)^{\mathbf{R}\mathrm{N}} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}/\mathrm{N}, w)\big]$$
$$\mathcal{E} \mapsto \mathcal{I}^{\mathrm{N}}$$

where $\mathcal{I}$ *is a factorization that has injective components and that is quasi-isomorphic to* $\mathcal{E}$.

Definition **3.46.** — *Let* $\mathcal{L}$ *be an invertible equivariant sheaf on* X *and let* $w \in \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}}$. *Let*

$$\mathrm{V}(\mathcal{L}) := \underline{\mathrm{Spec}}_{\mathrm{X}}(\mathrm{Sym}\,\mathcal{L})$$

denote the geometric vector bundle associated to $\mathcal{L}$. *It carries an action of* $\mathrm{G} \times \mathbf{G}_m$ *where* G *acts via the equivariant structure on* $\mathcal{L}$ *and* $\mathbf{G}_m$ *dilates the fibers of the bundle. The section,* w, *defines a regular* *function,* $f_w \in \Gamma(\mathrm{V}(\mathcal{L}), \mathcal{O}_{\mathrm{V}(\mathcal{L})}(1))^{\mathrm{G}\times\mathbf{G}_m}$ *where* (1) *denotes the projection character,* $\mathrm{G}\times\mathbf{G}_m \to \mathbf{G}_m$*. Finally, let* $\mathrm{U}(\mathcal{L})$ *denote the complement of the zero section in* $\mathrm{V}(\mathcal{L})$*. Let* $\pi : \mathrm{U}(\mathcal{L}) \to \mathrm{X}$ *denote the projection. It is equivariant with respect to the projection,* $\mathrm{G}\times\mathbf{G}_m \to \mathbf{G}_m$*.*

Lemma **3.47.** — *The pull back functor,*

$$\pi^* : \mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X} \to \mathrm{Qcoh}_{\mathrm{G}\times\mathbf{G}_m}\,\mathrm{U}(\mathcal{L}),$$

is an equivalence. Moreover, π^* *induces equivalences between subcategories of coherent and locally-free equivariant sheaves.*

Proof. — The variety, $\mathrm{U}(\mathcal{L})$, is a $\mathbf{G}_m$-torsor over X. Thus, the fppf quotient of $\mathrm{U}(\mathcal{L})$ by $\mathbf{G}_m$ is X. The statement of the lemma is a consequence of faithfully-flat descent. In other words, the global quotient stack $[\mathrm{U}(\mathcal{L})/\mathbf{G}_m]$ is represented by X, and therefore they have the same sheaf theory. □

Lemma **3.48.** — *The pull back functor,*

$$\pi^* : \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{Fact}\big(\mathrm{U}(\mathcal{L}), \mathrm{G}\times\mathbf{G}_m, f_w\big),$$

is an equivalence of dg-categories. Moreover, π^* *restricts to equivalences,*

$$\begin{aligned}
&\pi^* : \mathsf{Inj}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{Inj}\big(\mathrm{U}(\mathcal{L}), \mathrm{G}\times\mathbf{G}_m, f_w\big),\\
&\pi^* : \mathsf{Vect}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{Vect}\big(\mathrm{U}(\mathcal{L}), \mathrm{G}\times\mathbf{G}_m, f_w\big),\\
&\pi^* : \mathsf{fact}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{fact}\big(\mathrm{U}(\mathcal{L}), \mathrm{G}\times\mathbf{G}_m, f_w\big),\\
&\pi^* : \mathsf{vect}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{vect}\big(\mathrm{U}(\mathcal{L}), \mathrm{G}\times\mathbf{G}_m, f_w\big).
\end{aligned}$$

Proof. — This is an immediate consequence of Lemma 3.47. □

The following definitions seem to have no natural extension to the case of general equivariant line bundles. They will be essential later in the paper.

Definition **3.49.** — *Let* X *and* Y *be smooth varieties and let* $w \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ *and* $v \in \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}})$*. We set*

$$w \boxplus v := w \otimes 1 + 1 \otimes v \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \otimes_k \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) \cong \Gamma(\mathrm{X}\times\mathrm{Y}, \mathcal{O}_{\mathrm{X}\times\mathrm{Y}}).$$

We will have to deal with two potentials, $w, v \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$, that are semi-invariant with respect to different characters of different groups. The largest group for which $w \boxplus v$ is semi-invariant is as follows.

Definition **3.50.** — *Let* G *and* H *be affine algebraic groups and let* $\chi : \mathrm{G} \to \mathbf{G}_m$ *and* $\chi' : \mathrm{H} \to \mathbf{G}_m$ *be characters. Define a character of* $\mathrm{G} \times \mathrm{H}$ *by*

$$\begin{aligned}\chi' - \chi : \mathrm{G} \times \mathrm{H} &\to \mathbf{G}_m \\ (g, h) &\mapsto \chi(g)^{-1}\chi'(h).\end{aligned}$$

Let $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}$ *be the kernel of* $\chi' - \chi$ *or equivalently the fiber product of* G *and* H *over* $\mathbf{G}_m$.

Definition **3.51.** — *Let* X *be a smooth variety equipped with an action of an affine algebraic group,* G*, and let* Y *be a smooth variety equipped with an action of an affine algebraic group,* H*. Let* $\chi : \mathrm{G} \to \mathbf{G}_m$ *and* $\chi' : \mathrm{H} \to \mathbf{G}_m$ *be characters. Let* $w \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}(\chi))^{\mathrm{G}}$ *and* $v \in \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}(\chi'))^{\mathrm{H}}$.

We have a dg-functor

$$\boxtimes : \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Fact}(\mathrm{Y}, \mathrm{H}, v) \to \mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)$$

called the **exterior product**. *It is defined as*

$$\mathcal{E} \boxtimes \mathcal{F} := \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}\big) \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}\big).$$

Explicitly, we have

$$\begin{aligned}(\mathcal{E} \boxtimes \mathcal{F})_{-1} &:= \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}(\mathcal{E}_{-1} \boxtimes \mathcal{F}_0 \oplus \mathcal{E}_0 \boxtimes \mathcal{F}_{-1}) \\ (\mathcal{E} \boxtimes \mathcal{F})_0 &:= \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\mathcal{E}_0 \boxtimes \mathcal{F}_0 \oplus \mathcal{E}_{-1}(\chi) \boxtimes \mathcal{F}_{-1}\big).\end{aligned}$$

Lemma **3.52.** — *Assume that* $\chi' - \chi$ *is not torsion. Let* $\mathcal{E}^1 \in \mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)]$, $\mathcal{F}^1 \in \mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{Y}, \mathrm{H}, v)]$ *and let* $\mathcal{E}^2 \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$, $\mathcal{F}^2 \in \mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{Y}, \mathrm{H}, v)]$. *Taking exterior products induces a natural isomorphism:*

$$\begin{aligned}\boxtimes : \bigoplus_{t\in\mathbf{Z}} &\mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X},\mathrm{G},w)]}\big(\mathcal{E}^1, \mathcal{E}^2[-t]\big) \otimes_k \mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{Y},\mathrm{H},v)]}\big(\mathcal{F}^1, \mathcal{F}^2[t]\big) \\ &\to \mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},w\boxplus v)]}\big(\mathcal{E}^1 \boxtimes \mathcal{F}^1, \mathcal{E}^2 \boxtimes \mathcal{F}^2\big).\end{aligned}$$

Proof. — We may assume that all factorizations are locally-free in order to simplify notation for the derived functors in the proof. We suppress the subscripts on Hom's and tensor products to help control notational girth.

We have the following chain of isomorphisms

$$\begin{aligned}&\mathrm{Hom}\big(\mathcal{E}^1 \boxtimes \mathcal{F}^1, \mathcal{E}^2 \boxtimes \mathcal{F}^2\big) \\ &:= \mathrm{Hom}(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^1\big) \otimes \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^1\big), \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^2\big) \otimes \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^2\big) \\ &\cong \mathrm{Hom}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^1\big), \mathcal{H}om\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^1\big), \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^2\big) \otimes \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^2\big)\big)\big) \\ &\cong \mathrm{Hom}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^1\big), \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_1^*\mathcal{E}^2\big) \otimes \mathcal{H}om\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^1\big), \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\pi_2^*\mathcal{F}^2\big)\big)\big)\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\cong \mathrm{Hom}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}(\pi_1^*\mathcal{E}^1), \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}(\pi_1^*\mathcal{E}^2) \otimes \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)\big)\\
&\cong \mathrm{Hom}\big(\pi_1^*\mathcal{E}^1, \mathrm{Ind}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}(\pi_1^*\mathcal{E}^2) \otimes \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)\big)\big)\\
&\cong \mathrm{Hom}\big(\pi_1^*\mathcal{E}^1, \pi_1^*\mathcal{E}^2 \otimes \mathrm{Ind}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)\big)\\
&\cong \mathrm{Hom}\Big(\pi_1^*\mathcal{E}^1, \pi_1^*\mathcal{E}^2 \otimes \bigoplus_{l\in\mathbf{Z}} \pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)(l(\chi'-\chi))\Big).
\end{aligned}
$$

The second line is by definition. The third line is Corollary 3.28 i.e. tensor-Hom adjunction. The fourth line can be seen by appealing to Lemma 3.31 and associativity of tensor product using the fact that $\mathcal{F}^1$ is locally-free of finite rank to pull out a dual and put it back in. Note that $\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}$ commutes with duals so the order of operations is not germane. The fifth line follows from the fact that the functors Res and π_i^* are both monoidal, so they commute with $\otimes$ and $\mathcal{H}om$. The sixth line uses the adjunction of Corollary 3.43. Since we have assumed that $\chi' - \chi$ is not torsion, we have an isomorphism

$$
\mathrm{G}\times\mathrm{H}/\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H} \cong \mathbf{G}_m.
$$

As this quotient is affine, Ind is exact and $\mathbf{R}\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \cong \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}$. The seventh line is the projection formula for the induction functor, Lemma 3.44. The eighth line uses Lemma 2.18.

Let $q : \mathrm{Y} \to \operatorname{Spec} k$ and $p : \mathrm{X} \to \operatorname{Spec} k$ be the structure morphisms. Continuing with the isomorphisms from Equation (3.2) and using morphism spaces in $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$, we have

$$
\begin{aligned}
&\mathrm{Hom}(\mathcal{E}^1 \boxtimes \mathcal{F}^1, \mathcal{E}^2 \boxtimes \mathcal{F}^2)\\
&\quad\cong \mathrm{Hom}\Big(\mathcal{E}^1, \Big(\mathbf{R}\pi_{1*}\pi_1^*\mathcal{E}^2 \otimes \bigoplus_{l\in\mathbf{Z}} \pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)(l(\chi'-\chi))\Big)^{\mathbf{R}\mathrm{H}}\Big)\\
&\quad\cong \mathrm{Hom}\Big(\mathcal{E}^1, \mathcal{E}^2 \otimes \Big(\mathbf{R}\pi_{1*} \bigoplus_{l\in\mathbf{Z}} \pi_2^*\mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)(l(\chi'-\chi))\Big)^{\mathbf{R}\mathrm{H}}\Big)\\
&\quad\cong \mathrm{Hom}\Big(\mathcal{E}^1, \mathcal{E}^2 \otimes \Big(\mathbf{R}\pi_{1*}\pi_2^* \bigoplus_{l\in\mathbf{Z}} \mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)(l(\chi'-\chi))\Big)^{\mathbf{R}\mathrm{H}}\Big)\\
&\quad\cong \mathrm{Hom}\Big(\mathcal{E}^1, \mathcal{E}^2 \otimes \Big(p^*\mathbf{R}q_* \bigoplus_{l\in\mathbf{Z}} \mathcal{H}om(\mathcal{F}^1, \mathcal{F}^2)(l(\chi'-\chi))\Big)^{\mathbf{R}\mathrm{H}}\Big)\\
&\quad\cong \mathrm{Hom}\Big(\mathcal{E}^1, \mathcal{E}^2 \otimes_k \bigoplus_{l\in\mathbf{Z}} \mathrm{Hom}(\mathcal{F}^1, \mathcal{F}^2[2l])(-l\chi) \oplus \mathrm{Hom}(\mathcal{F}^1, \mathcal{F}^2[2l+1])(-l\chi)[-1]\Big)
\end{aligned}
$$

$$\cong \mathrm{Hom}\Big(\mathcal{E}^1, \bigoplus_{t\in\mathbf{Z}} \mathcal{E}^2[-t] \otimes_k \mathrm{Hom}\big(\mathcal{F}^1, \mathcal{F}^2[t]\big)\Big)$$
$$\cong \bigoplus_{t\in\mathbf{Z}} \mathrm{Hom}\big(\mathcal{E}^1, \mathcal{E}^2[-t]\big) \otimes_k \mathrm{Hom}\big(\mathcal{F}^1, \mathcal{F}^2[t]\big).$$

The first line uses that the right adjoint to π_1^* is the composition $(\mathbf{R}\pi_{1*})^{\mathbf{R}\mathrm{H}}$ by Corollary 2.24. The second line morally uses the projection formula. However, we have not provided a projection formula in this general context. We can work around this by deriving the two projection formulas from Lemmas 2.8 and 2.16 and rewriting the functor $\pi_1^* = (\pi_1')^* \circ \mathrm{Res}_r$ where $r : \mathrm{G} \times \mathrm{H} \to \mathrm{G}$ denotes the projection homomorphism and $\pi_1' : \mathrm{X} \times \mathrm{Y} \to \mathrm{X}$ denotes the $\mathrm{G} \times \mathrm{H}$ equivariant projection where H acts trivially on X. The fourth line uses flat base change, Lemma 2.19. The fifth line uses Lemma 2.26 to pull the invariants inside p^*. The sixth line comes from substitution of the isomorphism,

$$(\mathbf{3.3}) \qquad \Big(\mathbf{R}q_* \bigoplus_{l\in\mathbf{Z}} \mathcal{H}om\big(\mathcal{F}^1, \mathcal{F}^2\big)\big(l(\chi' - \chi)\big)\Big)^{\mathbf{R}\mathrm{H}}$$
$$\cong \bigoplus_{l\in\mathbf{Z}} \mathrm{Hom}\big(\mathcal{F}^1, \mathcal{F}^2[2l]\big)(-l\chi) \oplus \mathrm{Hom}\big(\mathcal{F}^1, \mathcal{F}^2[2l+1]\big)(-l\chi)[-1].$$

Equation (3.3) is a consequence of Lemma 3.34 and the identity $(\chi') = [2]$. The sixth line uses that $\mathcal{E}^2 \otimes \bullet$ commutes with coproducts and a straightforward identification of the twists with shifts using $(\chi) = [2]$. The final line follows since $\mathcal{E}^1$ is a coherent factorization. By Proposition 3.15 it is a compact object, and therefore, $\mathrm{Hom}(\mathcal{E}^1, \bullet)$ commutes with coproducts. The total isomorphism gives an inverse to $\boxtimes$. □

Finally, let us define a version of an integral transformation for factorizations.

Definition **3.53.** — *Let* $\mathcal{P} \in \mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, (-w) \boxplus v)$. *Equip* Y *with the trivial* G *action to give it a* $\mathrm{G} \times \mathrm{H}$ *action in full. View* $\pi_2 : \mathrm{X} \times \mathrm{Y} \to \mathrm{Y}$ *as* $\mathrm{G} \times \mathrm{H}$*-equivariant. Set*

$$\Phi_{\mathcal{P}}^{\mathrm{X}\to\mathrm{Y}} : \mathsf{Fact}(\mathrm{X}, \mathrm{G}, w) \to \mathsf{Fact}(\mathrm{Y}, \mathrm{H}, v)$$
$$\mathcal{E} \mapsto \big(\pi_{2*}\big(\pi_1^*\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathrm{Ind}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}^{\mathrm{G}\times\mathrm{H}} \mathcal{P}\big)\big)^{\mathrm{G}}.$$

We will also denote the associated functor on the derived categories by

$$\Phi_{\mathcal{P}}^{\mathrm{X}\to\mathrm{Y}} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{Y}, \mathrm{H}, v)\big]$$
$$\mathcal{E} \mapsto \big(\mathbf{R}\pi_{2*}\big(\mathbf{L}\pi_1^*\mathcal{E} \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathrm{Ind}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}^{\mathrm{G}\times\mathrm{H}} \mathcal{P}\big)\big)^{\mathbf{R}\mathrm{G}}.$$

The object $\mathcal{P}$ *is called the* **kernel** *of* $\Phi_{\mathcal{P}}^{\mathrm{X}\to\mathrm{Y}}$.

View $\mathcal{F}$ *as a factorization of* $0 \in \Gamma(X, \mathcal{O}_X(\chi))^G$ *as* $\Upsilon\mathcal{F}$. *Define the factorization,*

$$\nabla(\mathcal{F}) := \mathrm{Ind}_G^{G\times\mathbf{G}_m G} \Delta_*\mathcal{F} := \Upsilon \, \mathrm{Ind}_G^{G\times\mathbf{G}_m G} \Delta_*\mathcal{F}.$$

Set

$$\nabla := \nabla(\mathcal{O}_X).$$

Lemma **3.54.** — *There is a natural transformation of dg-functors*

$$\Phi_{\nabla(\mathcal{F})} \to \bullet \otimes_{\mathcal{O}_X} \mathcal{F}$$

inducing an isomorphism of derived functors,

$$\Phi_{\nabla(\mathcal{F})} \cong \bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{F} : \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big] \to \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(X, G, w)\big].$$

In particular, ∇ *is the kernel of the identity functor.*

Proof. — For any $\mathcal{E} \in \mathsf{Fact}(X, G, w)$, we have a natural morphism

$$\begin{aligned}\big(\pi_{2*}\big(\pi_1^*\mathcal{E} \otimes_{\mathcal{O}_{X\times X}} \mathrm{Ind}_G^{G\times G} \Delta_*\mathcal{F}\big)\big)^G &\mapsto \big(\pi_{2*} \mathrm{Ind}_G^{G\times G} \Delta_*\big(\Delta^* \mathrm{Res}_G^{G\times G} \pi_1^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}\big)\big)^G \\ &\cong \big(\pi_{2*} \mathrm{Ind}_G^{G\times G} \Delta_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})\big)^G \\ &\cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}.\end{aligned}$$

The first line is from the projection formula for Δ^*, Δ_*, Lemma 2.8, and $\mathrm{Res}_G^{G\times G}$, $\mathrm{Ind}_G^{G\times G}$, Lemma 2.16, applied component-wise to a factorization. The second line comes from the isomorphism

$$\Delta^* \mathrm{Res}_G^{G\times G} \pi_1^* \cong \Delta^*\pi_1^* \cong (\pi_1 \circ \Delta)^* \cong \mathrm{Id}$$

where for the first isomorphism we view π_1 as G-equivariant with respect to the diagonal action of G on $X \times X$. For the third line, we use that

$$\big(\pi_{2*} \mathrm{Ind}_G^{G\times G} \Delta_*\big)^G \cong \mathrm{Id}$$

as the functor, $(\pi_{2*} \mathrm{Ind}_G^{G\times G} \Delta_*)^G$, is right adjoint to $\Delta^* \mathrm{Res}_G^{G\times G} \pi_2^* \cong \mathrm{Id}$. Combining the natural morphisms gives the natural transformation

$$\Phi_{\nabla(\mathcal{F})} \to \bullet \otimes_{\mathcal{O}_X} \mathcal{F}.$$

The statement for the derived functors follows via the same argument, replacing the usual functors by their derived versions, and noting that derived projection formula is an isomorphism by Lemma 3.38. □

Lemma **3.55.** — *Let $p : X \to \operatorname{Spec} k$ be the structure map. There is a natural transformation of dg-functors*

$$\left(p_* \Delta^* (\mathcal{E}^\vee \boxtimes \mathcal{F})\right)^G \to \operatorname{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})$$

inducing a natural isomorphism

$$\left(\mathbf{R}p_* \mathbf{L}\Delta^* (\mathcal{E}^\vee \boxtimes \mathcal{F})\right)^{\mathbf{R}G} \cong \mathbf{R}\operatorname{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})$$

if we assume $\mathcal{E} \in D^{\mathrm{abs}}[\mathsf{fact}(X, G, w)]$.

Proof. — We have

$$\begin{aligned}\left(p_* \Delta^* \mathcal{E}^\vee \boxtimes \mathcal{F}\right)^G &= \left(p_* \Delta^* \left(\operatorname{Res}^{G\times G}_{G\times_{\mathbf{G}_m} G} \pi_1^* \mathcal{E}^\vee \otimes_{\mathcal{O}_{X\times X}} \operatorname{Res}^{G\times G}_{G\times_{\mathbf{G}_m} G} \pi_2^* \mathcal{F}\right)\right)^G \\ &\cong \left(p_* (\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F})\right)^G \\ &\to \left(p_* \mathcal{H}om_X(\mathcal{E}, \mathcal{F})\right)^G \\ &\cong \operatorname{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F}).\end{aligned}$$

The first line is by definition. The second line follows from by distributing Δ^* across the tensor product then observing that we have an isomorphism $\operatorname{Res}^{G\times G}_{G\times_{\mathbf{G}_m} G} \pi_1^* \cong (\pi_1')^*$ where $\pi_1' : X \times X \to X$ is equivariant with respect to the first projection $G \times_{\mathbf{G}_m} G \to G$ and similarly, $\operatorname{Res}^{G\times G}_{G\times_{\mathbf{G}_m} G} \pi_2^* \cong (\pi_2')^*$. Finally, $\pi_1 \circ \Delta \cong \pi_2 \circ \Delta \cong 1_X$. The third line follows from the natural map

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_X(\mathcal{E}, \mathcal{F}).$$

The fourth line is induced from the isomorphism of functors

$$\left(p_* \mathcal{H}om_X(\mathcal{E}, \mathcal{F})\right)^G \cong \operatorname{Hom}_{\mathsf{Fact}(X,G,w)}(\mathcal{E}, \mathcal{F})$$

of Lemma 3.34.

The statement for the derived functors follows via analogous arguments replacing the usual functors by their derived version and using Lemma 3.31 to know that the natural map

$$\mathcal{E}^{\mathbf{L}\vee} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathbf{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F})$$

is an isomorphism if $\mathcal{E}$ is coherent. □

Definition **3.56.** — *The* ***trace functor*** *on* $D^{\mathrm{abs}}[\mathsf{Fact}(X \times X, G \times_{\mathbf{G}_m} G, (-w) \boxplus w)]$ *is the functor*

$$\begin{aligned}\mathbf{L}\mathrm{Tr} := \left(\mathbf{R}p_* \mathbf{L}\Delta^*\right)^{\mathbf{R}G} &: D^{\mathrm{abs}}\left[\mathsf{Fact}\left(X \times X, G \times_{\mathbf{G}_m} G, (-w) \boxplus w\right)\right] \\ &\to D^{\mathrm{b}}(\operatorname{Qcoh} \operatorname{Spec} k).\end{aligned}$$

Lemma **3.57.** — *Assume that* $(\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G})/\mathrm{G}\cong \mathrm{K}_\chi$ *is finite. There is an isomorphism of functors*

$$\mathrm{Tr}\cong \mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},(-w)\boxplus w)}\left(\nabla^{\mathbf{L}\vee},\bullet\right)$$

on $\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},(-w)\boxplus w)]$.

Proof. — As $(\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G})/\mathrm{G}\cong \mathrm{K}_\chi$ is finite, $\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}}$ preserves coherent G-equivariant sheaves. For coherent factorizations, dualization is an anti-equivalence by Lemma 3.30.

Now, we have

$$\begin{aligned}
\mathbf{L}\mathrm{Tr} &= \left(\mathbf{R}p_*\mathbf{L}\Delta^*\right)^{\mathbf{R}\mathrm{G}}\\
&\cong \mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X},\mathrm{G},0)}\left(\mathcal{O}_\mathrm{X},\mathbf{L}\Delta^*(\bullet)\right)\\
&\cong \mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X},\mathrm{G},0)}\left(\mathbf{L}\Delta^*(\bullet)^{\mathbf{L}\vee},\mathcal{O}_\mathrm{X}\right)\\
&\cong \mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},w\boxplus(-w))}\left((\bullet)^{\mathbf{L}\vee},\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}}\Delta_*\mathcal{O}_\mathrm{X}\right)\\
&\cong \mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},(-w)\boxplus w)}\left(\nabla^{\mathbf{L}\vee},\bullet\right).
\end{aligned}$$

The first line is a definition. The second line is from the isomorphism of functors,

$$\mathbf{R}\mathrm{Hom}_{\mathsf{Fact}(\mathrm{X},\mathrm{G},0)}(\mathcal{O}_\mathrm{X},\bullet)\cong(\mathbf{R}p_*)^{\mathbf{R}\mathrm{G}}.$$

The third line uses that Δ^* commutes with duals and $\bullet$ is assumed to be coherent. The fourth line is adjunction between $\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}}\Delta_*$ and $\mathbf{L}\Delta^*$. The fifth line uses dualization, coherence of $\bullet$, and the definition of ∇. □

In the process of proving a generation statement for categories of factorizations, we will want to make use of some geometry. As such, we need an alternate, more geometric, characterization of these factorization categories. This characterization is due, in various generality, to Eisenbud [Eis80], Buchweitz [Buc86], Orlov [Orl09, Orl12], Polishchuk-Vaintrob [PV10], and [Pos11].

Let us recall the definition of the singularity category.

Definition **3.58.** — *Let* Y *be a scheme of finite type over* k *and let* G *be an affine algebraic group acting on* Y. *Assume that* Y *has enough locally-free* G*-equivariant sheaves. The* G*-equivariant* **singularity category**, *or* G*-equivariant* **stable category**, *of* Y *is the Verdier quotient*

$$\mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y}):=\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\mathrm{Y})/\operatorname{perf}_{\mathrm{G}}\mathrm{Y}$$

where $\operatorname{perf}_{\mathrm{G}}\mathrm{Y}$ *is the thick subcategory of locally-free* G*-equivariant sheaves of finite rank on* Y. *Let* Z *be a closed* G*-invariant subset of* Y, *then we let* $\mathrm{D}^{\mathrm{sg}}_{\mathrm{Z},\mathrm{G}}(\mathrm{Y})$ *be the kernel of the functor,* $j^*:\mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y})\to\mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{U})$.

Assume we have a smooth variety X equipped with an action of an affine algebraic group G and an invariant section $w \in \Gamma(X, \mathcal{L})^G$ for an invertible equivariant sheaf, $\mathcal{L}$. Set $Y = Z_w$ to be the vanishing locus of w. Let $i : Y \to X$ denote the inclusion.

Lemma **3.59.** — *The scheme,* Y*, has enough locally-free* G*-equivariant sheaves. Moreover, every coherent* G*-equivariant sheaf on* Y *admits an epimorphism from* $i^*\mathcal{V}$ *where* $\mathcal{V}$ *is locally-free of finite rank.*

Proof. — As X is smooth, it has enough locally-free G-equivariant sheaves by Theorem 2.29. Given any coherent G-equivariant sheaf on Y, $\mathcal{E}$, we can find a locally-free G-equivariant sheaf of finite rank, $\mathcal{V}$, and an epimorphism, $\psi : \mathcal{V} \to i_*\mathcal{E}$. The morphism, $i^*\psi : i^*\mathcal{V} \to i^*i_*\mathcal{E} \cong \mathcal{E}$, remains an epimorphism as i^* is right exact. □

Consider the functor,

$$\begin{aligned} \mathrm{cok} : [\mathsf{vect}(X, G, w)] &\to D^{\mathrm{sg}}_G(Y) \\ \mathcal{E} &\mapsto \mathrm{cok}\, \phi_0^{\mathcal{E}}. \end{aligned}$$

Lemma **3.60.** — *Assume that* w *is not identically zero on any component of* X*. The functor,* cok*, is well-defined and exact.*

Proof. — This is a special case of [PV10, Lemma 3.12]. □

Lemma **3.61.** — *Assume that* w *is not identically zero on any component of* X*. Let* Z *be a closed* G*-invariant subset of* Y*. The functor,* cok*, descends to the absolute derived category,*

$$\mathrm{cok} : D^{\mathrm{abs}}_Z[\mathsf{vect}(X, G, w)] \to D^{\mathrm{sg}}_{Z,G}(Y).$$

Proof. — Let us treat the situation $Z = Y$ first. In the case where G is trivial, this is [Orl12, Proposition 3.2]. The same argument works with the inclusion of G. We recall the argument for the convenience of the reader. Let

$$0 \to \mathcal{G} \xrightarrow{q} \mathcal{E} \xrightarrow{p} \mathcal{F} \to 0$$

be an exact sequence of factorizations and let $\mathcal{T}$ be the totalization. Recall that

$$\begin{aligned} \mathcal{T}_{-1} &:= \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{E}_0 \oplus \mathcal{F}_{-1} \\ \mathcal{T}_0 &:= \mathcal{G}_0 \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{F}_0 \\ \phi_0^{\mathcal{T}} &:= \begin{pmatrix} \phi_0^{\mathcal{G}} \otimes_{\mathcal{O}_X} \mathcal{L} & 0 & 0 \\ q_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} & -\phi_{-1}^{\mathcal{E}} & 0 \\ 0 & p_0 & \phi_0^{\mathcal{F}} \end{pmatrix} \end{aligned}$$

$$\phi_{-1}^{\mathcal{T}} := \begin{pmatrix} \phi_{-1}^{\mathcal{G}} \otimes_{\mathcal{O}_X} \mathcal{L} & 0 & 0 \\ q_0 \otimes_{\mathcal{O}_X} \mathcal{L} & -\phi_0^{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{L} & 0 \\ 0 & p_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} & \phi_{-1}^{\mathcal{F}} \end{pmatrix}.$$

Consider the associated exact sequence over $\operatorname{coh}_G X$

$$0 \to \mathcal{G}_0 \xrightarrow{\begin{pmatrix} q_0 \\ \phi_{-1}^{\mathcal{G}} \end{pmatrix}} \mathcal{E}_0 \oplus \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\begin{pmatrix} p_0 & 0 \\ -\phi_{-1}^{\mathcal{E}} & q_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \end{pmatrix}}$$
$$\mathcal{F}_0 \oplus \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\begin{pmatrix} \phi_{-1}^{\mathcal{F}} & p_1 \otimes_{\mathcal{O}_X} \mathcal{L} \end{pmatrix}} \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \to 0$$

and let $\mathcal{U}$ be the cokernel of the map $\mathcal{G}_0 \to \mathcal{E}_0 \oplus \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}$. Let $(\alpha_0, \alpha_1) : \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{E}_0 \to \mathcal{U}$ be the epimorphism and $\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} : \mathcal{U} \to \mathcal{F}_0 \oplus \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}$ be the monomorphism. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \oplus \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \arrow[r, "\begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi_0^{\mathcal{G}} \otimes_{\mathcal{O}_X} \mathcal{L} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 0 \\ 1_{\mathcal{E}_0} & 0 \\ 0 & 1_{\mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}} \end{pmatrix}"'] & \mathcal{U} \oplus \mathcal{G}_0 \otimes_{\mathcal{O}_X} \mathcal{L} \arrow[d, "\begin{pmatrix} \beta_0 & 0 \\ \beta_1 & 0 \\ 0 & 1_{\mathcal{G}_0 \otimes_{\mathcal{O}_X} \mathcal{L}} \end{pmatrix}"] \\
\mathcal{F}_{-1} \oplus \mathcal{E}_0 \oplus \mathcal{G}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \arrow[r, "\phi_0^{\mathcal{T}}"] \arrow[d, "\begin{pmatrix} 1_{\mathcal{F}_{-1}} & 0 & 0 \end{pmatrix}"'] & \mathcal{F}_0 \oplus \mathcal{E}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{G}_0 \otimes_{\mathcal{O}_X} \mathcal{L} \arrow[d, "\begin{pmatrix} \phi_{-1}^{\mathcal{F}} & p_1 \otimes_{\mathcal{O}_X} \mathcal{L} & 0 \end{pmatrix}"] \\
\mathcal{F}_{-1} \arrow[r, "w"] & \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L}
\end{tikzcd}$$

with columns being short exact sequences. Thus, we have an exact sequence of cokernels, as coherent sheaves on Y,

$$0 \to \operatorname{cok} \begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi_0^{\mathcal{G}} \otimes_{\mathcal{O}_X} \mathcal{L} \end{pmatrix} \to \operatorname{cok} \phi_0^{\mathcal{T}} \to i^* \mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \to 0.$$

As $i^*(\mathcal{F}_{-1} \otimes_{\mathcal{O}_X} \mathcal{L})$ is trivial in $D_G^{sg}(Y)$, we have an isomorphism

$$\operatorname{cok} \begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi_0^{\mathcal{G}} \otimes_{\mathcal{O}_X} \mathcal{L} \end{pmatrix} \cong \operatorname{cok} \phi_0^{\mathcal{T}}$$

in $\mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y})$. We also have a commutative diagram

$$\begin{tikzcd}
\mathcal{G}_0 \arrow[r, "w"] \arrow[d, "\begin{pmatrix} q_0 \\ \phi^{\mathcal{G}}_{-1} \end{pmatrix}"'] & \mathcal{G}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \arrow[d, "\begin{pmatrix} 0 \\ 1_{\mathcal{G}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L}} \end{pmatrix}"] \\
\mathcal{E}_0 \oplus \mathcal{G}_{-1} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \arrow[r, "\begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi^{\mathcal{G}}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \end{pmatrix}"] \arrow[d, "(\alpha_0 \quad \alpha_1)"'] & \mathcal{U} \oplus \mathcal{G}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \arrow[d, "(1_{\mathcal{U}} \quad 0)"] \\
\mathcal{U} \arrow[r, "1_{\mathcal{U}}"] & \mathcal{U}
\end{tikzcd}$$

with columns being short exact sequences. Thus, we have an isomorphism of coherent sheaves

$$i^*\mathcal{G}_0 \cong \operatorname{cok}\begin{pmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi^{\mathcal{G}}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \end{pmatrix}.$$

Thus, $\operatorname{cok}\left(\begin{smallmatrix} \alpha_0 & \alpha_1 \\ 0 & \phi^{\mathcal{G}}_0 \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{L} \end{smallmatrix}\right) \cong \operatorname{cok}\phi^{\mathcal{T}}_0$ is trivial in $\mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y})$. This proves the statement when $\mathrm{Z} = \mathrm{Y}$.

Now the general case follows from the case where $\mathrm{Z} = \mathrm{Y}$. Indeed, it is clear that cok commutes with restriction to open subsets. Thus, we have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{abs}}[\mathsf{vect}(\mathrm{X}, \mathrm{G}, w)] \arrow[r, "\operatorname{cok}"] \arrow[d, "j^*"'] & \mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y}) \arrow[d, "j^*"] \\
\mathrm{D}^{\mathrm{abs}}[\mathsf{vect}(\mathrm{U}, \mathrm{G}, w|_{\mathrm{U}})] \arrow[r, "\operatorname{cok}"] & \mathrm{D}^{\mathrm{sg}}_{\mathrm{G}}(\mathrm{Y} \cap \mathrm{U})
\end{tikzcd}$$

where $j : \mathrm{U} = \mathrm{X} \setminus \mathrm{Z} \to \mathrm{X}$ is the inclusion. Thus, cok induces a functor between the kernels of j^*. On the factorization side, this is $\mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}[\mathsf{vect}(\mathrm{X}, \mathrm{G}, w)]$ while on the singularity side this is $\mathrm{D}^{\mathrm{sg}}_{\mathrm{Z},\mathrm{G}}(\mathrm{Y})$. □

Definition **3.62.** — *Define the functor*

$$\begin{aligned} \mathbf{L}\operatorname{cok} : \mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}\big[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)\big] &\to \mathrm{D}^{\mathrm{sg}}_{\mathrm{Z},\mathrm{G}}(\mathrm{Y}) \\ \mathcal{E} &\mapsto \operatorname{cok}\mathcal{V} \end{aligned}$$

where $\mathcal{V}$ is a factorization that has locally-free components and is quasi-isomorphic to $\mathcal{E}$.

In the other direction, we use the functor Υ.

Lemma **3.63.** — *Assume that* w *is not identically zero on any component of* X. *The functor,* Υ, *descends further to a functor*

$$\Upsilon : D^{sg}_{G}(Y) \to D^{abs}\big[\mathsf{fact}(X, G, w)\big].$$

Moreover, if Z *is a closed* G*-invariant subset of* Y, *then* Υ *induces a functor*

$$\Upsilon : D^{sg}_{Z,G}(Y) \to D^{abs}_{Z}\big[\mathsf{fact}(X, G, w)\big].$$

Proof. — We treat the first statement. We need to check that Υ annihilates $\mathrm{perf}_G\, Y$. By Lemma 3.59, it suffices to show that it annihilates $i^*\mathcal{V}$ for $\mathcal{V}$ a locally-free G-equivariant sheaf of finite rank on X. For a coherent G-equivariant sheaf on X, $\mathcal{E}$, define a factorization, $\mathcal{H}_{\mathcal{E}} := (\mathcal{E}, \mathcal{E}, w, 1_{\mathcal{E}})$. There is a short exact sequence

$$0 \to \mathcal{H}_{\mathcal{V}} \otimes \mathcal{L}^{-1} \to \mathcal{H}_{\mathcal{V}} \to \Upsilon\big(i^*\mathcal{V}\big) \to 0.$$

Thus, $\Upsilon(i^*\mathcal{V})$ is quasi-isomorphic to the cone $\mathcal{H}_{\mathcal{V}} \otimes \mathcal{L}^{-1} \to \mathcal{H}_{\mathcal{V}}$. It is straightforward to see that any $\mathcal{H}_{\mathcal{V}}$ is contractible. Thus, $\Upsilon(i^*\mathcal{V})$ is zero in $D^{abs}[\mathsf{fact}(X, G, w)]$.

It is clear that Υ commutes with restriction to open subsets. Thus, we have a commutative diagram of functors

$$\begin{tikzcd}
D^{sg}_{G}(Y) \arrow[r, "\Upsilon"] \arrow[d, "j_U^*"'] & D^{abs}[\mathsf{vect}(X, G, w)] \arrow[d, "j_V^*"] \\
D^{sg}_{G}(U) \arrow[r, "\Upsilon"] & D^{abs}[\mathsf{vect}(V, G, w|_V)]
\end{tikzcd}$$

where $j_U : U = Y \setminus Z \to X$ and $j_V : V = X \setminus Z \to X$ are the inclusions. Thus, Υ induces a functor between the kernels of j_U^* and j_V^*. On the factorization side, this is $D^{abs}_{Z}[\mathsf{vect}(X, G, w)]$ while on the singularity side this is $D^{sg}_{Z,G}(Y)$. □

Proposition **3.64.** — *Let* X *be a smooth variety equipped with an action of an affine algebraic group* G *and an invariant section* $w \in \Gamma(X, \mathcal{L})^G$ *for an invertible equivariant sheaf,* $\mathcal{L}$. *Let* Y *be the vanishing locus of* w *and let* Z *be a closed* G*-invariant subset of* Y. *Assume that* w *is not identically zero on any component of* X. *The functor,*

$$\Upsilon : D^{sg}_{Z,G}(Y) \to D^{abs}_{Z}\big[\mathsf{fact}(X, G, w)\big],$$

is essentially surjective.

Proof. — Let us check that $\Upsilon \circ \mathbf{L}\mathrm{cok} \cong \mathrm{Id}$. Recall that, for a coherent G-equivariant sheaf on X, $\mathcal{E}$, we define a factorization, $\mathcal{H}_{\mathcal{E}} := (\mathcal{E}, \mathcal{E}, w, 1_{\mathcal{E}})$. There is a short exact sequence of factorizations

$$0 \to \mathcal{H}_{\mathcal{V}_{-1}} \to \mathcal{V} \to \Upsilon\, \mathrm{cok}\, \mathcal{V} \to 0$$

for a factorization with locally-free components. As $\mathcal{H}_{\mathcal{V}}$ is contractible, $\mathcal{V}$ is quasi-isomorphic to $\Upsilon \operatorname{cok} \mathcal{V}$. $\square$

Remark **3.65.** — One can prove that Υ is an equivalence by using arguments in the proof [Pos11, Theorem 2.7] and accounting for a group action, see also [Orl12, Theorem 3.5] and [PV10, Theorem 3.14]. We skip this, as only essential surjectivity is necessary for the generation arguments of Section 4.

We finish by recording an observation concerning how Υ interacts with exterior products.

Lemma **3.66.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups acting on, respectively,* X *and* Y. *Let* $w \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}(\chi))^{\mathrm{G}}$ *and* $v \in \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}(\chi'))^{\mathrm{H}}$ *for characters* $\chi : \mathrm{G} \to \mathbf{G}_m$ *and* $\chi' : \mathrm{H} \to \mathbf{G}_m$. *Let* $i_w : \mathrm{Z}_w \to \mathrm{X}$ *be the zero locus of* w, $i_v : \mathrm{Z}_v \to \mathrm{Y}$ *be the zero locus of* v, *and* $i_{w \boxplus v} : \mathrm{Z}_{w \boxplus v} \to \mathrm{X} \times \mathrm{Y}$ *be the zero locus of* $w \boxplus v$.

For any $\mathcal{E} \in \operatorname{coh}_{\mathrm{G}} \mathrm{Z}_w$ *and* $\mathcal{F} \in \operatorname{coh}_{\mathrm{H}} \mathrm{Z}_v$, *there are natural isomorphisms of* $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}$-*equivariant factorizations of* $w \boxplus v$,

$$(\Upsilon \mathcal{E}) \boxtimes (\Upsilon \mathcal{F}) \cong \Upsilon \operatorname{Res}^{\mathrm{G} \times \mathrm{H}}_{\mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}} (i_{w*}\mathcal{E} \boxtimes i_{v*}\mathcal{F}).$$

Proof. — It is straightforward to check that both of these factorizations are

$$\Upsilon \operatorname{Res}^{\mathrm{G} \times \mathrm{H}}_{\mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}} \big(\pi_1^* i_{w*}\mathcal{E} \otimes_{\mathcal{O}_{\mathrm{X} \times \mathrm{Y}}} \pi_2^* i_{v*}\mathcal{F}\big). \qquad \square$$

4. Generation of equivariant derived categories

To identify the internal Hom dg-categories for equivariant factorizations, we will need to prove a generation statement for our candidate categories. In this section, we lay the groundwork and establish results to which we will appeal in Section 5.

For a singular variety, X, equipped with a G-action, we want to find a nice set of generators for the bounded derived category of coherent G-equivariant sheaves, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}_{\mathrm{G}} \mathrm{X})$. One natural approach would be to study generation in a compactly-generated triangulated category whose category of compact objects is exactly $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}_{\mathrm{G}} \mathrm{X})$. Such categories do exist. Since $\mathrm{D}^{\mathrm{b}}(\operatorname{coh}_{\mathrm{G}} \mathrm{X})$ admits an enhancement to a dg-category, we could use the derived category of dg-modules over the enhancement. Or, a more geometric construction due to Krause, [Kra05], uses the homotopy category of injective complexes of quasi-coherent sheaves, in the non-equivariant setting. This could be extended to handle our situation.

However, this is not the approach we choose. Instead, we follow the method of Rouquier in [Rou08] and focus on $\mathrm{D}^{\mathrm{b}}(\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X})$, the bounded derived category of all quasi-coherent G-equivariant sheaves on X. The category, $\mathrm{D}^{\mathrm{b}}(\operatorname{Qcoh}_{\mathrm{G}} \mathrm{X})$, is not compactly-generated as it does not possess all coproducts. However, the definition of a compact object is still valid and useful for $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$. Indeed [Rou08, Proposition 6.15] implies that the category of compact objects of $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$ is exactly $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{X})$. A further advantage of studying $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$ comes from the fact that local cohomology of a coherent G-equivariant, or quasi-coherent sheaf, is always bounded and quasi-coherent, though usually never coherent.

Let us recall some notions of generation.

Definition **4.1.** — *Given a triangulated category, $\mathcal{T}$, we say that a subcategory, $\mathcal{S}$, is* **thick** *if it is triangulated and closed under summands.*

Let $\mathcal{S}'$ be another subcategory. We say that a subcategory, $\mathcal{S}$, **generates** *$\mathcal{S}'$, if the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$, and closed under finite coproducts and summands, contains $\mathcal{S}'$. If $\mathcal{S}' = \mathcal{T}$, we shall often say that $\mathcal{S}$ generates.*

We say that $\mathcal{S}$ **generates** *$\mathcal{S}'$* **up to infinite coproducts** *if the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$, and closed under arbitrary coproducts and summands, contains $\mathcal{S}'$. If $\mathcal{S}' = \mathcal{T}$, we shall often say that $\mathcal{S}$ generates up to infinite coproducts.*

In addition, recall that an object C *of $\mathcal{T}$ is called* **compact** *if* $\mathrm{Hom}_{\mathcal{T}}(\mathrm{C}, \bullet)$ *commutes with all coproducts.*

A triangulated category, $\mathcal{T}$, is **compactly-generated** *if it is co-complete, the compact objects form a set, and* $\mathrm{Hom}_{\mathcal{T}}(\mathrm{C}, \mathrm{X}) = 0$ *for all compact objects,* C, *of $\mathcal{T}$ implies that* $\mathrm{X} \cong 0$.

The following is a now-standard result on compactly-generated triangulated categories.

Lemma **4.2.** — *Assume $\mathcal{T}$ is a co-complete triangulated category and the compact objects in $\mathcal{T}$ form a set. Then, $\mathcal{T}$ is compactly-generated if and only if the compact objects generate up to infinite coproducts.*

Proof. — See [Nee92] for a proof. □

The following result generalizes one direction of Lemma 4.2.

Lemma **4.3.** — *Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{C}, \mathcal{C}'$ be a subcategory of compact objects of $\mathcal{T}$. If $\mathcal{C}$ generates $\mathcal{C}'$ up to infinite coproducts, then $\mathcal{C}$ generates $\mathcal{C}'$.*

Proof. — See [BV03, Proposition 2.2.4] or [Rou08, Corollary 3.13]. □

Let X be a separated, reduced scheme of finite type over k and let G be an affine algebraic group acting on X, $\sigma : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$. We record some generation results about the category, $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{X})$. Their statements and proofs are in the style of Rouquier, [Rou08], see also the arguments in [LP11].

Definition **4.4.** — *Let $\mathcal{E}$ be a quasi-coherent* G*-equivariant sheaf on* X. *Let* Z *be a* G*-invariant subscheme of* X *determined by a sheaf of ideals, $\mathcal{I}_{\mathrm{Z}}$. We say that* E *is* **scheme-theoretically sup-**

ported *on* Z *if* $\mathcal{I}_{\mathrm{Z}}\mathcal{E} = 0$. *We say that* $\mathcal{E}$ *is* ***set-theoretically supported*** *on* Z *if* $j^*\mathcal{E} = 0$ *for the inclusion* $j : \mathrm{X} \setminus \mathrm{Z} \to \mathrm{X}$.

Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{X})$ *be the triangulated subcategory of* $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{X})$ *consisting of complexes whose cohomology sheaves are set-theoretically supported on* Z.

Remark **4.5.** — Let $l : \mathrm{Z} \to \mathrm{X}$ be the inclusion of Z into X. Then, a quasi-coherent G-equivariant sheaf is scheme-theoretically supported on Z if and only if it is in the essential image of l_*.

Lemma **4.6.** — *Let* Z *be a* G*-invariant closed subscheme of* X. *Let* $\mathcal{S}, \mathcal{S}'$ *be subcategories of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{X})$. *If* $\mathcal{S}$ *generates* $\mathcal{S}'$ *up to infinite coproducts in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{X})$, *then* $\mathcal{S}$ *generates* $\mathcal{S}'$.

Proof. — We apply Lemma 4.3. The compact objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{X})$ are exactly the objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{X})$ by Proposition 6.15 of [Rou08]. □

We also record the following useful statement.

Lemma **4.7.** — *Any quasi-coherent* G*-equivariant sheaf,* $\mathcal{E}$, *is generated up to infinite coproducts by its coherent* G*-equivariant subsheaves.*

Proof. — Any quasi-coherent G-equivariant sheaf, $\mathcal{E}$, is the colimit of its coherent G-equivariant subsheaves, see [Tho97, Lemma 1.4]. The colimit fits to into an exact sequence,

$$0 \to \bigoplus_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ coherent}}} \mathcal{F} \to \bigoplus_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ coherent}}} \mathcal{F} \to \operatorname{colim} \mathcal{F} \cong \mathcal{E} \to 0.$$

Here the morphism,

$$\bigoplus_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ coherent}}} \mathcal{F} \to \bigoplus_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ coherent}}} \mathcal{F},$$

is defined as follows. Given two coherent equivariant subsheaves, $\mathcal{F}$ and $\mathcal{F}'$, the morphism $\mathcal{F} \to \mathcal{F}'$ equals

$$\begin{cases} 0 & \text{if } \mathcal{F} \not\subseteq \mathcal{F}' \\ -i & \text{if } i : \mathcal{F} \hookrightarrow \mathcal{F}' \text{ is a proper inclusion} \\ 1 & \text{if } \mathcal{F} = \mathcal{F}'. \end{cases}$$

As such, $\mathcal{E}$ is isomorphic to a cone over an endomorphism of a coproduct of coherent equivariant sheaves. Thus, $\mathcal{E}$ is generated, up to infinite coproducts, by its coherent G-equivariant subsheaves. □

Let Z be a G-invariant closed subset of X and $l : \mathrm{Z} \to \mathrm{X}$ be the inclusion.

Lemma **4.8.** — *The category,* $\mathrm{D}^{\mathrm{b}}_{\mathrm{Z}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$*, is generated up to infinite coproducts by the image of* $l_* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{Z}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{X})$.

Proof. — If we can generate the cohomology sheaves of a bounded complex, then we can generate said complex. So we may reduce to generating all quasi-coherent G-equivariant sheaves that are set-theoretically supported on Z. By Lemma 4.7, it suffices to generate all coherent G-equivariant sheaves that are set-theoretically supported on Z. However, for a coherent sheaf set-theoretically supported on Z, there is an n such that $\mathcal{I}_{\mathrm{Z}}^{n}\mathcal{E} = 0$. Thus, we have a filtration

$$0 = \mathcal{I}_{\mathrm{Z}}^{n}\mathcal{E} \subset \mathcal{I}_{\mathrm{Z}}^{n-1}\mathcal{E} \subset \cdots \subset \mathcal{I}_{\mathrm{Z}}\mathcal{E} \subset \mathcal{E}.$$

There are exact triangles

$$\mathcal{I}_{\mathrm{Z}}^{s}\mathcal{E} \to \mathcal{I}_{\mathrm{Z}}^{s-1}\mathcal{E} \to \mathcal{F}_s \to \mathcal{I}_{\mathrm{Z}}^{s}\mathcal{E}[1]$$

with $\mathcal{F}_s$ scheme-theoretically supported on Z. Thus, we see can generate a coherent G-equivariant sheaf using coherent G-equivariant sheaves scheme-theoretically supported on Z finishing the argument. □

Before continuing with the course of the argument, let us recall the definition of local cohomology for equivariant sheaves. For the arguments of this section, local cohomology complexes provide an efficient means of chopping complexes up with respect to their support.

Let Z be a G-invariant closed subset of X and $\mathcal{E}$ be a quasi-coherent G-equivariant sheaf on X. Set

$$\mathcal{H}_{\mathrm{Z}}\mathcal{E}(\mathrm{U}) := \left\{ e \in \Gamma(\mathrm{U}, \mathcal{E}) \mid \exists\, n,\ \mathcal{I}_{\mathrm{Z}}^{n} e = 0 \right\}$$
$$\mathcal{Q}_{\mathrm{Z}}\mathcal{E} := j_* j^* \mathcal{E}$$

where $j : \mathrm{X} \setminus \mathrm{Z} \to \mathrm{X}$ is the inclusion of the complement of Z. There is a left exact sequence

$$0 \to \mathcal{H}_{\mathrm{Z}}\mathcal{E} \to \mathcal{E} \to \mathcal{Q}_{\mathrm{Z}}\mathcal{E}.$$

Moreover, if $\mathcal{E}$ is flasque, there is a short exact sequence

$$0 \to \mathcal{H}_{\mathrm{Z}}\mathcal{E} \to \mathcal{E} \to \mathcal{Q}_{\mathrm{Z}}\mathcal{E} \to 0.$$

The quasi-coherent sheaf, $\mathcal{H}_{\mathrm{Z}}\mathcal{E}$, inherits the G-equivariant structure of $\mathcal{E}$. Let

$$\mathbf{R}\mathcal{H}_{\mathrm{Z}} : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$$
$$\mathbf{R}\mathcal{Q}_{\mathrm{Z}} : \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$$

be the associated right-derived functors. Note that there is a triangle of exact functors

$$\mathbf{R}\mathcal{H}_{\mathrm{Z}} \to \mathrm{Id} \to \mathbf{R}\mathcal{Q}_{\mathrm{Z}} \to \mathbf{R}\mathcal{H}_{\mathrm{Z}}[1]. \tag{4.1}$$

We now use the above discussion to reduce generation arguments to the G-invariant irreducible case.

Lemma **4.9.** — *Let* $\mathrm{X} = \mathrm{Z}_1 \cup \mathrm{Z}_2$ *be the decomposition of* X *into two* G*-invariant closed subsets,* Z_1 *and* Z_2*. Let* $l_i : \mathrm{Z}_i \to \mathrm{X}$ *denote the inclusion of* Z_i *into* X*. The objects in the essential image of the pushforward,* $l_{i*} : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{Z}_i) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{X})$*, for* $i = 1, 2$ *generate* $\mathrm{D}^{\mathrm{b}}(\mathrm{Qcoh}_{\mathrm{G}}\, \mathrm{X})$ *up to infinite coproducts.*

Proof. — We appeal to the exact triangle in Equation (4.1) to see that $\mathcal{E}$ is generated by $\mathbf{R}\mathcal{Q}_{\mathrm{Z}_1}\mathcal{E}$ and $\mathbf{R}\mathcal{H}_{\mathrm{Z}_1}\mathcal{E}$. Note that $\mathbf{R}\mathcal{Q}_{\mathrm{Z}_1}\mathcal{E}$ is supported on the complement of Z_1. As $\mathrm{X} = \mathrm{Z}_1 \cup \mathrm{Z}_2$, $\mathbf{R}\mathcal{Q}_{\mathrm{Z}_1}\mathcal{E}$ is set-theoretically supported on Z_2 while $\mathbf{R}\mathcal{H}_{\mathrm{Z}_1}\mathcal{E}$ is set-theoretically supported on Z_1. Applying Lemma 4.8, finishes the argument. □

We will need to pass to the singular locus so we record a simple lemma.

Lemma **4.10.** — *Let* $\sigma : \mathrm{G} \times \mathrm{X} \to \mathrm{X}$ *be an action of an affine algebraic group,* G*, on a reduced, separated scheme of finite type,* X*. Let* Sing X *denote the closed subset of* X *defined by*

$$\mathrm{Sing}\,\mathrm{X} := \{x \in \mathrm{X} \mid \mathcal{O}_{\mathrm{X},x} \textit{ is not regular}\}.$$

Equip Sing X *with the reduced, induced structure sheaf. Then, the action of* G *on* X *restricts to* Sing X.

Proof. — It suffices to verify that $\sigma_g := \sigma(g, \bullet) : \mathrm{X} \to \mathrm{X}$ preserves Sing X for each $g \in \mathrm{G}$. However, σ_g is an automorphism of X and hence must preserve Sing X. □

We will need to use normality of a variety which is not guaranteed by the assumptions of the proceeding lemmas. We take a moment to comment on lifting the action of G to the normalization in an equivariant manner.

Lemma **4.11.** — *Let* $\nu : \widetilde{\mathrm{X}} \to \mathrm{X}$ *be the normalization of* X*. There is a unique action of* G *on* $\widetilde{\mathrm{X}}$ *making* ν G*-equivariant.*

Proof. — Since G is smooth, $\mathrm{G} \times \widetilde{\mathrm{X}}$ is normal. The map $\sigma \circ (1 \times \nu) : \mathrm{G} \times \widetilde{\mathrm{X}} \to \mathrm{X}$ is dominant and therefore factors uniquely through ν. Let $\tilde{\sigma} : \mathrm{G} \times \widetilde{\mathrm{X}} \to \widetilde{\mathrm{X}}$ be the unique lift. The uniqueness of the lift also allows one to verify that $\widetilde{\sigma}$ is an action of G on X. With this lift, $\nu : \widetilde{\mathrm{X}} \to \mathrm{X}$ becomes G-equivariant. □

Lemma **4.12.** — *Let* $f : \mathrm{X} \to \mathrm{Y}$ *be a* G*-equivariant morphism such that* X *possesses an* f*-ample family of equivariant line bundles,* $\mathcal{L}_\alpha, \alpha \in \mathrm{A}$*. The full subcategory of* $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\, \mathrm{X})$ *consisting* *of objects of the form*

$$\mathcal{L}_\alpha \otimes f^* \mathcal{E}$$

for $\mathcal{E} \in \mathrm{coh}_{\mathrm{G}} \mathrm{Y}$ *and* $\alpha \in \mathrm{A}$ *generates all coherent* G*-equivariant sheaves of locally-finite projective dimension in* $\mathrm{Qcoh}\,\mathrm{X}$. *Moreover, if* Y *possesses enough locally-free* G*-equivariant sheaves of finite rank, then the full subcategory of* $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}} \mathrm{X})$ *consisting of objects of the form*

$$\mathcal{L}_\alpha \otimes f^* \mathcal{V}$$

for $\mathcal{V} \in \mathrm{coh}_{\mathrm{G}} \mathrm{Y}$ *locally-free and* $\alpha \in \mathrm{A}$ *generates all coherent* G*-equivariant sheaves of locally-finite projective dimension in* $\mathrm{Qcoh}\,\mathrm{X}$.

Proof. — Let $\mathcal{E}$ be a coherent G-equivariant sheaf of locally-finite projective dimension in $\mathrm{Qcoh}\,\mathrm{X}$. There is a finite set $\mathrm{A}' \subseteq \mathrm{A}$ such that

$$\bigoplus_{\alpha \in \mathrm{A}'} \mathcal{L}_\alpha \otimes f^* f_* \big(\mathcal{L}_\alpha^{-1} \otimes \mathcal{E}\big) \to \mathcal{E}$$

is an epimorphism as $\mathcal{L}_\alpha$ is an f-ample family. For each α, there exists a coherent G-equivariant subsheaf, $\mathcal{F}_\alpha$, of $f_*(\mathcal{L}_\alpha^{-1} \otimes \mathcal{E})$ such that the restriction of the co-unit morphism remains an epimorphism

$$\bigoplus_{\alpha \in \mathrm{A}'} \mathcal{L}_\alpha \otimes f^* \mathcal{F}_\alpha \to \mathcal{E}.$$

If we assume that Y possesses enough locally-free G-equivariant sheaves of finite rank, there is a locally-free G-equivariant sheaf, $\mathcal{V}_\alpha$, on Y and an epimorphism, $\mathcal{V}_\alpha \to \mathcal{F}_\alpha$. Pulling back and composing, we have an epimorphism

$$\bigoplus_{\alpha \in \mathrm{A}'} \mathcal{L}_\alpha \otimes f^* \mathcal{V}_\alpha \to \mathcal{E}.$$

Taking kernels and iterating this process we may construct an exact sequence

$$\cdots \to \mathcal{G}_s \to \cdots \to \mathcal{G}_1 \to \mathcal{E} \to 0$$

where each $\mathcal{G}_i$ is a sum of objects of the $\mathcal{L}_\alpha \otimes f^* \mathcal{F}_\alpha$ for some finite set of $\alpha \in \mathrm{A}'$. Moreover, if Y possesses enough locally-free equivariant sheaves, we may take $\mathcal{E}$ to be locally-free.

Let $\mathcal{K}_s$ be the kernel of $\mathcal{G}_s \to \mathcal{G}_{s-1}$. We have a short exact sequence

$$0 \to \mathcal{G}_s \to \cdots \to \mathcal{G}_1 \to \mathcal{E} \to 0.$$

This represents an element of

$$\mathrm{Ext}^s_{\mathrm{Qcoh}_{\mathrm{G}} \mathrm{X}}(\mathcal{E}, \mathcal{K}_s).$$

As $\mathcal{E}$ is an object of locally-finite projective dimension in $\mathrm{Qcoh}\,\mathrm{X}$, from Lemma 2.32, there is an s_0 such that

$$\mathrm{Ext}^s_{\mathrm{Qcoh}_G \mathrm{X}}(\mathcal{E}, \mathcal{K}_s) = 0$$

for $s \geq s_0$. Take s larger than s_0. Then, there is a quasi-isomorphism,

$$\mathcal{K}_s[s] \oplus \mathcal{E} \simeq \mathcal{G}_s \to \cdots \to \mathcal{G}_1.$$

Thus, $\mathcal{E}$ is generated by objects of the form

$$\mathcal{L}_\alpha \otimes f^*\mathcal{E}$$

for $\mathcal{E} \in \mathrm{coh}_{\mathrm{G}}\,\mathrm{Y}$ and $\alpha \in \mathrm{A}$. If Y has enough equivariant locally-free sheaves, then $\mathcal{E}$ is generated by objects of the form

$$\mathcal{L}_\alpha \otimes f^*\mathcal{E}$$

for $\mathcal{E} \in \mathrm{coh}_{\mathrm{G}}\,\mathrm{Y}$ locally-free and $\alpha \in \mathrm{A}$. □

Next, we demonstrate how to produce a set of generators from a set of generators of the singular locus of X.

Lemma **4.13.** — *Let* X *be a divisorial variety. Let* Sing X *be the singular locus of* X *with its reduced, induced structure sheaf. Let* $l : \mathrm{Sing}\,\mathrm{X} \to \mathrm{X}$ *denote the inclusion. Let* Y *be a closed subset of* X *that is* G*-invariant. Then, the subcategory, whose objects are*

- $\nu_*\mathcal{V}$ *where* $\mathcal{V}$ *is a locally-free* G*-equivariant sheaves of finite rank on* $\widetilde{\mathrm{X}}$ *plus*
- *the objects in the essential image of the pushforward,*

$$l_* : \mathrm{D^b}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{Y} \cap \mathrm{Sing}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{X}),$$

generate the subcategory $\mathrm{D^b_Y}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$ *up to infinite coproducts.*

Moreover, if one assumes that X *has enough locally-free* G*-equivariant sheaves, then the subcategory, whose objects are*

- *locally-free* G*-equivariant sheaves of finite rank on* X*, plus*
- *the objects in the essential image of the pushforward,*

$$l_* : \mathrm{D^b}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{Y} \cap \mathrm{Sing}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{X}),$$

generates $\mathrm{D^b_Y}(\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{X})$ *up to infinite coproducts.*

Proof. — To generate a bounded complex, it suffices to generate its cohomology sheaves. Therefore, we may reduce to generating quasi-coherent G-equivariant sheaves set-theoretically supported on Y up to infinite coproducts. By Lemma 4.7, it then suffices to generate coherent G-equivariant subsheaves set-theoretically supported on Y up to infinite coproducts. Let $\mathcal{E}$ be a coherent G-equivariant sheaf. Complete the unit of the adjunction, $\mathcal{E} \to \nu_* \nu^* \mathcal{E}$ to an exact triangle

$$\mathcal{E} \to \nu_* \nu^* \mathcal{E} \to \mathcal{D} \to \mathcal{E}[1].$$

Since ν is an isomorphism on U, $\mathcal{D}$ is set-theoretically supported on $\operatorname{Sing} X \cap Y$. Since $\mathcal{D}$ is coherent it is generated by the essential image of l_* by Lemmas 4.6 and 4.8. Thus, to generate $\mathcal{E}$ it suffices to generate $\nu_* \nu^* \mathcal{E}$. Note also that if $\mathcal{E}$ is a locally-free sheaf of finite rank, then we generate $\nu_* \nu^* \mathcal{E}$ as we are allowed to use $\mathcal{E}$.

Set $Z = \nu^{-1}(\operatorname{Sing} X)$ and $U = \widetilde{X} \setminus Z$. If $\mathcal{V}$ is a locally-free G-equivariant sheaf of finite rank on $\widetilde{X}$, then we have an exact triangle,

$$\mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V} \to \mathcal{V} \to \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \mathcal{V} \to \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V}[1].$$

Applying ν_*, we have another exact triangle,

$$\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V} \to \nu_* \mathcal{V} \to \nu_* \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \mathcal{V} \to \nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V}[1].$$

The set-theoretic support of $\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V}$ is contained in $\operatorname{Sing} X \cap Y$ as $\nu(Z) = \operatorname{Sing} X$. By Lemma 4.8, $\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \mathcal{V}$ is generated up to infinite coproducts by the essential image of l_*. Thus, $\nu_* \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \mathcal{V}$ is generated up to infinite coproducts by the full subcategory consisting of $\nu_* \mathcal{V}$ where $\mathcal{V}$ is a locally-free G-equivariant on $\widetilde{X}$ and the essential image of l_*.

Let $\mathcal{E}$ be a coherent G-equivariant sheaf on X supported on Y. We have a triangle,

$$\mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E} \to \nu^* \mathcal{E} \to \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E} \to \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}[1].$$

Applying ν_*, we get another triangle,

$$\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E} \to \nu_* \nu^* \mathcal{E} \to \nu_* \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E} \to \nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}[1],$$

we see that to generate $\nu_* \nu^* \mathcal{E}$ it suffices to generate $\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}$ and $\nu_* \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}$. The complex, $\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}$, is set-theoretically supported on $\operatorname{Sing} X \cap Y$. By Lemma 4.8, $\nu_* \mathbf{R}\mathcal{H}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}$ is generated up to infinite coproducts by the essential image of l_*. Thus, we reduce to generating $\nu_* \mathbf{R}\mathcal{Q}_{Z \cap \nu^{-1}(Y)} \nu^* \mathcal{E}$.

As ν is an affine morphism, the pullback of an ample family remains an ample family. Using Theorem 2.29, we may construct an exact complex

$$\cdots \to \mathcal{V}_s \to \cdots \to \mathcal{V}_2 \to \mathcal{V}_1 \to \nu^* \mathcal{E} \to 0$$

where each $\mathcal{V}_i$ is a locally-free G-equivariant sheaf of finite rank. Apply j^* where $j : U = \widetilde{X} \setminus (Z \cap \nu^{-1}(Y)) \to \widetilde{X}$ is the inclusion. As j^* is exact, the complex

$$\cdots \to j^* \mathcal{V}_s \to \cdots \to j^* \mathcal{V}_2 \to j^* \mathcal{V}_1 \to j^* \nu^* \mathcal{E} \to 0$$

remains exact. Let $\mathcal{K}_s$ be the kernel of the map $j^*\mathcal{V}_s \to j^*\mathcal{V}_{s-1}$. The exact sequence

$$0 \to \mathcal{K}_s \to j^*\mathcal{V}_s \to \cdots \to j^*\mathcal{V}_2 \to j^*\mathcal{V}_1 \to j^*\nu^*\mathcal{E} \to 0$$

represents an element of $\mathrm{Ext}^s_{\mathrm{Qcoh}_G U}(j^*\nu^*\mathcal{E}, \mathcal{K}_s)$. As $j^*\nu^*\mathcal{E}$ is supported on the smooth subset, $U \supset \tilde{X} \setminus Z$, this vanishes for $s \geq s_0$, for some s_0, by Lemma 2.32. Consequently, there is a quasi-isomorphism

$$j^*\nu^*\mathcal{E} \oplus \mathcal{K}_s[s] \simeq j^*\mathcal{V}_s \to \cdots \to j^*\mathcal{V}_2 \to j^*\mathcal{V}_1.$$

Applying $\nu_*\mathbf{R}j_*$, we see that $\nu_*\mathbf{R}\mathcal{Q}_{Z\cap\nu^{-1}(Y)}\nu^*\mathcal{E}$ is generated by $\nu_*\mathbf{R}\mathcal{Q}_{Z\cap\nu^{-1}(Y)}\mathcal{V}_i$ for $1 \leq i \leq s$. We have already observed that we can generate $\nu_*\mathbf{R}\mathcal{Q}_{Z\cap\nu^{-1}(Y)}\mathcal{V}$ up to infinite coproducts when $\mathcal{V}$ is locally-free of finite rank. We conclude that we can generate $\nu_*\mathbf{R}\mathcal{Q}_{Z\cap\nu^{-1}(Y)}\nu^*\mathcal{E}$ using the subcategory consisting of ν-pushforwards of G-equivariant invertible sheaves on $\tilde{X}$ and the essential image of l_* up to infinite coproducts finishing the argument.

If we assume that X has enough locally-free G-equivariant sheaves, then we can repeat the previous argument replacing $\tilde{X}$ by X. □

Corollary **4.14.** — *Assume that* X *possesses enough locally-free* G*-equivariant sheaves. Let* $\mathrm{Sing}\, X$ *be the singular locus of* X *with its reduced, induced structure sheaf. Let* $l : \mathrm{Sing}\, X \to X$ *denote the inclusion. Let* Y *be a closed subset. The subcategory,* $D^b_Y(\mathrm{coh}_G X)$*, is generated by all locally-free* G*-equivariant sheaves of finite rank on* X *and all objects in the essential image of* $l_* : D^b(\mathrm{coh}_G \mathrm{Sing}\, X \cap Y) \to D^b(\mathrm{coh}_G X)$.

Proof. — The second part of Lemma 4.13 states that the subcategory consisting of all locally-free coherent G-equivariant sheaves and the essential image of l_* generates $D^b_Y(\mathrm{Qcoh}_G X)$ up to infinite coproducts. Thus, by Lemma 4.6, the subcategory consisting of all locally-free G-equivariant sheaves of finite rank on X and the essential image of l_* generates $D^b_Y(\mathrm{coh}_G X)$. □

Remark **4.15.** — One may use induction by iteratively passing to singular loci to produce a slightly smaller generating subcategory for $D^b(\mathrm{coh}_G X)$.

Definition **4.16.** — *Assume that* X *has enough* G*-equivariant locally-free sheaves. Let* U *be an open* G*-invariant subset of* X *and let* $\mathrm{Perf}_{U,G} X$ *be the full subcategory of* $D^b(\mathrm{Qcoh}_G X)$ *whose restriction to* $D^b(\mathrm{Qcoh}_G U)$ *is quasi-isomorphic to a bounded complex of locally-free* G*-equivariant sheaves. Let* $\mathrm{perf}_{U,G} X$ *be the subcategory of* $\mathrm{Perf}_{U,G} X$ *consisting of complexes quasi-isomorphic to bounded complexes of coherent sheaves.*

Lemma **4.17.** — *Assume that* X *has enough* G*-equivariant locally-free sheaves. The category,* $\mathrm{Perf}_{U,G} X$*, is generated up to infinite coproducts by locally-free* G*-equivariant sheaves of finite rank and* *the image of*

$$l_* : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}} \mathrm{Y}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}} \mathrm{X})$$

where $\mathrm{Y} = \mathrm{X} \setminus \mathrm{U}$.

Proof. — Let $\mathcal{E} \in \mathrm{Perf}_{\mathrm{U},\mathrm{G}} \mathrm{X}$. Using the assumption that G has enough locally-free G-equivariant sheaves and a standard argument (see for the example the proof of Lemma 4.12), we may construct a bounded complex of locally-free sheaves $\mathcal{P}$ and a morphism

$$\mathcal{P} \to \mathcal{E}$$

whose cone is a quasi-coherent sheaf that is locally-free on U. Since, by Lemma 4.7, we may generate bounded complexes of locally-free sheaves $\mathcal{P}$ up to infinite coproducts with locally-free coherent sheaves, it suffices to generate this cone. We continue with the assumption that $\mathcal{E}$ is a quasi-coherent sheaf.

There is an exact triangle

$$\mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{E} \to \mathcal{E} \to \mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{E} \to \mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{E}[1].$$

It suffices to generate $\mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{E}$ and $\mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{E}$. We can generate $\mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{E}$ up to infinite coproducts by the image of l_* by Lemma 4.8. Thus, we reduce to generating $\mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{E}$.

Using the assumption of having enough G-equivariant locally-free sheaves, we may construct an exact complex

$$\cdots \to \mathcal{V}_s \to \cdots \to \mathcal{V}_1 \to \mathcal{E} \to 0$$

with $\mathcal{V}_i$ being locally-free G-equivariant sheaves. Apply j^* to get an exact complex

$$\cdots \to j^*\mathcal{V}_s \to \cdots \to j^*\mathcal{V}_1 \to j^*\mathcal{E} \to 0.$$

Let $\mathcal{K}_s$ be the kernel of the map $j^*\mathcal{V}_s \to j^*\mathrm{V}_{s-1}$. The exact sequence

$$0 \to \mathcal{K}_s \to j^*\mathcal{V}_s \to \cdots \to j^*\mathcal{V}_1 \to j^*\mathcal{E} \to 0$$

represents an element of $\mathrm{Ext}^s_{\mathrm{Qcoh}_{\mathrm{G}} \mathrm{U}}(j^*\mathcal{E}, \mathcal{K}_s)$. As $j^*\mathcal{E}$ is perfect, this vanishes for $s \geq s_0$, for some s_0, by Lemma 2.32. Assuming $s \geq s_0$, there is a quasi-isomorphism

$$j^*\mathcal{E} \oplus \mathcal{K}_s[s] \simeq j^*\mathcal{V}_s \to \cdots \to j^*\mathcal{V}_2 \to j^*\mathcal{V}_1.$$

Pushing this forward via $\mathbf{R}j_*$ shows that $\mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{E}$ is generated by $\mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{V}$ for $\mathcal{V}$ locally-free. Thus, we reduce to generating $\mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{V}$ for $\mathcal{V}$ locally-free. But, for such a $\mathcal{V}$, there is an exact triangle,

$$\mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{V} \to \mathcal{V} \to \mathbf{R}\mathcal{Q}_{\mathrm{Z}}\mathcal{V} \to \mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{V}[1]$$

and we may generate $\mathbf{R}\mathcal{H}_{\mathrm{Z}}\mathcal{V}$ and $\mathcal{V}$ up to infinite coproducts by locally-free G-equivariant sheaves of finite rank and the image of l_* by Lemma 4.8. $\square$

Corollary **4.18.** — *Assume that* X *has enough* G*-equivariant locally-free sheaves. The category,* $\mathrm{perf}_{\mathrm{U},\mathrm{G}}\,\mathrm{X}$*, is generated by locally-free* G*-equivariant sheaves of finite rank and the image of*

$$l_* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{Y}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{X})$$

where $\mathrm{Y} = \mathrm{X} \setminus \mathrm{U}$.

Proof. — This follows from Lemma 4.17 by applying Lemma 4.6. □

The following lemma shows that generators restrict under changing of the group.

Lemma **4.19.** — *Let* X *be a separated, reduced, divisorial scheme of finite type equipped with a* G *action. Assume that* $\mathrm{G/H}$ *is affine. Then,* $\mathrm{D^b}(\mathrm{coh_H}\,\mathrm{X})$ *is generated by the essential image of* $\mathrm{Res}^{\mathrm{G}}_{\mathrm{H}} : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_H}\,\mathrm{X})$.

Proof. — Recall that $\mathrm{Res}^{\mathrm{G}}_{\mathrm{H}}$ factors as $\iota^* \circ \alpha^*$ where $\alpha : \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} \to \mathrm{X}$ is induced by the action of G on X and $\iota : \mathrm{X} \to \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}$ is induced by the unit element of G. The functor, $\iota^* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_H}\,\mathrm{X})$, is an equivalence by Lemma 2.13 so it suffices to show that the image of $\alpha^* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X})$ generates. We factor α as

$$\begin{tikzcd}
\mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & \mathrm{G/H} \times \mathrm{X} \arrow[dl, "p"] \\
\mathrm{X} &
\end{tikzcd}$$

where

$$\begin{aligned}
\Phi : \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} &\to \mathrm{G/H} \times \mathrm{X} \\
(g, x) &\mapsto \big(g\mathrm{H}, \sigma(g, x)\big)
\end{aligned}$$

and p is the projection. The morphism, Φ, is an isomorphism so we reduce to checking that the image of $p^* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G/H} \times \mathrm{X})$ generates.

Let us handle the case that $\dim \mathrm{X} = 0$. Under our standing assumptions X is reduced, therefore X is smooth. Since $\mathrm{G/H}$ is affine, $\mathcal{O}_{\mathrm{G/H}}$ is ample and is naturally equivariant. Lemma 4.12 applies directly to show that the essential image of p^* generates

Now assume we have proven the statement for X with all components of X having dimension $< n$ and assume that $\dim \mathrm{X} = n$. By Corollary 4.14, $\mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G/H} \times \mathrm{X})$ is generated by $\nu'_* \mathcal{V}$ where $\mathcal{V}$ are locally-free G-equivariant sheaves of finite rank, $\nu' : \widetilde{\mathrm{G/H} \times \mathrm{X}} \to \mathrm{G/H} \times \mathrm{X}$ is the normalization, and the essential image of

$$l_* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{Sing}\,\mathrm{G/H} \times \mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G/H} \times \mathrm{X}).$$

Since G/H is smooth $\mathrm{Sing}(\mathrm{G}/\mathrm{H} \times \mathrm{X}) = \mathrm{G}/\mathrm{H} \times \mathrm{Sing}\,\mathrm{X}$. Applying the induction hypothesis, the essential image of

$$p^* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{Sing}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{Sing}\,\mathrm{G}/\mathrm{H} \times \mathrm{X})$$

generates. Thus, the essential image of

$$p^* : \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh_G}\,\mathrm{G}/\mathrm{H} \times \mathrm{X})$$

generates the essential image of l_*. We are left to generate the coherent G-equivariant sheaves, $\nu'_*\mathcal{V}$, for $\mathcal{V}$ locally-free G-equivariant sheaves of finite rank on the normalization.

Since G/H is smooth, $\widetilde{\mathrm{G}/\mathrm{H} \times \mathrm{X}} \cong \mathrm{G}/\mathrm{H} \times \widetilde{\mathrm{X}}$. We have a commutative diagram.

$$\begin{tikzcd}
\mathrm{G}/\mathrm{H} \times \widetilde{\mathrm{X}} \arrow[r, "1 \times \nu"] \arrow[d, "\tilde{p}"'] & \mathrm{G}/\mathrm{H} \times \mathrm{X} \arrow[d, "p"] \\
\widetilde{\mathrm{X}} \arrow[r, "\nu"] & \mathrm{X}
\end{tikzcd}$$

Applying Lemma 4.12, since $\mathcal{O}_{\mathrm{G}/\mathrm{H}\times\widetilde{\mathrm{X}}}$ is $\widetilde{p}$-ample, any locally-free G-equivariant sheaf of finite rank, $\mathcal{V}$, is generated by the essential image of $\widetilde{p}^*$.

Therefore, $\nu'_*\mathcal{V} = (1 \times \nu)_*\mathcal{V}$ is generated by the essential image of $(1 \times \nu)_* \circ \widetilde{p}^*$. As p is flat,

$$(1 \times \nu)_* \circ \widetilde{p}^* \cong p^* \circ \nu_*.$$

Thus, $\nu'_*\mathcal{V}$ is generated by the essential image of p^* finishing the proof. □

The next proposition demonstrates that exterior products generate in the equivariant setting.

Proposition **4.20.** — *Let* G *and* H *be affine algebraic groups, and* X *and* Y *be separated, reduced, divisorial schemes of finite type equipped with actions* $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$ *and* $\mathrm{H} \times \mathrm{Y} \to \mathrm{Y}$. *The subcategory consisting of* $\mathcal{E} \boxtimes \mathcal{F}$ *for* $\mathcal{E} \in \mathrm{coh_G}\,\mathrm{X}$ *and* $\mathcal{F} \in \mathrm{coh_H}\,\mathrm{Y}$ *generates* $\mathrm{D^b}(\mathrm{Qcoh}_{\mathrm{G}\times\mathrm{H}}\,\mathrm{X} \times \mathrm{Y})$ *up to infinite coproducts.*

Proof. — By Lemma 4.7, it suffices to generate all coherent $\mathrm{G} \times \mathrm{H}$-equivariant sheaves up to infinite coproducts.

We proceed by induction on the dimension of $\mathrm{X} \times \mathrm{Y}$. Assume that $\dim \mathrm{X} \times \mathrm{Y} = 0$. The morphism,

$$h := f \times g : \mathrm{X} \times \mathrm{Y} \to \mathrm{Spec}\,k \times \mathrm{Spec}\,k \cong \mathrm{Spec}\,k,$$

coming from the product of the structure maps, $f : \mathrm{X} \to \mathrm{Spec}\,k$ and $g : \mathrm{Y} \to \mathrm{Spec}\,k$, is affine and $\mathrm{G} \times \mathrm{H}$-equivariant therefore $\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}$ is ample. By Lemma 2.32, any object of $\operatorname{coh} X \times Y$ has locally-finite projective dimension since $X \times Y$ is smooth. Applying Lemma 4.12, we see that the essential image of h^* generates $D^b(\operatorname{coh}_{G\times H} X \times Y)$. Moreover,

$$h^*(\mathcal{E} \boxtimes \mathcal{F}) \cong f^*\mathcal{E} \boxtimes g^*\mathcal{F}.$$

So validity of the claim in the case $X = Y = \operatorname{Spec} k$ implies validity of the claim for all $X \times Y$ of dimension zero. For a finite dimensional $G \times H$ representation, the evaluation morphism

$$\operatorname{Hom}_{\operatorname{Qcoh}_H \operatorname{Spec} k}\left(\operatorname{Res}_H^{G\times H} V, V\right) \otimes_k \operatorname{Res}_H^{G\times H} V \to V$$

is an epimorphism. Here, $\operatorname{Hom}_{\operatorname{Qcoh}_H}(\operatorname{Res}_H^{G\times H} V, V)$ is a G-representation. By Lemma 2.32 the category of G-representations has finite global dimension. Thus, there are enough exterior products to resolve any $G \times H$-representation finishing the base case of the induction.

Assume we have proven the statement whenever $\dim X \times Y < n$ and let us treat a product with $\dim X \times Y = n$. From Lemma 4.13, $D^b(\operatorname{Qcoh}_{G\times H} X \times Y)$ is generated up to infinite coproducts by $\nu_*\mathcal{V}$ for locally-free $G \times H$-equivariant sheaves of finite rank on the normalization $\widetilde{X \times Y}$ and the essential image of

$$l_* : D^b(\operatorname{coh}_{G\times H} \operatorname{Sing} X \times Y) \to D^b(\operatorname{coh}_{G\times H} X \times Y).$$

The singular locus of $X \times Y$ is the union of two closed subsets: $(\operatorname{Sing} X) \times Y$ and $X \times \operatorname{Sing} Y$. From Lemma 4.9, $D^b(\operatorname{Qcoh}_{G\times H} \operatorname{Sing}(X \times Y))$ is generated up to infinite coproducts by the images of $D^b(\operatorname{coh}_{G\times H}(\operatorname{Sing} X) \times Y)$ and $D^b(\operatorname{coh}_{G\times H} X \times \operatorname{Sing} Y)$ under pushforward. Using the induction hypothesis, exterior products generate both $D^b(\operatorname{coh}_{G\times H}(\operatorname{Sing} X) \times Y)$ and $D^b(\operatorname{coh}_{G\times H} X \times \operatorname{Sing} Y)$. Thus, the essential image of l_* is generated up to infinite coproducts by exterior products. Next, we turn to locally-free equivariant sheaves pushed forward from the normalization.

The normalization of $X \times Y$ is the product of the normalizations, $\widetilde{X} \times \widetilde{Y}$, [EGA IV.2, Corollary 6.14.3]. We have assumed that X and Y have ample families. Since normalization is affine, $\widetilde{X}$ and $\widetilde{Y}$ have ample families given by the pullbacks from X and Y, respectively. The exterior product of ample families is again an ample family. Since $\widetilde{X} \times \widetilde{Y}$ is normal, taking sufficient powers of each line bundle, we get an ample family where all the line bundles admit equivariant structures, [Tho97, Lemma 2.10]. Thus, for any locally-free G-equivariant sheaf, $\mathcal{V}$, there is an exact sequence of equivariant sheaves

$$\cdots \to \mathcal{F}_s \to \cdots \to \mathcal{F}_1 \to \mathcal{V} \to 0$$

where each $\mathcal{F}_i$ is an exterior product. The locally-free sheaf $\mathcal{V}$ has locally-finite projective dimension, and thus is a summand of the complex

$$\mathcal{F}_s \to \cdots \to \mathcal{F}_1$$

for s sufficiently large. We see that exterior products generate all locally-free equivariant sheaves on $\widetilde{X} \times \widetilde{Y}$. Pushing forward an exterior product under the normalization, map $\widetilde{X} \times \widetilde{Y} \to X \times Y$, yields another exterior product via the projection formula and flat base change. Thus, exterior products also generate $\nu_* \mathcal{V}$ for locally-free $G \times H$-equivariant sheaves of finite rank, $\mathcal{V}$, on the normalization. This finishes the proof. $\square$

Corollary **4.21.** — *Let* G *and* H *be affine algebraic groups,* X *and* Y *separated, reduced schemes of finite type equipped with actions* $G \times X \to X$ *and* $H \times Y \to Y$. *The subcategory consisting of* $\mathcal{E} \boxtimes \mathcal{F}$ *for* $\mathcal{E} \in \mathrm{coh}_G X$ *and* $\mathcal{F} \in \mathrm{coh}_H Y$ *generates* $D^b(\mathrm{coh}_{G \times H} X \times Y)$.

Proof. — This follows from Proposition 4.20 by applying Lemma 4.6. $\square$

Next, we turn our attention to showing that exterior products of factorizations generate the appropriate category. We will demonstrate such generation for exterior products in the singularity category and then use that to pass to factorizations.

Lemma **4.22.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups acting on, respectively,* X *and* Y. *Let* $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ *and* $v \in \Gamma(Y, \mathcal{O}_Y(\chi'))^H$ *for characters* $\chi : G \to \mathbf{G}_m$ *and* $\chi' : H \to \mathbf{G}_m$. *Let* $i_w : Z_w \to X$ *be the zero locus of* w, $i_v : Z_v \to Y$ *be the zero locus of* v, *and* $i_{w \boxplus v} : Z_{w \boxplus v} \to X \times Y$ *be the zero locus of* $w \boxplus v$. *Let* $l : \mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v \to Z_{w \boxplus v}$ *be the inclusion.*

Objects of the form $l_* \mathrm{Res}^{G \times H}_{G \times_{\mathbf{G}_m} H}(\mathcal{E} \boxtimes \mathcal{F})$ *for* $\mathcal{E} \in \mathrm{coh}_G \mathrm{Sing}\, Z_w$ *and* $\mathcal{F} \in \mathrm{coh}_H \mathrm{Sing}\, Z_v$ *generate* $D^{\mathrm{sg}}_{Z_w \times Z_v, G \times_{\mathbf{G}_m} H}(Z_{w \boxplus v})$.

Proof. — By Corollary 4.18, the inverse image of $D^{\mathrm{sg}}_{Z_w \times Z_v, G \times_{\mathbf{G}_m} H}(Z_{w \boxplus v})$ in $D^b(\mathrm{coh}_{G \times_{\mathbf{G}_m} H} Z_{w \boxplus v})$ is generated by locally-free G-equivariant sheaves and objects of $D^b(\mathrm{coh}_{G \times_{\mathbf{G}_m} H} Z_{w \boxplus v})$ set-theoretically supported on $Z_w \times Z_v$. By Corollary 4.14, locally-free G-equivariant sheaves of finite rank on $Z_{w \boxplus v}$ and objects in the image of l_* for the inclusion $l : \mathrm{Sing}\, Z_{w \boxplus v} \cap (Z_w \times Z_v) \to Z_{w \boxplus v}$ generate $D^b_{Z_w \times Z_v}(\mathrm{coh}_{G \times_{\mathbf{G}_m} H} Z_{w \boxplus v})$. So, in combination, we can generate $D^{\mathrm{sg}}_{Z_w \times Z_v, G \times_{\mathbf{G}_m} H}(Z_{w \boxplus v})$ using the essential image of l_*. It remains to check that exterior products generate $D^b(\mathrm{coh}_{G \times H} \mathrm{Sing}\, Z_{w \boxplus v} \cap (Z_w \times Z_v))$.

Note that $\mathrm{Sing}\, Z_{w \boxplus v} \cap (Z_w \times Z_v) = \mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v$. By Lemma 4.19, the essential image of

$$\begin{aligned} \mathrm{Res}^{G \times H}_{G \times_{\mathbf{G}_m} H} &: D^b(\mathrm{coh}_{G \times H} \mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v) \\ &\to D^b(\mathrm{coh}_{G \times_{\mathbf{G}_m} H} \mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v) \end{aligned}$$

generates. Notice also that $\mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v$ is divisorial simply by pulling back the ample family. Hence, we may apply Corollary 4.21, to see that $D^b(\mathrm{coh}_{G \times_{\mathbf{G}_m} H} \mathrm{Sing}\, Z_w \times \mathrm{Sing}\, Z_v)$ is generated by $\mathcal{E} \boxtimes \mathcal{F}$ for $\mathcal{E} \in \mathrm{coh}_G Z_w$ and $\mathcal{F} \in \mathrm{coh}_H Z_v$. $\square$

Lemma **4.23.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups acting on, respectively,* X *and* Y. *Let* $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ *and* $v \in \Gamma(Y, \mathcal{O}_Y(\chi'))^H$ *for characters* $\chi : G \to \mathbf{G}_m$ *and* $\chi' : H \to \mathbf{G}_m$. *Let* $i_w : Z_w \to X$ *be the zero locus of* w *and let* $i_v : Z_v \to Y$ *be the zero locus of* v. *The derived category of coherent factorizations supported on* $Z_w \times Z_v$, $D^{\mathrm{abs}}_{Z_w \times Z_v}[\mathsf{fact}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)]$, *is generated by exterior products.*

Proof. — By Lemma 4.22, objects of the form $l_* \operatorname{Res}^{G\times H}_{G\times_{\mathbf{G}_m} H} \mathcal{E} \boxtimes \mathcal{F}$ for $\mathcal{E} \in \operatorname{coh}_G \operatorname{Sing} Z_w$ and $\mathcal{F} \in \operatorname{coh}_H \operatorname{Sing} Z_v$ generate $D^{\mathrm{sg}}_{Z_w \times Z_v, G\times_{\mathbf{G}_m} H}(Z_{w\boxplus v})$. By Lemma 3.66, for any $\mathcal{E} \in \operatorname{coh}_G Z_w$ and $\mathcal{F} \in \operatorname{coh}_H Z_v$, there are natural isomorphisms of $G \times_{\mathbf{G}_m} H$-equivariant factorizations of $w \boxplus v$,

$$(\Upsilon\mathcal{E}) \boxtimes (\Upsilon\mathcal{F}) \cong \Upsilon \operatorname{Res}^{G\times H}_{G\times_{\mathbf{G}_m} H}(i_{w*}\mathcal{E} \boxtimes i_{v*}\mathcal{F}).$$

Finally, by Proposition 3.64, Υ is essentially surjective. Thus, $(\Upsilon\mathcal{E}) \boxtimes (\Upsilon\mathcal{F})$ for $\mathcal{E} \in \operatorname{coh}_G Z_w$ and $\mathcal{F} \in \operatorname{coh}_H Z_v$ generate $D^{\mathrm{abs}}_{Z_w \times Z_v}[\mathsf{fact}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)]$. □

5. Bimodule and functor categories for equivariant factorizations

5.1. *Morita products and functor categories for factorization categories.* — We now turn to studying tensor products and internal-homomorphism dg-categories of factorization categories in the homotopy category of dg-categories, $\operatorname{Ho}(\text{dg-cat}_k)$. The main references for background are [Kel06, Toë07].

Definition **5.1.** — *A dg-functor,* $f : \mathsf{C} \to \mathsf{D}$, *is a* ***quasi-equivalence*** *if*

$$H^\bullet(f) : H^\bullet(\operatorname{Hom}_{\mathsf{C}}(c, c')) \to H^\bullet(\operatorname{Hom}_{\mathsf{D}}(f(c), f(c')))$$

is an isomorphism for all $c, c' \in \mathsf{C}$ *and* $[f] : [\mathsf{C}] \to [\mathsf{D}]$ *is essentially surjective.*

Let $\operatorname{Ho}(\text{dg-cat})_k$ *denote the localization of* dg-cat_k *at the class of quasi-equivalences. This category is called* ***the homotopy category of dg-categories***. *If* C *and* D *are quasi-equivalent, we shall write* $\mathsf{C} \simeq \mathsf{D}$.

Definition **5.2.** — *Let* D *be a dg-category. The category of left* D*-modules, denoted* $\mathsf{D}\text{-Mod}$, *is the dg-category of dg-functors,* $\mathsf{D} \to \mathsf{C}(k)$ *where* $\mathsf{C}(k)$ *is the dg-category of chain complexes of vector spaces over* k. *The category of right* D*-modules is the category of left* D^{op}*-modules.*

Each object $d \in \mathsf{D}$ *provides a representable right module*

$$\begin{aligned} h_d : \mathsf{D}^{\mathrm{op}} &\to \mathsf{C}(k) \\ d' &\mapsto \operatorname{Hom}_{\mathsf{D}}(d', d). \end{aligned}$$

We denote the dg-Yoneda embedding by $h : \mathsf{D} \to \mathsf{D}^{\mathrm{op}}\text{-Mod}$.

The Verdier quotient of $[\mathsf{D}\text{-Mod}]$ *by the subcategory of acyclic modules is called the* ***derived category of*** D***-modules*** *and is denoted by* $D[\mathsf{D}\text{-Mod}]$. *The smallest thick subcategory of* $D[\mathsf{D}^{\mathrm{op}}\text{-Mod}]$ *containing the image of* $[h]$ *is called the* ***category of perfect*** D***-modules*** *and is denoted by* $\operatorname{perf}(\mathsf{D})$.

Remark **5.3.** — Throughout the paper, with the exception of the proof of Corollary 5.18, we will take C to be a **quasi-small** dg-category. A dg-category D is quasi-small if $[\mathsf{D}]$ is essentially small. In this case, we can choose a small full subcategory of D quasi-equivalent to D and work with that subcategory to define categories of modules and bimodules. This sidesteps certain set-theoretic issues in the quasi-small case. However, doing this in each example is tedious and not edifying. So we will suppress these arguments throughout the paper.

When C is not quasi-small, but only $\mathbf{U}$-small, one only considers $\mathbf{U}$-small dg-modules. We suppress any of the set-theoretic issues as we do not ascend to a higher universe in the proof of Corollary 5.18.

Definition **5.4.** — *Let* C *and* D *be two dg-categories. A* **quasi-functor** $a : \mathsf{C} \to \mathsf{D}$ *is a dg-functor*

$$a : \mathsf{C} \to \mathsf{D}^{\mathrm{op}}\text{-Mod}$$

such that for each $c \in \mathsf{C}$, $a(c)$ *is quasi-isomorphic to* h_d *for some* $d \in \mathsf{D}$*. Note that a quasi-functor corresponds to a bimodule* $a \in \mathsf{C} \otimes_k \mathsf{D}^{\mathrm{op}}\text{-Mod}$*. Also note, that any quasi-functor induces a functor on homotopy categories which we denote by* $[a] : [\mathsf{C}] \to [\mathsf{D}]$*. In particular, it makes sense to extend the definition of quasi-equivalence to quasi-functors.*

Lemma **5.5.** — *The isomorphism classes of morphisms from* C *to* D *in* $\mathrm{Ho}(\text{dg-cat})_k$ *are in bijection with isomorphism classes of quasi-functors from* C *to* D *viewed as objects of* $\mathrm{D}[\mathsf{C} \otimes_k \mathsf{D}^{\mathrm{op}}\text{-Mod}]$*. In particular, two dg-categories are quasi-equivalent if and only if they are related by a quasi-functor that is a quasi-equivalence.*

Proof. — This is an immediate consequence of the internal Hom constructed by Töen for $\mathrm{Ho}(\text{dg-cat})_k$, [Toë07, Theorem 6.1]. □

The following provides a useful language to keep track of dg-categories.

Definition **5.6.** — *Let* $\mathcal{T}$ *be a triangulated category. An* **enhancement** *of* $\mathcal{T}$ *is a dg-category,* C*, and an exact equivalence*

$$\epsilon : [\mathsf{C}] \to \mathcal{T}.$$

We recall the following result concerning dg-quotients.

Theorem **5.7.** — *Let* C *be a small dg-category and let* D *be a full dg-subcategory. There exists a dg-category* C/D*, unique in* $\mathrm{Ho}(\text{dg-cat}_k)$*, and dg-functor* $\xi : \mathsf{C} \to \mathsf{C}/\mathsf{D}$ *such that for any morphism* $\eta : \mathsf{C} \to \mathsf{A}$ *in* $\mathrm{Ho}(\text{dg-cat}_k)$ *with* $\eta|_{\mathsf{D}} = 0$ *there exists a morphism* $\lambda : \mathsf{C}/\mathsf{D} \to \mathsf{A}$ *with* $\eta \cong \lambda \circ \xi$*.*

Proof. — This is [Dri04, Theorem 3.4]. The objects of C/D in [Dri04, Section 3] are exactly the objects of C. Note that we use that k is a field here. □

Let X be a smooth variety, G be an affine algebraic group acting on X, $\mathcal{L}$ be an invertible G-equivariant sheaf on X, and $w \in \Gamma(X, \mathcal{L})^G$.

Definition **5.8.** — *Let* $D^{abs}\mathsf{vect}(X, G, w)$ *denote the dg-quotient as in Theorem 5.7 of* $\mathsf{vect}(X, G, w)$ *by* $\mathsf{acycvect}(X, G, w)$.

Corollary **5.9.** — *The dg-quotient* $D^{abs}\mathsf{vect}(X, G, w)$ *is an enhancement of* $D^{abs}[\mathsf{fact}(X, G, w)]$.

Proof. — The result is an immediate consequence of Theorem 5.7 and Proposition 3.14. □

Definition **5.10.** — *We will need the following factorization of* 0. *Let* $\mathcal{J}$ *be an injective resolution of* $\mathcal{O}_X$ *and consider the factorization,* $\mathcal{I}^{\mathcal{O}} := \Upsilon\mathcal{J}$, *of* 0.

Proposition **5.11.** — *The dg-category* $\mathsf{Inj}(X, G, w)$ *is an enhancement of* $D^{abs}[\mathsf{Fact}(X, G, w)]$. *The dg-category* $\mathsf{Inj}_{coh}(X, G, w)$ *is an enhancement of* $D^{abs}[\mathsf{fact}(X, G, w)]$. *There is an isomorphism in* $\mathrm{Ho}(\text{dg-cat}_k)$ *between* $\mathsf{Inj}_{coh}(X, G, -w)$ *and* $D^{abs}\mathsf{vect}(X, G, w)^{op}$.

If X *is affine and* G *is reductive, then, additionally,* $\mathsf{Vect}(X, G, w)$ *is an enhancement of* $D^{abs}[\mathsf{Fact}(X, G, w)]$ *and* $\mathsf{vect}(X, G, w)$ *is an enhancement of* $D^{abs}[\mathsf{fact}(X, G, w)]$.

Proof. — The first two statements follow from Proposition 3.11. While the final two follow from Proposition 3.14. For the third statement, consider the dg-functor,

$$\mathcal{H}om_X(\bullet, \mathcal{I}^{\mathcal{O}}) : \mathsf{vect}(X, G, w)^{op} \to \mathsf{Inj}_{coh}(X, G, -w),$$

which sends the subcategory $\mathsf{acycvect}(X, G, w)^{op}$ to acyclic factorizations with injective components. Thus, the induced functor

$$\mathcal{H}om_X(\bullet, \mathcal{I}^{\mathcal{O}}) : \mathsf{acycvect}(X, G, w)^{op} \to \mathsf{Inj}_{coh}(X, G, -w)$$

vanishes on homotopy categories. By [Dri04, Theorem 1.6.2] and Lemma 3.30, $\mathsf{Inj}_{coh}(X, G, -w)$ is a dg-quotient of $\mathsf{vect}(X, G, w)^{op}$ by $\mathsf{acycvect}(X, G, w)^{op}$. By uniqueness, there is an isomorphism in $\mathrm{Ho}(\text{dg-cat}_k)$ between $\mathsf{Inj}_{coh}(X, G, -w)$ and $D^{abs}\mathsf{vect}(X, G, w)^{op}$. □

Corollary **5.12.** — *There is an isomorphism in* $\mathrm{Ho}(\text{dg-cat}_k)$,

$$\mathsf{Inj}_{coh}(X, G, -w) \cong \mathsf{Inj}_{coh}(X, G, w)^{op}.$$

Proof. — The dg-functor

$$\mathcal{H}om_{\mathrm{X}}(\bullet, \mathcal{O}_{\mathrm{X}}) : \mathsf{vect}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}} \to \mathsf{vect}(\mathrm{X}, \mathrm{G}, -w)$$

is an equivalence of dg-categories that preserves the subcategories of acyclic locally-free factorizations. Thus, it induces a quasi-equivalence

$$\mathrm{D}^{\mathrm{abs}}\,\mathsf{vect}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}} \cong \mathrm{D}^{\mathrm{abs}}\,\mathsf{vect}(\mathrm{X}, \mathrm{G}, -w).$$

Applying Proposition 5.11 finishes the argument. □

Definition **5.13.** — *Let* $\mathsf{Inj}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations acyclic off of* Z. *Let* $\mathsf{Inj}_{\mathrm{coh},\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations acyclic off of* Z. *Let* $\overline{\mathsf{Inj}}_{\mathrm{coh},\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{Inj}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations acyclic off of* Z *and compact in* $[\mathsf{Inj}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)]$.

Let $\mathsf{Vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{Vect}(\mathrm{X}, \mathrm{G}, w)$ *consisting factorizations acyclic off of* Z. *Let* $\mathsf{vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{vect}(\mathrm{X}, \mathrm{G}, w)$ *consisting factorizations acyclic off of* Z. *Let* $\overline{\mathsf{Vect}}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *be the full subcategory of* $\mathsf{Vect}(\mathrm{X}, \mathrm{G}, w)$ *consisting of factorizations acyclic off of* Z *and compact in* $\mathrm{D}^{\mathrm{abs}}[\mathsf{Vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)]$.

Corollary **5.14.** — *The dg-category* $\mathsf{Inj}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *is an enhancement of* $\mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$. *The dg-category* $\mathsf{Inj}_{\mathrm{coh},\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *is an enhancement of* $\mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)]$.

If X *is affine and* G *is reductive, then, additionally,* $\mathsf{Vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *is an enhancement of* $\mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}[\mathsf{Fact}(\mathrm{X}, \mathrm{G}, w)]$ *and* $\mathsf{vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *is an enhancement of* $\mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}}[\mathsf{fact}(\mathrm{X}, \mathrm{G}, w)]$. *Moreover,* $\mathsf{Inj}_{\mathrm{coh},\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$ *is quasi-equivalent to* $\mathsf{vect}_{\mathrm{Z}}(\mathrm{X}, \mathrm{G}, w)$.

Proof. — This is an immediate consequence of Proposition 5.11 given the definitions above. □

Theorem **5.15.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups acting on, respectively,* X *and* Y. *Let* $w \in \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}(\chi))^{\mathrm{G}}$ *and* $v \in \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}(\chi'))^{\mathrm{H}}$ *for characters* $\chi : \mathrm{G} \to \mathbf{G}_m$ *and* $\chi' : \mathrm{H} \to \mathbf{G}_m$. *Let* $i_w : \mathrm{Z}_w \to \mathrm{X}$ *be the zero locus of* w *and let* $i_v : \mathrm{Z}_v \to \mathrm{Y}$ *be the zero locus of* v.

Assume that $\chi' - \chi$ *is not torsion. The dg-functor,*

$$\begin{aligned}\lambda_{w\boxplus v} : \mathsf{Inj}_{\mathrm{Z}_w\times\mathrm{Z}_v}&(\mathrm{X}\times\mathrm{Y}, \mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}, w\boxplus v)\\ &\to \left(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)\otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\right)^{\mathrm{op}}\text{-Mod}\\ \mathcal{I} \mapsto \mathrm{Hom}&_{\mathsf{Fact}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}, w\boxplus v)}(\bullet\boxtimes\bullet, \mathcal{I})\end{aligned}$$

induces an equivalence

$$\begin{aligned}\epsilon_{w \boxplus v} : \mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}_w \times \mathrm{Z}_v}\big[\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)\big] \\ \to \mathrm{D}\Big(\big(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\big)^{\mathrm{op}}\text{-Mod}\Big)\end{aligned}$$

satisfying

$$\epsilon_{w \boxplus v}(\mathcal{E} \boxtimes \mathcal{F}) \cong h_{\mathcal{E} \otimes_k \mathcal{F}}.$$

If, in addition, X *and* Y *are affine and* G *and* H *are reductive, then the dg-functor*

$$\begin{aligned}\lambda_{w \boxplus v} : \mathsf{Vect}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v) \\ \to \big(\mathsf{vect}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v)\big)^{\mathrm{op}}\text{-Mod}) \\ \mathcal{V} \mapsto \mathrm{Hom}_{\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)}(\bullet \boxtimes \bullet, \mathcal{V})\end{aligned}$$

induces an equivalence

$$\begin{aligned}\epsilon_{w \boxplus v} : \mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}_w \times \mathrm{Z}_v}\big[\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)\big] \\ \to \mathrm{D}\Big(\big(\mathsf{vect}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v)\big)^{\mathrm{op}}\text{-Mod}\Big)\end{aligned}$$

satisfying

$$\epsilon_{w \boxplus v}(\mathcal{E} \boxtimes \mathcal{F}) \cong h_{\mathcal{E} \otimes_k \mathcal{F}}.$$

Proof. — We just need to check that the induced functor,

$$\begin{aligned}\epsilon_{w \boxplus v} : \mathrm{D}^{\mathrm{abs}}_{\mathrm{Z}_w \times \mathrm{Z}_v}\big[\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)\big] \\ \cong \big[\mathsf{Inj}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)\big] \\ \xrightarrow{[\lambda_{w \boxplus v}]} \Big[\big(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\big)^{\mathrm{op}}\text{-Mod}\Big] \\ \to \mathrm{D}\Big(\big(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\big)^{\mathrm{op}}\text{-Mod}\Big)\end{aligned}$$

is an equivalence. Note that $\epsilon_{w \boxplus v}$ commutes with coproducts since the exterior products, $\mathcal{E} \boxtimes \mathcal{F}$, are compact in $\mathrm{D}^{\mathrm{abs}}[\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)]$ when $\mathcal{E} \in \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)$ and $\mathcal{F} \in \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)$. The triangulated category, $[\mathsf{Inj}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)]$, is compactly generated by Proposition 3.15 and the objects, $h_{\mathcal{E} \otimes \mathcal{F}}$, for a $\mathcal{E} \in \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)$ and $\mathcal{F} \in \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)$, form a compact set of generators for the category, $\mathrm{D}((\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v))^{\mathrm{op}}\text{-Mod})$.

Thus to check that $\epsilon_{w \boxplus v}$ is an equivalence it suffices to check that it takes a compact generating set to a compact generating set and is fully-faithful on those sets. Let us first show that there is a quasi-isomorphism of bimodules

$$\begin{aligned}h_{\mathcal{E} \otimes \mathcal{F}} := \mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}(\mathrm{X}, \mathrm{G}, w)}}(\bullet, \mathcal{E}) \otimes_k \mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}(\mathrm{Y}, \mathrm{H}, v)}}(\bullet, \mathcal{F}) \\ \simeq \mathrm{Hom}_{\mathsf{Fact}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v)}(\bullet \boxtimes \bullet, \mathcal{I}_{\mathcal{E} \boxtimes \mathcal{F}})\end{aligned}$$

where we have a morphism of factorizations $\mathcal{E} \boxtimes \mathcal{F} \to \mathcal{I}_{\mathcal{E}\boxtimes\mathcal{F}}$ whose cone is acyclic and where the components of $\mathcal{I}_{\mathcal{E}\boxtimes\mathcal{F}}$ have injective components. We have the natural morphism

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}(X,G,w)}}(\bullet, \mathcal{E}) \otimes_k \mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}(Y,H,v)}}(\bullet, \mathcal{F}) \\
&\xrightarrow{\boxtimes} \mathrm{Hom}_{\mathsf{Fact}(X\times Y, G\times_{\mathbf{G}_m} H, w\boxplus v)}(\bullet \boxtimes \bullet, \mathcal{E} \boxtimes \mathcal{F}) \\
&\to \mathrm{Hom}_{\mathsf{Fact}(X\times Y, G\times_{\mathbf{G}_m} H, w\boxplus v)}(\bullet \boxtimes \bullet, \mathcal{I}_{\mathcal{E}\boxtimes\mathcal{F}})
\end{aligned}$$

where the later morphism is given by composing with $\mathcal{E} \boxtimes \mathcal{F} \to \mathcal{I}_{\mathcal{E}\boxtimes\mathcal{F}}$. By Lemma 3.52, this is a quasi-isomorphism. Again, appealing to Lemma 3.52 shows that $\epsilon_{w\boxplus v}$ is fully-faithful on exterior products. It remains to check that exterior products are generators for $\mathrm{D}^{\mathrm{abs}}_{Z_w \times Z_v}[\mathsf{Fact}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)]$, but this is Lemma 4.23.

The statements with X and Y affine and G and H reductive follow via an analogous argument. Indeed, in this case, taking G invariants is exact and locally-free objects are projective so we can work with locally-free objects in the exact same manner. □

Definition **5.16.** — *Let* C *be a dg-category. The category* C-Mod *possesses the structure of a model category with* $f : F \to G$ *being a fibration, respectively a weak equivalence, if* $f(c) : F(c) \to G(c)$ *is an epimorphism in each degree, respectively a quasi-isomorphism, for each* $c \in \mathsf{C}$. *This determines the cofibrations: they are those morphisms satisfying the left lifting property with respect to all acyclic fibrations, i.e. those maps that are fibrations and weak equivalences.*

Any object of C-Mod *is fibrant. We let* $\widehat{\mathsf{C}}$ *be the subcategory of cofibrant objects in* C^{op}-Mod. *The dg-category* $\widehat{\mathsf{C}}$ *is an enhancement of* $\mathrm{D}[\mathsf{C}^{\mathrm{op}}\text{-Mod}]$. *We let* $\widehat{\mathsf{C}}_{\mathrm{pe}}$ *be the full sub-dg-category of* $\widehat{\mathsf{C}}$ *consisting of all objects that are compact in* $\mathrm{D}[\mathsf{C}^{\mathrm{op}}\text{-Mod}]$. *As any representable dg-module is cofibrant, we have a dg-functor*

$$h : \mathsf{C} \to \widehat{\mathsf{C}}_{\mathrm{pe}}.$$

Following the lead of Töen, we introduce the following product. Assume that C *is small and let* D *be another small dg-category over* k. *The* **Morita product** *of* C *and* D *is*

$$\mathsf{C} \circledast \mathsf{D} := (\widehat{\mathsf{C} \otimes_k \mathsf{D}})_{\mathrm{pe}}$$

viewed as an object of $\mathrm{Ho}(\text{dg-cat}_k)$. *Because we view it as an object of* $\mathrm{Ho}(\text{dg-cat}_k)$, *it is unique up to quasi-equivalence.*

Remark **5.17.** — The cofibrant objects of C^{op}-Mod are exactly the summands of semi-free dg-modules [FHT01]. One can check that summands of semi-free dg-modules have the appropriate lifting property. Furthermore, for any dg-module, M, there exists a semi-free dg-module, F, and an acyclic fibration, $F \to M$. If we assume that M is cofibrant, this must split.

Corollary **5.18.** — *Let* X *be a smooth variety,* G *be an affine algebraic group acting on* X*,* $\mathcal{L}$ *be an invertible* G*-equivariant sheaf on* X*, and* $w \in \Gamma(\mathrm{X}, \mathcal{L})^{\mathrm{G}}$*. Let* Y *be a smooth variety,* H *be an affine algebraic group acting* X*,* $\mathcal{L}'$ *be an invertible* H*-equivariant sheaf on* Y*, and* $v \in \Gamma(\mathrm{Y}, \mathcal{L}')^{\mathrm{H}}$*. There are isomorphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}&\mathsf{Inj}\big(\mathrm{U}(\mathcal{L}) \times \mathrm{U}\big(\mathcal{L}'\big), \mathrm{G} \times \mathrm{H} \times \mathbf{G}_m, f_w \boxplus f_v\big)\\ &\quad\cong \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, \widehat{w) \otimes_k} \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\end{aligned}$$

and

$$\begin{aligned}&\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \circledast \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)\\ &\quad\cong \overline{\mathsf{Inj}}_{\mathrm{coh}}\big(\mathrm{U}(\mathcal{L}) \times \mathrm{U}\big(\mathcal{L}'\big), \mathrm{G} \times \mathrm{H} \times \mathbf{G}_m, f_w \boxplus f_v\big).\end{aligned}$$

Assume in addition that X *and* Y *are affine and* G *and* H *are reductive. Then, there are isomorphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}&\mathsf{Vect}\big(\mathrm{U}(\mathcal{L}) \times \mathrm{U}\big(\mathcal{L}'\big), \mathrm{G} \times \mathrm{H} \times \mathbf{G}_m, f_w \boxplus f_v\big)\\ &\quad\cong \mathsf{vect}(\mathrm{X}, \mathrm{G}, \widehat{w) \otimes_k} \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v)\end{aligned}$$

and

$$\begin{aligned}&\mathsf{vect}(\mathrm{X}, \mathrm{G}, w) \circledast \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v)\\ &\quad\cong \overline{\mathsf{vect}}\big(\mathrm{U}(\mathcal{L}) \times \mathrm{U}\big(\mathcal{L}'\big), \mathrm{G} \times \mathrm{H} \times \mathbf{G}_m, f_w \boxplus f_v\big).\end{aligned}$$

In the special case that $\mathcal{L} = \mathcal{O}_{\mathrm{X}}(\chi)$ *and* $\mathcal{L}' = \mathcal{O}_{\mathrm{Y}}(\chi')$ *for characters* $\chi : \mathrm{G} \to \mathbf{G}_m$ *and* $\chi' : \mathrm{H} \to \mathbf{G}_m$*, if we assume that* χ *or* χ' *is not torsion, then there are isomorphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\mathsf{Inj}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v) \cong \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, \widehat{w) \otimes_k} \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)$$

and

$$\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w) \circledast \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v) \cong \overline{\mathsf{Inj}}_{\mathrm{coh}\,\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v).$$

Assume in addition that X *and* Y *are affine and* G *and* H *are reductive. Then, there are isomorphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\mathsf{Vect}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v) \cong \mathsf{vect}(\mathrm{X}, \mathrm{G}, \widehat{w) \otimes_k} \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v)$$

and

$$\mathsf{vect}(\mathrm{X}, \mathrm{G}, w) \circledast \mathsf{vect}(\mathrm{Y}, \mathrm{H}, v) \cong \overline{\mathsf{vect}}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, w \boxplus v).$$

Proof. — By Lemma 3.48, we have equivalences of dg-categories

$$\mathsf{Fact}(X, G, w) \cong \mathsf{Fact}\big(U(\mathcal{L}), G \times \mathbf{G}_m, f_w\big)$$
$$\mathsf{Fact}(Y, H, w) \cong \mathsf{Fact}\big(U\big(\mathcal{L}'\big), H \times \mathbf{G}_m, f_v\big).$$

Replacing $(X, G, \mathcal{L}, w)$ and $(Y, H, \mathcal{L}', v)$ by $(U(\mathcal{L}), G \times \mathbf{G}_m, \mathcal{O}_{U(\mathcal{L})}(1), f_w)$ and $(U(\mathcal{L}'), H \times \mathbf{G}_m, \mathcal{O}_{U(\mathcal{L}')}(1), f_v)$, we may assume that $\mathcal{L}$ and $\mathcal{L}'$ are (non-equivariantly) trivial as sheaves on X and Y, respectively, and continue the argument. Finally, as f_w and f_v are both linear along the fibers, Euler's formula using the fiber coordinates shows that f_w vanishes only along the singular locus of f_w and similarly for f_v. Thus, f_w and f_v both vanish along the singular locus of $f_w \boxplus f_v$. Consequently, factorizations supported away from $Z_{f_w} \times Z_{f_v}$ are automatically acyclic. Thus, we are reduced to proving the special case of the statement.

We have a dg-functor,

$$\lambda_{w\boxplus v} : \mathsf{Inj}_{Z_w \times Z_v}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)$$
$$\to \big(\mathsf{Inj}_{\mathrm{coh}}(X, G, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(Y, H, v)\big)^{\mathrm{op}}\text{-Mod}$$

and an inclusion

$$\mathsf{Inj}_{\mathrm{coh}}(X, G, \widehat{-w) \otimes_k} \mathsf{Inj}_{\mathrm{coh}}(Y, H, v)$$
$$\to \big(\mathsf{Inj}_{\mathrm{coh}}(X, G, w) \otimes_k \mathsf{Inj}_{\mathrm{coh}}(Y, H, v)\big)^{\mathrm{op}}\text{-Mod}\,.$$

We then have a dg-functor

$$a : \mathsf{Inj}_{Z_w \times Z_v}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)$$
$$\to \big(\mathsf{Inj}_{\mathrm{coh}}(X, G, \widehat{-w) \otimes_k} \mathsf{Inj}_{\mathrm{coh}}(Y, H, v)\big)^{\mathrm{op}}\text{-Mod}$$
$$M \mapsto \mathrm{Hom}_{(\mathsf{Inj}_{\mathrm{coh}}(X,G,w)\otimes_k \mathsf{Inj}_{\mathrm{coh}}(Y,H,v))^{\mathrm{op}}\text{-Mod}}(\bullet, M).$$

For any $N \in \mathsf{Inj}_{Z_w \times Z_v}(X \times Y, G \times_{\mathbf{G}_m} H, w \boxplus v)$, there exists an $M \in \mathsf{Inj}_{\mathrm{coh}}(X, G, \widehat{-w)\otimes_k}$ $\mathsf{Inj}_{\mathrm{coh}}(Y, H, v)$ and a quasi-isomorphism $f : M \to N$. The induced natural transformation

$$\mathrm{Hom}(\bullet, f) : \mathrm{Hom}(\bullet, M) \to \mathrm{Hom}(\bullet, N)$$

is a quasi-isomorphism if we restrict the argument to lie in $\mathsf{Inj}_{\mathrm{coh}}(X, G, \widehat{-w)\otimes_k}$ $\mathsf{Inj}_{\mathrm{coh}(Y,H,v)}$. Thus, $a(N)$ is quasi-isomorphic to h_M. Given M quasi-isomorphic to N and M′ quasi-isomorphic to N′, we have natural isomorphisms

$$\mathrm{Hom}_{D[(\mathsf{Inj}_{\mathrm{coh}}(X,G,\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(Y,H,v))^{\mathrm{op}}\text{-Mod}]}\big(a(N), a\big(N'\big)\big)$$
$$\cong \mathrm{Hom}_{D[(\mathsf{Inj}_{\mathrm{coh}}(X,G,\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(Y,H,v))^{\mathrm{op}}\text{-Mod}]}(h_M, h_{M'})$$

$$
\begin{aligned}
&\cong \mathrm{Hom}_{[\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v)]}\big(\mathrm{M},\mathrm{M}'\big)\\
&\cong \mathrm{Hom}_{\mathrm{D}[(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},w)\otimes_k\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v))^{\mathrm{op}}\text{-Mod}]}\big(\mathrm{M},\mathrm{M}'\big)\\
&\cong \mathrm{Hom}_{\mathrm{D}[(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},w)\otimes_k\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v))^{\mathrm{op}}\text{-Mod}]}\big(\mathrm{N},\mathrm{N}'\big)\\
&\cong \mathrm{Hom}_{[\mathsf{Inj}_{\mathrm{Z}_w\times\mathrm{Z}_v}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},w\boxplus v)]}\big(\mathrm{N},\mathrm{N}'\big)
\end{aligned}
$$

where the first isomorphism is due to the fact that $a(\mathrm{N})$ is quasi-isomorphic to h_{M} and $a(\mathrm{N}')$ is quasi-isomorphic to $h_{\mathrm{M}'}$, the second uses the Yoneda embedding, the third uses that $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v)$ is an enhancement of $\mathrm{D}[(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},w)\otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v))^{\mathrm{op}}\text{-Mod}]$, the fourth uses the assumed quasi-isomorphisms, and the final isomorphism uses that $\mathsf{Inj}_{\mathrm{Z}_w\times\mathrm{Z}_v}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},w\boxplus v)$ is an enhancement of $\mathrm{D}[(\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},w)\otimes_k \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v))^{\mathrm{op}}\text{-Mod}]$, i.e. Theorem 5.15.

Thus, a is a quasi-functor inducing a quasi-equivalence

$$
\mathsf{Inj}_{\mathrm{Z}_w\times\mathrm{Z}_v}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},w\boxplus v)\simeq \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v).
$$

The isomorphism in $\mathrm{Ho}(\text{dg-cat}_k)$

$$
\begin{aligned}
&\mathsf{Inj}_{\mathrm{Z}_w\times\mathrm{Z}_v}\big(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},(-w)\boxplus v\big)\\
&\qquad\cong \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},\widehat{-w)\otimes_k}\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v)
\end{aligned}
$$

induces an isomorphism between the compact objects,

$$
\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X},\mathrm{G},w)\circledast\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y},\mathrm{H},v)\cong\overline{\mathsf{Inj}}_{\mathrm{coh},\mathrm{Z}_w\times\mathrm{Z}_v}(\mathrm{X}\times\mathrm{Y},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H},w\boxplus v).
$$

In the case that X and Y are affine and G and H are reductive, an analogous argument suffices. Indeed, as noted before, taking G invariants is exact and locally-free objects are projective so we can work with locally-free objects in the exact same manner. □

Remark **5.19.** — In the case that $\mathcal{L}=\mathcal{O}_{\mathrm{X}}(\chi)$ and $\mathcal{L}'=\mathcal{O}_{\mathrm{Y}}(\chi')$, the quotient stack $[\mathrm{U}(\mathcal{O}_{\mathrm{X}}(\chi))\times\mathrm{U}(\mathcal{O}_{\mathrm{Y}}(\chi'))/(\mathrm{G}\times\mathrm{H}\times\mathbf{G}_m)]$ is isomorphic to $[\mathrm{X}\times\mathrm{Y}\times\mathbf{G}_m/(\mathrm{G}\times\mathrm{H})]$ via the morphism

$$
\begin{aligned}
\phi:\mathrm{U}\big(\mathcal{O}_{\mathrm{X}}(\chi)\big)\times\mathrm{U}\big(\mathcal{O}_{\mathrm{Y}}(\chi')\big)\cong\mathbf{G}_m\times\mathrm{X}\times\mathbf{G}_m\times\mathrm{Y}&\to\mathrm{X}\times\mathrm{Y}\times\mathbf{G}_m\\
(\alpha,x,\beta,y)&\mapsto\big(x,y,\alpha^{-1}\beta\big).
\end{aligned}
$$

The quotient stack $[\mathrm{X}\times\mathrm{Y}\times\mathbf{G}_m/(\mathrm{G}\times\mathrm{H})]$ is isomorphic to $[\mathrm{X}\times\mathrm{Y}/\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}]$ as the map

$$
\begin{aligned}
(\mathrm{G}\times\mathrm{H})\overset{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}{\times}(\mathrm{X}\times\mathrm{Y})&\to\mathrm{X}\times\mathrm{Y}\times\mathbf{G}_m\\
(g,h,x,y)&\mapsto\big(x,y,\chi(g)^{-1}\chi'(h)\big)
\end{aligned}
$$

is an isomorphism assuming that $\chi' - \chi : G \times H \to \mathbf{G}_m$ is not torsion. This gives a direct comparison for the two LG models describing the Morita product in the case $\mathcal{L} = \mathcal{O}_X(\chi)$ and $\mathcal{L}' = \mathcal{O}_Y(\chi')$.

One of the many great results of [Toë07] is the following. It provides a description of the continuous internal Hom dg-category in $\mathrm{Ho}(\text{dg-cat}_k)$.

Theorem **5.20.** — *Let* C *and* D *be small dg-categories over* k. *Then, there is an isomorphism in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\mathbf{R}\mathrm{Hom}_c(\widehat{\mathsf{C}}, \widehat{\mathsf{D}}) \cong \widehat{\mathsf{C}^{\mathrm{op}} \otimes_k \mathsf{D}}.$$

Given a module, $F \in \widehat{\mathsf{C}^{\mathrm{op}} \otimes_k \mathsf{D}}$, *the corresponding dg-functor,* $\Psi_F : \mathsf{C} \to \widehat{\mathsf{D}}$, *sends* $c \in \mathsf{C}$ *to* $F(c, \bullet) \in \widehat{\mathsf{D}}$. *This uniquely determines a dg-functor,* $\Psi_F : \widehat{\mathsf{C}} \to \widehat{\mathsf{D}}$, *for which* $[\Psi_F]$ *commutes with coproducts.*

Proof. — As stated, this result is [Toë07, Corollary 7.6]. $\square$

Remark **5.21.** — Töen's result is more general. The field, k, can be replaced by a commutative ring. The derivation of the tensor product, $\otimes_k$, is then required.

Applying Theorem 5.20, we can give the following description of the continuous internal Hom dg-category for equivariant factorizations.

Theorem **5.22.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups. Assume that* G *acts on* X *and* H *acts on* Y. *Let* $\chi : G \to \mathbf{G}_m$ *and* $\chi' : H \to \mathbf{G}_m$ *be characters and let* $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ *and* $v \in \Gamma(Y, \mathcal{O}_Y(\chi'))^H$.

There is an isomorphism in $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_c\big(\widehat{\mathsf{Inj}_{\mathrm{coh}}(X, G, w)}, \widehat{\mathsf{Inj}_{\mathrm{coh}}(Y, H, v)}\big)\\ &\quad\cong \mathsf{Inj}_{Z_w \times Z_v}\big(X \times Y, G \times_{\mathbf{G}_m} H, (-w) \boxplus v\big)\end{aligned}$$

such that the induced map on homotopy categories corresponding to $\mathcal{I} \in \mathsf{Inj}_{Z_w \times Z_v}(X \times Y, G \times_{\mathbf{G}_m} H, (-w) \boxplus v)$ *is* $\Phi_{\mathcal{I}}$.

If X *is affine and* G *is reductive, then there is an isomorphism in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_c\big(\widehat{\mathsf{vect}(X, G, w)}, \widehat{\mathsf{vect}(Y, H, v)}\big)\\ &\quad\cong \mathsf{Vect}_{Z_w \times Z_v}\big(X \times Y, G \times_{\mathbf{G}_m} H, (-w) \boxplus v\big)\end{aligned}$$

such that the induced map on homotopy categories corresponding to $\mathcal{P} \in \mathsf{Vect}_{Z_w \times Z_v}(X \times Y, G \times_{\mathbf{G}_m} H, (-w) \boxplus v)$ *is* $\Phi_{\mathcal{P}}$.

Proof. — We have isomorphisms in $\mathrm{Ho}(\text{dg-cat}_k)$,

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_c\big(\widehat{\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)}, \widehat{\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v)}\big) &\cong \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}} \widehat{\otimes_k} \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v) \\ &\cong \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, -w) \widehat{\otimes_k} \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y}, \mathrm{H}, v) \\ &\cong \mathsf{Inj}_{\mathrm{Z}_w \times \mathrm{Z}_v}\big(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, (-w) \boxplus v\big).\end{aligned}$$

The first line follows from Theorem 5.20. The second line comes from Proposition 5.11 which states that $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}}$ is quasi-equivalent to $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, -w)$. The third line is an application of Corollary 5.18.

Next, we need to check that the induced functor on homotopy categories for a given $\mathcal{I} \in \mathsf{Inj}_{\mathrm{Z}_w \times \mathrm{Z}_v}(\mathrm{X} \times \mathrm{Y}, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{H}, (-w) \boxplus v)$ is $\Phi_{\mathcal{I}}$ up to isomorphism. Recall that the isomorphism of $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}}$ and $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, -w)$ follows from the diagram of dg-functors

$$\begin{tikzcd}
\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}} & \mathsf{vect}(\mathrm{X}, \mathrm{G}, -w) \arrow[l, "\mathcal{H}om_{\mathrm{X}}(\bullet, \mathcal{I}^{\mathcal{O}})"'] \arrow[d, leftrightarrow, "\mathcal{H}om_{\mathrm{X}}(\bullet, \mathcal{O}_{\mathrm{X}})"] \\
\mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, -w) & \mathsf{vect}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}} \arrow[l, "\mathcal{H}om_{\mathrm{X}}(\bullet, \mathcal{I}^{\mathcal{O}})"']
\end{tikzcd}$$

The induced dg-functor on the image of

$$\mathcal{H}om_{\mathrm{X}}\big(\bullet, \mathcal{I}^{\mathcal{O}}\big) : \mathsf{vect}(\mathrm{X}, \mathrm{G}, -w) \to \mathsf{Inj}_{\mathrm{coh}}(\mathrm{X}, \mathrm{G}, w)^{\mathrm{op}}$$

is

$$\begin{aligned}\mathcal{H}om_{\mathrm{X}}\big(\mathcal{E}, \mathcal{I}^{\mathcal{O}}\big) \otimes \mathcal{J} \quad &\mapsto \mathrm{Hom}_{\mathsf{Fact}}\big(\mathcal{H}om_{\mathrm{X}}\big(\mathcal{E}^{\vee}, \mathcal{I}^{\mathcal{O}}\big) \boxtimes \mathcal{J}, \mathcal{I}\big) \\ \cong \quad &\mapsto \mathrm{Hom}_{\mathsf{Fact}}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_2^* \mathcal{J}, \mathcal{H}om_{\mathrm{X}\times\mathrm{Y}}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_1^* \mathcal{H}om_{\mathrm{X}}\big(\mathcal{E}^{\vee}, \mathcal{I}^{\mathcal{O}}\big), \mathcal{I}\big)\big) \\ \cong \quad &\mapsto \mathrm{Hom}_{\mathsf{Fact}}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_2^* \mathcal{J}, \mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_1^* \mathcal{E}^{\vee} \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathcal{I}\big) \\ \cong \quad &\mapsto \mathrm{Hom}_{\mathsf{Fact}}\big(\mathcal{J}, \pi_{2*} \mathrm{Ind}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}\big(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_1^* \mathcal{E}^{\vee} \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathcal{I}\big)\big) \\ \cong \quad &\mapsto \mathrm{Hom}_{\mathsf{Fact}}\big(\mathcal{J}, \pi_{2*}\big(\pi_1^* \mathcal{E}^{\vee} \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathrm{Ind}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \mathcal{I}\big)\big).\end{aligned}$$

The first line uses tensor-Hom adjunction, Proposition 3.27. The second line uses the natural isomorphism, $\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_1^* \mathcal{E}^{\vee} \otimes_{\mathcal{O}_{\mathrm{X}\times\mathrm{Y}}} \mathcal{I} \to \mathcal{H}om_{\mathrm{X}\times\mathrm{Y}}(\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \pi_1^* \mathcal{H}om_{\mathrm{X}}(\mathcal{E}^{\vee}, \mathcal{I}^{\mathcal{O}}), \mathcal{I}))$. The third line uses the adjunctions, $\pi_2^* \dashv \pi_{2*}$ and $\mathrm{Res}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}} \dashv \mathrm{Ind}^{\mathrm{G}\times\mathrm{H}}_{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{H}}$. The fourth line applies the projection formula, Lemma 2.16.

As $\mathcal{H}om_{\mathrm{X}}(\mathcal{E}, \mathcal{I}^{\mathcal{O}})$ is quasi-isomorphic to $\mathcal{E}^{\vee}$, from the aligned display, we see that the induced map on the homotopy categories is $\Phi_{\mathcal{I}}$. The case of X affine and G reductive is handled in an analogous, even simpler, manner. □

Proof of Theorem 1.1. — By Lemma 3.48, we have equivalences of dg-categories

$$\mathsf{Fact}(X, G, w) \cong \mathsf{Fact}\big(U(\mathcal{L}), G \times \mathbf{G}_m, f_w\big)$$
$$\mathsf{Fact}(Y, H, w) \cong \mathsf{Fact}\big(U(\mathcal{L}'), H \times \mathbf{G}_m, f_v\big).$$

Theorem 5.22 applied to $(U(\mathcal{L}), G \times \mathbf{G}_m, f_w)$ and $(U(\mathcal{L}'), H \times \mathbf{G}_m, f_v)$ gives the statement. ☐

5.2. *Hochschild invariants.* — In this section, we compute the Hochschild invariants in a simple case: G acting linearly on $\mathbf{A}^n$. We start out a bit more generally. Let G act on X and let $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$. For the whole of this section, we assume

$$\operatorname{Sing} Z_{(-w)\boxplus w} \subseteq Z_w \times Z_w$$

so we may remove the support restrictions in the results of Section 5.1.

Definition **5.23.** — *Let* C *be a small dg-category. The* **Hochschild cohomology** *of* C *is the graded vector space*

$$\bigoplus_{t \in \mathbf{Z}} \operatorname{Hom}_{D(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{C}\text{-Mod})}\big(\mathsf{C}, \mathsf{C}[t]\big).$$

where C *is the bimodule given by*

$$\mathsf{C}\big(c, c'\big) = \operatorname{Hom}_{\mathsf{C}}\big(c, c'\big).$$

When $\mathsf{C} = \mathsf{Inj}_{\mathrm{coh}}(X, G, w)$*, we denote the Hochschild cohomology by* $HH^\bullet(X, G, w)$.

We have a trace functor

$$\begin{aligned} \mathrm{Tr} : \mathsf{C}^{\mathrm{op}} \otimes \mathsf{C} &\to \mathsf{C}(k) \\ \big(c, c'\big) &\mapsto \operatorname{Hom}_{\mathsf{C}}\big(c, c'\big). \end{aligned}$$

This admits an extension to $\mathsf{C} \otimes \mathsf{C}^{\mathrm{op}}$ *-Mod by*

$$F \mapsto F \otimes_{\mathsf{C} \otimes \mathsf{C}^{\mathrm{op}}} \mathsf{C}.$$

The **Hochschild homology** *of* C *is defined to be the homology of*

$$\mathsf{C} \overset{\mathbf{L}}{\otimes}_{\mathsf{C}^{\mathrm{op}} \otimes \mathsf{C}} \mathsf{C}.$$

When $\mathsf{C} = \mathsf{Inj}_{\mathrm{coh}}(X, G, w)$*, we denote the Hochschild cohomology by* $HH_\bullet(X, G, w)$.

Lemma **5.24.** — *Let* X *and* Y *be smooth varieties and let* G *and* H *be affine algebraic groups acting on, respectively,* X *and* Y. *Let* $w \in \Gamma(X, \mathcal{O}_X(\chi))^G$ *and* $v \in \Gamma(Y, \mathcal{O}_Y(\chi'))^H$ *for characters* $\chi : G \to \mathbf{G}_m$ *and* $\chi' : H \to \mathbf{G}_m$. *Assume that* $\operatorname{Sing} Z_{(-w)\boxplus w} \subseteq Z_w \times Z_w$.

We have isomorphisms

$$\mathrm{HH}^t(X, G, w) \cong \mathrm{Hom}_{D^{\mathrm{abs}}[\mathsf{Fact}(X\times Y, G\times_{\mathbf{G}_m} G, w\boxplus(-w))]}\big(\nabla, \nabla[t]\big).$$

We also have isomorphisms

$$\mathrm{HH}_t(X, G, w) \cong \mathrm{H}^t(\mathbf{L}\mathrm{Tr}\,\nabla)$$

where $\mathbf{L}\mathrm{Tr}$ *is trace functor on* $D^{\mathrm{abs}}[\mathsf{Fact}(X \times Y, G \times_{\mathbf{G}_m} G, (-w) \boxplus w]$.

Proof. — By Theorem 5.15, we have an equivalence

$$\begin{aligned} &D^{\mathrm{abs}}\big[\mathsf{Fact}(X \times Y, G \times_{\mathbf{G}_m} G, w \boxplus (-w)\big] \\ &\quad \to D\big(\mathsf{Inj}_{\mathrm{coh}}(X, G, w)^{\mathrm{op}} \otimes \mathsf{Inj}_{\mathrm{coh}}(X, G, w)\text{-Mod}\big) \\ &\mathcal{P} \mapsto \mathbf{R}\mathrm{Hom}\big(\bullet \boxtimes \bullet^{\mathbf{L}\vee}, \mathcal{P}\big). \end{aligned}$$

The assumption on the singular support of $Z_{(-w)\boxplus w}$ allows us to remove the support condition.

We have natural quasi-isomorphisms

$$\begin{aligned} \mathbf{R}\mathrm{Hom}\big(\mathcal{E} \boxtimes \mathcal{F}^{\mathbf{L}\vee}, \nabla\big) &= \mathbf{R}\mathrm{Hom}\big(\mathcal{E} \boxtimes \mathcal{F}^{\mathbf{L}\vee}, \mathrm{Ind}_G^{G\times_{\mathbf{G}_m} G} \Delta_* \mathcal{O}_X\big) \\ &\simeq \mathbf{R}\mathrm{Hom}\big(\mathbf{L}\Delta^* \operatorname{Res}_G^{G\times_{\mathbf{G}_m} G} \mathcal{E} \boxtimes \mathcal{F}^{\mathbf{L}\vee}, \mathcal{O}_X\big) \\ &\simeq \mathbf{R}\mathrm{Hom}\big(\mathcal{E} \otimes \mathcal{F}^{\mathbf{L}\vee}, \mathcal{O}_X\big) \\ &\simeq \mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}). \end{aligned}$$

The first line is the definition of ∇. The second line is an application of the adjunctions $\operatorname{Res}_G^{G\times_{\mathbf{G}_m} G} \dashv \mathrm{Ind}_G^{G\times_{\mathbf{G}_m} G}$, Corollary 3.43, and $\mathbf{L}\Delta^* \dashv \Delta_*$, derived from Lemma 3.35. The third line comes from the identity $\mathbf{L}\Delta^* \circ \pi_i^* \cong \mathrm{Id}$ for $i = 1, 2$. The final line is tensor-Hom adjunction, Corollary 3.28, and the assumption that $\mathcal{F}$ is quasi-isomorphic to a coherent factorization so

$$\mathcal{F}^{\mathbf{L}\vee\mathbf{L}\vee} \cong \mathcal{F}.$$

We turn to the statement concerning Hochschild homology. Under the equivalence

$$\begin{aligned} &D^{\mathrm{abs}}\big[\mathsf{Fact}(X \times Y, G \times_{\mathbf{G}_m} G, (-w) \boxplus w\big] \\ &\quad \to D\big(\mathsf{Inj}_{\mathrm{coh}}(X, G, w) \otimes \mathsf{Inj}_{\mathrm{coh}}(X, G, w)^{\mathrm{op}}\text{-Mod}\big) \end{aligned}$$

the categorical trace corresponds to the trace functor

$$\left(\mathbf{R}p_*\mathbf{L}\Delta^*\right)^{\mathbf{R}G}$$

by Lemma 3.55. □

Remark **5.25.** — As transposing the two copies of X induces an equivalence

$$\begin{aligned}&\mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}, w\boxplus(-w)\big]\\ &\quad\cong \mathrm{D}^{\mathrm{abs}}\big[\mathsf{Fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},(-w)\boxplus w\big]\end{aligned}$$

which preserves the diagonal, we can compute Hochschild invariants in either derived category of factorizations.

The Hochschild cohomology is a subalgebra of a larger algebra.

Definition **5.26.** — *The* ***extended Hochschild cohomology*** *of* $(\mathrm{X},\mathrm{G},w)$ *is the* $\widehat{\mathrm{G}}\times\mathbf{Z}$*-graded k-algebra*

$$\bigoplus_{\rho\in\widehat{\mathrm{G}},t\in\mathbf{Z}}\mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{X}\times\mathrm{X},\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G},(-w)\boxplus w)]}\big(\nabla,\nabla(\rho)[t]\big).$$

We denote the extended Hochschild cohomology by $\mathrm{HH}^{\bullet}_e(\mathrm{X},\mathrm{G},w)$.

Remark **5.27.** — The ring $\mathrm{HH}^{\bullet}_e(\mathrm{X},\mathrm{G},w)$ is a factorization analog of generalized Hochschild cohomology of a variety X with support in $\mathrm{T}\in\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{X}\times\mathrm{X})$ and coefficients in $\mathrm{E}\in\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{X}\times\mathrm{X})$, $\mathrm{HH}^{\bullet}_{\mathrm{T}}(\mathrm{X},\mathrm{E})$ defined by Kuznetsov [Kuz10]. Here, we take E to be the diagonal and T to be the kernels of twist functors.

Lemma **5.28.** — *There is a natural isomorphism,*

$$\mathrm{HH}^t(\mathrm{X},\mathrm{G},w)\to\mathrm{HH}^{(0,t)}_e(\mathrm{X},\mathrm{G},w).$$

Proof. — This is clear. □

To compute $\mathrm{HH}^{\bullet}_e(\mathrm{X},\mathrm{G},w)$, we first must identify the complex

$$\mathbf{L}\Delta^*\,\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}}\,\Delta_*\mathcal{O}_{\mathrm{X}}$$

of coherent G-equivariant sheaves on X. Let K_χ be the kernel of χ.

Lemma **5.29.** — *There is a* $\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}$*-equivariant isomorphism,*

$$\begin{aligned}\Sigma:\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}\overset{\mathrm{G}}{\times}\mathrm{X}\times\mathrm{X}&\to\mathrm{K}_\chi\times\mathrm{X}\times\mathrm{X}\\ (g_1,g_2,x_1,x_2)&\mapsto\big(g_1g_2^{-1},\sigma(g_1,x_1),\sigma(g_2,x_2)\big),\end{aligned}$$

where $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$ *acts on* K_χ *via*

$$(g_1, g_2) \cdot g := g_1 g g_2^{-1}.$$

Proof. — The inverse morphism is

$$\begin{aligned} \mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X} &\to \mathrm{G} \times_{\mathbf{G}_m} \mathrm{G} \overset{\mathrm{G}}{\times} \mathrm{X} \times \mathrm{X} \\ (g, x_1, x_2) &\mapsto \left(g, e, \sigma\left(g^{-1}, x_1\right), x_2\right). \end{aligned}$$

□

Consider the $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$-equivariant subvariety defined by

$$\mathrm{O}(\Delta) := \left\{(g, x_1, x_2) \mid \sigma(g, x_2) = x_1\right\} \subset \mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X}.$$

Lemma **5.30.** — *Under the composition of the equivalence of Lemma 2.13 and the equivalence* Σ_*, *the* G*-equivariant sheaf* $\Delta_* \mathcal{O}_\mathrm{X}$ *corresponds to the structure sheaf of* $\mathrm{O}(\Delta)$ *in* $\mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X}$ *i.e.*

$$\iota^* \Sigma^* \mathcal{O}_{\mathrm{O}(\Delta)} \cong \Delta_* \mathcal{O}_\mathrm{X}.$$

Proof. — Recall that the equivalence of Lemma 2.13 is induced by ι^* where $\iota : \mathrm{X} \times \mathrm{X} \to \mathrm{G} \times_{\mathbf{G}_m} \mathrm{G} \overset{\mathrm{G}}{\times} \mathrm{X} \times \mathrm{X}$ is the inclusion along the identity. Note that $\Sigma \circ \iota$ remains the inclusion along the identity, but now of $\mathrm{X} \times \mathrm{X}$ into $\mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X}$. Since both Σ^* and ι^* are equivalences before deriving, they are exact. Thus, the statement of the lemma is equivalent to checking that the equation defining $\mathrm{O}(\Delta)$ restricts to the diagonal when we restrict to $\{e\} \times \mathrm{X} \times \mathrm{X}$. This is clear. □

From now on, we assume that K_χ is finite. Consider the coherent sheaf

$$\bigoplus_{g \in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)}$$

where

$$\Gamma^t(\sigma_g) := \left\{(x_1, x_2) \in \mathrm{X} \times \mathrm{X} \mid \sigma(g, x_2) = x_1\right\}$$

is the transpose of the graph of σ_g.

Lemma **5.31.** — *The coherent sheaf* $\bigoplus_{g \in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)}$ *possesses a natural* $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$*-equivariant structure such that there is an isomorphism of coherent* $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$*-equivariant sheaves*

$$\mathrm{Ind}_\mathrm{G}^{\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}} \Delta_* \mathcal{O}_\mathrm{X} \cong p_*(\mathcal{O}_{\mathrm{O}(\Delta)}) \cong \bigoplus_{g \in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)}.$$

where $p : \mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X} \to \mathrm{X} \times \mathrm{X}$ *is the projection.*

Proof. — The second isomorphism is clear from the (now) standing assumption that K_χ is finite and induces the natural equivariant structure on $\bigoplus_{g\in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)}$.

For the first isomorphism, we recall that, in general, $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}$ is the composition $\alpha_* \circ (\iota^*)^{-1}$ where $\iota : \mathrm{X} \to \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X}$ is the inclusion along the identity and $\alpha : \mathrm{G} \overset{\mathrm{H}}{\times} \mathrm{X} \to \mathrm{X}$ is the morphism induced by the action of G on X. In our case, we have the commutative diagram

$$\begin{tikzcd} \mathrm{G} \times_{\mathbf{G}_m} \mathrm{G} \overset{\mathrm{G}}{\times} \mathrm{X} \times \mathrm{X} \arrow[rr, "\Sigma"] \arrow[dr, "\alpha"'] & & \mathrm{K}_\chi \times \mathrm{X} \times \mathrm{X} \arrow[dl, "p"] \\ & \mathrm{X} \times \mathrm{X} & \end{tikzcd}$$

Now, by Lemma 5.30, we have

$$\left(\iota^*\right)^{-1} \Delta_* \mathcal{O}_{\mathrm{X}} \cong \Sigma^* \mathcal{O}_{\mathrm{O}(\Delta)}.$$

Applying α_* to both sides we get

$$\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}} \cong p_*(\mathcal{O}_{\mathrm{O}(\Delta)})$$

where the simplification on the right hand side comes either by flat base change for the isomorphism Σ or by using the isomorphism $\Sigma_*^{-1} = \Sigma^*$. □

From this point forward, we restrict our attention to $\mathrm{X} = \mathbf{A}^n$ equipped with a linear action of G such that K_χ is finite. It is easy to see that this implies that G is reductive. Write $\mathbf{A}^n = \operatorname{Spec} \operatorname{Sym}(\mathrm{V})$. Then, we have a right exact sequence

$$\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \xrightarrow{s} \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \to \Delta_* \mathcal{O}_{\mathbf{A}^n} \to 0$$

where the first morphism is

$$v \otimes f \mapsto f(v \otimes 1 - 1 \otimes v).$$

The potential $(-w) \boxplus w$ vanishes on $\Delta_* \mathcal{O}_{\mathbf{A}^n}$. Since X is affine and G is reductive, locally-free coherent equivariant sheaves are projective objects. Thus, there exists a morphism

$$t : \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \to \mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n}$$

making the diagram

$$\begin{tikzcd} \mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[r, "s"] \arrow[d, "(-w) \boxplus w"'] & \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[d, "(-w) \boxplus w"] \arrow[dl, "t"'] \\ \mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[r, "s"] & \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \end{tikzcd}$$

commute.

Similarly, given $g \in \mathrm{G}$, we can twist this diagram by σ_g as follows. We have a right exact sequence

$$\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \xrightarrow{s_g} \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \to \mathcal{O}_{\Gamma^t(\sigma_g)} \to 0$$

where the first morphism is

$$v \otimes f \mapsto f\left(g^{-1} \cdot v \otimes 1 - 1 \otimes v\right).$$

Here $g^{-1} \cdot v$ is the element of $\operatorname{Sym} \mathrm{V}$ given by the automorphism of rings dual to $\sigma_g : \mathbf{A}^n \to \mathbf{A}^n$. For $g \in \mathrm{K}_\chi$, $(-w) \boxplus w$ vanishes on $\mathcal{O}_{\Gamma^t(\sigma_g)}$ so there exists a

$$t_g : \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \to \mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n}$$

making the diagram

$$\begin{tikzcd}
\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[r, "s_g"] \arrow[d, "(-w) \boxplus w"'] & \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[d, "(-w) \boxplus w"] \arrow[dl, "t_g"'] \\
\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n} \arrow[r, "s_g"] & \mathcal{O}_{\mathbf{A}^n \times \mathbf{A}^n}
\end{tikzcd}$$

commute.

Lemma **5.32.** — *There are quasi-isomorphisms of* $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$*-equivariant factorizations,*

$$\bigoplus_{g \in \mathrm{K}_\chi} \mathcal{K}(s_g, t_g) \cong \bigoplus_{g \in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)} \cong \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}} \Delta_* \mathcal{O}_{\mathbf{A}^n}.$$

Proof. — The second isomorphism is already stated in Lemma 5.31. The first quasi-isomorphism follows from an immediate application of Proposition 3.20. $\square$

Since each $\mathcal{K}(s_g, t_g)$ is a factorization with locally-free components, to compute

$$\mathbf{L}\Delta^* \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}} \Delta_* \mathcal{O}_{\mathbf{A}^n}$$

we may compute

$$\Delta^* \left(\bigoplus_{g \in \mathrm{K}_\chi} \mathcal{K}(s_g, t_g) \right).$$

We record the following lemma as a reminder of the structure of $\Delta^* \mathcal{K}(s_g, t_g)$.

Lemma **5.33.** — *The factorization* $\Delta^*\mathcal{K}(s_g, t_g)$ *has components*

$$\Delta^*\mathcal{K}(s_g, t_g)_{-1} = \bigoplus_{l\geq 0} \Lambda^{2l+1}\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi)$$
$$\Delta^*\mathcal{K}(s_g, t_g)_0 = \bigoplus_{l\geq 0} \Lambda^{2l}\mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi)$$

and morphisms given by

$$\bullet \lrcorner \Delta^* s_g + \bullet \wedge \Delta^* t_g$$

where

$$\begin{aligned}\Delta^* s_g : \mathrm{V} \otimes_k \mathcal{O}_{\mathbf{A}^n} &\to \mathcal{O}_{\mathbf{A}^n} \\ v \otimes f &\mapsto f\big(g^{-1} \cdot v - v\big).\end{aligned}$$

Proof. — This is clear from the definition of the Koszul factorization, $\mathcal{K}(s_g, t_g)$. □

Definition **5.34.** — *Let* $g \in \mathrm{G}$. *Set*

$$\mathrm{V}_g := \big\{v \in \mathrm{V} \mid g^{-1} \cdot v = v\big\}.$$

The ideal sheaf of $(\mathbf{A}^n)^g$ *corresponds to* $\{g^{-1} \cdot f - f \mid f \in \operatorname{Sym}\mathrm{V}\}$. *This determines a subspace* $\mathrm{W}_g \subseteq \mathrm{V}$. *Note that there is an equivariant splitting* $\mathrm{V} = \mathrm{V}_g \oplus \mathrm{W}_g$.

Let $\kappa_g : \mathrm{G} \to \mathbf{G}_m$ *be the character corresponding to* $\Lambda^{\dim \mathrm{W}_g}\mathrm{W}_g$. *More precisely,* $\mathcal{O}_{\mathbf{A}^n}(\kappa_g)$ *is the invertible sheaf corresponding to the free graded module of rank* 1, $\Lambda^{\dim \mathrm{W}_g}\mathrm{W}_g \otimes_k \operatorname{Sym}\mathrm{V}$.

Lemma **5.35.** — *There is a quasi-isomorphism between* $\Delta^*\mathcal{K}(s_g, t_g)$ *and the Koszul factorization* $i_{g*}\mathcal{K}(0, \mathrm{d}w_g)$ *where*

$$i_g : \big(\mathbf{A}^n\big)^g \to \mathbf{A}^n$$

is the inclusion, 0 *is the morphism*

$$\mathrm{V}_g \otimes_k \mathcal{O}_{(\mathbf{A}^n)^g} \xrightarrow{0} \mathcal{O}_{(\mathbf{A}^n)^g},$$

and w_g *is the restriction of* w *to* $(\mathbf{A}^n)^g$.

Proof. — Consider the pullback of s_g and t_g to $(\mathbf{A}^n)^g \times (\mathbf{A}^n)^g$ via

$$i_g \times i_g : \big(\mathbf{A}^n\big)^g \times \big(\mathbf{A}^n\big)^g \to \mathbf{A}^n \times \mathbf{A}^n.$$

We have

$$(i_g \times i_g)^* s_g(v) = v \otimes 1 - 1 \otimes v$$

and a commutative diagram

$$\begin{tikzcd}
\mathrm{V}_g \otimes_k \mathcal{O}_{(\mathbf{A}^n)^g \times (\mathbf{A}^n)^g} \arrow[r, "(i_g \times i_g)^* s_g"] \arrow[d, "(-w_g) \boxplus w_g"'] & \mathcal{O}_{(\mathbf{A}^n)^g \times (\mathbf{A}^n)^g} \arrow[d, "(-w_g) \boxplus w_g"] \arrow[dl, "(i_g \times i_g)^* t_g"'] \\
\mathrm{V}_g \otimes_k \mathcal{O}_{(\mathbf{A}^n)^g \times (\mathbf{A}^n)^g} \arrow[r, "(i_g \times i_g)^* s_g"] & \mathcal{O}_{(\mathbf{A}^n)^g \times (\mathbf{A}^n)^g}
\end{tikzcd}$$

Let $\Delta_g : (\mathbf{A}^n)^g \to (\mathbf{A}^n)^g \times (\mathbf{A}^n)^g$ be the diagonal embedding. Then, $\Delta_g^*(i_g \times i_g)^* t_g = \mathrm{d}w_g$. As the diagram

$$\begin{tikzcd}
(\mathbf{A}^n)^g \arrow[r, "\Delta_g"] \arrow[d, "i_g"'] & (\mathbf{A}^n)^g \times (\mathbf{A}^n)^g \arrow[d, "i_g \times i_g"] \\
\mathbf{A}^n \arrow[r, "\Delta"] & \mathbf{A}^n \times \mathbf{A}^n
\end{tikzcd}$$

commutes, we have $i_g^* \Delta^* t_g = \Delta_g^*(i_g \times i_g)^* t_g = \mathrm{d}w_g$ while $i_g^* \Delta^* s_g = \Delta_g^*(i_g \times i_g)^* s_g = 0$. Thus,

$$i_g^* \Delta^* \mathcal{K}(s_g, t_g) \cong \mathcal{K}(0, \mathrm{d}w_g).$$

Now, associated to the adjunction $i_g^* \dashv i_{g*}$, we have a morphism

$$\pi : \Delta^* \mathcal{K}(s_g, t_g) \to i_{g*} i_g^* \Delta^* \mathcal{K}(s_g, t_g) \cong i_{g*} \mathcal{K}(0, \mathrm{d}w_g)$$

which we claim is a quasi-isomorphism.

To verify this claim, we check that the kernel of π, $\ker(\pi)$, is acyclic. The components of $\ker(\pi)$ are

$$\ker(\pi)_{-1} = \bigoplus_{\substack{l \geq 0, a > 0 \\ a+b=2l+1}} \Lambda^a \mathrm{W}_g \otimes_k \Lambda^b \mathrm{V}_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi)$$

$$\ker(\pi)_0 = \mathcal{I}_{(\mathbf{A}^n)^g} \oplus \bigoplus_{\substack{l \geq 0, a > 0 \\ a+b=2l}} \Lambda^a \mathrm{W}_g \otimes_k \Lambda^b \mathrm{V}_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi).$$

Let

$$\mathcal{J}^j := \ker\big(\bullet \lrcorner \Delta^* s_g\big) : \Lambda^j \mathrm{W}_g \otimes_k \mathcal{O}_{\mathbf{A}^n} \to \Lambda^{j-1} \mathrm{W}_g \otimes_k \mathcal{O}_{\mathbf{A}^n}$$

and $\mathcal{J}^0 := \mathcal{I}_{(\mathbf{A}^n)^g}$. As $\Delta^* s_g$ vanishes on V_g and $\Delta^* t_g$ has image in V_g, we have a filtration $F^j \ker(\pi)$. In the case $j = 2u$, it is

$$F^j \ker(\pi)_{-1} = \bigoplus_{\substack{b \geq j, a>0 \\ a+b=2l+1}} \Lambda^a W_g \otimes_k \Lambda^b V_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi)$$

$$F^j \ker(\pi)_0 = \Lambda^j V_g \otimes_k \mathcal{J}^j(u\chi) \oplus \bigoplus_{\substack{b \geq j, a>0 \\ a+b=2l}} \Lambda^a W_g \otimes_k \Lambda^b V_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi).$$

In the case $j = 2u+1$, it is

$$F^j \ker(\pi)_{-1} = \Lambda^j V_g \otimes_k \mathcal{J}^j(u\chi) \oplus \bigoplus_{\substack{b \geq j, a>0 \\ a+b=2l+1}} \Lambda^a W_g \otimes_k \Lambda^b V_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi)$$

$$F^j \ker(\pi)_0 = \bigoplus_{\substack{b \geq j, a>0 \\ a+b=2l}} \Lambda^a W_g \otimes_k \Lambda^b V_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(l\chi).$$

The associated graded factorization, $F^j \ker(\pi)/F^{j+1} \ker(\pi)$, is the totalization of the exact sequence

$$0 \to \Lambda^j V_g \otimes_k \Lambda^{\dim W_g} W_g \otimes_k \mathcal{O}_{\mathbf{A}^n} \xrightarrow{\bullet \lrcorner \Delta^* s_g} \cdots \xrightarrow{\bullet \lrcorner \Delta^* s_g} \Lambda^j V_g \otimes_k \Lambda^{j+1} W_g \otimes_k \mathcal{O}_{\mathbf{A}^n}$$
$$\xrightarrow{\bullet \lrcorner \Delta^* s_g} \Lambda^j V_g \otimes_k \mathcal{J}^j \to 0$$

where the final term is in degree $-\dim W_g$. Thus, $\ker(\pi)$ is filtered by acyclic complexes and hence acyclic. This implies that π is a quasi-isomorphism as desired. $\square$

Definition **5.36.** — *Let* $\kappa : G \to \mathbf{G}_m$ *be the character corresponding to* $\Lambda^n V$.

Lemma **5.37.** — *Assume that* K_χ *is finite. Then, there is an isomorphism*

$$HH_t\big(\mathbf{A}^n, G, w\big) \cong HH_e^{(\kappa, n+t)}\big(\mathbf{A}^n, G, w\big).$$

Proof. — We have,

$$\begin{aligned} HH_t\big(\mathbf{A}^n, G, w\big) &\cong \operatorname{Hom}\Big(\big(\operatorname{Ind}_G^{G\times_{\mathbf{G}_m} G} \Delta_* \mathcal{O}_X\big)^\vee, \operatorname{Ind}_G^{G\times_{\mathbf{G}_m} G} \Delta_* \mathcal{O}_X[t]\Big) \\ &\cong \operatorname{Hom}\Bigg(\bigoplus_{g\in K_\chi} \mathcal{K}(s_g, t_g)^\vee, \operatorname{Ind}_G^{G\times_{\mathbf{G}_m} G} \Delta_* \mathcal{O}_X[t]\Bigg) \end{aligned}$$

$$\cong \operatorname{Hom}\Big(\bigoplus_{g\in \mathrm{K}_\chi} \mathcal{K}(t_g^\vee, s_g^\vee), \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}}[t]\Big)$$
$$\cong \operatorname{Hom}\Big(\bigoplus_{g\in \mathrm{K}_\chi} \mathcal{O}_{\Gamma^t(\sigma_g)} \otimes_k \Lambda^n \mathrm{V}^\vee[-n], \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}}[t]\Big)$$
$$\cong \operatorname{Hom}\big(\operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}}, \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}} \otimes_k \Lambda^n \mathrm{V}[t+n]\big)$$
$$= \operatorname{Hom}\big(\operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}}, \operatorname{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_* \mathcal{O}_{\mathrm{X}}(\kappa)[t+n]\big)$$
$$= \mathrm{HH}_e^{(\kappa, n+t)}\big(\mathbf{A}^n, \mathrm{G}, w\big).$$

All morphisms are computed in $\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n \times \mathbf{A}^n, \mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}, (-w) \boxplus w)]$.

The first line follows from Lemma 3.57. The second line follows from Lemma 5.32. The third line is Lemma 3.21. The fourth line comes from Proposition 3.20. The fifth line is another application of Lemma 5.32. The six line is by definition as is the seventh line. □

Definition **5.38.** — *Let* $(r_1, \dots, r_c)$ *be a sequence of elements of a commutative ring,* R. *We let*

$$\mathrm{H}^\bullet(\mathbf{r})$$

denote the cohomology of the Koszul complex for $(r_1, \dots, r_c)$. *We call* $\mathrm{H}^\bullet(\mathbf{r})$ *the* **Koszul cohomology** *of* $(r_1, \dots, r_c)$.

In the case, $(r_1, \dots, r_c) = (\partial_1 w, \dots, \partial_n w)$ *for* $\mathrm{R} = k[x_1, \dots, x_n]$, *we denote the Koszul cohomology by* $\mathrm{H}^\bullet(\mathrm{d}w)$. *The* **Jacobian algebra** *of* w *is* $\mathrm{H}^0(\mathrm{d}w)$ *but we denote it by* $\mathrm{Jac}(w)$ *for transparency.*

Theorem **5.39.** — *Let* G *act linearly on* $\mathbf{A}^n$ *and let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^{\mathrm{G}}$. *Assume that* K_χ *is finite and* $\chi : \mathrm{G} \to \mathbf{G}_m$ *is surjective. Then,*

$$\mathrm{HH}_e^{(\rho,t)}\big(\mathbf{A}^n, \mathrm{G}, w\big)$$
$$\cong \Bigg(\bigoplus_{\substack{g\in\mathrm{K}_\chi, l\geq 0 \\ t-\dim \mathrm{W}_g = 2u}} \mathrm{H}^{2l}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u-l)\chi\big)$$
$$\oplus \bigoplus_{\substack{g\in\mathrm{K}_\chi, l\geq 0 \\ t-\dim \mathrm{W}_g = 2u+1}} \mathrm{H}^{2l+1}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u-l)\chi\big)\Bigg)^{\mathrm{G}}$$

If, additionally, we assume the support of $(\mathrm{d}w)$ *is* $\{0\}$, *then we have*

$$\begin{aligned}
&\mathrm{HH}_e^{(\rho,t)}(\mathbf{A}^n, \mathrm{G}, w) \\
&\cong \Bigg(\bigoplus_{\substack{g\in \mathrm{K}_\chi \\ t-\dim \mathrm{W}_g = 2u}} \mathrm{Jac}(w_g)(\rho - \kappa_g + u\chi) \\
&\quad \oplus \bigoplus_{\substack{g\in \mathrm{K}_\chi \\ t-\dim \mathrm{W}_g = 2u+1}} \mathrm{Jac}(w_g)(\rho - \kappa_g + u\chi) \Bigg)^{\mathrm{G}}.
\end{aligned}$$

Proof. — We have

$$\begin{aligned}
\mathrm{HH}_e^{(\rho,t)}(\mathbf{A}^n, \mathrm{G}, w) &:= \mathrm{Hom}_{\mathrm{D^{abs}}[\mathsf{fact}(\mathbf{A}^n\times\mathbf{A}^n, \mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}, (-w)\boxplus w)]}\big(\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_*\mathcal{O}_{\mathbf{A}^n}, \mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_*\mathcal{O}_{\mathbf{A}^n}(\rho)[t]\big) \\
&\cong \mathrm{Hom}_{\mathrm{D^{abs}}[\mathsf{fact}(\mathbf{A}^n\times\mathbf{A}^n, \mathrm{G}, (-w)\boxplus w)]}\big(\mathrm{Res}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_*\mathcal{O}_{\mathbf{A}^n}, \Delta_*\mathcal{O}_{\mathbf{A}^n}(\rho)[t]\big) \\
&\cong \mathrm{Hom}_{\mathrm{D^{abs}}[\mathsf{fact}(\mathbf{A}^n, \mathrm{G}, 0)]}\big(\mathbf{L}\Delta^* \mathrm{Res}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_*\mathcal{O}_{\mathbf{A}^n}, \mathcal{O}_{\mathbf{A}^n}(\rho)[t]\big) \\
&\cong \mathrm{Hom}\big(\mathbf{L}\Delta^* \mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}\times_{\mathbf{G}_m}\mathrm{G}} \Delta_*\mathcal{O}_{\mathbf{A}^n}, \mathcal{O}_{\mathbf{A}^n}(\rho)[t]\big) \\
&\cong \mathrm{Hom}\Big(\bigoplus_{g\in\mathrm{K}_\chi} i_{g*}\mathcal{K}(0, \mathrm{d}w_g), \mathcal{O}_{\mathbf{A}^n}(\rho)[t]\Big) \\
&\cong \mathrm{Hom}\Big(\mathcal{O}_{\mathbf{A}^n}, \bigoplus_{g\in\mathrm{K}_\chi} i_{g*}\mathcal{K}(0, \mathrm{d}w_g)^\vee(\rho)[t]\Big) \\
&\cong \mathrm{Hom}\Big(\mathcal{O}_{\mathbf{A}^n}, \bigoplus_{g\in\mathrm{K}_\chi} i_{g*}\mathcal{K}(\mathrm{d}w_g, 0)(\rho-\kappa_g)[t-\dim \mathrm{W}_g]\Big).
\end{aligned}$$

The first line is by definition. The second line is adjunction for Res and Ind, Lemma 3.42. The third line applies the adjunction, $\mathbf{L}\Delta^* \dashv \Delta_*$, Lemma 3.35. The fourth line is a slight notational respite obtained by viewing Δ as an equivariant for the diagonal embedding of G into $\mathrm{G} \times_{\mathbf{G}_m} \mathrm{G}$. The fifth line is Lemma 5.35. The sixth line is just the equivalence $(-)^\vee$. We justify the seventh line in the next paragraph.

Let $\widetilde{\mathrm{K}}(0, \mathrm{d}w_g)$ be the Koszul factorization on $\mathbf{A}^n$ associated to

$$\mathrm{V}_g \otimes_k \mathcal{O}_{\mathbf{A}^n} \xrightarrow{0} \mathcal{O}_{\mathbf{A}^n}$$

and

$$\mathcal{O}_{\mathbf{A}^n} \xrightarrow{\mathrm{d}w_g} \mathrm{V}_g \otimes_k \mathcal{O}_{\mathbf{A}^n}(\chi).$$

Using contraction with morphism,

$$\begin{aligned} \mathrm{W}_g \otimes_k \mathcal{O}_{\mathbf{A}^n} &\to \mathcal{O}_{\mathbf{A}^n} \\ w \otimes_k f &\mapsto f w, \end{aligned}$$

we have a exact sequence of Koszul factorizations,

$$\begin{aligned} 0 &\to \Lambda^{\dim \mathrm{W}_g} \mathrm{W}_g \otimes_k \widetilde{\mathrm{K}}(0, \mathrm{d}w_g) \to \cdots \to \mathrm{W}_g \otimes_k \widetilde{\mathrm{K}}(0, \mathrm{d}w_g) \to \widetilde{\mathrm{K}}(0, \mathrm{d}w_g) \\ &\to i_{g*}\mathcal{K}(0, \mathrm{d}w_g) \to 0. \end{aligned}$$

Hence, $\mathcal{K}(0, \mathrm{d}w_g)^\vee$ is quasi-isomorphic to the totalization of the complex

$$0 \leftarrow \Lambda^{\dim \mathrm{W}_g} \mathrm{W}_g^\vee \otimes_k \widetilde{\mathrm{K}}(0, \mathrm{d}w_g)^\vee \leftarrow \cdots \leftarrow \mathrm{W}_g^\vee \otimes_k \widetilde{\mathrm{K}}(0, \mathrm{d}w_g)^\vee \leftarrow 0.$$

This is, in turn quasi-isomorphic to $i_{g*}\mathcal{K}(\mathrm{d}w_g, 0) \otimes_k \Lambda^{\dim \mathrm{W}_g} \mathrm{W}_g^\vee[-\dim \mathrm{W}_g]$.

The factorization, $\mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)$, has components

$$\begin{aligned} \mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)_{-1} &= \bigoplus_{l \geq 0} \Lambda^{2l+1} \mathrm{V}_g^\vee \otimes_k \mathcal{O}_{(\mathbf{A}^n)^g}\big(\rho - \kappa_g - (l+1)\chi\big) \\ \mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)_0 &= \bigoplus_{l \geq 0} \Lambda^{2l} \mathrm{V}_g^\vee \otimes_k \mathcal{O}_{(\mathbf{A}^n)^g}(\rho - \kappa_g - l\chi) \end{aligned}$$

with morphisms given by contraction with $\mathrm{d}w_g$. The cohomology of $\mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)$ is

$$\begin{aligned} \mathrm{H}^{2u}\big(\mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)\big) &\cong \bigoplus_{l \geq 0} \mathrm{H}^{2l}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u - l)\chi\big) \\ \mathrm{H}^{2u+1}\big(\mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)\big) &\cong \bigoplus_{l \geq 0} \mathrm{H}^{2l+1}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u - l)\chi\big). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\mathrm{Hom}\Big(\mathcal{O}_{\mathbf{A}^n}, \bigoplus_{g \in \mathrm{K}_\chi} i_{g*}\mathcal{K}(\mathrm{d}w_g, 0)(\rho - \kappa_g)[t - \dim \mathrm{W}_g]\Big) \\ &\cong \Big(\bigoplus_{\substack{g \in \mathrm{K}_\chi, l \geq 0 \\ t - \dim \mathrm{W}_g = 2u}} \mathrm{H}^{2l}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u - l)\chi\big) \\ &\quad \oplus \bigoplus_{\substack{g \in \mathrm{K}_\chi, l \geq 0 \\ t - \dim \mathrm{W}_g = 2u+1}} \mathrm{H}^{2l+1}(\mathrm{d}w_g)\big(\rho - \kappa_g + (u - l)\chi\big)\Big)^{\mathrm{G}}. \end{aligned}$$

If $(\mathrm{d}w)$ has support $\{0\}$, then so does $(\mathrm{d}w_g)$ for all g. So all Koszul complexes only have cohomology in homological degree zero. $\square$

Remark **5.40.** — By specializing to appropriate graded pieces, one can use Theorem 5.39 to extract both $\mathrm{HH}^\bullet(\mathbf{A}^n, \mathrm{G}, w)$ and $\mathrm{HH}_\bullet(\mathbf{A}^n, \mathrm{G}, w)$.

Corollary **5.41.** — *Let* $\mathbf{A}^n = \operatorname{Spec}(\operatorname{Sym} \mathrm{V})$ *carry a* $\mathbf{G}_m$ *action with weight* (-1). *Let* $w \in \operatorname{Sym} \mathrm{V}$ *be homogeneous of degree* d. *Then, we have isomorphisms*

$$\mathrm{HH}_t\big(\mathbf{A}^n, \mathbf{G}_m, w\big) \cong \begin{cases} \operatorname{Jac}(w)_{d(\frac{n+t}{2})-n} & t \neq 0 \\ \operatorname{Jac}(w)_{d(\frac{n}{2})-n} \oplus k^{\oplus d-1} & t = 0. \end{cases}$$

Proof. — We have $\kappa = -n$. In this case, $\mathrm{K}_\chi \cong \mathbf{Z}/d\mathbf{Z}$. If $g \neq e$, then $\mathrm{V}_g = \{0\}$, thus $\kappa_g = -n$ and $\dim \mathrm{W}_g = n$. For $g = e$, we have $\kappa_e = 0$ and $\dim \mathrm{W}_e = 0$. Applying Lemma 5.37 and Theorem 5.39, we have

$$\mathrm{HH}_t\big(\mathbf{A}^n, \mathbf{G}_m, w\big) \cong \operatorname{Jac}(w)_{-n+d(\frac{n+t}{2})} \oplus \bigoplus_{g \neq e} \operatorname{Jac}(w_g)_{d\frac{t}{2}}.$$

We have $\operatorname{Jac}(w_g) \cong k(0)$ so the latter term only contributes to $t = 0$. □

Remark **5.42.** — This computation was first done by Căldăraru and Tu, [CT10, Example 6.4]. It is also performed, independently, by Polishchuk and Vaintrob [PV11].

6. Implications for Hodge theory

In this section, we give two applications of the ideas and computations of the previous sections to Hodge theory. To fully state the results, we recall some of the functoriality of Hochschild homology. Recall that $\operatorname{perf}(\mathsf{C})$ consists of all compact objects in $\mathrm{D}(\mathsf{C}^{\mathrm{op}}\text{-Mod})$.

Proposition **6.1.** — *Let* C *and* D *be saturated dg-categories over* k. *Let* F *be an object of* $\operatorname{perf}(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{D})$. *Then, there is a homomorphism of vector spaces,*

$$\mathrm{F}_\bullet : \mathrm{HH}_\bullet(\mathsf{C}) \to \mathrm{HH}_\bullet(\mathsf{D}).$$

Moreover, the assignment, $\mathrm{F} \mapsto \mathrm{F}_\bullet$, *is natural in the following sense. Let* $\mathrm{F}_1 \in \operatorname{perf}(\mathsf{B}^{\mathrm{op}} \otimes \mathsf{C})$ *and* $\mathrm{F}_2 \in \operatorname{perf}(\mathsf{C}^{\mathrm{op}} \otimes \mathsf{D})$ *and let* $\mathrm{F}_2 \circ \mathrm{F}_1$ *denote the* B-D *bimodule corresponding to the tensor product* $\mathrm{F}_1 \overset{\mathbf{L}}{\otimes}_{\mathsf{C}} \mathrm{F}_2$. *Then,* $(\mathrm{F}_2 \circ \mathrm{F}_1)_\bullet \cong \mathrm{F}_{2\bullet} \circ \mathrm{F}_{1\bullet}$.

Proof. — This is [PV12, Lemma 1.2.1]. □

Definition **6.2.** — *Let* C *and* D *be saturated dg-categories over* k. *Let* F *be an object of* $\operatorname{perf}(\mathsf{C}^{\mathrm{op}} \otimes_k \mathsf{D})$. *We will call the linear map,* $\mathrm{F}_\bullet$, *the* ***pushforward*** *by* F.

For an object $\mathrm{E} \in \mathrm{perf}(\mathsf{C})$*, we get an induced map,*

$$\mathrm{E}_\bullet : k[0] \cong \mathrm{HH}_\bullet(k) \to \mathrm{HH}_\bullet(\mathsf{C}).$$

The map, $\mathrm{E}_\bullet$*, is called the* **Chern character map** *and the element* $\mathrm{E}_\bullet(1)$ *is called the* **Chern character** *of* E*. The map*

$$\mathrm{E} \mapsto \mathrm{E}_\bullet(1)$$

is denoted by ch.

There is also a natural pairing on Hochschild homology.

Proposition **6.3.** — *Let* C *be saturated dg-category over* k*. There is a natural pairing*

$$\langle \cdot, \cdot \rangle : \mathrm{HH}_\bullet(\mathsf{C}) \otimes_k \mathrm{HH}_\bullet(\mathsf{C}) \to k$$

satisfying

$$\chi\left(\bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathrm{perf}(\mathsf{C})}(\mathrm{E}_1, \mathrm{E}_2[i])\right) = \langle \mathrm{ch}(\mathrm{E}_1), \mathrm{ch}(\mathrm{E}_2) \rangle$$

for $\mathrm{E}_1, \mathrm{E}_2 \in \mathrm{perf}(\mathsf{C})$.

Proof. — This pairing is constructed for smooth and proper dg-algebras in [Shk07, Section 1.2]. In this case, the equality

$$\chi\left(\bigoplus_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathrm{perf}(\mathsf{C})}(\mathrm{E}_1, \mathrm{E}_2[i])\right) = \langle \mathrm{ch}(\mathrm{E}_1), \mathrm{ch}(\mathrm{E}_2) \rangle$$

is a special case of [Shk07, Theorem 1.3.1]. The pairing is also defined for a general saturated dg-category in [PV12, Section 1.2]. As any saturated dg-category is Morita equivalent to a smooth and proper dg-algebra, the naturality of the pairing extends the result from algebras to categories. □

Definition **6.4.** — *Let* C *be a saturated dg-category. We shall call the pairing*

$$\langle \cdot, \cdot \rangle : \mathrm{HH}_\bullet(\mathsf{C}) \otimes_k \mathrm{HH}_\bullet(\mathsf{C}) \to k$$

the **categorical pairing** *on Hochschild homology.*

We will also need the following result due to Polishchuk and Vaintrob.

Theorem **6.5.** — *Let* $\mathbf{A}^n$ *carry a linear action of* G*, an algebraic group, and let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^{\mathrm{G}}$*. Assume that* K_χ *is finite and* $\chi : \mathrm{G} \to \mathbf{G}_m$ *is surjective. Furthermore, assume that* $(\mathrm{d}w)$ *is supported at* $\{0\} \in \mathbf{A}^n$*. For a character,* $\rho : \mathrm{G} \to \mathbf{G}_m$*, the twist functor,*

$$(\rho) : \mathrm{D}^{\mathrm{abs}}\,\mathsf{fact}\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathrm{D}^{\mathrm{abs}}\,\mathsf{fact}\big(\mathbf{A}^n, \mathrm{G}, w\big),$$

induces a pushforward map,

$$(\rho)_\bullet : \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{G}, w\big)$$

which is multiplication by $\rho(g)^{-1}$ *on* $\mathrm{Jac}(w_g)$ *for* $g \in \mathrm{K}_\chi$*. In other words, the decomposition of Theorem 5.39 is exactly the eigenspace decomposition for the action of* $\widehat{\mathrm{G}}$ *on* $\mathrm{HH}_\bullet(\mathbf{A}^n, \mathrm{G}, w)$*.*

Proof. — This is part of [PV11, Theorem 2.6.1], albeit stated in the notation used in this paper. □

6.1. *Another look at Griffiths' theorem.* — In this section, we recall a celebrated result of Griffiths, reproved and understood in categorical language as a combination of Theorem 5.39, the Hochschild-Kostant-Rosenberg isomorphism, and a theorem of Orlov [Orl09].

Definition **6.6.** — *Let* Z *be a smooth complex projective hypersurface in* $\mathbf{P}^{n-1}_{\mathbf{C}}$ *defined by* $w \in \mathbf{C}[x_1, \dots, x_n]$*. An element of* $H^{2(n-2-k)}(\mathrm{Z}; \mathbf{C})$ *is called* **primitive** *if it cups trivially with* H^k*, where* H *is the class of a hyperplane section. We write*

$$\mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}; \mathbf{C})$$

for the subspace of primitive classes. We will write

$$\mathrm{H}^{\bullet,\bullet}_{\mathrm{prim}}(\mathrm{Z})$$

for the intersections of $\mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}; \mathbf{C})$ *with each bi-graded piece of the Dolbeault cohomology of* Z*.*

In our context, by the Lefschetz Hyperplane Theorem, all primitive cohomology classes lie in the middle dimensional cohomology, $\mathrm{H}^{n-2}(\mathrm{Z}; \mathbf{C})$. Furthermore, all elements are primitive when n is odd. When n is even, all Dolbeault classes of type $(p, n-2-p)$, $\mathrm{H}^{p,n-2-p}(\mathrm{Z})$, with $p \neq \frac{n-2}{2}$ are primitive, while $\mathrm{H}^{\frac{n-2}{2},\frac{n-2}{2}}_{\mathrm{prim}}(\mathrm{Z})$ are just those classes lying in the kernel of the cup product with H. The following description is due to Griffiths.

Theorem **6.7.** — *There is an isomorphism,*

$$\mathrm{H}^{p,n-2-p}_{\mathrm{prim}}(\mathrm{Z}) \cong \mathrm{Jac}(w)_{d(n-1-p)-n}.$$

Proof. — This is [Gri69, Theorem 8.1]. □

Comparing Griffiths' result with Theorem 5.39 we see a striking similarity. Indeed, $\mathrm{Jac}(w)_{d(n-1-p)-n}$, is also the summand of $\mathrm{HH}_{n-2-2p}(\mathbf{A}^n, \mathbf{G}_m, w)$ corresponding to $g = e$. This is not a coincidence. To give a precise comparison, we will need to recall two results.

Definition **6.8.** — *Let* Z *be a smooth, projective variety. Let* $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z})$ *denote the dg-category of bounded below chain complexes of injective sheaves on* Z *with bounded and coherent cohomology. We denote the Hochschild homology of* $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z})$ *by* $\mathrm{HH}_\bullet(\mathrm{Z})$.

Definition **6.9.** — *The* ***Mukai pairing*** *on* $\mathrm{H}^\bullet(\mathrm{Z}; \mathbf{C})$ *is*

$$(v, v')_{\mathrm{M}} := \int_{\mathrm{Z}} v^\vee \cdot v' \cdot \mathrm{td}(\mathrm{Z})$$

where $v^\vee = \sum_{p,q} (-1)^p v_{p,q}$ *if* $v = \sum_{p,q} v_{p,q}$ *is the Hodge decomposition.*

The first result we use is the Hochschild-Kostant-Rosenberg isomorphism. It allows one to reinterpret Dolbeault cohomology categorically.

Theorem **6.10.** — *Let* Z *be smooth projective variety. There are natural isomorphisms,*

$$\mathrm{HH}_t(\mathrm{Z}) \cong \bigoplus_{q-p=t} \mathrm{H}^q(\mathrm{Z}, \Omega^p_{\mathrm{Z}}) \cong \bigoplus_{q-p=t} \mathrm{H}^{p,q}(\mathrm{Z}).$$

We denote the isomorphism by $\phi_{\mathrm{HKR}} : \mathrm{HH}_\bullet(\mathrm{Z}) \to \mathrm{H}^\bullet(\mathrm{Z}; \mathbf{C})$. *Under the HKR isomorphism, we have*

$$\langle \alpha, \alpha' \rangle = \big(\phi_{\mathrm{HKR}}(\alpha), \phi_{\mathrm{HKR}}(\alpha')\big)_{\mathrm{M}}.$$

The Chern character and classical Chern character agree under the HKR isomorphism

$$\phi_{\mathrm{HKR}}\big(\mathrm{ch}(\mathcal{E})\big) = \mathrm{ch}_{class}(\mathcal{E}).$$

Furthermore, for an integral functor, $\Phi_{\mathcal{K}} : \mathrm{D^b}(\mathrm{coh}\,\mathrm{X}) \to \mathrm{D^b}(\mathrm{coh}\,\mathrm{X})$, *the action of* $\Phi_{\mathcal{K}\bullet}$ *under the HKR isomorphism is the cohomological integral transform,* $\Phi^{\mathrm{H}}_{\mathcal{K}}$, *associated to* $\mathrm{ch}_{class}(\mathcal{K}) \in \mathrm{H}^\bullet(\mathrm{X} \times \mathrm{Y}; \mathbf{C})$.

Proof. — The HKR isomorphism in the affine case is due to [HKR62]. In this generality, it is due to Swan [Swa96, Corollary 2.6] and Kontsevich [Kon03], see also [Yek02]. The preservation of the Chern character was stated in [Mar01] and proven as [Cal05, Theorem 4.5]. The equality of the pairings is [Ram10, Theorem 1]. The equality

$$\phi_{\mathrm{HKR}} \circ \Phi_{\mathcal{K}\bullet} = \Phi^{\mathrm{H}}_{\mathcal{K}} \circ \phi_{\mathrm{HKR}}$$

is a consequence of [Ram10, Theorem 2] and the definition of Φ^{muk}_* in [Ram10]. □

Definition **6.11.** — *Let* Z *be a smooth, projective variety. Define the endofunctor,*

$$\{1\} := \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}} \circ \mathrm{T}_{\mathcal{O}(1)} : \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z}) \to \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z}),$$

where

$$\mathrm{T}_{\mathcal{O}(1)}(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{Z}}} \mathcal{O}_{\mathrm{Z}}(1)$$

and, for $i \in \mathbf{Z}$,

$$\mathrm{L}_{\mathcal{O}_{\mathrm{Z}}(i)}(\mathcal{E}) := \mathrm{Cone}\Big(\mathrm{Hom}\big(\widetilde{\mathcal{O}}_{\mathrm{Z}}(i), \mathcal{E}\big) \otimes_k \widetilde{\mathcal{O}}_{\mathrm{Z}}(i) \to \mathcal{E}\Big)$$

where $\widetilde{\mathcal{O}}_{\mathrm{Z}}(i)$ *is an injective resolution of* $\mathcal{O}_{\mathrm{Z}}(i)$. *Let*

$$\varsigma\big(\mathcal{O}_{\mathrm{Z}}(i)\big) : \mathrm{Id} \to \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}(i)}$$

denote the induced natural transformation.

The second result we use is a theorem of Orlov [Orl09], generalized mildly to account for a larger grading group. Let G be an Abelian affine algebraic group acting on $\mathbf{A}^n$. We assume that G has rank one so that

$$\mathrm{G} \cong \mathbf{G}_m \times \mathrm{G}_{\mathrm{tors}}$$

for $\mathrm{G}_{\mathrm{tors}}$ a finite Abelian group.

Definition **6.12.** — *We say that* G *acts* **positively** *on* $\mathbf{A}^n$ *if with respect to the induced* $\mathbf{G}_m$-*action all nonzero linear functions on* $\mathbf{A}^n$ *have positive degree.*

We have a $\mathbf{G}_m$-equivariant isomorphism $\omega_{\mathbf{A}^n} \cong \mathcal{O}_{\mathbf{A}^n}(\mathrm{N})$ for N equal to the sum of the degrees of x_i if $\mathbf{A}^n = \mathrm{Spec}\, k[x_1, \ldots, x_n]$.

Theorem **6.13.** — *Let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^{\mathrm{G}}$ *for a character* $\chi : \mathrm{G} \to \mathbf{G}_m$ *with* $\chi|_{\mathbf{G}_m} = d > 0$. *Let* Y *be the zero locus of* w *on punctured affine space* $\mathbf{A}^n \setminus \{0\}$. *If* $\mathrm{G} = \mathbf{G}_m$ *and* $\mathrm{N} = n$, *let* Z *denote the projective hypersurface determined by* w.

Assume w *is not zero and that* Y *is smooth. Further, assume that* G *acts positively.*

- *If* $d < \mathrm{N}$, *then there exists morphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\Phi : \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y})$$
$$\Phi^! : \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y}) \to \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big)$$

and a semi-orthogonal decomposition

$$\begin{aligned}&\mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{Y})\\&\quad=\Big\langle \bigoplus_{\alpha|_{\mathbf{G}_m}=d-\mathrm{N}} \mathcal{O}_{\mathrm{Y}}(\alpha), \ldots, \bigoplus_{\alpha|_{\mathbf{G}_m}=-1} \mathcal{O}_{\mathrm{Y}}(\alpha), [\Phi]\big(\big[\mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big)\big]\big)\Big\rangle.\end{aligned}$$

Moreover, if $\mathrm{G} = \mathbf{G}_m$ *and* $\mathrm{N} = n$*, there are quasi-isomorphisms of bimodules*

$$\begin{aligned}\Phi^! \circ \{1\} \circ \Phi &\cong (1)\\ \Phi^! \circ \Phi &\cong \nabla\end{aligned}$$

and

$$\big[\Phi^!\big]\mathcal{O}_{\mathrm{Z}}(i) \cong 0$$

for $d - \mathrm{N} \leq i \leq -1$*.*

- *If* $d = \mathrm{N}$*, then there exists inverse morphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}\Phi &: \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y})\\ \Psi &: \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y}) \to \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big).\end{aligned}$$

If, in addition, $\mathrm{G} = \mathbf{G}_m$ *and* $\mathrm{N} = n$*, there is a quasi-isomorphism of bimodules*

$$\{1\} \circ \Phi \cong \Phi \circ (1).$$

Moreover, for each $s \in k[x_1, \ldots, x_n]$ *homogeneous of degree* i*, the natural transformations of exact functors,*

$$\begin{aligned}s &: \mathrm{Id}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n, \mathbf{G}_m, w)]} \to (i)\\ s &: \mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z})} \to \mathrm{T}_{\mathcal{O}_{\mathrm{Z}}(i)}\end{aligned}$$

satisfy the identity

$$\begin{aligned}\Phi(s) &= \varsigma(\mathcal{O}_{\mathrm{Z}}) \circ \cdots \circ \varsigma\big(\mathcal{O}_{\mathrm{Z}}(i-1)\big) \circ s : \mathrm{Id} \to \Phi \circ (i) \circ \Phi^{-1}\\ &\cong \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}} \circ \cdots \mathrm{L}_{\mathcal{O}_{\mathrm{Z}(i-1)}} \circ \mathrm{T}_{\mathcal{O}_{\mathrm{Z}(i)}}.\end{aligned}$$

- *If* $d > \mathrm{N}$*, then there exists morphisms in* $\mathrm{Ho}(\text{dg-cat}_k)$

$$\begin{aligned}\Psi^! &: \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y})\\ \Psi &: \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y}) \to \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^n, \mathrm{G}, w\big)\end{aligned}$$

and a semi-orthogonal decomposition

$$\begin{aligned}&\mathrm{D}^{\mathrm{abs}}\big[\mathsf{fact}\big(\mathbf{A}^n, \mathrm{G}, w\big)\big]\\&\quad=\Big\langle \bigoplus_{\alpha|_{\mathbf{G}_m}=-1+d-\mathrm{N}} k(\alpha), \ldots, \bigoplus_{\alpha|_{\mathbf{G}_m}=0} k(\alpha), [\Psi]\big(\big[\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y})\big]\big)\Big\rangle.\end{aligned}$$

Moreover, if $\mathrm{G} = \mathbf{G}_m$ *and* $\mathrm{N} = n$*, there are quasi-isomorphisms of bimodules*

$$\Psi^! \circ (1) \circ \Psi \cong \{1\}$$
$$\Psi^! \circ \Psi \cong \Delta_* \mathcal{O}_{\mathrm{Z}}$$

and

$$[\Psi^!] k(j) \cong 0$$

for $d - \mathrm{N} - 1 \geq j \geq 0$.

Proof. — In the case that $\mathbf{G}_m = \mathrm{G}$, in [Orl09, Theorem 2.13], Orlov constructs the triangulated functors and the semi-orthogonal decompositions of the triangulated categories. The isomorphisms on the level of triangulated functors were constructed in [BFK11, Proposition 5.8]. Căldăraru and Tu [CT10, Theorem 5.9] lifted these functors to dg-functors between appropriate enhancements. We indicate the extension to G as in the statement of the theorem.

Consider the following diagram of dg-categories:

$$\begin{tikzcd} & \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U}) \arrow[dl, "\Upsilon_i"'] \arrow[dr, "\pi_i"] & \\ \mathsf{Inj}_{\mathrm{coh}}(\mathbf{A}^n, \mathrm{G}, w) & & \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}}(\mathrm{Y}) \arrow[ul, bend right, "\omega_i"'] \end{tikzcd}$$

Here U is zero locus of w in $\mathbf{A}^n$. The dg-category $\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U})$ consists of bounded below complexes of injective G-equivariant sheaves on U whose cohomology lies in $\mathbf{G}_m$-degrees $\geq i$, is bounded, and finitely-generated. Let Υ_i denote the restriction of Υ to $\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U})$. Finally, let π the restriction along the inclusion $\mathrm{Y} \to \mathrm{U}$, π_i the restriction of π to $\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U})$, and let ω_i denote the functor,

$$\omega_i(\mathcal{F}) := \bigoplus_{\substack{\alpha \in \widehat{\mathrm{G}} \\ \alpha|_{\mathbf{G}_m} \geq i}} \mathrm{H}^0\big(\mathrm{Y}, \mathcal{F}(\alpha)\big).$$

Note that, as ω_i is right adjoint to π_i at the level of the Abelian category of equivariant sheaves, the corresponding dg-functors are also adjoint.

Next, define D_i to be the quasi-essential image of ω_i, in particular D_i is closed under quasi-isomorphism, and $\mathsf{P}_{\geq i}$ to be the full dg-subcategory of $\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U})$ containing the injective resolutions of $\mathcal{O}_{\mathrm{U}}(\alpha)$ for $\alpha|_{\mathbf{G}_m} \leq i$. Finally, let T_i be the full dg-subcategory containing all $\mathcal{F}$ that satisfy

$$\mathrm{H}^{\bullet}\big(\mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U})}(\mathcal{F}, \mathcal{P})\big) = 0$$

for all $\mathcal{P} \in \mathsf{P}_{\geq i}$.

As $\pi \circ \omega_i = \mathrm{Id}$, the restriction of π_i to D_i is a quasi-equivalence and ω_i its inverse. Following arguments of [Orl09], which we suppress, the restriction of Υ to T_i is a quasi-equivalence. Let ν_i be the inverse to $\Upsilon|_{\mathsf{T}_i}$ in $\mathrm{Ho}(\text{dg-cat}_k)$. One then sets

$$\Phi_i := \pi \circ \nu_i, \Phi := \Phi_1$$
$$\Phi_i^! := \Upsilon \circ \omega_i, \Phi^! := \Phi_1^!$$
$$\Psi_i := \Upsilon \circ \omega_i, \Psi := \Psi_1$$
$$\Psi_i^! := \pi \circ \nu_{i-d+n}, \Psi^! := \Psi_1^!.$$

The proofs of the existence of the semi-orthogonal decompositions follow along the same arguments of [Orl09] using the fact that

$$\mathbf{R}\mathrm{Hom}_{\mathsf{Qcoh}\,\mathrm{U}}(\bullet, \mathcal{O}_{\mathrm{U}}) : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{U})^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}_{\mathrm{G}}\,\mathrm{U})$$

is an equivalence satisfying

$$\mathbf{R}\mathrm{Hom}_{\mathsf{Qcoh}\,\mathrm{U}}(k, \mathcal{O}_{\mathrm{U}}) \cong k(\nu)[-n]$$

for $\nu \in \widehat{\mathrm{G}}$ with $\nu|_{\mathbf{G}_m} = \mathrm{N}$.

In the case $\mathbf{G}_m = \mathrm{G}$ and $n = \mathrm{N}$, we have an equivalence $\mathrm{Qcoh}_{\mathrm{G}}\,\mathrm{Y} \cong \mathrm{Qcoh}\,\mathrm{Z}$. The statements that

$$\left[\Phi^!\right]\mathcal{O}_{\mathrm{Z}}(i) \cong 0$$

for $d - \mathrm{N} \leq i \leq -1$ and

$$\left[\Psi^!\right]k(j) \cong 0$$

for $d - \mathrm{N} - 1 \geq j \geq 0$ follow immediately from [Orl09].

The only remaining statement to check is that concerning the existence of quasi-isomorphisms between the stated bimodules. It suffices to show that the corresponding dg-functors are naturally quasi-isomorphic.

Now, consider the following dg-functor,

$$\mathrm{M} : \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq i}(\mathrm{U}) \to \mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq 0}(\mathrm{U})$$
$$\mathcal{E} \mapsto \mathrm{Cone}(\mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}_{\mathrm{G}}, \geq 1}(\mathrm{U})}\left(\widetilde{\mathcal{O}}_{\mathrm{U}}, \mathcal{E}(1)\right) \otimes_k \left(\widetilde{\mathcal{O}}_{\mathrm{U}} \xrightarrow{ev} \mathcal{E}(1)\right)$$

where $(\widetilde{\mathcal{O}}_{\mathrm{U}}$ is an injective resolution of $\mathcal{O}_{\mathrm{U}}$. Note that we have a natural transformation $\eta : (1) \to \mathrm{M}$.

Consider the diagram

$$\begin{tikzcd}
\mathsf{Inj}_{\mathrm{coh}_G,\geq i}(\mathrm{U}) \arrow[r, "\mathrm{M}"] & \mathsf{Inj}_{\mathrm{coh}_G,\geq 0}(\mathrm{U}) \arrow[d, "\pi"] \\
\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z}) \arrow[u, "\omega_1"] \arrow[r, "\mathrm{L}_{\mathcal{O}_Z} \circ \mathrm{T}_{\mathcal{O}_Z(1)}"] & \mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z})
\end{tikzcd}$$

The composition equals

$$(\pi \circ \mathrm{M} \circ \omega_i)(\mathcal{E}) := \mathrm{Cone}\Big(\mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}_G,\geq 0}(\mathrm{U})}\big(\widetilde{\mathcal{O}}_{\mathrm{U}}, \omega_i\mathcal{E}(1)\big) \otimes_k \pi\widetilde{\mathcal{O}}_{\mathrm{U}} \\ \xrightarrow{\mathrm{ev}} (\pi \circ \omega_i)\big(\mathcal{E}(1)\big)\Big).$$

Using the adjunction, $\pi \dashv \omega_i$, and the identity, $\pi \circ \omega_i \cong \mathrm{Id}$, the composition is isomorphic to

$$\mathrm{Cone}\Big(\mathrm{Hom}_{\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Z})}\big(\widetilde{\mathcal{O}}_{\mathrm{Z}}, \mathcal{E}(1)\big) \otimes_k \widetilde{\mathcal{O}}_{\mathrm{Z}} \xrightarrow{\mathrm{ev}} \mathcal{E}(1)\Big) = \mathcal{E}\{1\}.$$

Thus, we have a natural isomorphism

$$\pi \circ \mathrm{M} \circ \omega_i \cong \{1\}. \tag{6.1}$$

We will use this equation in both cases.

Now, assume that $d \leq n$ and consider the composition

$$\Phi^! \circ \{1\} \circ \Phi = \Upsilon \circ \omega_1 \circ \{1\} \circ \pi \circ \nu_1.$$

We can substitute

$$\Upsilon \circ \omega_1 \circ \{1\} \circ \pi \circ \nu_1 \cong \Upsilon \circ \omega_1 \circ \pi \circ \mathrm{M} \circ \omega_1 \circ \pi \circ \nu_1.$$

Since the image of ν_1 lies in D_1 by [Orl09], we have

$$\omega_1 \circ \pi \circ \nu_1 \cong \nu_1.$$

One can check, as in [BFK11, Lemma 5.7], that $\mathrm{M} \circ \nu_1$ has quasi-essential image in D_1, thus we have a natural quasi-isomorphism

$$\mathrm{M} \circ \nu_1 \to \omega_1 \circ \pi \circ \mathrm{M} \circ \nu_1.$$

This gives a natural quasi-isomorphism

$$\Phi^! \circ \{1\} \circ \Phi \simeq \Upsilon \circ \mathrm{M} \circ \nu_1.$$

The composition

$$\Upsilon \circ (1) \circ \nu_1 \xrightarrow{\Upsilon(\eta_{\nu_1})} \Upsilon \circ \mathrm{M} \circ \nu_1$$

is a quasi-isomorphism for all objects of $\mathsf{Inj}_{\mathrm{coh}}(\mathbf{A}^n, \mathbf{G}_m, w)$ as $\Upsilon(\widetilde{\mathcal{O}}_{\mathrm{U}})$ is acyclic. Thus, using the above and Equation (6.1), we have a quasi-isomorphism

$$\Phi^! \circ \{1\} \circ \Phi \simeq \Upsilon \circ \mathrm{M} \circ \nu_1 \simeq \Upsilon \circ (1) \circ \nu_1 = (1).$$

Now, assume that $d \geq n$ and consider the composition

$$\Psi^! \circ (1) \circ \Psi = \pi \circ \nu_{1-d+n} \circ (1) \circ \Upsilon \circ \omega_1.$$

One has a natural quasi-isomorphism

$$(1) \circ \Upsilon = \Upsilon \circ (1) \xrightarrow{\Upsilon(\eta)} \Upsilon \circ \mathrm{M}.$$

Thus,

$$\pi \circ \nu_{1-d+n} \circ (1) \circ \Upsilon \circ \omega_1 \cong \pi \circ \nu_{1-d+n} \circ \Upsilon \circ \mathrm{M} \circ \omega_1.$$

As $\mathsf{D}_1 \subset \mathsf{T}_{1-d+n}$ by [Orl09] and $\mathrm{M}(\mathsf{D}_1)$ lies in D_1, we have

$$\pi \circ \nu_{1-d+n} \circ \Upsilon \circ \mathrm{M} \circ \omega_1 \cong \pi \circ \mathrm{M} \circ \omega_1 \cong \{1\}$$

where the last quasi-isomorphism is Equation (6.1).

Finally, let us assume that $d = \mathrm{N} = n$ and $\mathrm{G} = \mathbf{G}_m$. Let $s \in k[x_1, \ldots, x_n]$ be homogeneous of degree 1, the natural transformations of exact functors,

$$s : \mathrm{Id}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n, \mathbf{G}_m, w)]} \to (1)$$
$$s : \mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z})} \to \mathrm{T}_{\mathcal{O}_{\mathrm{Z}}(1)}.$$

Let $\mathcal{E}$ be an object of $\mathrm{Inj}_{\mathrm{coh}}(\mathrm{Z})$ and consider $s : \mathcal{E} \to \mathrm{T}_{\mathcal{O}_{\mathrm{Z}}(1)}(\mathcal{E})$. Applying ω_1 gives a morphism

$$\omega_1(s) : \omega_1(\mathcal{E}) \to \omega_1(\mathcal{E})_{\geq 2}(1).$$

Composing with the inclusion

$$\omega_1(\mathcal{E})_{\geq 2}(1) \to \omega_1(\mathcal{E})(1)$$

equals

$$s : \omega_1(\mathcal{E}) \to \omega_1(\mathcal{E})(1).$$

Apply the dg-functor

$$\mathrm{L}_{\mathcal{O}_{\mathrm{U}}}(\mathcal{I}) := \mathrm{Cone}\big(\mathrm{Hom}(\widetilde{\mathcal{O}}_{\mathrm{U}}, \mathcal{I}) \otimes_k \widetilde{\mathcal{O}}_{\mathrm{U}} \to \mathcal{I}\big).$$

We get a map

$$\varsigma(\mathcal{O}_{\mathrm{U}})_{\omega_1(\mathcal{E})} \circ s : \omega_1(\mathcal{E}) \to \mathrm{L}_{\mathcal{O}_{\mathrm{U}}}\big(\omega_1(\mathcal{E})(1)\big).$$

Since $\omega_1(\mathcal{E})_{\geq 2}(1)$ is concentrated in homogeneous degrees ≥ 1, we have

$$\mathrm{H}^\bullet\big(\mathrm{Hom}\big(\widetilde{\mathcal{O}}_{\mathrm{U}}, \omega_1(\mathcal{E})_{\geq 2}(1)\big)\big) = \mathrm{H}^\bullet\big(\mathrm{Hom}\big(\mathcal{O}_{\mathrm{U}}, \omega_1(\mathcal{E})_{\geq 2}(1)\big)\big) = 0.$$

Thus, $\varsigma(\mathcal{O}_{\mathrm{U}})_{\omega_1(\mathcal{E})_{\geq 2}(1)} : \omega_1(\mathcal{E})_{\geq 2}(1) \to \mathrm{L}_{\mathcal{O}_{\mathrm{U}}}(\omega_1(\mathcal{E})_{\geq 2}(1))$, is a quasi-isomorphism. Applying π, gives

$$\Phi(s) = \varsigma(\mathcal{O}_{\mathrm{Z}})_{\mathcal{E}} \circ s : \mathcal{E} \to \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}} \circ \mathrm{T}_{\mathcal{O}_{\mathrm{Z}}(1)}(\mathcal{E})$$

on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z})$. It is straightforward to check there are isomorphisms

$$\Phi \circ (i) \circ \Phi^{-1} \cong \{i\} \cong \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}} \circ \cdots \circ \mathrm{L}_{\mathcal{O}_{\mathrm{Z}}(i-1)} \circ \mathrm{T}_{\mathcal{O}_{\mathrm{Z}}(i)}.$$

We have two algebra homomorphisms

$$\mathrm{S} \to \bigoplus_{i \in \mathbf{Z}} \mathrm{Nat}\big(\mathrm{Id}, \{i\}\big)$$

where Nat denotes natural transformations. The first is given by conjugation by Φ while the second is

$$s \mapsto \varsigma(\mathcal{O}_{\mathrm{Z}}) \circ \cdots \circ \varsigma\big(\mathcal{O}_{\mathrm{Z}}(i-1)\big) \circ s$$

for $s \in \mathrm{S}_i$. These agree on generators for S and hence agree overall. □

Remark **6.14.** — From the arguments above, it is clear that in the case $\mathrm{G} = \mathbf{G}_m$ and $d = \mathrm{N} = n$, that $\Phi(k[1]) \cong \mathcal{O}_{\mathrm{Z}}$.

Remark **6.15.** — One could also apply the results in [BFK12] on VGIT for equivariant factorizations. Or, one could directly lift the statements of [Orl09] using the results of [Ela11].

Remark **6.16.** — The case $\mathrm{G} \neq \mathbf{G}_m$ will be used in [BFK13]. Henceforth, we will only apply Theorem 6.13 under the assumption that $\mathrm{G} = \mathbf{G}_m$ act in the usual manner on $\mathbf{A}^n$.

Corollary **6.17.** — *Let w be a degree d homogeneous polynomial in $k[x_1, \ldots, x_n]$ with its standard grading. Let* Z *be the projective hypersurface defined by w. Assume that* Z *is smooth.*

- *If $d < n$, we have a commutative diagram of vector spaces*

$$\begin{tikzcd}
\mathrm{HH}_\bullet(Z) \arrow[r, "\{1\}_\bullet"] & \mathrm{HH}_\bullet(Z) \arrow[d, "\Phi^!_\bullet"] \\
\mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w) \arrow[u, "\Phi_\bullet"] \arrow[r, "(1)_\bullet"] & \mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w)
\end{tikzcd}$$

Moreover,

$$\Phi^!_\bullet \circ \Phi_\bullet = 1,$$

the functor $\Phi^!_\bullet$ is right adjoint to $\Phi_\bullet$ under the categorical pairing, and we have an orthogonal decomposition

$$\mathrm{HH}_\bullet(Z) = \mathrm{Im}\,\Phi_\bullet \oplus \bigoplus_{j=d-n-1}^{-1} \mathbf{C} \cdot \mathrm{ch}\big(\mathcal{O}_Z(j)\big).$$

- *If $d = n$, we have a commutative diagram of vector spaces*

$$\begin{tikzcd}
\mathrm{HH}_\bullet(Z) \arrow[r, "\{1\}_\bullet"] & \mathrm{HH}_\bullet(Z) \\
\mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w) \arrow[u, "\Phi_\bullet"] \arrow[r, "(1)_\bullet"] & \mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w) \arrow[u, "\Phi_\bullet"]
\end{tikzcd}$$

and $\Phi_\bullet$ is an isomorphism.
- *If $d > n$, we have a commutative diagram of vector spaces*

$$\begin{tikzcd}
\mathrm{HH}_\bullet(Z) \arrow[r, "\{1\}_\bullet"] \arrow[d, "\Psi_\bullet"] & \mathrm{HH}_\bullet(Z) \\
\mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w) \arrow[r, "(1)_\bullet"] & \mathrm{HH}_\bullet(\mathbf{A}^n, \mathbf{G}_m, w) \arrow[u, "\Psi^!_\bullet"]
\end{tikzcd}$$

Moreover,

$$\Psi^!_\bullet \circ \Psi_\bullet = 1,$$

the functor $\Psi^!_\bullet$ is right adjoint to $\Psi_\bullet$ under the categorical pairing, and we have an orthogonal decomposition

$$\mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathbf{G}_m, w\big) = \mathrm{Im}\,\Psi_\bullet \oplus \bigoplus_{j=0}^{d-n-1} \mathbf{C} \cdot \mathrm{ch}\big(k(j)\big).$$

Proof. — All statements but the adjunction and orthogonal decomposition are immediate consequences of Theorem 6.13 and the functoriality for pushforwards, [PV12, Section 1]. We check the adjunctions.

We only provide an argument for the case $d > n$. The case $d < n$ is analogous. We have a splitting

$$\mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathbf{G}_m, w\big) = \mathrm{Im}\,\Psi_\bullet \oplus \ker \Psi_\bullet^! . \tag{6.2}$$

Counting dimensions, we also have an orthogonal decomposition

$$\mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathbf{G}_m, w\big) = \mathrm{Im}\,\Psi_\bullet \oplus \bigoplus_{j=0}^{d-n-1} \mathbf{C} \cdot \mathrm{ch}\big(k(j)\big).$$

Thus,

$$\ker \Psi_\bullet^! = \bigoplus_{j=0}^{d-n-1} \mathbf{C} \cdot \mathrm{ch}\big(k(j)\big)$$

and the splitting of Equation (6.2) is orthogonal with respect to the Mukai pairing. The adjunction now follows via a straightforward linear algebra argument. □

Remark **6.18.** — For the case, $d \leq n$, the argument can be significantly simplified using [Kuz11, Theorem 7.1]. This result guarantees a splitting of $\mathrm{HH}_\bullet(\mathrm{X})$ for any semi-orthogonal decomposition of $\mathrm{D^b}(\mathrm{coh}\,\mathrm{X})$ at the triangulated level without having to prove anything at the level of dg-categories. For the sake of this utility, we will appeal to this result in Section 6.2.

Definition **6.19.** — *Let* $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ *be a linear endomorphism of a vector space,* V*, over* $\mathbf{C}$*, and let* $\lambda \in \mathbf{C}$*. We denote the* λ*-eigenspace of* T *by* $\mathrm{E}_\lambda(\mathrm{T})$.

Lemma **6.20.** — *Under the HKR isomorphism, Theorem 6.10, there is an equality*

$$\phi_{\mathrm{HKR}}\big(\mathrm{E}_1\big(\{1\}_\bullet\big)\big) = \mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}; \mathbf{C}).$$

Proof. — Let us first observe that

$$\phi_{\mathrm{HKR}}^{-1}\big(\mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}; \mathbf{C})\big) \subseteq \mathrm{E}_1\big(\{1\}_\bullet\big).$$

It easy to check, cf. [Huy05, Exercise 5.37], that, for $v \in \mathrm{H}^\bullet(\mathrm{Z}; \mathbf{C})$,

$$\mathrm{T}^{\mathrm{H}}_{\mathcal{O}_\mathrm{Z}(1)}(v) = v \cdot \mathrm{ch}_{class}\big(\mathcal{O}_\mathrm{Z}(1)\big).$$

If we assume that v is primitive, then

$$v \cdot \mathrm{ch}_{class}\big(\mathcal{O}_\mathrm{Z}(1)\big) = v.$$

It is also easy to verify, cf. [Huy05, Exercise 8.15], that

$$L^{H}_{\mathcal{O}_Z}(v) = v - \big(\mathrm{ch}_{class}(\mathcal{O}_Z), v\big)_M \mathrm{ch}_{class}(\mathcal{O}_Z).$$

By definition, the pairing is expressed as

$$\big(\mathrm{ch}_{class}(\mathcal{O}_Z), v\big)_M = \int_Z \mathrm{ch}_{class}(\mathcal{O}_Z)^{\vee} \cdot v \cdot \mathrm{td}(Z) = \int_Z v \cdot \mathrm{td}(Z).$$

As the Todd class, $\mathrm{td}(Z)$, is of the form $1 + Hp(H)$ for some polynomial p, and v is primitive, we have

$$\int_Z v \cdot \mathrm{td}(Z) = \int_Z v.$$

However, by the Lefschetz Hyperplane Theorem, primitive classes cannot have top dimensional components. Hence,

$$\int_Z v = 0$$

and

$$L^{H}_{\mathcal{O}_Z}(v) = v.$$

As the cohomological integral transform $\{1\}^H$ corresponds to $\{1\}_\bullet$ under the HKR isomorphism, Theorem 6.10, we see that $\phi_{\mathrm{HKR}}^{-1}(v) \in E_1(\{1\}_\bullet)$.

Next, let $\bigoplus_{i=0}^{n-2} \mathbf{C} \cdot H^i$ be the subspace of $H^\bullet(Z;\mathbf{C})$ corresponding to powers of the hyperplane class, H. Assume that $v = \sum_{i=0}^{n-2} a_i H^i$ lies in $\phi_{\mathrm{HKR}}(E_1(\{1\}_\bullet)) = E_1(\{1\}^H)$. Then,

$$a_0 = a_0 - \big(\mathrm{ch}_{class}(\mathcal{O}_Z), v\big)_M$$

$$a_i = \sum_{j=0}^{i} \frac{a_j}{(i-j)!}, \qquad i > 0.$$

This immediately implies that $v = 0$. As the induced map on cohomology $\{1\}^H$ preserves the splitting

$$H^\bullet(Z;\mathbf{C}) = H^\bullet_{\mathrm{prim}}(Z;\mathbf{C}) \oplus \bigoplus_{i=0}^{n-2} \mathbf{C} \cdot H^i,$$

we see that

$$E_1\big(\{1\}^H\big) = H^\bullet_{\mathrm{prim}}(Z;\mathbf{C})$$

and

$$\phi_{\mathrm{HKR}}^{-1}\big(H^\bullet_{\mathrm{prim}}(Z;\mathbf{C})\big) = E_1\big(\{1\}_\bullet\big). \qquad \square$$

Theorem **6.21.** — *Let w be a homogeneous polynomial of degree d in $\mathbf{C}[x_1, \dots, x_n]$. Assume that w defines a smooth projective hypersurface, Z.*

- *If we assume $d \leq n$, then the linear map, $\Phi_\bullet$, induces an isomorphism,*
$$\Phi_\bullet : E_1\big(\{1\}_\bullet\big) \to E_1\big((1)_\bullet\big).$$
- *If we assume $d \geq n$, then the linear map, $\Psi^!_\bullet$, induces an isomorphism,*
$$\Psi^!_\bullet : E_1\big(\{1\}_\bullet\big) \to E_1\big((1)_\bullet\big).$$

In particular, Orlov's theorem and the HKR isomorphism provide isomorphisms,
$$H^{p,n-2-p}_{\mathrm{prim}}(Z) \cong \mathrm{Jac}(w)_{d(n-1-p)-n}.$$

Proof. — Let us treat the case $d \leq n$ first. Let $v \in E_1(\{1\}_\bullet)$. By Lemma 6.20, $\phi_{\mathrm{HKR}}(v) \in H^\bullet_{\mathrm{prim}}(Z; \mathbf{C})$. Thus, v is orthogonal to $\mathrm{ch}(\mathcal{O}_Z(j))$ under the Mukai pairing for each $j \in \mathbf{Z}$. By Corollary 6.17, we have an orthogonal decomposition
$$\mathrm{HH}_\bullet(Z) = \Phi_\bullet \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathbf{G}_m, w\big) \oplus \bigoplus_{j=d-n}^{-1} \mathbf{C} \cdot \mathrm{ch}\big(\mathcal{O}_Z(j)\big).$$
Write $v = \Phi_\bullet v' \oplus v''$ with respect to this decomposition. Thus, for $j \in \mathbf{Z}$,
$$0 = (\phi_{\mathrm{HKR}}(v), \phi_{\mathrm{HKR}}\big(\mathrm{ch}\big(\mathcal{O}_Z(j)\big)\big)_M = \big\langle v, \mathrm{ch}\big(\mathcal{O}_Z(j)\big)\big\rangle = \big\langle v'', \mathrm{ch}\big(\mathcal{O}_Z(j)\big)\big\rangle$$
as $\phi_{\mathrm{HKR}}(v) \in H^\bullet_{\mathrm{prim}}(Z; \mathbf{C})$ and $\phi_{\mathrm{HKR}}(\mathrm{ch}(\mathcal{O}_Z(j)) = \mathrm{ch}_{class}(\mathcal{O}_Z(j)) \in \bigoplus_{i=0}^{n-2} \mathbf{C} \cdot H^i$ are orthogonal with respect to the Mukai pairing. Due to their exceptionality, the set of vectors $\mathrm{ch}(\mathcal{O}_Z(d-n)), \dots, \mathrm{ch}(\mathcal{O}_Z(-1))$ forms an orthonormal basis for $\bigoplus_{j=d-n}^{-1} \mathbf{C} \cdot \mathrm{ch}(\mathcal{O}_Z(j))$. Consequently, $v'' = 0$.

Using Corollary 6.17 repeatedly, we have
$$(1)_\bullet\big(v'\big) = \Phi^!_\bullet\{1\}_\bullet \Phi_\bullet v' = \Phi^!_\bullet \Phi_\bullet v' = v'$$
i.e. $v' \in E_1((1)_\bullet)$. Thus, $\Phi_\bullet$ maps $E_1((1)_\bullet)$ monomorphically into $E_1(\{1\}_\bullet)$. Counting dimensions, we see this is an isomorphism.

Now, let us turn our attention to $d \geq n$. By Theorem 6.13, we have an orthogonal decomposition
$$\mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathbf{G}_m, w\big) = \Psi_\bullet \mathrm{HH}_\bullet(Z) \oplus \bigoplus_{j=0}^{d-n-1} \mathbf{C} \cdot \mathrm{ch}\big(k(j)\big).$$
Assume that $v \in E_1(\{1\}_\bullet)$. Write
$$\big((1)_\bullet \circ \Psi_\bullet\big)(v) = \Psi_\bullet v \oplus v'$$

with respect to this decomposition. Let us compute

$$\langle \mathrm{ch}(k(j)), v' \rangle$$

for some $j \in \mathbf{Z}$. By orthogonality, we have

$$\langle \mathrm{ch}(k(j)), v' \rangle = \langle \mathrm{ch}(k(j)), ((1)_\bullet \circ \Psi_\bullet)(v) \rangle.$$

Since (-1) is inverse to (1) and $\Psi_\bullet \dashv \Psi^!_\bullet$, from Corollary 6.17, we have

$$\langle \mathrm{ch}(k(j)), ((1)_\bullet \circ \Psi_\bullet)(v) \rangle = \langle (\Psi^!_\bullet \circ (-1)_\bullet)(\mathrm{ch}(k(j))), v \rangle.$$

Using the functorial properties of pushforwards, we have

$$(\Psi^!_\bullet \circ (-1)_\bullet)(\mathrm{ch}(k(j))) = \mathrm{ch}(\Psi^!(k(j-1))).$$

It is easy to check, in Orlov's equivalence, that $\Psi^! k(j-1)$ lies in the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathrm{Z})$ generated by the objects $\mathcal{O}_{\mathrm{Z}}(j), j \in \mathbf{Z}$. Note that we do *not* need to pass to direct summands. Thus,

$$\mathrm{ch}(\Psi^! k(j-1)) \in \bigoplus_{j=0}^{n-2} \mathbf{C} \cdot \mathrm{ch}(\mathcal{O}_{\mathrm{Z}}(j)) = \phi_{\mathrm{HKR}}^{-1}\left(\bigoplus_{j=0}^{n-2} \mathbf{C} \cdot \mathrm{H}^j \right).$$

By Lemma 6.20, $\phi_{\mathrm{HKR}}(v) \in \mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}; \mathbf{C})$. Thus, v is orthogonal to $\mathrm{ch}(\Psi^! k(j-1))$ for any $j \in \mathbf{Z}$. Therefore, $v' = 0$ and we have a well-defined monomorphism

$$\Psi_\bullet : \mathrm{E}_1(\{1\}_\bullet) \to \mathrm{E}_1((1)_\bullet).$$

Counting dimensions finishes the argument.

Now, to see that

$$\mathrm{H}^{p,n-2-p}_{\mathrm{prim}}(\mathrm{Z}) \cong \mathrm{Jac}(w)_{d(n-1-p)-n},$$

notice that by Corollary 5.41 and Theorem 6.5, we have an isomorphism

$$\mathrm{E}_1((1)_\bullet) \cap \mathrm{HH}_t(\mathbf{A}^n, \mathbf{G}_m, w) \cong \mathrm{Jac}_{d(\frac{n+t}{2})-n}.$$

By Theorem 6.21, we have an isomorphism

$$\mathrm{E}_1(\{1\}_\bullet) \cap \mathrm{HH}_t(\mathrm{Z}) \cong \mathrm{E}_1((1)_\bullet) \cap \mathrm{HH}_t(\mathbf{A}^n, \mathbf{G}_m, w).$$

From Lemma 6.20 and Theorem 6.10, we have an isomorphism

$$\mathrm{H}^\bullet_{\mathrm{prim}}(\mathrm{Z}) \cap \bigoplus_{q-p=t} \mathrm{H}^{p,q}(\mathrm{Z}) \cong \mathrm{E}_1(\{1\}_\bullet) \cap \mathrm{HH}_t(\mathrm{Z}).$$

Since we only have primitive cohomology in the middle degree, we must have $p + q = n - 2$. Solving for t gives $t = n - 2 - 2p$. Plugging in gives the statement. $\square$

Remark **6.22.** — One can also define $\mathrm{H}^{\bullet}_{\mathrm{prim}}(\mathrm{Z})$ as the orthogonal to $\sum_{i\in\mathbf{Z}} k \cdot \mathrm{ch}(\mathcal{O}_{\mathrm{Z}}(i))$ with respect to the categorical pairing. This extends Theorem 6.21 to other algebraically closed fields of characteristic zero.

Remark **6.23.** — In addition to having interesting Eigenspaces, the determinant of $\{1\}_{\bullet}$ is the geometric genus of the hypersurface.

Definition **6.24.** — *Let* Z *be a smooth, projective hypersurface. Let* $\mathcal{K}$ *be the object*

$$\mathcal{K} := \mathcal{I}_{\Delta} \otimes_{\mathcal{O}_{\mathrm{Z}\times\mathrm{Z}}} \pi_1^*\mathcal{O}_{\mathrm{Z}}(1)[1].$$

Define the graded ring

$$\mathrm{S}(\mathrm{Z}) := \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z}\times\mathrm{Z})}\big(\Delta_*\mathcal{O}_{\mathrm{Z}}, \mathcal{K}^{*i}\big)$$

where K^{*i} *denotes* i*-th self-convolution* $\mathcal{K}$*, cf. [Huy05, Section 5.1].*

Lemma **6.25.** — *Assume that* Z *is Calabi-Yau. There is an isomorphism of functors*

$$\{1\} \cong \Phi_{\mathcal{K}} : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z})$$

and an injective homomorphism of graded rings

$$\mathrm{Jac}(w) \to \mathrm{S}(\mathrm{Z})$$

where w *is the defining polynomial of* Z*.*

Proof. — It is straightforward to check that we have a quasi-isomorphism of kernels, $\mathcal{K} \cong \{1\}$. Using Orlov's equivalence from Theorem 6.13, we get a isomorphism of graded rings

$$\bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n\times\mathbf{A}^n, \mathbf{G}_m\times_{\mathbf{G}_m}\mathbf{G}_m, (-w)\boxplus w)]}\big(\nabla, \nabla(i)\big)$$
$$\to \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathrm{Z}\times\mathrm{Z})}\big(\Delta_*\mathcal{O}_{\mathrm{Z}}, \mathcal{K}^{*i}\big).$$

There is a natural homomorphism of graded rings

$$k[x_1, \ldots, x_n] \to \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n\times\mathbf{A}^n, \mathbf{G}_m\times_{\mathbf{G}_m}\mathbf{G}_m, (-w)\boxplus w)]}\big(\nabla, \nabla(i)\big)$$

given by multiplying by a polynomial. By Theorem 5.39, this induces a monomorphism

$$\mathrm{Jac}(w) \to \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n\times\mathbf{A}^n, \mathbf{G}_m\times_{\mathbf{G}_m}\mathbf{G}_m, (-w)\boxplus w)]}\big(\nabla, \nabla(i)\big).$$

The total composition is the desired homomorphism $\mathrm{Jac}(w) \to \mathrm{S}(\mathrm{Z})$. $\square$

Remark **6.26.** — A natural question to ask of Griffiths' Residue Theorem is: where do all the other graded pieces of the Jacobian algebra go? Lemma 6.25 provides the answer in terms of the derived category of Z for a Calabi-Yau hypersurface. The whole Jacobian algebra sits as a graded subring of morphisms in $\mathrm{D^b}(\mathrm{coh}\,\mathrm{Z} \times \mathrm{Z})$ from the identity functor to powers of $\{1\}$. Certain powers of $\{1\}$ are shifts of the Serre functor. Those graded pieces of the Jacobian algebra then appear in $\mathrm{HH}_\bullet(\mathrm{Z}) \cong \mathrm{H}^\bullet(\mathrm{Z}; \mathbf{C})$.

In the Fano case, we have to replace $\mathrm{S}(\mathrm{Z})$ with the graded algebra

$$\bigoplus_{i \geq 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{coh}\,\mathrm{Z}\times\mathrm{Z})}\big(\mathcal{P}, \mathcal{P} * \{i\} * \mathcal{P}\big)$$

where $\mathcal{P} = \Phi \circ \Phi^!$ is the kernel associated to the inclusion of $\mathrm{D^{abs}}[\mathsf{fact}(\mathbf{A}^n, \mathbf{G}_m, w)] \to \mathrm{D^b}(\mathrm{coh}\,\mathrm{Z})$ as an admissible subcategory, [Kuz11].

In the general type case, we have different kernels, $\mathcal{K}_i = \Psi^! \circ (i) \circ \Psi$, for each i. The natural repository for the Jacobian algebra is the graded vector space

$$\bigoplus_{i \geq 0} \mathrm{Hom}_{\mathrm{D^b}(\mathrm{coh}\,\mathrm{Z}\times\mathrm{Z})}(\Delta_* \mathcal{O}_{\mathrm{Z}}, \mathcal{K}_i).$$

In each situation, we have a categorical realization of Griffiths' fundamental result that sees the entire Jacobian algebra.

6.2. *Using equivariant factorizations to study algebraic cycles.* — In this section we examine how algebraic classes behave under variation of the group action. Using Theorem 6.5, the induction functor, and functoriality of push-forwards, Proposition 6.1, one can precisely relate the algebraic classes under induction and restriction of the group action. The following is essentially due to Polishchuk and Vaintrob.

Proposition **6.27.** — *Let* $\mathbf{A}^n$ *carry a linear action of* G, *an Abelian algebraic group, and let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^{\mathrm{G}}$. *Assume that* K_χ *is finite and* $\chi : \mathrm{G} \to \mathbf{G}_m$ *is surjective. Furthermore, assume that* $(\mathrm{d}w)$ *is supported at* $\{0\} \in \mathbf{A}^n$. *Let* $\phi : \mathrm{H} \to \mathrm{G}$ *be an injective homomorphism of affine algebraic groups and assume that* $\chi \circ \phi$ *is surjective. Consider the functors,*

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathsf{vect}\big(\mathbf{A}^n, \mathrm{H}, w\big) \to \mathsf{vect}\big(\mathbf{A}^n, \mathrm{G}, w\big)$$
$$\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} : \mathsf{vect}\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathsf{vect}\big(\mathbf{A}^n, \mathrm{H}, w\big),$$

and the induced maps,

$$\mathrm{Ind}_{\mathrm{H}\bullet}^{\mathrm{G}} : \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{H}, w\big) \to \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{G}, w\big)$$
$$\mathrm{Res}_{\mathrm{H}\bullet}^{\mathrm{G}} : \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{G}, w\big) \to \mathrm{HH}_\bullet\big(\mathbf{A}^n, \mathrm{H}, w\big).$$

The composition is the linear map satisfying

$$\mathrm{Ind}_{\mathrm{H}_\bullet}^{\mathrm{G}} \circ \mathrm{Res}_{\mathrm{H}_\bullet}^{\mathrm{G}} : \mathrm{HH}_\bullet(\mathbf{A}^n, \mathrm{G}, w) \to \mathrm{HH}_\bullet(\mathbf{A}^n, \mathrm{G}, w)$$
$$v \mapsto \begin{cases} |\mathrm{G}/\mathrm{H}|v & v \in \mathrm{Jac}(w_g) \textit{ with } g \in \mathrm{K}_{\chi\circ\phi} \\ 0 & v \in \mathrm{Jac}(w_g) \textit{ with } g \notin \mathrm{K}_{\chi\circ\phi}. \end{cases}$$

Proof. — Let K denote the kernel of $\widehat{\phi} : \widehat{\mathrm{G}} \to \widehat{\mathrm{H}}$. For $c \in \mathrm{K}$, $c(g) = 1$ if and only if $g \in \mathrm{K}_{\chi\circ\phi}$. From Lemma 2.16, we have an isomorphism of functors, $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \circ \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} \cong p_* p^*$, where $p : \mathrm{G}/\mathrm{H} \times \mathbf{A}^n \to \mathbf{A}^n$ is the projection. Therefore, $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \circ \mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} \cong \bigoplus_{c\in\mathrm{K}} (c)$. Note that $\bigoplus_{c\in\mathrm{K}} (c)$ can be factored as a composition

$$\mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w) \xrightarrow{\kappa} \coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w) \xrightarrow{\oplus} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$$

where κ maps to the factor corresponding to c by the autoequivalence, (c), and $\oplus$ is the functor that takes $\coprod \mathcal{E}_c$ to $\oplus\mathcal{E}_c$. Here $\coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$ denotes the category whose objects are $|\mathrm{K}|$-tuples of objects from $\mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$ and whose morphisms are $|\mathrm{K}|$-tuples of morphisms $\mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$. Denote an object of $\coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$ by $\oplus_{c\in\mathrm{K}} \mathcal{E}_c e_c$ where we think of e_c as orthogonal idempotents.

A generator of $\mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$ exists by Lemma 4.14, Proposition 3.64, and the assumption that the support of $(\mathrm{d}w)$ is $\{0\}$. Choose a generator, $\mathcal{G}$, and let A denote its dg-endomorphism complex. If we take $\oplus\mathcal{G}e_c$ as our generator of $\coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$, we see its dg-endomorphism complex is $\tilde{\mathrm{A}} = \mathrm{A}e_1 \oplus \cdots \oplus \mathrm{A}e_c$ where e_c are (closed) orthogonal idempotents. It is easy to see that $\tilde{\mathrm{A}} \overset{\mathbf{L}}{\otimes}_{\tilde{\mathrm{A}}^e} \tilde{\mathrm{A}} \cong \bigoplus_{c\in\mathrm{K}} (\mathrm{A} \overset{\mathbf{L}}{\otimes}_{\mathrm{A}^e} \mathrm{A}) e_c$. Thus, $\mathrm{HH}_\bullet(\coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w))$ is isomorphic to $\bigoplus_{c\in\mathrm{K}} \mathrm{HH}_\bullet(\mathbf{A}^n, \mathrm{G}, w) e_c$.

Theorem 6.5 says that the action on the component of $\mathrm{HH}_\bullet(\coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w))$ corresponding to $\mathrm{Jac}(w_g)$ is multiplication by $c(g)^{-1}$.

In terms of $\tilde{\mathrm{A}}$ and A, $\oplus : \coprod_{c\in\mathrm{K}} \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w) \to \mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$ corresponds to the summing map $\tilde{\mathrm{A}} \to \mathrm{A}$ which takes $\oplus a_c e_c$ to $\sum a_c$. It is easy to see the induced action on Hochschild homology is again summation.

Now, we see that if $g \in \mathrm{K}_{\chi\circ\phi}$, then each c acts trivially and the summand corresponding to $\mathrm{Jac}(w_g)$ gets multiplied by $|\mathrm{K}| = |\mathrm{G}/\mathrm{H}|$. If $g \notin \mathrm{K}_{\chi\circ\phi}$, then $c(g)$ is nonzero and $\sum_{c\in\mathrm{K}} c(g) = 0$. $\square$

Next, we prove a lemma that allows us to lift algebraic cycles via induction.

Lemma **6.28.** — *Let* $\mathbf{A}^n$ *carry a linear action of* G*, an Abelian algebraic group, and let* $w \in \Gamma(\mathbf{A}^n, \mathcal{O}_{\mathbf{A}^n}(\chi))^{\mathrm{G}}$. *Assume that* K_χ *is finite and* $\chi : \mathrm{G} \to \mathbf{G}_m$ *is surjective. Furthermore, assume that* $(\mathrm{d}w)$ *is supported at* $\{0\} \in \mathbf{A}^n$. *Assume that the image of the Chern character,*

$$\mathrm{ch} : \mathrm{K}_0(\mathbf{A}^{nr}, \mathrm{G}, w^{\boxplus r}) \to \mathrm{HH}_0(\mathbf{A}^{nr}, \mathrm{G}, w^{\boxplus r}),$$

spans, over $\mathbf{C}$, *for all* $r \geq 1$. *Furthermore, assume that*

$$\left(\mathrm{Jac}(w_g)(-\kappa_e - \kappa_g - u\chi)\right)^{\mathrm{G}} = 0$$

for $t \neq 0$ *and* $g \neq e$ *where either* $2u = \dim \mathrm{W}_g + t - n$ *or* $2u + 1 = \dim \mathrm{W}_g + t - n$. *Then, the image of the Chern character,*

$$\mathrm{ch} : \mathrm{K}_0\left(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\right) \to \mathrm{HH}_0\left(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\right),$$

also spans, over $\mathbf{C}$.

Proof. — If $\mathsf{C}_1, \ldots, \mathsf{C}_n$ are saturated dg-categories, then it is straightforward to verify

$$\bigotimes_{i=1}^{r} \mathrm{HH}_\bullet(\mathsf{C}_i) \cong \mathrm{HH}_\bullet(\mathsf{C}_1 \circledast \cdots \circledast \mathsf{C}_r)$$

where the isomorphism is given by taking tensor products over k. Thus, by Corollary 5.18,

$$\mathrm{HH}_\bullet\left(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\right) \cong \mathrm{HH}_\bullet\left(\mathbf{A}^{n}, \mathrm{G}, w\right)^{\otimes r}.$$

In particular,

$$\mathrm{HH}_0\left(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\right) \cong \bigoplus_{i_1 + \cdots + i_r = 0} \mathrm{HH}_{i_1}\left(\mathbf{A}^{n}, \mathrm{G}, w\right) \otimes_k \cdots \otimes_k \mathrm{HH}_{i_r}\left(\mathbf{A}^{n}, \mathrm{G}, w\right). \tag{6.3}$$

To verify the claim, we need to find a basis of $\mathrm{HH}_\bullet(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r})$ which are Chern characters of objects of $\mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r})]$. We proceed by induction on r.

The base case, $r = 1$, is covered under the assumptions of the lemma. Assume the lemma is true for all products of size $< r$, and consider the case of r. Under the isomorphism of Equation (6.3), it is enough to find a basis of decomposable vectors, i.e. those expressible as tensor products of elements of $\mathrm{HH}_\bullet(\mathbf{A}^{n}, \mathrm{G}, w)$. Let

$$v := v_1 \otimes_k \cdots \otimes_k v_n \in \mathrm{HH}_0\left(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\right)$$

be a decomposable vector. We have two cases: one,

$$\text{some } v_i \in \mathrm{HH}_0\left(\mathbf{A}^{n}, \mathrm{G}, w\right),$$

and, two,

$$\text{no } v_i \in \mathrm{HH}_0\left(\mathbf{A}^{n}, \mathrm{G}, w\right).$$

Let us consider case one first. In this case,

$$v_1 \otimes_k \cdots \otimes_k \widehat{v_i} \otimes_k \cdots \otimes_k v_n \in \mathrm{HH}_0\left(\left(\mathbf{A}^{n}\right)^{\times r-1}, \mathrm{G}^{\times_{\mathbf{G}_m} r-1}, w^{\boxplus r-1}\right),$$

under the isomorphism of Equation (6.3). By induction, there exists a factorization, $\mathcal{E} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^{n(r-1)}, \mathrm{G}^{\times_{\mathbf{G}_m} r-1}, w^{\boxplus r-1})]$, with

$$\mathrm{ch}(\mathcal{E}) = v_1 \otimes_k \cdots \otimes_k \widehat{v_i} \otimes_k \cdots \otimes_k v_n$$

and $\mathcal{E}' \in \mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathbf{A}^n, \mathrm{G}, w)]$ with $\mathrm{ch}(\mathcal{E}') = v_i$. Then,

$$\mathrm{ch}\big(\mathcal{E} \boxtimes \mathcal{E}'\big) = v_1 \otimes_k \cdots \otimes_k v_n.$$

This covers the first case.

Let us move to the second case. Note that, since we have assumed

$$\big(\mathrm{Jac}(w_g)(-\kappa_e - \kappa_g - u\chi)\big)^{\mathrm{G}} = 0$$

for $t \neq 0$ and $g \neq e$, all of the $v_i \in \mathrm{HH}_0(\mathbf{A}^n, \mathrm{G}, w)$ lie in the untwisted sector corresponding to $g = e$. Consider, the diagonal homomorphism, $\phi : \mathrm{G} \to \mathrm{G}^{\times_{\mathbf{G}_m} r}$. By Proposition 6.27, we know the map,

$$\big(\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}} \circ \mathrm{Res}_{\mathrm{G}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}}\big)_{\bullet} : \mathrm{HH}_0\big(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big),$$

applied to v is

$$\big(\mathrm{Ind}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}} \circ \mathrm{Res}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}}\big)(v) = \big|\big(\mathrm{G}^{\times_{\mathbf{G}_m} r}\big)/\mathrm{G}\big| v.$$

By assumption, we can find an $\mathcal{E} \in \mathrm{D}^{\mathrm{abs}}[\mathsf{fact}(\mathrm{A}^{\otimes n}, \mathrm{M}, w^{\boxplus n})]$ with $\mathrm{ch}(\mathcal{E}) = \mathrm{Res}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}}(v)$. By Proposition 6.1, we get

$$\begin{aligned}\mathrm{ch}\big(\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}} \mathcal{E}\big) = \mathrm{Ind}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}}\big(\mathrm{ch}(\mathcal{E})\big) &= \big(\mathrm{Ind}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}} \circ \mathrm{Res}_{\mathrm{G}_{\bullet}}^{\mathrm{G}^{\times_{\mathbf{G}_m} r}}\big)(v) \\ &= \big|\big(\mathrm{G}^{\times_{\mathbf{G}_m} r}\big)/\mathrm{G}\big| v.\end{aligned}$$

Thus, over $\mathbf{C}$, we can find a spanning set of decomposable vectors in the image of the Chern class map. □

Remark **6.29.** — If we could define an appropriate rational structure on the Hochschild homology of $\mathsf{vect}(\mathbf{A}^n, \mathrm{G}, w)$, the arguments of Lemma 6.28 would generalize to show the following statement. Assume that

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}, \mathrm{G}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}, \mathrm{G}, w^{\boxplus r}\big)_{\mathbf{Q}},$$

spans, over $\mathbf{Q}$, for all $r \geq 1$. Furthermore, assume that

$$\big(\mathrm{Jac}(w_g)(-\kappa_e - \kappa_g - u\chi)\big)^{\mathrm{G}} = 0$$

for $t \neq 0$ and $g \neq e$. Then, the image of the Chern character,

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}, \mathrm{G}^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big)_{\mathbf{Q}},$$

also spans, over $\mathbf{Q}$. As such, this gives a bootstrap procedure for proving the Hodge conjecture for Morita products of factorization categories by proving it for simpler grading groups. In fact, recent work of Blanc [Bla12] may yield the appropriate rational structure.

Corollary **6.30.** — *Consider $\mathbf{A}^n_{\mathbf{C}}$ with the standard $\mathbf{G}_m$-action. Let w be the Fermat cubic or quartic polynomial. Then, the image of*

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big)$$

spans over $\mathbf{C}$.

Proof. — The result is a consequence of the splitting result for Hochschild homology of derived categories under semi-orthogonal decomposition, [Kuz09, Theorem 7.3].

We do this by applying Lemma 6.28 for $\mathrm{G} = \mathbf{G}_m$. To do so, we must check that

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big)$$

spans. Appealing to Theorem 6.13, we have a semi-orthogonal decomposition,

$$\begin{aligned}&\mathrm{D^b}(\mathrm{coh}\, \mathrm{Z}_{w^{\boxplus r}})\\ &\quad = \big\langle \mathcal{O}_{\mathrm{Z}_{w^{\boxplus r}}}(-rn+d), \ldots, \mathcal{O}_{\mathrm{Z}_{w^{\boxplus r}}}(-1), \mathrm{D^{abs}}\big[\mathsf{fact}\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m, w^{\boxplus r}\big)\big]\big\rangle,\end{aligned}$$

where $\mathrm{Z}_{w^{\boxplus r}}$ is the associated projective hypersurface. Kuznetsov's result then states we have a decomposition

$$\mathrm{HH}_0(\mathrm{Z}_{w^{\boxplus r}}) = \bigoplus_{i=-rn+d}^{-1} \mathbf{C} \cdot \mathrm{ch}\big(\mathcal{O}_{\mathrm{Z}_{w^{\boxplus r}}}(i)\big) \oplus \mathrm{HH}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m, w^{\boxplus r}\big).$$

Ran [Ran80] proved that for $d = 3, 4$, the image of

$$\mathrm{ch} : \mathrm{K}_0\big(\mathrm{D^b}(\mathrm{coh}\, \mathrm{Z}_{w^{\boxplus r}})\big) \to \mathrm{HH}_0(\mathrm{Z}_{w^{\boxplus r}})$$

spans $\mathrm{HH}_0(\mathrm{Z}_{w^{\boxplus r}})$ over $\mathbf{C}$. Using Proposition 6.1, we deduce that the image of

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m, w^{\boxplus r}\big)$$

spans over $\mathbf{C}$. The vanishing condition on the twisted sectors of the Hochschild homology follows as the fixed locus of any $g \notin \mathbf{G}_m$ is the origin of $\mathbf{A}^n$. This verifies the hypotheses of Lemma 6.28 so we may conclude that the image of

$$\mathrm{ch} : \mathrm{K}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big) \to \mathrm{HH}_0\big(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big)$$

spans over $\mathbf{C}$ for all $r \geq 1$. $\square$

Remark **6.31.** — One may rephrase the conclusion of Corollary 6.30 as: the Hodge conjecture over $\mathbf{Q}$ is true for $\mathrm{D}^{\mathrm{abs}}\,\mathsf{fact}(\mathbf{A}^{nr}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r})$.

We can apply Lemma 6.28 to reprove the Hodge conjecture for arbitrary self-products of a certain K3 surface closely related to the Fermat cubic fourfold. We first recall a result of Kuznetsov.

Proposition **6.32.** — *Let* X *be the Fermat cubic fourfold in* $\mathbf{P}^5$. *There exists a unique K3 surface,* Y*, such that there is a semi-orthogonal decomposition,*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{X}) = \langle \mathcal{O}_{\mathrm{X}}(-3), \mathcal{O}_{\mathrm{X}}(-2), \mathcal{O}_{\mathrm{X}}(-1), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y})\rangle.$$

Proof. — The Fermat cubic fourfold is a Pfaffian cubic. Thus, the existence of Y is consequence of Kuznetsov's results on Homological Projective Duality, see [Kuz10] for the statement. As mentioned previously, Ran proved that the image of

$$\mathrm{ch} : \mathrm{K}_0(\mathrm{X}) \to \mathrm{HH}_0(\mathrm{X})$$

spans over $\mathbf{Q}$, [Ran80]. Using the splitting of Hochschild homology and naturality of pushforwards in Hochschild homology, we deduce that the image of

$$\mathrm{ch} : \mathrm{K}_0(\mathrm{Y}) \to \mathrm{HH}_0(\mathrm{Y})$$

spans over $\mathbf{Q}$. In particular, since Y is a K3 surface, it must have Picard rank 20. If we have two such K3's surfaces, Y_1 and Y_2, with

$$\begin{aligned}\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{X}) &= \langle \mathcal{O}_{\mathrm{X}}(-3), \mathcal{O}_{\mathrm{X}}(-2), \mathcal{O}_{\mathrm{X}}(-1), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y}_1)\rangle \\ &= \langle \mathcal{O}_{\mathrm{X}}(-3), \mathcal{O}_{\mathrm{X}}(-2), \mathcal{O}_{\mathrm{X}}(-1), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y}_2)\rangle.\end{aligned}$$

Then, we must have an equivalence,

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y}_1) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y}_2).$$

However, K3 surfaces with Picard rank more than 11 do not have non-trivial Fourier-Mukai partners [HLOY04, Corollary 2.7.1]. □

Corollary **6.33.** — *Let* Y *be the K3 surface appearing in Proposition 6.32. The Hodge conjecture holds for all self-products,* $\mathrm{Y}^{\times r}$*,* $r \geq 1$.

Proof. — By [Kuz11, Theorem 7.1], the projection functor, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{X}) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathrm{Y})$, lifts to a dg-functor between enhancements. It is then straightforward to check that Theorem 6.13 induces a quasi-equivalence between $\mathsf{Inj}_{\mathrm{coh}}(\mathbf{A}^6_{\mathbf{C}}, \mathbf{G}_m, w)$ and $\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y})$, where

$$w = x_1^3 + \cdots + x_6^3.$$

Thus, we have quasi-equivalences,

$$\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y})^{\circledast r} \simeq \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^6_{\mathbf{C}}, \mathbf{G}_m, w\big)^{\circledast r} \simeq \mathsf{Inj}_{\mathrm{coh}}\big(\mathbf{A}^{6r}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r}\big).$$

The final quasi-equivalence is Corollary 5.18. Toën, [Toë07, Section 8], proves that there is a quasi-equivalence

$$\mathsf{Inj}_{\mathrm{coh}}(\mathrm{Y})^{\circledast r} \simeq \mathsf{Inj}_{\mathrm{coh}}\big(\mathrm{Y}^{\times r}\big).$$

We know the Hodge conjecture for $\mathsf{Inj}_{\mathrm{coh}}(\mathbf{A}^{6r}_{\mathbf{C}}, \mathbf{G}_m^{\times_{\mathbf{G}_m} r}, w^{\boxplus r})$ is true by Corollary 6.30. □

Remark **6.34.** — In the initial version of this paper, we claimed that Corollary 6.33 was a new case of the Hodge conjecture. After the first version was released, we were informed by P. Stellari that this case is already known, see [RM08]. We happily thank Stellari for this communication.

Remark **6.35.** — Ran's work was extended by N. Aoki, [Aok83]. Aoki's work relies on that of T. Shioda, [Shi79]. Shioda proves that the Hodge conjecture holds for Fermat hypersurfaces as long as a certain arithmetic condition is satisfied. Aoki gives a reinterpretation of this arithmetic condition. One can directly construct factorizations whose Chern characters span the classes, $\mathfrak{D}_d^{m-1}$, studied by Shioda-Aoki. Using their arithmetic argument, one can then prove *directly* that the Hodge conjecture holds for categories of $\mathbf{G}_m$-equivariant factorizations of Fermat potentials of degree d in $\mathbf{A}^{2m}_{\mathbf{C}}$ when d is prime, $d = 4$, or every prime divisor of d is greater than $m+1$. We omit the details, though they can be found in the initial arXived version of this article.

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