# ON TWO GEOMETRIC REALIZATIONS OF AN AFFINE HECKE ALGEBRA

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ABSTRACT

The article is a contribution to the local theory of geometric Langlands duality. The main result is a categorification of the isomorphism between the (extended) affine Hecke algebra associated to a reductive group G and Grothendieck group of equivariant coherent sheaves on Steinberg variety of Langlands dual group $G^\vee$; this isomorphism due to Kazhdan–Lusztig and Ginzburg is a key step in the proof of tamely ramified local Langlands conjectures.

The paper is a continuation of the author's joint work with Arkhipov, it relies on the technical material developed in a joint work with Yun.

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## 1. Introduction and statement of the result

**1.1.** *Affine Hecke algebra and its two categorifications.* — Let $k$ be a field, and let $\mathrm{F} = k((t)) \supset \mathrm{O} = k[[t]]$ be the field of functions on the punctured formal disc over $k$ and its ring of integers. Let G be a split reductive linear algebraic group over $k$; let $\mathrm{B} \subset \mathrm{G}$ be a Borel subgroup, and $\mathrm{I} \subset \mathrm{G}(\mathrm{F})$ be the corresponding Iwahori subgroup (thus I is the preimage of B under the evaluation map $\mathrm{G}(\mathrm{O}) \to \mathrm{G}$).

If $k$ is finite then the group G(F) is a locally compact topological group, I is its open compact subgroup, and the space $\overline{\mathbf{H}}$ of $\mathbf{C}$-valued finitely supported functions on the two-sided quotient $\mathrm{I}\backslash\mathrm{G}(\mathrm{F})/\mathrm{I}$ carries an algebra structure under convolution; this is the Iwahori-Matsumoto Hecke algebra. Also $\overline{\mathbf{H}} = \mathbf{H} \otimes_{\mathbf{Z}[q^{\pm 1}]} \mathbf{C}$ where $\mathbf{H}$ is the (extended) affine Hecke algebra and the homomorphism $\mathbf{Z}[q^{\pm 1}] \to \mathbf{C}$ sends $q$ to $|k|$.

Based on Grothendieck "sheaf-function" correspondence principle, one can consider the category of $l$-adic complexes (or perverse sheaves) on an $\mathbf{F}_q$-scheme (or on its base change to an algebraically closed field) as the categorical counterpart, or categorification, of the space of functions on the set of $\mathbf{F}_q$-points of the scheme; in particular, the space of functions is a quotient of the Grothendieck group of the category. This approach yields a certain derived category of étale sheaves (respectively, constructible sheaves or D-modules) which should be viewed as a categorification of the affine Hecke algebra $\mathbf{H}$.

On the other hand, as was discovered by Kazhdan and Lusztig (and independently by Ginzburg), the affine Hecke algebra can be realized as the Grothendieck group of equivariant coherent sheaves on the Steinberg variety of the Langlands dual group, thus the corresponding derived category of coherent sheaves provides another categorification of $\mathbf{H}$.

The goal of the present paper is to construct an equivalence between the two triangulated categories which categorify the affine Hecke algebra. A step in this direction has been made in the previous works [1], [12], where a geometric theory of the anti-spherical

(Whittaker) module over $\mathbf{H}$ was developed;[1] in the present paper we extend this analysis to the affine Hecke algebra itself.

The possibility to realize the affine Hecke algebra $\mathbf{H}$ and the "anti-spherical" module over it as Grothendieck groups of (equivariant) coherent sheaves on varieties appearing in the Springer theory for $G^{\vee}$ plays a key role in Kazhdan–Lusztig's proof of classification of irreducible representations of $\overline{\mathbf{H}}$, which constitutes a particular case of local Langlands conjecture, see [35] and exposition in [24].[2] Thus one may hope that the categorification of these realizations proposed here can contribute to the geometric Langlands program. In fact, since the result of the paper was announced, it has been applied and generalized by several authors working in that area, see [28], [10], [19]; see also [13] for a survey of some applications and related results and [33] for a discussion of related dualities in gauge theory. Let us point out that existence of (some variant of) such a categorification was proposed as a conjecture by V. Ginzburg, see Introduction to [24].

**1.2.** *Statement of the result.* — Let us now describe our result in more detail.

**1.2.1.** *Categories of $l$-adic sheaves (the "Galois side").* — Recall the well known group schemes $\mathbf{G_O} \supset \mathbf{I}$ over $k$ (of infinite type) such that $\mathbf{G_O}(k) = G(O)$, $\mathbf{I}(k) = I$, and a group ind-scheme $\mathbf{G_F}$ with $\mathbf{G_F}(k) = G(F)$. We let $\mathbf{I}^0$ be the pro-unipotent radical of $\mathbf{I}$; if $k = \mathbf{F}_{p^a}$ then $I^0 = \mathbf{I}^0(k)$ is the pro-$p$ radical of I. We also have the quotient ind-varieties: the affine Grassmannian $\mathcal{G}\mathfrak{r}$, the affine flag variety $\mathcal{F}\ell = \mathbf{G_F}/\mathbf{I}$ and the extended affine flag variety $\widetilde{\mathcal{F}\ell} = \mathbf{G_F}/\mathbf{I}^0$, see e.g. [29], Appendix, §A.5. Thus $\mathcal{G}\mathfrak{r}$, $\mathcal{F}\ell$, $\widetilde{\mathcal{F}\ell}$ are direct limits of finite dimensional varieties with transition maps being closed embeddings, in the case of $\mathcal{G}\mathfrak{r}$ and $\mathcal{F}\ell$ all the finite dimensional varieties in the direct system are projective. We have $\mathcal{G}\mathfrak{r}(k) = G(F)/G(O)$, $\mathcal{F}\ell(k) = G(F)/I$, $\widetilde{\mathcal{F}\ell}(k) = G(F)/I^0$.

*From now on we assume that the base field $k$ is algebraically closed.*

Let $D(\widetilde{\mathcal{F}\ell})$, $D(\mathcal{F}\ell)$, $D(\mathcal{G}\mathfrak{r})$ be the constructible derived categories of $l$-adic sheaves ($l \neq char(k)$; see [25, §1.1.2], [8, §2.2.14–2.2.18]; and [29, §A.2], for a (straightforward) generalization of the definition of an $l$-adic complex to a certain class of ind-schemes) on the respective spaces.[3]

The protagonists of this paper are as follows. Let $D_{II} = D_I(\mathcal{F}\ell)$ be the $\mathbf{I}$-equivariant derived category of $l$-adic sheaves on $\mathcal{F}\ell$; $D_{I^0I} = D_{I^0}(\mathcal{F}\ell)$ be the $\mathbf{I}^0$-equivariant derived category of $l$-adic sheaves on $\mathcal{F}\ell$, and let $D_{I^0I^0}$ be the full subcategory in the $\mathbf{I}^0$-equivariant derived category of $\widetilde{\mathcal{F}\ell}$ consisting of complexes whose cohomology is

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[1] In *loc. cit.* the group G is assumed to be simple. However, its arguments apply also to the case of a general reductive group G.

[2] In fact, some of the key ideas of this theory already appeared in an earlier work of Lusztig [39], [40] where certain modules over the affine Hecke algebra were realized via K-groups of Springer fibers; also the relation between the $q$-deformation of the K-group and dilation equivariance was described in *loc. cit.*

[3] If $char(k) = 0$ we can also consider the corresponding derived categories of D-modules and if $k = \mathbf{C}$ we can work with the derived category of constructible sheaves in the classical topology. All our results, excepts for some statements in Section 11 which explicitly involve a Frobenius action, hold in these settings, the proofs are identical.

monodromic (or weakly equivariant, see [47]) with respect to the right $T = \mathbf{I}/\mathbf{I}^0$ action with unipotent monodromy.

The categories $D_{II}$ and $D_{I^0I^0}$ are equipped with an associative product operation provided by convolution; $D_{II}$ is unital while $D_{I^0I^0}$ lacks the unit object.[4] We have commuting actions of $D_{I^0I^0}$ and $D_{II}$ on $D_{I^0I}$ by left and right convolution respectively. The convolution operation will be denoted by $*$.

Let $\mathcal{P}_{II} \subset D_{II}$, $\mathcal{P}_{I^0I} \subset D_{I^0I}$, $\mathcal{P}_{I^0I^0} \subset D_{I^0I^0}$ be the subcategories of perverse sheaves. A standard argument (see e.g. [9, Proposition 1.5] for the first equivalence, the second one follows by a similar argument using e.g. [23, Corollary A.4.7]) shows that

$$(\mathbf{1})\qquad \begin{aligned} D^b(\mathcal{P}_{I^0I}) &\cong D_{I^0I}, \\ D^b(\mathcal{P}_{I^0I^0}) &\cong D_{I^0I^0}, \end{aligned}$$

while the natural functor $D^b(\mathcal{P}_{II}) \to D_{II}$ is not an equivalence.

**1.2.2.** *The dual side.* — Let $G^\vee$ be the Langlands dual group over the field $\overline{\mathbf{Q}}_l$. The goal of the paper is to provide a description for the above categories in terms of $G^\vee$. To formulate the answer we need to recall the following construction.

Let $X \to Y$, $X' \to Y$ be morphisms of algebraic varieties. We will assume that $X, X', Y$ are varieties over a field $k$, $Y$ is smooth and morphisms $X \to Y$, $X' \to Y$ are proper.

One can consider the *derived fiber product* $X\overset{L}{\times}_Y X'$ which is a *differential graded scheme* (DG-scheme for short), and the triangulated category $\mathrm{DGCoh}(X\overset{L}{\times}_Y X')$.

If $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{X'}) = 0$ for $i > 0$ then the derived fiber product reduces to the ordinary fiber product and $\mathrm{DGCoh}(X\overset{L}{\times}_Y X') = D^b(\mathrm{Coh}(X \times_Y X'))$.

The triangulated category $\mathrm{DGCoh}(X\overset{L}{\times}_Y X)$ has a natural monoidal structure provided by convolution. The category $D^b(\mathrm{Coh}(X))$ is naturally a module category for the monoidal category $\mathrm{DGCoh}(X\overset{L}{\times}_Y X)$. [For example, when $X$ is a finite set and $Y$ is a point the induced structures on the Grothendieck group amount to matrix multiplication and the action of $n \times n$ matrices on $n$-vectors respectively]. More generally, the category $\mathrm{DGCoh}(X\overset{L}{\times}_Y X')$ has two commuting actions: the action of $\mathrm{DGCoh}(X\overset{L}{\times}_Y X)$ on the left and an action of $\mathrm{DGCoh}(X'\overset{L}{\times}_Y X')$ on the right.

[4] Notice that convolution with an object of $D_{I^0I^0}$ involves direct image under a non-proper morphism, thus convolution could be defined in two different ways, using either direct image or direct image with compact support; we use the version with the ordinary direct image. However, the convolution diagram involved in the definition of convolution in $D_{I^0I^0}$ is a composition of a T bundle and a proper morphism, while the sheaves to which we apply the direct image are T-monodromic. The direct image under projection to the base of a T-monodromic sheaf on a principal T-bundle can be expressed as derived invariants of monodromy, cf. Lemma 44, while direct image with proper support is a Verdier dual operation. Since derived invariants with respect to a symmetric algebra action is a self-dual operation, up to homological shift, we see that the two definitions produce equivalent monoidal categories.

Given an action of an affine algebraic group H on $X, X', Y$ compatible with the maps, one gets equivariant versions of the above statements.

We will apply this in the following situation. We let $Y = \mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$, $X = \tilde{\mathfrak{g}}^\vee = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B}, x \in \mathfrak{b}\}$, $X' = \tilde{\mathcal{N}} = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B}, x \in rad(\mathfrak{b})\}$, where $\mathcal{B}$ is the flag variety for $G^\vee$ parametrizing Borel subalgebras in $\mathfrak{g}^\vee$.

A standard complete intersection argument shows that $\mathrm{Tor}^{\mathcal{O}_Y}_{>0}(\mathcal{O}_X, \mathcal{O}_{X'}) = 0$ for $X = \tilde{\mathfrak{g}}^\vee$, $Y = \mathfrak{g}^\vee$ and $X' = \tilde{\mathfrak{g}}^\vee$ or $X' = \tilde{\mathcal{N}}$, thus the corresponding derived fiber products coincide with the usual fiber product of schemes. However, it fails for $X = X' = \tilde{\mathcal{N}}$, $Y = \mathfrak{g}^\vee$, so the derived fiber product $\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}$ is essentially different from $\tilde{\mathcal{N}} \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}$.

We set $\mathrm{St} = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee$, $\mathrm{St}' = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}$.

**1.2.3.** *Statement of the result.* — We now formulate the main result of the paper.

For an algebraic variety X and a closed subset $Z \subset X$ we will let $\mathrm{Coh}_Z(X)$ denote the full subcategory in $\mathrm{Coh}(X)$ consisting of sheaves set-theoretically supported on Z. For a map $f : X \to Y$ and a closed subset $Z \subset Y$ we will abbreviate $\mathrm{Coh}_{f^{-1}(Z)}(X)$ to $\mathrm{Coh}_Z(X)$.

*Theorem* **1.** — *There exist natural equivalences of categories:*

$$\Phi_{I^0I^0} : D_{I^0I^0} \cong D^b\big(\mathrm{Coh}^{G^\vee}_{\mathcal{N}}(\mathrm{St})\big), \tag{2}$$

$$\Phi_{I^0I} : D_{I^0I} \cong D^b\big(\mathrm{Coh}^{G^\vee}(\mathrm{St}')\big), \tag{3}$$

$$\Phi_{II} : D_{II} \cong \mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}). \tag{4}$$

*Equivalences* (2) *and* (4) *are compatible with the convolution product, while* (3) *is compatible with the action of the categories from* (2) *and* (4).

**1.3.** *The action on the Iwahori-Whittaker category.* — It was pointed out above that the monoidal category of DG coherent (equivariant) sheaves on a fiber product $X \overset{L}{\times}_Y X$ admits a natural action on the derived category of (equivariant) coherent sheaves on X. In particular, monoidal category $\mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$ acts on $D^b(\mathrm{Coh}(\tilde{\mathcal{N}}))$, while $D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}))$ acts on $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$.

To describe the corresponding structures on the loop group side, recall the category of *Iwahori-Whittaker* sheaves. The quotient of $\mathbf{I}^0$ by its commutant is the sum of copies of the additive group indexed by vertices of the affine Dynkin graph. Fix an additive character $\psi$ of $\mathbf{I}^0$ which is trivial on the summand of $\mathbf{I}^0/(\mathbf{I}^0)'$ corresponding to the affine root(s) and is non zero on the other summands. We denote by $D^{\mathrm{I}}_{\mathrm{IW}}$ the $\mathbf{I}$-equivariant derived category of $l$-adic sheaves on the principal homogeneous space $\mathbf{G_F}/(\mathbf{I}^0)'$ which satisfies the $\psi$-equivariance condition with respect to the right action of $\mathbf{I}^0/(\mathbf{I}^0)'$, see [1]. (Conventions here differ from those of [1] by switching the roles of left and right multiplication.) We let $D^{\mathrm{I}^0}_{\mathrm{IW}}$ denote the category of $\mathbf{I}$ monodromic sheaves with unipotent monodromy on $\mathbf{G_F}/(\mathbf{I}^0)'$ which are $\psi$-equivariant with respect to the right action of $\mathbf{I}^0/(\mathbf{I}^0)'$, this is a particular case of the category considered in [23] (again, one needs to switch left with right to get from the present setting to that of [23]).

The categories $D_{I^0I^0}$, $D_{II}$ act on $D_{IW}^{I^0}$, $D_{IW}^{I}$ respectively by convolution.

*Theorem* **2.** — *There exist equivalences of categories*

$$\Phi_{IW}^{I} : D^b\big(\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})\big) \xrightarrow{\sim} D_{IW}^{I}, \tag{5}$$

$$\Phi_{IW}^{I^0} : D^b\big(\mathrm{Coh}_{\mathcal{N}}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)\big) \xrightarrow{\sim} D_{IW}^{I^0}, \tag{6}$$

*satisfying the following compatibilities: the equivalence* $\Phi_{IW}^{I^0}$ *is compatible with the action of* $D^b\mathrm{Coh}_{\mathcal{N}}^{G^\vee}(\mathrm{St})$ *coming from the action of* $D_{I^0I^0}$ *on* $D_{IW}^{I^0}$ *and equivalence* (2).

*The equivalence* $\Phi_{IW}^{I}$ *is compatible with the action of* $\mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}}\overset{L}{\times}_{\mathfrak{g}^\vee}\tilde{\mathcal{N}})$ *coming from the action of* $D_{II}$ *on* $D_{IW}^{I}$ *and equivalence* (4).

Another useful compatibility between the equivalences in Theorems 1 and 2 is stated at the end of Section 10.

A variant of equivalence (5) has been established in [1], and (6) can obtained by a similar argument, see below. More precisely, in [1] a functor $F : D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})) \to D_{II}$ is constructed, below we construct its "monodromic" counterpart $\Phi_{diag} : D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}}^\vee)) \to \hat{D}$, where $\hat{D}$ is a certain "pro"-completion of $D_{I^0I^0}$ and $\widehat{\tilde{\mathfrak{g}}}^\vee$ (a version of) the formal neighborhood of $\tilde{\mathcal{N}}$ in $\tilde{\mathfrak{g}}^\vee$ (see Section 2.1 for a precise definition). One can consider the composition of F with either left or right Whittaker averaging, both compositions turn out to be equivalences, the proofs of these two facts are parallel. In [1] we worked with left Whittaker averaging, while here we work with the right one (this allows us to work with modules over the monoidal category $D_{II}$ rather than modules over its opposite).

**1.4.** *Description of the strategy: the Hecke algebra perspective.* — Some of the constructions exploited here are sheaf-theoretic analogs of known results in the theory of affine Hecke algebras.

Recall that $\mathbf{H}$ has a standard basis $t_w$ indexed by elements $w$ in the extended affine Weyl group W.

Let $\Lambda$ be the coweight lattice of G and $\Lambda^+ \subset \Lambda$ be the set of dominant weights. There exists a unique collection of elements $\theta_\lambda \in \mathbf{H}$, $\lambda \in \Lambda$, such that $\theta_\lambda\theta_\mu = \theta_{\lambda+\mu}$ for all $\lambda, \mu \in \Lambda$ and $\theta_\lambda = T_\lambda$ for $\lambda \in \Lambda^+$. The categorification of the elements $\theta_\lambda$ are the so-called *Wakimoto sheaves*, see [1] and Section 3.3 below.

The elements $\theta_\lambda$ span a commutative subalgebra $A \subset \mathbf{H}$ which contains the center $Z(\mathbf{H})$ of the affine Hecke algebra. Categorification of the center is provided by the work of Gaitsgory [29]. Categorification of the formula expressing central elements as linear combinations of $\theta_\lambda$ is the fact that central sheaves of [29] admit a filtration whose associated graded is a sum of Wakimoto sheaves, see [1] and Section 3.5 below. This filtration plays a key role in our construction, see [1] and Section 4.2, yielding a categorification of:

$$A \cong K^0\big(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)\big) \xrightarrow{\delta_*} K^0\big(\mathrm{Coh}^{G^\vee}(\mathrm{St})\big), \qquad (\mathbf{7})$$

where $\delta : \tilde{\mathfrak{g}}^\vee \to \mathrm{St}$ is the diagonal embedding.

Another ingredient important to us is the *q-analog of the Schur anti-symmetrizer, or anti-spherical projector* $\xi = \sum_{w \in W_f} (-1)^{\ell(w)} t_w$. Its relevance to representation theory of $p$-adic groups comes from the fact that the left ideal $\overline{\mathbf{H}}\xi$ is canonically isomorphic to I invariants in the space of *Whittaker functions* on G(F), while its relation to canonical basis in the affine Hecke algebra, thus to perverse sheaves on $\mathcal{F}\ell$ goes back to [38].

The categorical counterpart of $\xi$ is the maximal projective object in the category of perverse sheaves on $G/B \cong \mathbf{G_O}/\mathbf{I}$ equivariant with respect to $\mathbf{I}^0$, it is discussed in Section 5. Under the equivalence with the coherent sheaves category that object corresponds to the structure sheaf of Steinberg variety.

Let $\mathbf{H}_{perf} \subset \mathbf{H}$ be the two-sided ideal generated by $\xi$. The full subcategory $D^{G^\vee}_{perf}(\mathrm{St}) \subset D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}))$ of perfect complexes can be considered as a categorification of $\mathbf{H}_{perf}$. Furthermore, it is easy to see that $\mathbf{H}_{perf}$ is freely generated by $\xi$ as a module over $A \otimes_Z A$. This allows one to deduce an equivalence between the two categorifications of $\mathbf{H}_{perf}$ from the categorification of (7). The subcategory $D^{G^\vee}_{perf}(\mathrm{St})$ is dense in $D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}))$ in an appropriate sense, which allows to extend the equivalence from the subcategory to the whole category.

In this text we follow the original plan conceived more than a decade ago and treat the issues of homological algebra by ad hoc methods, using explicit DG models for triangulated categories of constructible sheaves based on generalized tilting sheaves. While the properties of tilting sheaves established in the course of the argument are (in the author's opinion) of an independent interest, it is likely that recent advances in homotopy algebra can be used to develop an alternative approach.

## 2. Outline of the argument

**2.1.** *Further notations and conventions.* — We let $B \supset N$ be a Borel subgroup in G and its unipotent radical, and $N^\vee \subset B^\vee$ be similar subgroups in $G^\vee$; we assume that B is the image of $\mathbf{I}$ under the evaluation map $\mathbf{G_O} \to G$.

Let $\Lambda$ be the coweight lattice of G, i.e. the coweight lattice of the abstract Cartan group of G; thus $\Lambda$ is identified with the weight lattice of (the abstract Cartan group of) $G^\vee$. We let $W_f$ denote the Weyl group and $W = W_f \ltimes \Lambda$ the extended affine Weyl group; $\ell : W \to \mathbf{Z}_{\geq 0}$ is the length function, and $\Lambda^+ \subset \Lambda$ is the set of dominant coweights, $w_0 \in W_f$ is the longest element.

We let $W^f \subset W$ be the subset of minimal length representatives of right cosets $W/W_f$. Notice that $\Lambda^+ \subset W^f$.

We let $\mathcal{B} = \mathrm{G}^\vee/\mathrm{B}^\vee$ be the flag variety. The set of isomorphism classes of $\mathrm{G}^\vee$-equivariant line bundles on $\mathcal{B}$ is identified with $\Lambda$; for $\lambda \in \Lambda$ we let $\mathcal{O}_{\mathcal{B}}(\lambda)$ be the corresponding line bundle. Recall that $\mathcal{O}(\lambda)$ is semi-ample iff $\lambda \in \Lambda^+$. For $\lambda \in \Lambda^+$ we let $\mathrm{V}_\lambda$ denote the corresponding irreducible G-module, thus $\mathrm{V}_\lambda = \Gamma(\mathcal{B}, \mathcal{O}(\lambda))$.

The ind-schemes $\mathcal{F}\ell = \mathbf{G_F}/\mathbf{I}$, $\widetilde{\mathcal{F}\ell} = \mathbf{G_F}/\mathbf{I}^0$ and the categories $\mathrm{D_{II}} \supset \mathcal{P}_{\mathrm{II}}$, $\mathrm{D_{I^0I}} \supset \mathcal{P}_{\mathrm{I^0I^0}}$, $\mathrm{D_{I^0I^0}} \supset \mathcal{P}_{\mathrm{I^0I}}$ were introduced above. We abbreviate $\mathcal{P} = \mathcal{P}_{\mathrm{I^0I}}$ and let $\hat{\mathcal{P}}$, $\hat{\mathrm{D}}$ be the pro-completions of $\mathrm{D_{I^0I^0}}$ and $\mathcal{P}_{\mathrm{I^0I^0}}$ respectively, see Section 3.

Let $\pi : \widetilde{\mathcal{F}\ell} \to \mathcal{F}\ell$ be the projection.

The $\mathbf{I}$ orbits on $\mathcal{F}\ell$ are indexed by W, for $w \in \mathrm{W}$ we let $j_w : \mathcal{F}\ell_w \to \mathcal{F}\ell$ be the embedding of the corresponding orbit. We have $\dim(\mathcal{F}\ell_w) = \ell(w)$.

We have standard objects $j_{w!} := j_{w!}(\underline{\overline{\mathbf{Q}}_\ell}[\ell(w)])$ and costandard object $j_{w*} = j_{w*}(\underline{\overline{\mathbf{Q}}_\ell}[\ell(w)])$ in $\mathcal{P}$. Their counterparts in $\hat{\mathrm{D}}$ are the free monodromic (co)standard objects $\nabla_w$, $\Delta_w$, see Sections 3.1, 3.2.

We also consider the Iwahori-Whittaker categories $\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}} \supset \mathcal{P}^{\mathrm{I}}_{\mathrm{IW}}$, $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}} \supset \mathcal{P}^{\mathrm{I}^0}_{\mathrm{IW}}$, the pro-completions $\hat{\mathrm{D}}_{\mathrm{IW}}$, $\hat{\mathcal{P}}_{\mathrm{IW}}$ of, respectively, $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$, $\mathcal{P}^{\mathrm{I}^0}_{\mathrm{IW}}$, (co)standard objects $j^{\mathrm{IW}}_{w!}, j^{\mathrm{IW}}_{w*} \in \mathcal{P}^{\mathrm{I}}_{\mathrm{IW}}$ and free monodromic (co)standard object $\Delta^{\mathrm{IW}}_w$, $\nabla^{\mathrm{IW}}_w \in \hat{\mathcal{P}}_{\mathrm{IW}}$, $w \in \mathrm{W}^f$ (see Section 3 for further details).

Recall that $\mathrm{St} = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee$, let $p_{\mathrm{Spr},1} : \mathrm{St} \to \tilde{\mathfrak{g}}^\vee$, $p_{\mathrm{Spr},2} : \mathrm{St} \to \tilde{\mathfrak{g}}^\vee$ be the two projections. Also $\mathrm{St}' = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}$ with two projections $p'_{\mathrm{Spr},1} : \mathrm{St}' \to \tilde{\mathfrak{g}}^\vee$, $p'_{\mathrm{Spr},2} : \mathrm{St}' \to \tilde{\mathcal{N}}$. Let $\widehat{\tilde{\mathfrak{g}}^\vee} = \tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \widehat{\mathfrak{g}^\vee}$, $\widehat{\mathrm{St}} = \mathrm{St} \times_{\mathfrak{g}^\vee} \widehat{\mathfrak{g}^\vee}$, where $\widehat{\mathfrak{g}^\vee}$ is the spectrum[5] of the completion of the ring of functions $\mathcal{O}(\mathfrak{g}^\vee)$ at the ideal of the point 0.

For an algebraic group H acting on an (ind)-scheme X we let $\mathrm{D_H(X)}$ denote the equivariant derived category of H-equivariant constructible sheaves on X and let $\mathrm{Perv_H(X)} \subset \mathrm{D_H(X)}$ be the subcategory of perverse sheaves. Given a subgroup $\mathrm{K} \subset \mathrm{H}$ we have the functor of restricting the equivariance $\mathrm{Res}^{\mathrm{H}}_{\mathrm{K}} : \mathrm{D_H(X)} \to \mathrm{D_K(H)}$ and the left adjoint functor $\mathrm{Av}^{\mathrm{H}}_{\mathrm{K}} : \mathrm{D_K(X)} \to \mathrm{D_H(X)}$ (the latter can be thought of as the !-direct image for the morphism of stacks $\mathrm{X/K} \to \mathrm{X/H}$). In particular, we have a functor $\mathrm{Av}^{\mathbf{I}}_{\mathbf{I}^0} : \mathrm{D_{I^0I}} \to \mathrm{D_{II}}$ (to unburden typography we will write $\mathrm{Av}^{\mathrm{I}}_{\mathrm{I}^0}$).

In order to introduce a similar functor involving Iwahori-Whittaker sheaves we fix an Iwahori subgroup $\mathbf{I}_- \subset \mathbf{G_O}$ which is opposite to (in general position with) the subgroup $\mathbf{I}$. We also fix a nondegenerate additive character $\psi_-$ of $\mathbf{I}^0_-$. The pair $(\mathbf{I}^0_-, \psi_-)$ is conjugate to $(\mathbf{I}^0, \psi)$ by an element in G(F) which is unique up to right multiplication by an element in $\mathrm{I}^0$. Thus the categories $\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}}$, $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$ are canonically equivalent to the derived categories $\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}_-}$, $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}_-}$ of right $(\mathbf{I}_-, \psi_-)$-equivariant sheaves. We define

[5] Alternatively we could work with completion defined as a formal scheme, the resulting category of coherent sheaves would be equivalent. In more detail, by [34, Théoreme 5.4.1] the scheme $\widehat{\tilde{\mathfrak{g}}^\vee}$, $\widehat{\mathrm{St}}$ is the inductive limit in the category of schemes over $\mathfrak{g}^\vee$ of nilpotent thickenings of $\tilde{\mathcal{N}}$ in $\tilde{\mathfrak{g}}^\vee$ (respectively, $\mathrm{St}'$ in St). By [34, Théoreme 5.1.4(1)] the category of coherent sheaves on $\widehat{\tilde{\mathfrak{g}}^\vee}$ is equivalent to the category of coherent sheaves on the formal scheme completion of $\tilde{\mathcal{N}}$ in $\tilde{\mathfrak{g}}^\vee$ and similarly for $\widehat{\mathrm{St}}$, this readily extends to the category of $\mathrm{G}^\vee$-equivariant sheaves; cf. the discussion at the end of Introduction to [20].

the functors $\mathrm{Av}^{\mathrm{I}^0}_{\mathrm{IW}} : \mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}_-} \cong \mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}} \to \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$, $\mathrm{Av}^{\mathrm{IW}} : \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0} \to \mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$ by setting $\mathrm{Av}^{\mathrm{I}^0}_{\mathrm{IW}} = \mathrm{Av}^{\mathbf{I}^0}_{\mathbf{I}^0\cap\mathbf{I}^0_-}$, $\mathrm{Av}^{\mathrm{IW}} = \mathrm{Av}^{\mathbf{I}^0_-,\psi_-}_{\mathbf{I}^0\cap\mathbf{I}^0_-}$. Here we used that $\psi|_{\mathbf{I}^0\cap\mathbf{I}^0_-}$ is trivial; the restriction of equivariance functor is omitted from notation, and $\mathrm{Av}^{\mathbf{I}^0_-,\psi_-}_{\mathbf{I}^0\cap\mathbf{I}^0_-}$ is the left adjoint to the restriction of equivariance functor from $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}_-}$ to the corresponding $\mathbf{I}^0\cap\mathbf{I}^0_-$-equivariant category. The result of [16] implies that we get the same functor $\mathrm{Av}^{\mathrm{IW}}$ if we replace $\mathrm{Av}^{\mathbf{I}^0_-,\psi_-}_{\mathbf{I}^0\cap\mathbf{I}^0_-}$ by the corresponding right adjoint to the restriction of equivariance functor: the Whittaker averaging functor is clean. We also have a similarly defined functor on I-equivariant categories: ${}^{\mathrm{I}}\mathrm{Av}_{\mathrm{IW}} : \mathrm{D}_{\mathrm{II}^0} \to \mathrm{D}^{\mathrm{I}}_{\mathrm{IW}}$.

Notice that the definition of $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$ involves the left action of $\mathrm{I}^0$, while $\mathrm{Av}^{\mathrm{I}^0}_{\mathrm{IW}}$, $\mathrm{Av}^{\mathrm{IW}}$ have to do with the right action; when the action used may not be clear from the context we use notation $\mathrm{Av}^{left}_{\mathrm{I}^0}$, $\mathrm{Av}^{right}_{\mathrm{I}^0}$ to distinguish between the two.

**2.2.** *Idea of the argument: structural aspects.* — The functor from the coherent category to the constructible one stems from certain natural structures on the constructible category. To describe the mechanism of obtaining such a functor from the additional structures on the target category it is convenient to use the concept of a triangulated category $\mathcal{C}$ *over* a stack X.

**2.2.1.** *Linear structure over a stack.* — We refer to [31] for the notion of an abelian category over an algebraic stack, and to [32] for a generalization to triangulated (or rather homotopy theoretic) context. For our present purposes it suffices to use the following simplified version of this concept. Let S be an algebraic stack and $\mathcal{C}$ a triangulated category (in all our example $\mathrm{S} = \mathrm{X}/\mathrm{G}$ where X is a quasi-projective algebraic variety and G is a reductive algebraic group). The subcategory of perfect complexes $\mathrm{D}_{perf}(\mathrm{S}) \subset \mathrm{D}^b(\mathrm{Coh}(\mathrm{S}))$ is a triangulated tensor category under the usual tensor product of coherent sheaves. By an S-*linear structure* on $\mathcal{C}$ we will mean an action of the tensor category $\mathrm{D}_{perf}(\mathrm{S})$ on $\mathcal{C}$ compatible with the triangulated structure.

We now list basic classes of examples of such a structure to be used below.

(1) If $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$ is an affine scheme, then for an R-linear abelian category $\mathcal{A}$ the triangulated category $\mathrm{D}^b(\mathcal{A})$ acquires a natural S-linear structure.
(2) Let $\mathrm{S} = pt/\mathrm{H}$ where H is a linear algebraic group. If an abelian category $\mathcal{A}$ is a module category for the tensor category Rep(H) of algebraic (finite dimensional) representations acting by exact functors, then $\mathrm{D}^b(\mathcal{A})$ is an S-linear triangulated category.
(3) Combining the first two examples, assume now that $\mathrm{S} = \mathrm{Spec}(\mathrm{R})/\mathrm{H}$ is a quotient of an affine scheme by a linear algebraic group action. Let $\mathcal{A}$ be an abelian category which is a module category for Rep(H) acting on $\mathcal{A}$ by exact functors. Then we can define a new (in general not abelian) "*deequivariantized*" category $\mathcal{A}_{deeq}$ by setting $\mathrm{Ob}(\mathcal{A}_{deeq}) = \mathrm{Ob}(\mathcal{A})$, $\mathrm{Hom}_{deeq}(\mathrm{X}, \mathrm{Y}) =$

$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(X, \underline{\mathcal{O}}_H(Y))$ where $\mathrm{Ind}(\mathcal{A})$ stands for the category of Ind-objects in $\mathcal{A}$ and $\underline{\mathcal{O}}_H \in \mathrm{Ind}(\mathrm{Rep}(H))$ denotes the space of regular functions on H with H acting by left translations, see Section 4.2.1 for further details.

Then $\mathcal{A}_{deeq}$ is a category enriched over the category of algebraic (not necessarily finite dimensional) representations of H. Then an R-linear structure on $\mathcal{A}_{deeq}$ which is compatible with the H-action induces an S-linear structure on $D^b(\mathcal{A})$. To see this observe that $\mathrm{Hom}_{\mathrm{Coh}^H(\mathrm{Spec}(R))}(V \otimes \mathcal{O}, V' \otimes \mathcal{O}) = (V' \otimes V^* \otimes R)^H$, thus an equivariant R-linear structure on $\mathcal{A}_{deeq}$ induces an action of the tensor category $\mathrm{Coh}_{fr}(S)$ on $\mathcal{A}$ by exact functors; here $\mathrm{Coh}_{fr}(S) \subset \mathrm{Coh}(S) = \mathrm{Coh}^H(\mathrm{Spec}(R))$ is the full subcategory consisting of objects $V \otimes \mathcal{O}_{\mathrm{Spec}(R)}$, $V \in \mathrm{Rep}(H)$. Since $D_{perf}(S)$ is the Karoubi (idempotent) completion of the homotopy category of finite complexes $\mathrm{Ho}(\mathrm{Coh}_{fr}(S))$, the action of $\mathrm{Coh}_{fr}(S)$ on $\mathcal{A}$ induces an S-linear structure on $D^b(\mathcal{A})$ (notice that $D^b(\mathcal{A})$ is necessarily Karoubian).

(4) Suppose we are given an open embedding of algebraic stacks $S \hookrightarrow S'$ and a category $\mathcal{C}$ with an $S'$-linear structure; assume for simplicity that $S'$ is a quotient of a quasi-projective variety over a field of characteristic zero by an action of a reductive group and S comes from an invariant open subvariety therein. By results of [46] (see also [43, §2.1.4, esp. proof of Lemma 2.6][6]) we have

$$D_{perf}(S) \cong \mathrm{Idem}\big(D_{perf}(S')/D_{perf}(S')_{\partial S'}\big) \tag{8}$$

where Idem denotes the Karoubi (idempotent) completion and $D_{perf}(S')_{\partial S'}$ is the full subcategory of perfect complexes on $S'$ whose restriction to S vanishes. Thus if $\mathcal{C}$ is a Karoubian (idempotent complete) category, then an $S'$-linear structure on $\mathcal{C}$ such that $D_{perf}(S')_{\partial S'}$ acts by zero induces an S-linear structure on $\mathcal{C}$.

(5) One can use a variant of Serre's description of the category of coherent sheaves on a projective variety as a quotient of the category of graded modules over the homogeneous coordinate ring to devise a procedure for constructing an S-linear structure for more general stacks S.

Suppose that $S = X/H$ where X is a quasi-projective variety with an action of an affine algebraic group H. Assume that a linearization of the action, i.e. a linear action of H on the linear space $\mathbf{A}^{n+1}$ together with an equivariant locally closed embedding $X \to \mathbf{P}^n$ is fixed. Let $C \subset \mathbf{A}^{n+1}$ be the cone over the closure $\overline{X}$ of X in $\mathbf{P}^n$. Then C is an affine variety acted upon by $H \times \mathbf{G}_m$ and we have an open embedding $S \to S' = C/(H \times \mathbf{G}_m)$. Using (8) we see that an S-linear structure on $\mathcal{C} = D^b(\mathcal{A})$ can be constructed by providing $\mathcal{A}$ with a $\mathrm{Rep}(H \times \mathbf{G}_m)$ action by exact functors, introducing an R-linear structure on

[6] The result is only claimed in *loc. cit.* for a subscheme in a scheme but the case of stacks of the described type follows by the same argument.

$\mathcal{A}_{deeq}$ where $R = \mathcal{O}(C)$ is the homogeneous coordinate ring of the projective variety $\overline{X}$, and verifying that the resulting $S'$-linear structure sends $D_{perf}(S')_{\partial S'}$ to zero.

*Remark* **3.** — Most of the statements in the main Theorem of the paper assert an equivalence between (a subcategory of) $D^b(\mathrm{Coh}(S))$ for an algebraic stack S and $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$ (with the exception of (4) which involves coherent sheaves on a DG-stack and an equivariant derived category of constructible sheaves).

We first construct the S-linear structure on $\mathcal{C} = D^b(\mathcal{A})$ and then consider the action on a particular object of $\mathcal{C}$ to get an equivalence. The construction of the action almost follows the pattern of example (5). The difference is as follows. We have $S = X/G^\vee$ where X admits an affine equivariant map to $\mathcal{B}^2$. Though $\mathcal{B}^2$ is a projective variety there is no preferred choice of an equivariant projective embedding, so to keep things more canonical we work with the "multi-homogeneous" coordinate ring and consider open embeddings of our stacks into $Y/(G^\vee \times T^{\vee 2})$ for an appropriate affine variety Y. A more essential difference is that while $\mathrm{Rep}(G^\vee)$ acts by exact functors on our abelian category $\mathcal{A}$, the action of $\mathrm{Rep}(T^{\vee 2})$ is only defined on the triangulated category $\mathcal{C}$, it is not compatible with the natural *t*-structure on $\mathcal{C} = D^b(\mathcal{A})$.

An additional argument based on properties of tilting modules is needed to deal with this issue (see Section 4.4.2).

**2.2.2.** *The list of structures.* — We concentrate on the equivalence (2), the equivalence (3) is similar, and (4) will be deduced formally from (2).

Consider the following sequence of maps

$$St/G^\vee \rightrightarrows \tilde{\mathfrak{g}}^\vee/G^\vee \to \mathfrak{g}^\vee/G^\vee \to pt/G^\vee.$$

Moving from right to left in this sequence, we successively equip $\hat{D}$ with the linear structure for the corresponding stack.

The $pt/G^\vee$-linear structure comes from an action of the tensor category $\mathrm{Rep}(G^\vee)$ on the abelian category $\mathcal{P}_{I^0 I}$. Such an action was defined in [29] where the *central sheaves* categorifying the canonical basis in the center of the affine Hecke algebra were constructed; an extension of the action to $\mathcal{P}_{I^0 I^0}$ is sketched in Section 3.5 below.

By a version of the Tannakian formalism, lifting an action of the tensor category $\mathrm{Rep}(G^\vee)$ to a $\mathfrak{g}^\vee/G^\vee$-linear structure amounts to equipping the $\mathrm{Rep}(G^\vee)$ action with a tensor endomorphism. Such an endomorphism comes from the *logarithm of monodromy* acting on central sheaves: recall that the central sheaves are constructed by nearby cycles which carry a monodromy automorphism.

We now discuss the two structures of a stack over $\tilde{\mathfrak{g}}^\vee/G^\vee$. The starting point here is the familiar observation that for a representation V of $G^\vee$ the trivial vector bundle $V \otimes \mathcal{O}_{\mathcal{B}}$ with fiber V on the flag variety $\mathcal{B} = G^\vee/B^\vee$ carries a canonical filtration whose associated graded is a sum of line bundles. This filtration can be lifted to a similar filtration for $V \otimes \mathcal{O}_{\tilde{\mathfrak{g}}^\vee}$. Under our equivalences this filtration corresponds to a filtration on (monodromic) central sheaves by (monodromic) *Wakimoto sheaves* (the non-monodromic version was presented in [1], and the monodromic generalization is presented below in Section 3.3). It turns out that the filtration defines a monoidal functor $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)) \to \hat{D}$. We then get two commuting actions of $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$ on $D_{I^0I^0}$ from the left and the right action of the monoidal category $\hat{D}$ on itself; combining the two actions we see that $\hat{D}$ is naturally a category over $\tilde{\mathfrak{g}}^{\vee 2}/G^\vee$. Since $\mathrm{Rep}(G^\vee)$ acts by *central* functors and the tensor endomorphism is compatible with the central structure, we conclude that the $\tilde{\mathfrak{g}}^{\vee 2}/G^\vee$ linear structure factors through the one for the fiber square $(\tilde{\mathfrak{g}}^\vee \times_{\mathfrak{g}^\vee} \tilde{\mathfrak{g}}^\vee)/G^\vee = \mathrm{St}/G^\vee$.

More precisely, we get the monoidal functor $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)) \to \hat{D}$ from the filtration following a strategy similar to the one in Example (5) above. The first term of the filtration (the "lowest weight arrow") determines a functor from $D^{G^\vee\times T^\vee}_{perf}(C)$ where $C$ is a certain affine scheme with an action of $G^\vee \times T^\vee$ with an open $G^\vee \times T^\vee$-equivariant embedding $G^\vee/U^\vee \to C$. The fact that the lowest weight arrow extends to a filtration satisfying certain properties implies that complexes supported on $\partial(G^\vee/U^\vee) = C \setminus (G^\vee/U^\vee)$ act by zero. These ideas have already been used in [1].

The fact that the action of the log monodromy endomorphisms on the category $D_{I^0I^0}$ of monodromic sheaves is *nilpotent*, allows us to show that the $\mathrm{St}/G^\vee$-linear structure on $D_{I^0I^0}$, $\hat{D}$ factors through a canonical $\widehat{\mathrm{St}}/G^\vee$-linear structure, where $\widehat{\mathrm{St}}$ is formal completion of St at the preimage of $\mathcal{N}$.

Once the $\widehat{\mathrm{St}}/G^\vee$-linear structure on $\hat{D}$ is constructed, any object $M \in \hat{D}$ defines a functor $D^{G^\vee}_{perf}(\widehat{\mathrm{St}}) \to \hat{D}$, $\mathcal{F} \mapsto \mathcal{F}(M)$. We use the functor (denoted by $\Phi_{perf}$) corresponding to the choice $M = \hat{\Xi}$ where $\hat{\Xi}$ is a certain tilting pro-object discussed in Section 5. This choice can be motivated by the requirement of compatibility with the equivalence $\Phi^{I^0}_{IW}$: the object $\hat{\Xi}$ is obtained from the unit object in $\hat{D}$ by projection to $\hat{D}_{IW}$ composed with its adjoint, on the dual side this corresponds to the sheaf $pr^*_{\mathrm{Spr},2} pr_{\mathrm{Spr},2*}(\delta_*(\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee})) \cong \mathcal{O}_{\widehat{\mathrm{St}}}$, where $\delta : \tilde{\mathfrak{g}}^\vee \to \mathrm{St}$ is the diagonal embedding. Thus the compatibility implies that $\Phi_{perf}(\mathcal{O}) \cong \hat{\Xi}$. The object $\hat{\Xi}$ can also be thought of as a categorification of the element $\xi$ in the affine Hecke algebra, thus it is closely related to Whittaker model, see Section 1.4.

The fact that $\Phi_{perf}$ constructed this way is compatible with projection to $D^I_{IW}$ follows from the properties of $\hat{\Xi}$.

We then establish the equivalence $\Phi^{I^0}_{IW}$ as in [1]. Together with compatibilities between $\Phi_{perf}$ and $\Phi^{I^0}_{IW}$ this implies that $\Phi_{perf}$ is a full embedding.

Once $\Phi_{perf}$ is constructed and shown to be full, we get functors in the opposite direction $\widehat{\Psi} : \hat{D} \to D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}}))$, $\Psi : D_{I^0I^0} \to D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}))$. (The logic here is reminiscent of arguments in functional analysis where a map between spaces of (smooth rapidly decreasing) test functions induces a map between spaces of generalized functions going in the opposite direction.) We show existence of $\Psi$, $\widehat{\Psi}$ and check that they are equivalences based on a general result relating the categories $D^b(\mathrm{Coh}(X))$ and $D_{perf}(X)$ for an algebraic stack X. We show that $D^b(\mathrm{Coh}(X))$ embeds into the category of functors

$D_{perf}(X) \to \mathrm{Vect}$ and characterize the image of this embedding. The characterization makes use of the standard $t$-structure on the derived category of coherent sheaves. In order to apply the general criterion in our situation we show that, although the functor $\Phi_{perf}$ is not $t$-exact with respect to the natural $t$-structures on the two triangulated categories, it satisfies a weaker compatibility (see Section 8).

At this point the equivalence (2) is constructed, it remains to check its compatibility with the convolution monoidal structure. We use presentation of $\hat{D}$ as homotopy category of complexes of free-monodromic tilting (pro)sheaves introduced in [23] and recalled below. Using the observation that convolution of two free monodromic tilting sheaves is also a free monodromic tilting sheaf we get an explicit monoidal structure on the category of tilting complexes, which is identified with the monoidal structure on $\hat{D}$. It turns out that $\widehat{\Psi}$ sends a free monodromic tilting sheaf to a coherent sheaf (rather than a complex). Thus the monoidal structure on the equivalence $\Phi_{I^0I^0}$ follows from compatibility with the action on $\hat{D}_{IW}$, since a sheaf in $D^b(\mathrm{Coh}^{G^\vee}(\widehat{St}))$ can be uniquely reconstructed from the endo-functor of $D^b(\mathrm{Coh}^{G^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))$ given by convolution with $\mathcal{F}$.

**2.3.** *Description of the content.* — Sections 3 and 5 mostly recall the results of [23] while Section 4 recalls the material of [1] and extends it to the present slightly more general setting.

As was indicated above, it is technically convenient to enlarge both categories in (2) and construct the equivalence

$$\hat{D} \cong D^b\big(\mathrm{Coh}^{G^\vee}(\widehat{St})\big). \tag{9}$$

In Section 3 we recall the definition of $\hat{D}$ and an extension of the formalism of tilting sheaves to this setting. We also present a "monodromic" generalization of central sheaves [29].

Section 4 provides a generalization of the main result of [1] to the monodromic setting. Namely, it establishes a monoidal functor $\Phi_{diag}$ from the derived category of equivariant coherent sheaves on the formal completion $\widehat{\tilde{\mathfrak{g}}^\vee}$ of $\tilde{\mathfrak{g}}^\vee$ at $\tilde{\mathcal{N}}$ to $\hat{D}$. (The composition of this functor with the equivalence (9) which will be established later is the direct image under the diagonal embedding $\tilde{\mathfrak{g}}^\vee \to St$, see Lemma 43(b).) A variation of the argument allows us to define the action of the tensor category $(D^{G^\vee}_{perf}(St), \otimes_{\mathcal{O}})$ on $D_{I^0I^0}$ and $\hat{D}$.

We also consider the projection of $\hat{D}$ to the Iwahori-Whittaker category $\hat{D}_{IW}$ and show that the composition of $\Phi_{diag}$ with this projection induces an equivalence $\Phi^{I^0}_{IW}$ : $D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \xrightarrow{\sim} \hat{D}_{IW}$.

Section 5 is devoted to a particular object $\hat{\Xi} \in \widehat{\mathcal{P}}$ which will correspond to the structure sheaf of $\widehat{St}$ under the equivalence.

In Section 6 we define a functor $\Phi_{perf}$ from the subcategory of perfect complexes $D^{G^\vee}_{perf}(\widehat{St}) \subset D^b(\mathrm{Coh}^{G^\vee}(\widehat{St}))$ to $\hat{D}$ by sending an object in the tensor category $(D^{G^\vee}_{perf}(\widehat{St}), \otimes_{\mathcal{O}})$ to the result of its action on $\hat{\Xi}$.

We then make a step towards establishing monoidal structure on our functors: the functor $\Phi_{perf}$ allows to define an action of $D^{G^\vee}_{perf}(\widehat{St})$ on $D^{I^0}_{IW}$, while the category $D^b(\mathrm{Coh}^{G^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))$ also carries such an action; we use properties of $\hat{\Xi}$ to show that $\Phi^{I^0}_{IW}$ is compatible with these module structures. Here (in contrast with the previous paragraph) $D^{G^\vee}_{perf}(\widehat{St})$ is equipped with the convolution product (notice that the subcategory $D^{G^\vee}_{perf}(\widehat{St}) \subset D^b(\mathrm{Coh}^{G^\vee}(\widehat{St}))$ is easily seen to be closed under convolution).

This compatibility allows us to deduce that $\Phi_{perf}$ is a full embedding and endow it with the structure of a monoidal functor.

In Section 8 we check a property of $\Phi_{perf}$ with respect to the natural $t$-structures on the two categories. In Section 7 we give a general criterion allowing to extend an equivalence from the category of perfect complexes to the bounded derived category of coherent sheaves.

In Section 9 we show that the criterion of Section 7 applies, by virtue of properties established in Section 8, to the present situation yielding (9). We then deduce (3) and (4) by means of a general lemma describing the equivariant constructible category via the monodromic one. Section 10 deals with technicalities on DG-models for convolution monoidal categories of sheaves needed to equip our equivalences with a monoidal structure. The final Section 11 describes additional properties of our functors in relation to the Frobenius automorphism (where $k = \bar{\mathbf{F}}_q$) and $t$-structures, as well as conjectural generalizations and relation to Hodge D-modules.

## 3. Monodromic sheaves and pro-object

**3.1.** *Generalities on monodromic sheaves.* — Objects of $\mathcal{P}_{I^0I^0}$ are by definition perverse sheaves monodromic with respect to both the left and the right action of T on $\widetilde{\mathcal{F}\ell}$. Thus we get two actions of the group $\Lambda \times \Lambda$ by automorphisms of the identity functor of $\mathcal{P}_{I^0I^0}$ coming respectively from the left and the right monodromy. Both actions on each object are unipotent.

Let $\hat{\mathcal{P}}$ be the category of pro-objects M in $\mathcal{P}_{I^0I^0}$ such that the coinvariants of the left (equivalently, right) monodromy action belongs to $\mathcal{P}$. It is easy to see from the definitions that $\hat{\mathcal{P}}$ is identified with the heart of the natural $t$-structure on the pro-completion of the derived category $D_{I^0I^0}$ introduced in [23, Appendix A]. Furthermore, [23, Corollary A.4.7] implies that the category $\hat{D} = D^b(\hat{\mathcal{P}})$ is identified with that completion.[7] Thus the construction of *loc. cit.* shows that $\hat{D}$ is monoidal and contains $D_{I^0I^0}$ as a full subcategory closed under the convolution product. An object $\mathcal{F} \in \hat{D}$ belongs to $D_{I^0I^0}$ iff the

[7] This way to define $\hat{D}$ relies on the formalism of triangulated subcategories in the category of pro-objects in the derived category of constructible sheaves developed in [23, Appendix A] by Z. Yun. Alternatively one could first define the category $\hat{\mathcal{P}}$ as a subcategory in the category of pro-objects in $\mathcal{P}$ and use free monodromic tilting objects to equip $\hat{D} := D^b(\hat{\mathcal{P}})$ with a monoidal structure.

monodromy automorphisms of $\mathcal{F}$ are unipotent. The formalism of *loc. cit.* applies also to Iwahori-Whittaker sheaves yielding the definition of an abelian category $\hat{\mathcal{P}}_{\mathrm{IW}}$ and triangulated category $\hat{\mathrm{D}}_{\mathrm{IW}} \cong \mathrm{D}^b(\hat{\mathcal{P}}_{\mathrm{IW}})$, so that $\hat{\mathcal{P}}_{\mathrm{IW}}$ is a full subcategory in the category of pro-objects in $\mathcal{P}_{\mathrm{IW}}^{\mathrm{I}^0}$ consisting of pro-objects with finite coinvariants of monodromy automorphisms, while $\mathrm{D}_{\mathrm{IW}}^{\mathrm{I}^0}$ is a full subcategory on $\hat{\mathrm{D}}_{\mathrm{IW}}$ consisting of objects where monodromy automorphisms are unipotent. Convolution action of $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ on $\mathrm{D}_{\mathrm{IW}}^{\mathrm{I}^0}$ extends to an action of $\hat{\mathrm{D}}$ on $\hat{\mathrm{D}}_{\mathrm{IW}}$.

Let $\mathcal{E}$ be the *free prounipotent* rank one local system on T (see [23]), thus $\mathcal{E} = \varprojlim \mathcal{E}_n$ where $\mathcal{E}_n$ is the local system whose fiber at the unit element $1_{\mathrm{T}} \in \mathrm{T}$ is identified with the quotient of the group algebra of tame fundamental group $\pi_1^{tame}(\mathrm{T})$ by the $n$-th power of augmentation ideal, where the action of monodromy coincides with the natural structure of $\pi_1^{tame}(\mathrm{T})$ module. Let $\widetilde{\mathcal{F}\ell}_w$ denote the preimage of $\mathcal{F}\ell_w$ in $\widetilde{\mathcal{F}\ell}$. The quotient $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_w$ is a torsor over T, choosing an arbitrary trivialization of the torsor we get a projection $\widetilde{\mathcal{F}\ell}_w \to \mathrm{T}$ which we denote $pr_w$. Set $\Delta_w = j_{w!}pr_w^*(\mathcal{E})[\dim \widetilde{\mathcal{F}\ell}_w]$, $\nabla_w = j_{w*}pr_w^*(\mathcal{E})[\dim \widetilde{\mathcal{F}\ell}_w]$. The objects $\Delta_w$, $\nabla_w$ are defined uniquely up to a non-unique isomorphism, we call them a free-monodromic standard and costandard object respectively. One similarly defines $\Delta_w^{\mathrm{IW}}, \nabla_w^{\mathrm{IW}} \in \hat{\mathcal{P}}_{\mathrm{IW}}$.

**3.2.** *More on monodromic (co)standard pro-sheaves.* — A free prounipotent local system on $\widetilde{\mathcal{F}\ell}_w$ is defined uniquely up to a non-unique isomorphism, thus so are the (co)standard sheaves $\Delta_w$, $\nabla_w$. We now present geometric data allowing to fix these objects up to a canonical isomorphism.

Fix a maximal torus $\mathrm{T} \subset \mathrm{B}$ (recall that $\mathbf{I}$ maps to B under the evaluation map $\mathbf{G_O} \to \mathrm{G}$); we get a canonical identification of T with the abstract Cartan group of G, thus the group of coweights of T is identified with $\Lambda$. Thus for $w = \lambda \in \Lambda \subset \mathrm{W}$ the choice of a uniformizer $t \in \mathrm{F}$ defines an element $t_\lambda = \lambda(t) \in \mathrm{T}_\mathrm{F} \subset \mathrm{G(F)}$; its image in $\widetilde{\mathcal{F}\ell} = \mathbf{G}/\mathbf{I}^0$ lies in the orbit of I corresponding to $\lambda$. This yields the choice of a point $\overline{\lambda(t)} \in \mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_\lambda$ which gives a trivialization of the T-torsor, and hence the choice of objects $\Delta_\lambda$, $\nabla_\lambda$ defined uniquely up to a unique isomorphism. We use the same notation for those canonically defined objects and the objects defined earlier uniquely up to a non-unique isomorphism.

*Lemma* **4.** — *(a) We have isomorphisms* $\Delta_{w_1} * \Delta_{w_2} \cong \Delta_{w_1w_2}$, $\nabla_{w_1} * \nabla_{w_2} \cong \nabla_{w_1w_2}$ *when* $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

*(b) Assume that* $w_1 = \lambda_1, w_2 = \lambda_2 \in \Lambda^+$ *and let* $\Delta_{\lambda_i}, \nabla_{\lambda_i}$, $(i = 1, 2)$ *be the canonically defined objects as above. We have canonical isomorphisms* $\Delta_{\lambda_1} * \Delta_{\lambda_2} \cong \Delta_{\lambda_1+\lambda_2}$, $\nabla_{\lambda_1} * \nabla_{\lambda_2} \cong \nabla_{\lambda_1+\lambda_2}$, *which satisfy the associativity identity for a triple* $\lambda_1, \lambda_2, \lambda_3$.

*(c)* $\Delta_0 = \nabla_0$ *is the unit object in* $\hat{\mathrm{D}}$*; and we have a canonical isomorphism* $\nabla_w * \Delta_{w^{-1}} \cong \nabla_0$.

*(d) We have* $\Delta_{w_1} * \nabla_{w_2} \in \hat{\mathcal{P}}$, $\nabla_{w_1} * \Delta_{w_2} \in \hat{\mathcal{P}}$ *for all* $w_1, w_2 \in \mathrm{W}$.

*(e)* $\pi_*(\nabla_w) \cong j_{w*}$, $\pi_*(\Delta_w) \cong j_{w!}$ *canonically.*

*Proof.* — (a) and the first claim in (c) follow from [23, Lemma 4.3.3], [23, Corollary 4.2.2]. A noncanonical isomorphism in the second statement in (c) follows from the similar non-monodromic statement $j_{w*} * j_{w^{-1}!} \cong j_{e!} \cong j_{e*}$ by using (10), (11) below and the observation that any object X in $\hat{\mathrm{D}}$ with $\pi_*(\mathrm{X}) \cong j_{e*}$ is isomorphic to $\Delta_0$. The non-monodromic statement is standard, in the case of a finite dimensional flag variety it goes back to [7]; to check it directly one can reduce to the case when $w$ is a simple reflection, then it amounts to an easy calculation based on the fact that $\mathrm{H}^*_c(\mathbf{A}^1 \setminus \{0\}) = 0$.

Now given $\lambda_1, \lambda_2 \in \Lambda^+$ consider the locally closed subvariety in the convolution diagram: $\widetilde{\mathcal{F}\ell}_{\lambda_1} \boxtimes^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}_{\lambda_2} \to \widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$. Using the above trivializations of the T torsors $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_{\lambda_1}$, $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_{\lambda_2}$, $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$ we can identify the quotient of $\widetilde{\mathcal{F}\ell}_{\lambda_1} \boxtimes^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}_{\lambda_2}$ by $\mathbf{I}^0$ with $\mathrm{T}\times\mathrm{T}$ and the quotient of $\widetilde{\mathcal{F}\ell}_{\lambda_1+\lambda_2}$ by $\mathbf{I}^0$ with T; the quotient of the convolution map is readily seen to be the multiplication map $\mathrm{T}\times\mathrm{T}\to\mathrm{T}$. Since the convolution $\mathcal{E} *_{\mathrm{T}} \mathcal{E}$ is canonically isomorphic to $\mathcal{E}[-\dim \mathrm{T}]$ (here $*_{\mathrm{T}}$ denotes convolution of sheaves on the group T) we get the desired canonical isomorphism. Verification of the associativity identity is straightforward.

Part (d) follows once we know that the functors $\mathcal{M} \mapsto \mathcal{M} * \nabla_w$, $\mathcal{M} \mapsto \nabla_w * \mathcal{M}$ are right exact, while $\mathcal{M} \mapsto \mathcal{M} * \Delta_w$, $\mathcal{M} \mapsto \Delta_w * \mathcal{M}$ are left exact. These follow from their nonmonodromic analogues by virtue of (10), (11), which are standard, see e.g. [1, §5.1] (in *loc. cit.* only $w_1$, $w_2$ of a special form are considered, but the argument applies generally). Alternatively, the statement follows from (4.4), (4.5) in the proof of [23, Proposition 4.3.4].

Finally, part (e) easily follows from the fact that cohomology of the free prounipotent local system on T is zero in degrees other than $r = \dim(\mathrm{T})$ and $r$-th cohomology is one dimensional. □

**3.3.** *Wakimoto pro-sheaves.* — Recall *Wakimoto sheaves* $\mathrm{J}_\lambda \in \mathcal{P}_{\mathrm{II}}$ characterized by: $\mathrm{J}_\lambda * \mathrm{J}_\mu \cong \mathrm{J}_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$ and $\mathrm{J}_\lambda = j_{\lambda*}$ for $\lambda \in \Lambda_+$, see [1, 3.2]. The following monodromic version follows directly from Lemma 4(b,c).

*Corollary* **5.** — *There exists a monoidal functor* $\Theta : \mathrm{Rep}(\mathrm{T}\check{}) \to \hat{\mathrm{D}}$ *sending a dominant character* $\lambda$ *to* $\nabla_\lambda$ *and an anti-dominant character* $\mu$ *to* $\Delta_\mu$. *Such a functor is defined uniquely up to a unique isomorphism.*

The image of a character $\lambda$ of $\mathrm{T}\check{}$ under this functor will be called a free monodromic Wakimoto sheaf and will be denoted by $\mathcal{J}_\lambda$.

Some of the basic properties of Wakimoto sheaves are as follows.

*Lemma* **6.** — *We have:*

(a) $\mathcal{J}_\lambda \in \hat{\mathcal{P}} \subset \hat{\mathrm{D}}$.
(b) $\mathrm{Hom}^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) = 0$ *for* $\mu \not\preceq \lambda$ *where* $\preceq$ *is the standard partial order on (co)weights.*
(c) $\pi_*(\mathcal{J}_\lambda) \cong \mathrm{J}_\lambda$ *canonically.*

*Proof.* — (a) follows from Lemma 4(d). (b) is clear since

$$\mathrm{Hom}^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) \cong \mathrm{Hom}^\bullet(\mathcal{J}_{\lambda+\eta}, \mathcal{J}_{\mu+\eta}) = \mathrm{Hom}^\bullet(\nabla_{\lambda+\eta}, \nabla_{\mu+\eta}),$$

where $\eta \in \Lambda$ is chosen so that $\lambda+\eta, \mu+\eta \in \Lambda^+$. The latter Hom space vanishes when $\mu \not\preceq \lambda$ because in this case $\widetilde{\mathcal{F}\ell}_{\mu+\eta}$ is not contained in the closure of $\widetilde{\mathcal{F}\ell}_{\lambda+\eta}$. The special case of part (c) when $\pm\lambda \in \Lambda^+$ is contained in Lemma 4(e). To deduce the general case we use isomorphisms:

$$(\mathbf{10}) \qquad \mathcal{F} * \pi_*(\mathcal{G}) \cong \pi_*(\mathcal{F} * \mathcal{G}) \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}}, \quad \mathcal{F}, \mathcal{G} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0},$$

$$(\mathbf{11}) \qquad \mathcal{F} * \mathrm{Res}^{\mathrm{I}}_{\mathrm{I}^0}(\mathcal{G}) \cong \pi_*(\mathcal{F}) * \mathcal{G} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}}, \quad \mathcal{F} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}, \mathcal{G} \in \mathrm{D}_{\mathrm{II}},$$

which are easily checked using base change and transitivity of direct image. Given $\lambda \in \Lambda$ we write it as $\lambda = \lambda_+ - \lambda_-$, $\lambda_\pm \in \Lambda^+$ and apply (10) to $\mathcal{F} = \mathcal{J}_{\lambda_+}$, $\mathcal{G} = \mathcal{J}_{-\lambda_-}$. Applying then (11) to $\mathcal{F} = \mathcal{J}_{\lambda_+}$, $\mathcal{G} = \mathrm{J}_{-\lambda_-}$ we get statement (c). □

**3.4.** *Generalized tilting pro-objects.* — Recall that an object of $\mathcal{P}$ is called *tilting* if it carries a standard and also a costandard filtration; here a filtration is called (co)standard if its associated graded is a sum of (co)standard objects, see e.g. [9].

An object of $\hat{\mathcal{P}}$ is called *free-monodromic tilting* if it carries a free-monodromic standard and also a free-monodromic costandard filtration; here a filtration is called (co)standard if its associated graded is a sum of free-monodromic (co)standard objects, see [23].

Let $\mathcal{T} \subset \mathcal{P}$ be the full subcategory of tilting objects and $\hat{\mathcal{T}} \subset \hat{\mathcal{P}}$ denote the full subcategory of free-monodromic tilting objects [23, Definition A.7.1(1)].

Let $\mathrm{Ho}(\mathcal{T})$, $\mathrm{Ho}(\hat{\mathcal{T}})$ denote the homotopy category of bounded complexes of objects in $\mathcal{T}$, $\hat{\mathcal{T}}$ respectively.

The next proposition summarizes the properties of tilting objects that will be used in the argument.

*Proposition* **7.** — *(a) The natural functors* $\mathrm{Ho}(\mathcal{T}) \to \mathrm{D}^b(\mathcal{P}) = \mathrm{D}$, $\mathrm{Ho}(\hat{\mathcal{T}}) \to \mathrm{D}^b(\hat{\mathcal{P}}) = \hat{\mathrm{D}}$ *are equivalences.*

*(b) The convolution of two object in $\hat{\mathcal{T}}$ lies in $\hat{\mathcal{T}}$, thus* $\mathrm{Ho}(\hat{\mathcal{T}})$ *has a natural monoidal structure. The natural functor* $\mathrm{Ho}(\hat{\mathcal{T}}) \to \hat{\mathrm{D}}$ *is a monoidal equivalence.*

*(c) More generally, assume that $\mathcal{F}, \mathcal{G} \in \hat{\mathrm{D}}$ are represented by bounded complexes $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ of objects in $\hat{\mathcal{P}}$, such that $\mathcal{F}^i * \mathcal{G}^j \in \hat{\mathcal{P}}$. Then $\mathcal{F} * \mathcal{G}$ is represented by the total complex of the bicomplex $\mathcal{F}^i * \mathcal{G}^j$. The same statement holds for $\mathcal{F} \in \hat{\mathrm{D}}$ and $\mathcal{G} \in \hat{\mathrm{D}}_{\mathrm{IW}}$ or $\mathcal{G} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ represented by $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ with $\mathcal{F}^i * \mathcal{G}^j \in \hat{\mathcal{P}}_{\mathrm{IW}}$ (respectively, $\mathcal{F}^i * \mathcal{G}^j \in \mathcal{P}_{\mathrm{I}^0\mathrm{I}}$). Given three complexes $\mathcal{F}_1^\bullet$, $\mathcal{F}_2^\bullet$, $\mathcal{G}^\bullet$ the two isomorphisms between $\mathcal{F}_1 * \mathcal{F}_2 * \mathcal{G}$ and the object represented by the complex $\mathcal{C}^d = \bigoplus_{i+j+l=d} \mathcal{F}_1^i * \mathcal{F}_2^j * \mathcal{G}^l$ coincide.*

*Proof.* — The first statement in (a) appears in [9, Proposition 1.5], the second one (whose proof is similar) is a particular case of [23, Proposition B.1.7].

The first statement in (b) follows from [23, Proposition 4.3.4], while the second statement in (b) and (c) follow from [23, Proposition B.3.1] applied to the functor of push-forward under the convolution (or triple convolution) map and the twisted product of the corresponding class of sheaves, cf. also [23, Remark B.3.2]. □

*Remark* **8.** — Implicit in Proposition 7(a) is Ext vanishing:

$$\mathrm{Ext}^{>0}(\hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2) = 0 = \mathrm{Ext}^{>0}(\mathrm{T}_1, \mathrm{T}_2)$$

for $\mathrm{T}_1, \mathrm{T}_2 \in \mathcal{T}$, $\hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2 \in \hat{\mathcal{T}}$. A stronger statement will be used later:

$$\mathrm{Ext}^{>0}(\hat{\mathrm{M}}_1, \hat{\mathrm{M}}_2) = 0 = \mathrm{Ext}^{>0}(\mathrm{M}_1, \mathrm{M}_2)$$

where $\mathrm{M}_1, \mathrm{M}_2 \in \mathcal{P}_{\mathrm{I}^0\mathrm{I}}$, $\mathrm{M}_1$ admits a standard filtration while $\mathrm{M}_2$ admits a costandard filtration, $\hat{\mathrm{M}}_1, \hat{\mathrm{M}}_2 \in \hat{\mathcal{P}}$, $\hat{\mathrm{M}}_1$ admits a free-monodromic standard filtration, while $\hat{\mathrm{M}}_2$ admits a free-monodromic costandard filtration. The proof is immediate from $\mathrm{Ext}^{>0}(\Delta_{w_1}, \nabla_{w_2}) = 0 = \mathrm{Ext}^{>0}(j_{w_1!}, j_{w_2*})$.

*Proposition* **9.** — *An object* $\mathrm{M} \in \hat{\mathrm{D}}$ *admits a free-monodromic (co)standard filtration iff* $\pi_*(\mathrm{M}) \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ *lies in* $\mathcal{P}$ *and admits a (co)standard filtration.*

*Proof.* — The "only if" direction follows from Lemma 4(e), while the "if" direction is checked in [23, Lemma A.7.2]. □

*Corollary* **10.** — *An object* $\mathrm{M} \in \hat{\mathrm{D}}$ *lies in* $\hat{\mathcal{T}}$ *iff* $\pi_*(\mathrm{M}) \in \mathcal{T}$.

*Proposition* **11.** — *(a) For* $\mathrm{T} \in \hat{\mathcal{T}}$ *the functors* $\mathcal{F} \mapsto \mathrm{T} * \mathcal{F}$ *and* $\mathcal{F} \mapsto \mathcal{F} * \mathrm{T}$ *are t-exact (i.e. send* $\mathcal{P}_{\mathrm{I}^0\mathrm{I}^0}$ *to* $\mathcal{P}_{\mathrm{I}^0\mathrm{I}^0}$ *and* $\hat{\mathcal{P}}$ *to* $\hat{\mathcal{P}}$*).*

*(b) For any* $w \in \mathrm{W}$ *there exists a unique (up to an isomorphism) indecomposable object* $\mathrm{T}_w \in \mathcal{T}$ *whose support is the closure of* $\mathcal{F}\ell_w$*. There also exists a unique indecomposable object* $\hat{\mathrm{T}}_w \in \hat{\mathcal{T}}$ *whose support is the closure of* $\widetilde{\mathcal{F}\ell}_w$*. We have* $\pi_*(\hat{\mathrm{T}}_w) \cong \mathrm{T}_w$*.*

*(c) For* $\mathrm{T} \in \hat{\mathcal{T}}$ *and* $w \in \mathrm{W}$ *the objects* $\Delta_w * \mathrm{T}, \mathrm{T} * \Delta_w \in \hat{\mathcal{P}}$ *have a free-monodromic standard filtration, while the objects* $\nabla_w * \mathrm{T}, \mathrm{T} * \nabla_w \in \hat{\mathcal{P}}$ *have a free-monodromic costandard filtration.*

*Proof.* — Parts (a,c) follows from the proof of [23, Proposition 4.3.4]. The first statement in (b) is standard, see e.g. [9, Proposition 1.4]. The second one then follows from [23, §A.7] which shows that the functor $\mathcal{M} \mapsto \pi_*(\mathcal{M})$ induces a bijection between isomorphism classes of indecomposable objects in $\hat{\mathcal{T}}$ and $\mathcal{T}$: [23, Lemma A.7.2] shows that $\pi_* : \hat{\mathcal{T}} \to \mathcal{T}$, by [23, Lemma A.7.3] it induces a surjective map on isomorphism classes of objects, and [23, Lemma A.7.4] implies that this map is also injective, as it shows that for $\hat{T}_1, \hat{T}_2 \in \hat{\mathcal{T}}$ an isomorphism $\pi_*(\hat{T}_1) \cong \pi_*(\hat{T}_2)$ can be lifted to an isomorphism $\hat{T}_1 \cong \hat{T}_2$. Alternatively, the second statement in (b) follows from [23, Corollary 5.2.2]. □

*Corollary* **12.** — *Convolution with a free-monodromic tilting object preserves the categories of objects admitting a free-monodromic (co)standard filtration.*

**3.5.** *Monodromic central sheaves.* — We need to extend the central functors of [29] to the monodromic setting.

**3.5.1.** *A brief summary of [29].* — Recall first the main result of [29]. In our present notation it reads as follows.

For $V \in \mathrm{Rep}(G^\vee)$ one defines an exact functor $\mathcal{Z}_V : \mathcal{P}_{II} \to \mathcal{P}_{II}$. One then constructs canonical isomorphisms

$$Z_V * \mathcal{F} \cong \mathcal{Z}_V(\mathcal{F}) \cong \mathcal{F} * Z_V, \quad \mathcal{F} \in \mathcal{P}_{II}; \tag{12}$$

$$\mathcal{Z}_{V\otimes W} \cong \mathcal{Z}_V \circ \mathcal{Z}_W, \tag{13}$$

where $Z_V = \mathcal{Z}_V(\delta_e)$, where $\delta_e = j_{e!} = j_{e*}$ is the skyscraper at the point $\mathcal{F}\ell_e$.

The two isomorphisms satisfy natural compatibilities (some are demonstrated in [30]) which amount to saying that $V \mapsto Z_V$ is a tensor functor from $\mathrm{Rep}(G^\vee)$ to Drinfeld center of $D_{II}$.

The goal of this subsection is to extend these results to the monodromic setting.

Construction of the functor $Z_V$ is based on existence of a certain deformation of the affine flag variety $\mathcal{F}\ell$ and the convolution diagrams.

Let C be a smooth algebraic curve over $k$ and fix a point $x_0 \in C(k)$ and set $C^0 = C \setminus \{x_0\}$. The ind-schemes $\mathcal{F}\ell_C$, $\mathcal{F}\ell_C^{(2)}$, $\mathrm{Conv}_C$, $\mathrm{Conv}'_C$ were constructed in [29]. They are defined as moduli spaces parametrizing the following collections of data:

$\mathcal{F}\ell_C = \{(x, \mathcal{E}, \phi, \beta)\}$, where $x \in C$, $\mathcal{E}$ is a G-bundle on C, $\phi$ is a trivialization of $\mathcal{E}$ on $C \setminus \{x\}$ and $\beta \in (G/B)^{\mathcal{E}}_{x_0}$ is a point in the fiber of the associated fibration with fiber G/B at $x_0$.

$\mathcal{F}\ell_C^{(2)} = \{(x, \mathcal{E}, \phi', \beta)\}$, where $x$, $\mathcal{E}$, $\beta$ are as above and $\phi'$ is a trivialization of $\mathcal{E}$ on $C \setminus \{x, x_0\}$.

$\mathrm{Conv}_C = \{(x, \mathcal{E}, \mathcal{E}', \phi, \psi, \beta, \beta')\}$ where $x$, $\mathcal{E}$, $\beta$, $\phi$ are as above, $\mathcal{E}'$ is another G bundle on C, $\psi$ is an isomorphism $\mathcal{E}|_{C\setminus\{x\}} \cong \mathcal{E}'|_{C\setminus\{x\}}$, while $\beta' \in (G/B)^{\mathcal{E}'}_{x_0}$.

$\mathrm{Conv}'_C = \{(x, \mathcal{E}, \mathcal{E}', \phi, \psi', \beta, \beta')\}$ where $x$, $\mathcal{E}$, $\phi$, $\mathcal{E}'$, $\beta$, $\beta'$ are as above, while $\psi'$ is an isomorphism $\mathcal{E}|_{C\setminus\{x,x_0\}} \cong \mathcal{E}'|_{C\setminus\{x,x_0\}}$.

These ind-schemes come with a map to C satisfying the following properties.

The preimage of $x_0$ in $\mathcal{F}\ell_C$ is identified with $\mathcal{F}\ell$, while the preimage of $C \setminus \{x_0\}$ is identified with $G/B \times \mathcal{G}\mathfrak{r}_{C^0}$, where $\mathcal{G}\mathfrak{r}_{C^0}$ is the *Beilinson-Drinfeld global Grassmannian*; thus the fiber of $\mathcal{F}\ell_C$ over $y \in C^0(k)$ is (noncanonically) isomorphic to $G/B \times \mathcal{G}\mathfrak{r}$.

The preimage of $x_0$ in $\mathcal{F}\ell_C^{(2)}$ is identified with $\mathcal{F}\ell$, while the preimage of $C^0$ is identified with $\mathcal{F}\ell \times \mathcal{G}\mathfrak{r}_{C^0}$; thus the fiber of $\mathcal{F}\ell_C^{(2)}$ over $y \in C^0(k)$ is (noncanonically) isomorphic to $\mathcal{F}\ell \times \mathcal{G}\mathfrak{r}$.

To spell out the properties of $\mathrm{Conv}_C$, $\mathrm{Conv}'_C$ recall the convolution space $\mathcal{F}\ell \times^{\mathbf{I}} \mathcal{F}\ell$, which is the fibration over $\mathcal{F}\ell$ with fiber $\mathcal{F}\ell$ associated with the natural principal $\mathbf{I}$ bundle over $\mathcal{F}\ell$ using the action of $\mathbf{I}$ on $\mathcal{F}\ell$. We have the projection map $pr_1 : \mathcal{F}\ell \times^{\mathbf{I}} \mathcal{F}\ell \to \mathcal{F}\ell$ and the convolution map $conv : \mathcal{F}\ell \times^{\mathbf{I}} \mathcal{F}\ell \to \mathcal{F}\ell$ coming from multiplication map of the group $\mathbf{G_F}$.

The fiber of both $\mathrm{Conv}_C$ and $\mathrm{Conv}'_C$ over $x_0$ is $\mathcal{F}\ell \times^{\mathbf{I}} \mathcal{F}\ell$; the preimage of $C^0$ in $\mathrm{Conv}_C$ is the product $((G/B) \times^{\mathbf{I}} \mathcal{F}\ell) \times \mathcal{G}\mathfrak{r}_{C^0}$, while the preimage of $C^0$ in $\mathrm{Conv}'_C$ is identified with $(\mathcal{F}\ell \times^{\mathbf{I}} (G/B)) \times \mathcal{G}\mathfrak{r}_{C^0}$.

One has canonical ind-proper morphisms $conv_C : \mathrm{Conv}_C \to \mathcal{F}\ell^{(2)}$, $conv'_C : \mathrm{Conv}'_C \to \mathcal{F}\ell^{(2)}$ whose fiber over $x_0$ is the convolution map $conv$.

Starting from $V \in \mathrm{Rep}(\check{G})$ one can use the geometric Satake isomorphism to produce a semi-simple perverse sheaf $S(V)$ on $\mathcal{G}\mathfrak{r}_{C^0}$. For $\mathcal{F} \in \mathrm{Perv}(\mathcal{F}\ell)$ one gets a sheaf $\mathcal{F} \boxtimes S(V)$ on $\mathcal{F}\ell \times \mathcal{G}\mathfrak{r}_{C^0} \subset \mathcal{F}\ell_C^{(2)}$. Taking nearby cycles of that sheaf with respect to a local coordinate at $x_0$ one obtains a sheaf $\mathcal{Z}_V(\mathcal{F})$ on $\mathcal{F}\ell$.

The spaces $\mathrm{Conv}_C$, $\mathrm{Conv}'_C$ and the maps $conv_C$, $conv'_C$ are used in [29] to show that the functor $\mathcal{Z}_V|_{D_{II}}$ is isomorphic to both left and right convolution with a certain object $Z_V \in \mathcal{P}_{II}$.

**3.5.2.** *The monodromic case.* — A straightforward modification of the definition from [29] yields spaces $\widetilde{\mathcal{F}\ell}_C$, $\widetilde{\mathcal{F}\ell}_C^{(2)}$, $\widetilde{\mathrm{Conv}}_C$, $\widetilde{\mathrm{Conv}}'_C$, whose definition repeats the definition of ind-schemes in Section 3.5.1 with the only difference that $\beta$, $\beta'$ are replaced by $\tilde{\beta} \in (G/U)^{\mathcal{E}}_{x_0}$, $\tilde{\beta}' \in (G/U)^{\mathcal{E}'}_{x_0}$. The following facts about these ind-schemes are proven by an argument similar to that of [29] which deals with parallel statements about ind-schemes from Section 3.5.1.

The ind-schemes $\widetilde{\mathcal{F}\ell}_C$, $\widetilde{\mathcal{F}\ell}_C^{(2)}$, $\widetilde{\mathrm{Conv}}_C$, $\widetilde{\mathrm{Conv}}'_C$ come with a map to $C$ satisfying the following properties.

The preimage of $x_0$ in $\widetilde{\mathcal{F}\ell}_C$ is identified with $\widetilde{\mathcal{F}\ell}$, while the preimage of $C \setminus \{x_0\}$ is identified with $G/U \times \mathcal{G}\mathfrak{r}_{C^0}$; thus the fiber of $\widetilde{\mathcal{F}\ell}_C$ over $y \in C^0(k)$ is (noncanonically) isomorphic to $G/U \times \mathcal{G}\mathfrak{r}$.

The preimage of $x_0$ in $\widetilde{\mathcal{F}\ell}_C^{(2)}$ is identified with $\widetilde{\mathcal{F}\ell}$, while the preimage of $C \setminus \{x_0\}$ is identified with $\widetilde{\mathcal{F}\ell} \times \mathcal{G}\mathfrak{r}_{C^0}$; thus the fiber of $\widetilde{\mathcal{F}\ell}_C^{(2)}$ over $y \in C^0(k)$ is (noncanonically) isomorphic to $\widetilde{\mathcal{F}\ell} \times \mathcal{G}\mathfrak{r}$.

We will now use the convolution space $\widetilde{\mathcal{F}\ell} \times^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}$, which is a fibration over $\widetilde{\mathcal{F}\ell}$ with fiber $\widetilde{\mathcal{F}\ell}$ associated with the natural principal $\mathbf{I}^0$ bundle over $\widetilde{\mathcal{F}\ell}$ using the action of $\mathbf{I}^0$ on $\widetilde{\mathcal{F}\ell}$. We have the projection map $pr_1 : \widetilde{\mathcal{F}\ell} \times^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell} \to \widetilde{\mathcal{F}\ell}$ and the convolution map $\widetilde{conv} : \widetilde{\mathcal{F}\ell} \times^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell} \to \widetilde{\mathcal{F}\ell}$ coming from multiplication map of the group $\mathbf{G_F}$.

The fiber of both $\widetilde{\mathrm{Conv}}_{\mathrm{C}}$ and $\widetilde{\mathrm{Conv}}'_{\mathrm{C}}$ over $x_0$ is $\widetilde{\mathcal{F}\ell} \times^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}$; the preimage of $\mathrm{C}^0$ in $\widetilde{\mathrm{Conv}}_{\mathrm{C}}$ is the product $((\mathrm{G}/\mathrm{U}) \times^{\mathbf{I}^0} \widetilde{\mathcal{F}\ell}) \times \mathcal{G}\mathfrak{r}_{\mathrm{C}^0}$, while the preimage of $\mathrm{C}^0$ in $\widetilde{\mathrm{Conv}}'_{\mathrm{C}}$ is identified with $(\widetilde{\mathcal{F}\ell} \times^{\mathbf{I}} (\mathrm{G}/\mathrm{U})) \times \mathcal{G}\mathfrak{r}_{\mathrm{C}^0}$.

One has canonical morphisms $\widetilde{conv}_{\mathrm{C}} : \widetilde{\mathrm{Conv}}_{\mathrm{C}} \to \widetilde{\mathcal{F}\ell}^{(2)}_{\mathrm{C}}$, $\widetilde{conv}'_{\mathrm{C}} : \widetilde{\mathrm{Conv}}'_{\mathrm{C}} \to \widetilde{\mathcal{F}\ell}^{(2)}_{\mathrm{C}}$ whose fiber over $x_0$ is the convolution map $\widetilde{conv}$.

The main technical difference with the setting of [29] recalled in the previous subsection is that in contrast with the maps $conv_{\mathrm{C}}$, $conv'_{\mathrm{C}}$ the maps $\widetilde{conv}_{\mathrm{C}}$, $\widetilde{conv}'_{\mathrm{C}}$ are *not ind-proper*.

For $\mathrm{V} \in \mathrm{Rep}(\check{\mathrm{G}})$ and $\mathcal{F} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ we can form a complex $\mathcal{F} \boxtimes \mathrm{S}(\mathrm{V})$ on $\widetilde{\mathcal{F}\ell} \times \mathcal{G}\mathfrak{r}_{\mathrm{C}^0} \subset \widetilde{\mathcal{F}\ell}^{(2)}_{\mathrm{C}}$. Taking nearby cycles with respect to a local coordinate on C near $x_0$ we get a complex which we denote $\hat{\mathcal{Z}}_{\mathrm{V}}(\mathcal{F})$.

The functor $\hat{\mathcal{Z}}_{\mathrm{V}}$ obviously extends to the category $\hat{\mathrm{D}}$. We set $\hat{\mathrm{Z}}_{\mathrm{V}} = \hat{\mathcal{Z}}_{\mathrm{V}}(\Delta_e)$.

*Proposition* **13.** — *(a) Recall that* $\pi : \widetilde{\mathcal{F}\ell} \to \mathcal{F}\ell$ *is the projection. Then we have* $\hat{\mathcal{Z}}_{\mathrm{V}}(\pi^*\mathcal{F}) \cong \pi^*(\mathcal{Z}_{\mathrm{V}}(\mathcal{F}))$ *canonically.*

*(b)* $\hat{\mathcal{Z}}_{\mathrm{V}}$ *is canonically isomorphic to the functors of both left and right convolution with* $\hat{\mathrm{Z}}_{\mathrm{V}}$.

*(c) The map* $\mathrm{V} \mapsto \hat{\mathrm{Z}}_{\mathrm{V}}$ *extends to a central functor* $\mathrm{Rep}(\check{\mathrm{G}}) \to \hat{\mathrm{D}}$, *i.e. to a tensor functor from* $\mathrm{Rep}(\check{\mathrm{G}})$ *to the Drinfeld center of* $\hat{\mathrm{D}}$.

*(d) We have a canonical isomorphism* $\pi_*(\hat{\mathrm{Z}}_{\mathrm{V}}) \cong \mathrm{Z}_{\mathrm{V}}$.

*Proof.* — (a) follows from the fact that nearby cycles commute with pull-back under a smooth morphism.

The proof of (b,c) is parallel to the argument of [29] and [30] respectively, with the following modification. The argument of *loc. cit.* uses that the convolution maps and its global counterparts (denoted presently by $conv_{\mathrm{C}}$, $conv'_{\mathrm{C}}$) are proper in order to apply the fact that nearby cycles commute with direct image under a proper map. The maps $\widetilde{conv}$, $\widetilde{conv}_{\mathrm{C}}$, $\widetilde{conv}'_{\mathrm{C}}$ are not proper, thus we do not a'priori have an isomorphism between the direct image under $\widetilde{conv}_{\mathrm{C}}$ or $\widetilde{conv}'_{\mathrm{C}}$ of nearby cycles of a sheaf and nearby cycles of its direct image. However, we do have a canonical map in one direction. If we start from a sheaf on $\widetilde{\mathcal{F}\ell}$ which is the pull-back of a sheaf on $\mathcal{F}\ell$, then the map is an isomorphism because the sheaves in question are pull-backs under a smooth map of ones considered in [29]. Since all objects of $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ can be obtained from objects in the image of the pull-back functor $\mathrm{D}_{\mathrm{II}} \to \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ by successive extensions, the map in question is an isomorphism for any $\mathcal{F} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$, and claims (b,c) follows.

Using (b) we see that $\hat{\mathcal{Z}}_{\mathrm{V}}(\pi^*\mathcal{F}) \cong \pi^*(\mathcal{F} * \pi_*(\hat{\mathrm{Z}}(\mathrm{V})))$; thus (d) follows from (a). □

**3.5.3.** *Monodromy endomorphisms.* — Being defined as (the inverse limit of) nearby cycles sheaves, the objects $\hat{\mathrm{Z}}_{\mathrm{V}}$, $\mathrm{V} \in \mathrm{Rep}(\check{\mathrm{G}})$ carry a canonical *monodromy automorphism*. It is known that the monodromy automorphism acting on the sheaf $\mathrm{Z}_{\mathrm{V}}$ is unipotent, it follows that the one acting on $\hat{Z}_V$ is pro-unipotent. We let $m_V : \hat{Z}_V \to \hat{Z}_V$ denote the logarithm of monodromy.

It will be useful to have an alternative description of this endomorphism. Consider the action of $\mathbf{G}_m$ on $\widetilde{\mathcal{F}\ell}$ by *loop rotation*. Since each $\mathbf{I} \times \mathbf{I}$ orbit on $\widetilde{\mathcal{F}\ell}$ is invariant under this action, every object of $\mathcal{P}_{I^0I^0}$ is $\mathbf{G}_m$ monodromic with unipotent monodromy. Thus every $\mathcal{F} \in \mathcal{P}_{I^0I^0}$ acquires a canonical logarithm of monodromy endomorphism which we denote by $\mu_{\mathcal{F}}$. By passing to the limit we also get a definition of $\mu_{\mathcal{F}}$ for $\mathcal{F} \in \hat{\mathcal{P}}$.

*Proposition* **14.** — *(a) We have* $m_V = -\mu_{\hat{Z}_V}$.

*(b) The logarithm of monodromy defines a tensor endomorphism of the functor* $\hat{Z}$, *i.e. we have* $m_{V\otimes W} = m_V * \mathrm{Id}_{\hat{Z}_W} + \mathrm{Id}_{\hat{Z}_V} * m_W$.

*Proof.* — (a) follows by the argument of [1, 5.2], while (b) is parallel to [29, Theorem 2]. □

**3.5.4.** *Filtration of central sheaves by Wakimoto sheaves.* — It will be convenient to fix a total ordering on $\Lambda$ compatible with addition and the standard partial order. This allows to make sense of an object in an abelian category with a filtration indexed by $\Lambda$ and of its associated graded.

Recall that the object $\mathcal{J}_\lambda$ was defined canonically up to a unique isomorphism starting from a fixed uniformizer $t$ of the local field F, while the central functor $\mathcal{Z}_V$ was defined using an algebraic curve C with a point $x_0$ together with a fixed isomorphism between F and the field of functions on the punctured formal neighborhood of $x_0$ in C. In the next proposition we assume that $t$ is given by a local étale coordinate. We abbreviate $Z_{V_\lambda}$, $\hat{Z}_{V_\lambda}$ to $Z_\lambda$, $\hat{Z}_\lambda$ respectively.

*Proposition* **15.** — *(a) For any* $\lambda$ *there exists a canonical surjective morphism* $\varpi_\lambda : \hat{Z}_\lambda \to \mathcal{J}_\lambda$. *It is compatible with convolution in the following way: the composition of* $\varpi_{\lambda+\mu}$ *with the canonical map* $\hat{Z}_\lambda * \hat{Z}_\mu \to \hat{Z}_{\lambda+\mu}$ *coming from the canonical map* $V_\lambda \otimes V_\mu = \Gamma(\mathcal{O}_{\mathcal{B}}(\lambda)) \otimes \Gamma(\mathcal{O}_{\mathcal{B}}(\mu)) \to V_{\lambda+\mu} = \Gamma(\mathcal{O}_{\mathcal{B}}(\lambda+\mu))$ *equals* $\varpi_\lambda * \varpi_\mu$.

*(b) The surjection* $\varpi_\lambda$ *extends to a unique filtration on* $\hat{Z}_\lambda$ *indexed by* $\Lambda$ *with associated graded isomorphic to a sum of Wakimoto sheaves* $\mathcal{J}_\mu$.

*(c) The filtration on* $\hat{Z}_\lambda$ *is compatible with the monoidal structure on the functor* $V \mapsto \hat{Z}_V$, *making* $V \mapsto gr(\hat{Z}_V)$ *a monoidal functor.*

*Proof.* — (a) follows from the following standard geometric facts. Let $(\mathcal{F}\ell_C^{(2)})_\lambda$ be the closure of $\mathcal{F}\ell_e \times (\mathcal{G}\mathfrak{r}_{C^0})_\lambda \subset \mathcal{F}\ell_C^{(2)}$, where $e \in W$ is the unit element and $(\mathcal{G}\mathfrak{r}_{C^0})_\lambda$ is the locally closed subscheme in the Beilinson-Drinfeld global Grassmannian $\mathcal{G}\mathfrak{r}_{C^0}$ whose intersection with a fiber of the projection to $C^0$ is the $\mathbf{G_O}$ orbit $\mathcal{G}\mathfrak{r}_\lambda$ (recall that such a fiber is identified with $\mathcal{G}\mathfrak{r}$). Then $\mathcal{F}\ell_\lambda \subset \mathcal{F}\ell$ (where $\mathcal{F}\ell$ is identified with the fiber of $\mathcal{F}\ell_C^{(2)}$ over $x_0$) is contained in the smooth locus of $(\mathcal{F}\ell_C^{(2)})_\lambda$, it is open in $(\mathcal{F}\ell_C^{(2)})_\lambda \times_C$

$\{x_0\}$. It follows that $Z_\lambda$ which is by definition the nearby cycles of $\delta_e \boxtimes IC_\lambda$ (where $\delta_e$ is the skyscraper at $\mathcal{F}\ell_e$) is constant on $\mathcal{F}\ell_\lambda$ which is open in its support; see [1, 3.3.1, Lemma 9]. Likewise, considering the preimage $(\widetilde{\mathcal{F}\ell}^{(2)}_C)_\lambda$ of $(\mathcal{F}\ell^{(2)}_C)_\lambda$ in $\widetilde{\mathcal{F}\ell}^{(2)}_C$ we see that $\widetilde{\mathcal{F}\ell}_\lambda$ is open in the support of $\hat{Z}_\lambda$ and the restriction of $\hat{Z}_\lambda$ to $\widetilde{\mathcal{F}\ell}_\lambda$ is a free pro-unipotent local system (shifted by $\dim(\widetilde{\mathcal{F}\ell}_\lambda)$). This yields a surjection as in (a). To see existence of a canonical choice of the surjection it suffices to see that the stalk of $\hat{Z}_\lambda$ over the point $\overline{\lambda(t)}$ has a canonical generator as a topological $\pi_1(T)$ module. This follows from the fact that the section $(1_{\widetilde{\mathcal{F}\ell}}, \lambda_{\mathcal{G}_\mathfrak{r}}) : C^0 \to \widetilde{\mathcal{F}\ell}_C$ extends to $C$ and its value at $x_0$ is $\lambda(t)_{\widetilde{\mathcal{F}\ell}}$.

Uniqueness of the filtration in (b) follows from the fact that $\mathrm{Hom}^\bullet(\mathcal{J}_\lambda, \mathcal{J}_\mu) = 0$ for $\mu \not\preceq \lambda$ (Lemma 6(b)). Together with the isomorphism $\mathcal{J}_\lambda * \mathcal{J}_\mu \cong \mathcal{J}_{\lambda+\mu}$ this also implies compatibility with convolution and the monoidal property. Existence of the filtration is equivalent to the fact that $\mathcal{J}_\mu * \hat{Z}_\lambda$ admits a free-monodromic costandard filtration when $\mu$ is deep in the dominant chamber (more precisely, when $\mu + \nu$ is dominant for any weight $\nu$ of $V_\lambda$). This follows from Proposition 9 and the corresponding fact about the sheaves $Z_\lambda$ established in [1, §3.6]. □

*Remark.* — It is shown in [1] that the multiplicity of $J_\mu$ as a subquotient of $Z_V$ equals the multiplicity of the weight $\mu$ in representation V. It is clear that the same multiplicity also equals the multiplicity of $\mathcal{J}_\mu$ as a subquotient of $\hat{Z}_V$. This is also a consequence of (9), since that equivalence sends $\hat{Z}_V$ to $V \otimes \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}}$ which admits a filtration whose associated graded is a direct sum of line bundles on $\widehat{\tilde{\mathfrak{g}}}$ with the above multiplicities.

The objects $Z_\lambda$, $\hat{Z}_\lambda$ can be thought of as a categorification of the central elements $S_\lambda$ in the affine Hecke algebra introduced by Lusztig in [38]; the filtration by Wakimoto sheaves with the above multiplicities categorifies formula (8.2) of *loc. cit.*

**3.5.5.** *Torus monodromy.* — Every sheaf in $\mathcal{P}_{I^0I^0}$ is monodromic with respect to $T \times T$ with unipotent monodromy, since every irreducible object in $\mathcal{P}_{I^0I^0}$ is equivariant. Thus taking logarithm of monodromy we get an action of $\mathrm{Sym}(\mathfrak{t} \oplus \mathfrak{t})$ on $\mathcal{P}_{I^0I^0}$ by endomorphisms of the identity functor.

*Lemma* **16.** — *(a) The action of the two copies of* $\mathfrak{t}$ *on* $\Delta_w$, $\nabla_w$ *differ by twist with the element* $\bar{w} \in W_f$, *where we use the notation* $w \mapsto \bar{w}$ *for the projection* $W \to W_f$. *In particular, the left action of* $\mathfrak{t}$ *on the objects* $\Delta_\lambda$, $\nabla_\lambda$, $\lambda \in \Lambda$, *coincides with the right one.*

*(b) The left action of* $\mathfrak{t}$ *on the objects* $\mathcal{J}_\lambda$, $\lambda \in \Lambda$, $\hat{Z}_\mu$, $\mu \in \Lambda^+$ *coincides with the right one.*

*(c) The action of loop rotation monodromy on* $\Delta_\lambda$, $\nabla_\lambda$, $\mathcal{J}_\lambda$ *coincides with the image of coweight* $d\lambda \in \mathfrak{t}$ *under the above action of* $\mathfrak{t}$.

*Proof.* — Let $x_w$ be a point in $\widetilde{\mathcal{F}\ell}_w$ such that the orbit of $x_w$ with respect to the left and the right action of T coincide. Then restriction from $\widetilde{\mathcal{F}\ell}_w$ to $T(x_w)$ is an equivalence between $\mathbf{I} \times \mathbf{I}$ unipotently monodromic sheaves on $\widetilde{\mathcal{F}\ell}_w$ and unipotent local systems on $T(x_w)$. Also for $t \in T$ we have $t(x) = x(\bar{w}(t))$ in the self-explanatory notation. This implies (a).

The statement about $\mathcal{J}_\lambda$ in (b) for $\pm\lambda \in \Lambda^+$ follows from (a), this yields the general case because of compatibility of torus monodromy with convolution.

The statement about $\hat{Z}_\lambda$ in (b) follows from the construction with nearby cycles, since the action of $T^2$ on $\widetilde{\mathcal{F}\ell} \times \mathcal{G}r_{C^0}$ (where $T^2$ acts trivially on the second factor) extends to an action on $\widetilde{\mathcal{F}\ell}_C$.

Finally, part (c) is a consequence of the following observation. Let R denote the loop rotation action of $\mathbf{G}_m$ on $\widetilde{\mathcal{F}\ell}$. Then for $\lambda \in \Lambda$ let $h_\lambda : \mathbf{G}_m \to T$ be the corresponding homomorphism (see Section 3.2). Then we have $R(s)(x_\lambda) = h_\lambda(s)(x_\lambda)$. ☐

## 4. Construction of functors

**4.1.** *A functor from* $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$. — Recall that $\widehat{\tilde{\mathfrak{g}}}^\vee$ denotes the formal completion of $\tilde{\mathfrak{g}}^\vee$ at $\tilde{\mathcal{N}}$.

In this subsection we construct a monoidal functor $\Phi_{diag} : D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}}^\vee)) \to \hat{D}$. The functor we presently construct is compatible with the equivalence $\Phi : D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})) \cong \hat{D}$ that will be established in Section 9 as follows: $\Phi_{diag} \cong \Phi \circ \delta_*$, where $\delta : \widehat{\tilde{\mathfrak{g}}}^\vee \to \widehat{\mathrm{St}}$ is the diagonal embedding, see Lemma 43(b).

The construction is parallel to that of [1, §3], so we only recall the main ingredients of the construction referring the reader to [1] for details.

Following the strategy outlined in Section 2.2, we first list compatibilities satisfied by the functor $\Phi_{diag}$ which characterize it uniquely.

**4.1.1.** *Line bundles and Wakimoto sheaves.* — Recall that for $\lambda \in \Lambda$ the corresponding line bundle on $\mathcal{B}$ is denoted by $\mathcal{O}_{\mathcal{B}}(\lambda)$, while $\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}(\lambda)$ is its pull-back to $\widehat{\tilde{\mathfrak{g}}}^\vee$. The functor $\Phi_{diag}$ satisfies:

$$\Phi_{diag}(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}(\lambda)) \cong \mathcal{J}_\lambda.$$

This isomorphism is compatible with the monoidal structure on the two categories, i.e. it provides a tensor isomorphism between the functor $\Theta$ (see Corollary 5) and the composition of $\Phi_{diag}$ with the tensor functor $\lambda \mapsto \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}(\lambda)$.

**4.1.2.** *Twists by representations and central functors.* — We have a tensor functor $\mathrm{Rep}(G^\vee) \to \mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}}^\vee)$ sending a representation V to $V \otimes \mathcal{O}$. Composition of $\Phi_{diag}$ with this functor is isomorphic to the tensor functor $V \mapsto \hat{Z}_V$ (see Section 3.5).

**4.1.3.** *The lowest weight arrow.* — We have a familiar morphism of $G^\vee$-equivariant vector bundles on $\mathcal{B}$: $\mathcal{O}_{\mathcal{B}} \otimes V_\lambda \to \mathcal{O}_{\mathcal{B}}(\lambda)$. We can pull it back to $\widehat{\tilde{\mathfrak{g}}}^\vee$ to get a morphism in $\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}}^\vee)$. The functor $\Phi_{diag}$ sends this arrow to the map $\varpi_\lambda$ (notations of Proposition 15).

**4.1.4.** *Log monodromy endomorphism.* — Notice that for $x \in \mathfrak{g}^\vee$, $\mathcal{F} \in \mathrm{Coh}^{G^\vee}(\mathfrak{g}^\vee)$ the centralizer of $x$ in $G^\vee$ acts on the fiber $\mathcal{F}_x$ of $\mathcal{F}$ at $x$. Differentiating this action one gets the action of the Lie algebra of the centralizer $\mathfrak{z}(x)$. In particular, $x \in \mathfrak{z}(x)$ produces a canonical endomorphism of $\mathcal{F}_x$, it is easy to see that it comes from a uniquely defined endomorphism of $\mathcal{F}$, which we denote by $\mathfrak{m}_{\mathcal{F}}$ (in [1] we used notation $N^{taut}_{\mathcal{F}}$). It is clear that restricting $\mathfrak{m}$ to sheaves of the form $\mathcal{F} = V \otimes \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}^\vee}}$ one gets a tensor endomorphism of the tensor functor $V \mapsto V \otimes \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}^\vee}}$.

We require that $\Phi_{diag}$ sends $\mathfrak{m}_{V\otimes\mathcal{O}}$ to the monodromy endomorphism $m_V$.

**4.1.5.** *Projection to* $\mathfrak{t}^{\vee 2}$ *and torus monodromy.* — We have a canonical map $\tilde{\mathfrak{g}}^\vee \to \mathfrak{t}^\vee$, thus the category $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$ is canonically an $\mathcal{O}(\mathfrak{t}^\vee)$-linear category, i.e. $\mathfrak{t}^{\vee *} = \mathfrak{t}$ acts on it by endomorphisms of the identity functor. This induces a pro-nilpotent action of $\mathfrak{t}$ on $D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$.

According to Section 3.5.5, we have two commuting pronilpotent $\mathfrak{t}$ actions on $\hat{\mathcal{P}}$ and hence on $\hat{D}$. The functor $\Phi_{diag}$ intertwines the action of $\mathfrak{t}$ described in the previous paragraph with either of the two monodromy actions.

**4.2.** *Monoidal functor from sheaves on the diagonal.* — We use a version of homogeneous coordinate ring construction and Serre description of the category of coherent sheaves on a projective variety.

Let $C_{\tilde{\mathfrak{g}}^\vee}$ be the preimage of $\tilde{\mathfrak{g}}^\vee \subset \mathfrak{g}^\vee \times \mathcal{B}$ under the morphism $\mathfrak{g}^\vee \times G^\vee/U^\vee \to \mathfrak{g}^\vee \times \mathcal{B}$. Let $\overline{G^\vee/U^\vee}$ denote the affine closure of $G^\vee/U^\vee$. Notice that $G^\vee/U^\vee$ can be realized as a locally closed subscheme, namely as the orbit of a highest weight vector in the space V of a representation of G. Moreover, if the representation V is chosen appropriately, the closure of $G^\vee/U^\vee$ in V is isomorphic to $\overline{G^\vee/U^\vee}$. Define the action of the abstract Cartan $\mathfrak{t}^\vee$ on V such that $t \in \mathfrak{t}^\vee$ acts on an irreducible summand with highest weight $\lambda$ by the scalar $\langle\lambda, t\rangle$. Then define a closed subscheme[8] $\overline{C}_{\tilde{\mathfrak{g}}^\vee} \subset \mathfrak{g}^\vee \times \mathfrak{t}^\vee \times \overline{G^\vee/U^\vee}$ by the equation $x(v) = t(v)$, $x \in \mathfrak{g}^\vee$, $t \in \mathfrak{t}^\vee$, $v \in \overline{G^\vee/U^\vee} \subset V$. It is easy to see that $C_{\tilde{\mathfrak{g}}^\vee}$ is an open subscheme in $\overline{C}_{\tilde{\mathfrak{g}}^\vee}$. More precisely, without loss of generality we can assume that representation V is multiplicity free, i.e. it is a sum of pairwise non-isomorphic irreducible representations; then $C_{\tilde{\mathfrak{g}}^\vee}$ is identified with the intersection of $\overline{C}_{\tilde{\mathfrak{g}}^\vee}$ with the open set of vectors which have a nonzero projection to each irreducible factor.

We leave the proof of the following statement to the reader.

*Proposition* **17.** — *(A) The scheme* $\overline{C}_{\tilde{\mathfrak{g}}^\vee}$ *does not depend on the choice of* V *subject to the above conditions.*

*(B) Consider the category of commutative rings over* $\mathcal{O}(\mathfrak{t}^\vee)$ *equipped with a* $G^\vee$ *action which fixes the image of* $\mathcal{O}(\mathfrak{t}^\vee)$.

[8] Here notations diverge from that of [1], there "hat" was used to denote the affine cone, while in the present paper it is used to denote completions.

*The following two functors on that category are canonically isomorphic:*

(1) $R \mapsto \mathrm{Hom}(\mathrm{Spec}(R), \overline{C}_{\tilde{\mathfrak{g}}^\vee})$ *where* Hom *stands for maps compatible with the* $G^\vee$ *action and the map to* $\mathfrak{t}^\vee$.
(2) $R \mapsto \{(E_V, \iota_V) \mid V \in \mathrm{Rep}(G^\vee)\}$. *Here for* $V \in \mathrm{Rep}(G^\vee)$, $E_V \in \mathrm{End}_R(V \otimes R)$ *and* $\iota_V$ *is a map of* R*-modules* $R \to V \otimes R$. *This data is subject to the requirements:*
   (a) *functoriality in* V*;*
   (b) $E_{V\otimes W} = E_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes E_W$*;*
   (c) $\iota_{V\otimes W} = \iota_V \otimes \iota_W$*;*
   (d) *The action of* $E_{V_\lambda}$ *on the image of* $\iota_{V_\lambda}$ *coincides with the action of the element in* R *which is the image of* $\lambda \in \mathfrak{t}^{\vee *}$ *under the map* $\mathfrak{t}^{\vee *} \to R$.

**4.2.1.** *Deequivariantization.* — (cf. Section 2.2.1(3)) We will make use of the following construction. Let $\mathcal{C}$ be an additive category linear over the field $k$, with an action of the tensor category Rep(H) of (finite dimensional algebraic) representation of H, where H is a reductive algebraic group over $k$. (Recall that $k$ is algebraically closed of characteristic zero; the definition is applicable under less restrictive assumptions.)

We can then define a new category $\mathcal{C}_{deeq}$ by setting $\mathrm{Ob}(\mathcal{C}_{deeq}) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}_{deeq}}(A, B) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A, \underline{\mathcal{O}}(H)(B))$, where $\mathrm{Ind}(\mathcal{C})$ is the category of Ind-objects in $\mathcal{C}$ and $\underline{\mathcal{O}}(H)$ is the object of Ind(Rep(H)) coming from the module of regular functions on H equipped with the action of H by left translations. Using that H is reductive over an algebraically closed field of characteristic zero we can write the Ind-object $\underline{\mathcal{O}}(H)$ as $\bigoplus_{\underline{V}\in \mathrm{IrrRep}(H)} \underline{V} \otimes V^*$, where IrrRep(H) is a set of representatives for isomorphism classes of irreducible H modules and for a representation $\underline{V} \in \mathrm{IrrRep}(H)$ we let V denote the underlying vector space. Thus we have

$$\mathrm{Hom}_{deeq}(X, Y) = \bigoplus_{\underline{V}\in \mathrm{IrrRep}(H)} \mathrm{Hom}(X, \underline{V}(Y)) \otimes V^*.$$

For example, if $\mathcal{C} = D^b(\mathrm{Coh}^H(X))$ where X is a scheme equipped with an H action then for $\mathcal{F}, \mathcal{G} \in \mathcal{C}$ we have $\mathrm{Hom}_{deeq}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{D^b(\mathrm{Coh}(X))}(\mathcal{F}, \mathcal{G})$.

When we need to make the group H explicit in the above definition we write $\mathrm{Hom}^H_{deeq}$ instead of $\mathrm{Hom}_{deeq}$.

The category $\mathcal{C}_{deeq}$ is enriched over H-modules, i.e. every Hom space carries the structure of an H-module compatible with composition. We refer the reader to [3] for further details and to [31] for a more general construction (cf. also [1], proof of Proposition 4).

This formalism comes in handy for deducing the following statement.

Let $\mathrm{Coh}^{G^\vee \times T^\vee}_{fr}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})$ be the full subcategory in $\mathrm{Coh}^{G^\vee \times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})$ consisting of objects of the form $V \otimes \mathcal{O}$, $V \in \mathrm{Rep}(G^\vee \times T^\vee)$. In other words, objects of $\mathrm{Coh}^{G^\vee \times T^\vee}_{fr}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})$ are representations of $G^\vee \times T^\vee$ and morphisms are given by $\mathrm{Hom}(V_1, V_2) = \mathrm{Hom}_{\mathrm{Coh}^{G^\vee \times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})}(V_1 \otimes \mathcal{O}, V_2 \otimes \mathcal{O})$. This is a tensor category under the usual tensor product of vector bundles.

*Corollary* **18.** — *Let $\mathcal{C}$ be a k-linear additive monoidal category. Suppose we are given*

*(1) A tensor functor* $F : \mathrm{Rep}(G^{\vee} \times T^{\vee}) \to \mathcal{C}$.

*(2) A tensor endomorphism* $E$ *of* $F|_{\mathrm{Rep}(G^{\vee})}$, $E_{V_1 \otimes V_2} = E_{V_1} \otimes \mathrm{Id}_{F(V_2)} + \mathrm{Id}_{F(V_1)} \otimes E_{V_2}$.

*(3) An action of* $\mathcal{O}(\mathfrak{t}^{\vee})$ *on* $F$ *by endomorphisms, so that for* $f \in \mathcal{O}(\mathfrak{t}^{\vee})$ *we have* $f_{V_1 \otimes V_2} = f_{V_1} \otimes \mathrm{Id}_{F(V_2)} = \mathrm{Id}_{F(V_1)} \otimes f_{V_2}$.

*(4) A "lowest weight arrow"* $\varpi_\lambda : F(V_\lambda) \to F(\lambda)$ *making the following diagrams commutative:*

$$\begin{array}{ccc} F(V_\lambda \otimes V_\mu) & \longrightarrow & F(V_{\lambda+\mu}) \\ {\scriptstyle \varpi_\lambda \otimes_{\mathcal{C}} \varpi_\mu} \downarrow & & \downarrow {\scriptstyle \varpi_{\lambda+\mu}} \\ F(\lambda) \otimes_{\mathcal{C}} F(\mu) & \xrightarrow{\sim} & F(\lambda+\mu) \end{array}$$

$$\begin{array}{ccc} F(V_\lambda) & \xrightarrow{\varpi_\lambda} & F(\lambda) \\ {\scriptstyle E_{V_\lambda}} \downarrow & & \downarrow {\scriptstyle \lambda} \\ F(V_\lambda) & \xrightarrow{\varpi_\lambda} & F(\lambda) \end{array}$$

*where the right vertical map is the action of the element* $\lambda \in \mathfrak{t} \subset \mathcal{O}(\mathfrak{t}^{\vee})$ *coming from (3).*

*Then the tensor functor* $F$ *extends uniquely to a tensor functor* $\mathrm{Coh}_{fr}^{G^{\vee}\times T^{\vee}}(C_{\tilde{\mathfrak{g}}^{\vee}}) \to \mathcal{C}$, *so that* $E$ *goes to the tautological endomorphism* $\mathfrak{m}$ *(see Section 4.1.4), the action of* $\mathfrak{t}$ *comes from the projection* $\overline{C}_{\tilde{\mathfrak{g}}^{\vee}} \to \mathfrak{t}^{\vee}$ *and the lowest weight arrow comes from the map described in Section 4.1.3.*

*Proof.* — Extending the functor F to a functor $\mathrm{Coh}_{fr}^{G^{\vee}\times T^{\vee}}(\overline{C}_{\tilde{\mathfrak{g}}^{\vee}}) \to \mathcal{C}$ is equivalent to providing a $G^{\vee} \times T^{\vee}$-equivariant homomorphism $\mathcal{O}(\overline{C}_{\tilde{\mathfrak{g}}^{\vee}}) \to \mathrm{Hom}_{deeq}^{G^{\vee}\times T^{\vee}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$, where we used the action of $\mathrm{Rep}(G^{\vee} \times T^{\vee})$ on $\mathcal{C}$ given by $V : X \mapsto F(V)X$. We now apply Proposition 17 to the ring $R := \mathrm{Hom}_{deeq}^{G^{\vee}\times T^{\vee}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$. The action described in (3) provides it with a structure of a ring over $\mathcal{O}(\mathfrak{t}^{\vee})$; the tensor endomorphism E from (2) yields the collection $(E_V)$ and the arrows $\varpi_\lambda$ induce the maps $\iota_V$ as in Proposition 17. The commutative diagrams in part (4) of the corollary imply identities (c,d) in Proposition 17. Thus existence of a unique functor F with above properties follows from Proposition 17. ☐

**4.2.2.** *The functor* $\Phi_{diag}$. — We now construct a monoidal functor $\Phi_{diag}^{fr}$ : $\mathrm{Coh}_{fr}^{G^{\vee}\times T^{\vee}}(\overline{C}_{\tilde{\mathfrak{g}}^{\vee}}) \to \hat{\mathcal{P}}$ (more precisely, a monoidal functor to $\hat{D}$ taking values in $\hat{\mathcal{P}}$).

The functor is provided by Corollary 18: we have a tensor functor from $\mathrm{Rep}(G^{\vee})$ to $\hat{D}$ coming from the central functors (Section 4.1.2), and another commuting one from $\mathrm{Rep}(T^{\vee})$ to $\hat{D}$ coming from Wakimoto sheaves (3.3); the logarithm of monodromy endomorphisms (Section 4.1.4) provide endomorphism E while the torus monodromy (Section 3.5.5) gives an action of $\mathfrak{t} = \mathfrak{t}^{\vee *}$ (notice that due to Lemma 16(b) we get the same action by using either left or right torus action). The morphisms described in Section 4.1.3

yield arrows $\varpi_\lambda$. The conditions of Corollary 18 are checked as follows. Condition (2) follows from Proposition 14(b). Condition (3) is clear from compatibility of the convolution map with the torus action. The first commutative diagram in condition (4) follows from Proposition 15(a), while the second one is obtained by comparing Proposition 14(a) with Lemma 16(c).

**4.3.** *"Coherent" description of the anti-spherical (generalized Whittaker) category.* — Consider the composition $\mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))\to \mathrm{Ho}(\hat{\mathcal{P}})\to\hat{D}$ where Ho denotes the homotopy category of complexes of objects in the given additive category and the first arrow is induced by $\Phi_{diag}^{fr}$; this composition will be denoted by $\Phi_{diag}^{\mathrm{Ho}}$.

Let $\mathrm{Acycl}\subset \mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))$ be the subcategory of complexes whose restriction to the open subscheme $C_{\tilde{\mathfrak{g}}^\vee}$ is acyclic.

*Proposition* **19.** — *The functor* $\Phi_{diag}^{\mathrm{Ho}}$ *sends the subcategory* Acycl *of acyclic complexes to zero.*

*Proof.* — Proposition follows from existence of a filtration on $\hat{Z}_\lambda$ with associated graded being the sum of Wakimoto sheaves (Proposition 15(b)) by an argument parallel to [1, 3.7]. □

The perfect derived category of modules over a positively graded algebra (by which we mean a **Z**-graded algebra with vanishing negative components and component of degree zero generated by the unit element) over a field of characteristic zero is well known to be equivalent to the homotopy category of free graded modules, the same applies to equivariant modules, where an algebra is assumed to be equipped with an action of a reductive group. Applying this to $\mathcal{O}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})$ we see that $D_{perf}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee})\cong \mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))$. Since $\tilde{\mathfrak{g}}^\vee$ is smooth, $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))=D_{perf}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)$, thus (8) shows that[9]

$$D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))\cong \mathrm{Idem}(\mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))/\mathrm{Acycl}).$$

Thus Proposition 19 yields a functor $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))\to\hat{D}$. The log monodromy action of $\mathfrak{t}$ on the identity functor of $\hat{D}$ is pro-unipotent, thus it extends canonically to an action of the completion of $\mathcal{O}(\mathfrak{t}^\vee)$ at the maximal ideal of 0. It is easy to deduce that the functor factors canonically through a functor $D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))\to\hat{D}$, we denote the latter functor by $\Phi_{diag}$.

A closely related functor $F: D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}}))\to D_{II}$ was constructed in [1, §3].

[9] In fact, $K^0(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))=K^0(\mathrm{Coh}^{B^\vee}(\mathfrak{b}^\vee))=K^0(\mathrm{Coh}^{T^\vee}(pt))$ is generated by the classes of equivariant line bundles, thus the functor $\mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))\to D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))$ induces a surjection on $K^0$. By a standard argument (see e.g. [44, Corollary 0.10]) this implies: $D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee))\cong \mathrm{Ho}(\mathrm{Coh}_{fr}^{G^\vee\times T^\vee}(\overline{C}_{\tilde{\mathfrak{g}}^\vee}))/\mathrm{Acycl}$.

*Lemma* **20.** — *Let* $i : \tilde{\mathcal{N}} \to \widehat{\tilde{\mathfrak{g}}}^{\check{}}$ *be the embedding. The following diagrams commute up to a natural isomorphism:*

$$\begin{array}{ccc} D^b(\mathrm{Coh}^{G^{\check{}}}(\widehat{\tilde{\mathfrak{g}}}^{\check{}})) & \xrightarrow{\Phi_{diag}} & \hat{D} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ D^b(\mathrm{Coh}^{G^{\check{}}}(\tilde{\mathcal{N}})) & \xrightarrow{\mathrm{Res}^{I}_{I^0}\circ F} & D_{I^0I} \end{array}$$

$$\begin{array}{ccc} D^b(\mathrm{Coh}^{G^{\check{}}}(\tilde{\mathcal{N}})) & \xrightarrow{F} & D_{II} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{I}_{I^0}\circ\pi^*[r]} \\ D^b(\mathrm{Coh}^{G^{\check{}}}(\widehat{\tilde{\mathfrak{g}}}^{\check{}})) & \xrightarrow{\Phi_{diag}} & \hat{D} \end{array}$$

*where* Res *stands for restriction of equivariance, and* $r = \mathrm{rank}(\mathfrak{g})$.

*Proof.* — To check commutativity of the first diagram it suffices to prove the similar commutativity for functors on the categories of finite complexes in $\mathrm{Coh}^{G^{\check{}}\times T^{\check{}}}_{fr}(\overline{C}_{\tilde{\mathfrak{g}}^{\check{}}})$. This follows from the isomorphisms $\pi_*(\hat{Z}_V) \cong Z_V$ (Proposition 13(d)), $\pi_*(\mathcal{J}_\lambda) \cong J_\lambda$ (Lemma 6(c)) which are easily seen to be compatible with monodromy endomorphism and lowest weight arrows.

Now using commutativity of the first diagram we get a natural transformation between the two compositions in the second diagram due to the isomorphisms:

$$\begin{aligned} &\mathrm{Hom}\big(\mathrm{Res}^{I}_{I^0}\pi^*\big(F(\mathcal{F})\big)[r], \Phi_{diag}\big(i_*(\mathcal{F})\big)\big) \\ &\quad\cong \mathrm{Hom}\big(\mathrm{Res}^{I}_{I^0}F(\mathcal{F})[r], \pi_*\Phi_{diag}\big(i_*(\mathcal{F})\big)\big) \\ &\quad\cong \mathrm{Hom}\big(\mathrm{Res}^{I}_{I^0}F(\mathcal{F})[r], \mathrm{Res}^{I}_{I^0}F\big(i^*i_*(\mathcal{F})\big)\big), \end{aligned}$$

which yield the desired arrow since $\mathrm{Id}[r]$ is a canonical direct summand in the functor $i^*i_*$. The constructed arrow is nonzero, hence it is an isomorphism for $\mathcal{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$, as both compositions are then isomorphic to the skyscraper $\delta_e = j_{e!} = j_{e*}$ and $\mathrm{Hom}(\delta_e, \delta_e)$ is one dimensional. Also it is easy to see that the arrow is compatible with action of the tensor category $\mathrm{Coh}^{G^{\check{}}\times T^{\check{}}}_{fr}(\overline{C}_{\tilde{\mathfrak{g}}^{\check{}}})$, thus it is an isomorphism for $\mathcal{F}$ in a generating set of $D^b(\mathrm{Coh}^{G^{\check{}}}(\tilde{\mathcal{N}}))$, hence it is an isomorphism for all $\mathcal{F}$. □

**4.3.1.** *Equivalences* $\widehat{\Phi}_{IW}$, $\Phi^{I^0}_{IW}$. — We are now ready to establish (6).

The functor $\mathrm{Av}^{IW} : D_{I^0I^0} \to D^{I^0}_{IW}$ introduced at the end of Section 2.1 extends to a functor between the completed categories $\hat{D}$, $\hat{D}_{IW}$ introduced in Section 3.1, we will use the same notation for this extension.

*Proposition* **21.** — *(a) The functor* $\widehat{\Phi}_{\mathrm{IW}} := \mathrm{Av}^{\mathrm{IW}} \circ \Phi_{diag} : \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \to \hat{\mathrm{D}}_{\mathrm{IW}}$ *is an equivalence.*

*(b) The functor* $\widehat{\Phi}$ *restricts to an equivalence* $\Phi^{\mathrm{I}^0}_{\mathrm{IW}} : \mathrm{D}^b(\mathrm{Coh}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee)) \to \mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$.

*Proof.* — We first show that $\mathrm{Av}^{\mathrm{IW}} \circ \Phi_{diag}$ is fully faithful. It suffices to show that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Av}^{\mathrm{IW}}\Phi_{diag}(\mathcal{F}), \mathrm{Av}^{\mathrm{IW}}\Phi_{diag}(\mathcal{G}))$$

when $\mathcal{F} = i_*(\mathcal{F}')$, $\mathcal{F}' \in \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}}))$, then the statement follows since the image of $i_*$ generates $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))$ under extensions, so we get the isomorphism for $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b(\mathrm{Coh}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))$. Passing to the limit we then get the isomorphism for all $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$.

Using the parallel statement in the non-monodromic setting proved in [1, §4] and the first commutative diagram in Lemma 20 (or rather the statement obtained from it by left–right swap) we get:

$$\begin{aligned}
\mathrm{Hom}\big(i_*(\mathcal{F}'), \mathcal{G}\big) &\cong \mathrm{Hom}\big(\mathcal{F}', i^*\mathcal{G}[-r]\big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}}}\big({}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}\mathrm{F}(\mathcal{F}'), {}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}\mathrm{F}(i^*\mathcal{G})[-r]\big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}}}\big({}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}\mathrm{F}(\mathcal{F}'), (\mathrm{Av}^{\mathrm{I}}_{\mathrm{I}^0})^{left}_* \mathrm{Av}^{\mathrm{IW}}(\Phi_{diag}(\mathcal{G}))[-r]\big) \\
&\cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}}\big((\mathrm{Res}^{\mathrm{I}}_{\mathrm{I}^0})^{left}({}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}\mathrm{F}(\mathcal{F}')), \mathrm{Av}^{\mathrm{IW}}(\Phi_{diag}(\mathcal{G}))[-r]\big) \\
&\cong \mathrm{Hom}\big(\mathrm{Av}^{\mathrm{IW}}\Phi_{diag}(\mathcal{F}), \mathrm{Av}^{\mathrm{IW}}\Phi_{diag}(\mathcal{G}')\big),
\end{aligned}$$

where we used that $i^*[-r]$ is *right* adjoint to $i_*$. Here $(\mathrm{Av}^{\mathrm{I}}_{\mathrm{I}^0})^{left}_*$ is the *right* adjoint to the restriction of equivariance functor which can be thought of as a direct image under the morphism of stacks $\mathbf{I}^0\backslash\widetilde{\mathcal{F}\ell} \to \mathbf{I}\backslash\widetilde{\mathcal{F}\ell}$ (recall that Av stands for the !-direct image under that morphism).

This shows that the functor is fully faithful. Again using the parallel statement in the non-monodromic setting and Lemma 20 we see that the essential image of $\Phi_{diag}$ contains the image of the functor of restricting the equivariance $\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}} \to \mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$, since $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$ is generated by irreducible perverse sheaves which are I-equivariant, the essential image contains $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$, this proves part (b). Any object in $\hat{\mathrm{D}}_{\mathrm{IW}}$ is an inverse limit of objects in $\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}$, moreover, its image under the functor between the categories of pro-objects $\mathrm{Pro}(\mathrm{D}^{\mathrm{I}^0}_{\mathrm{IW}}) \to \mathrm{Pro}(\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}})$ induced by the averaging functor lies in $\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}} \subset \mathrm{Pro}(\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}})$. This shows that such an object is isomorphic to the image of a pro-object in $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))$ whose image under $i^* : \mathrm{Pro}(\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee))) \to \mathrm{Pro}(\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}})))$ lies in $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}}))$. An object in $\mathrm{Pro}(\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee)))$ satisfying the latter property is easily seen to lie in $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \subset \mathrm{Pro}(\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\tilde{\mathcal{N}}}(\tilde{\mathfrak{g}}^\vee)))$, this implies essential surjectivity in part (a). □

**4.4.** $\hat{\mathrm{D}}$ *is a category over* $\widehat{\mathrm{St}}/\mathrm{G}^\vee$, $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ *is a category over* $\mathrm{St}'/\mathrm{G}^\vee$. — The goal of this section is to construct an action of the tensor category $\mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}})$ on $\hat{\mathrm{D}}$ and of $\mathrm{D}^{\mathrm{G}^\vee}_{perf}(\mathrm{St}')$ on $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$, both categories are equipped with the tensor structure coming from tensor product of perfect complexes.

**4.4.1.** *The action of the tensor categories* $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^\vee}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}})$, $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^\vee}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}'})$. — We let $\overline{\mathrm{C}}_{\mathrm{St}}$ be the preimage of diagonal under the map $\overline{\mathrm{C}}_{\tilde{\mathfrak{g}}^\vee}\times\overline{\mathrm{C}}_{\tilde{\mathfrak{g}}^\vee}\to\mathfrak{g}^\vee\times\mathfrak{g}^\vee$, and let $\overline{\mathrm{C}}_{\mathrm{St}'}$ be the preimage of 0 under the second projection to $\mathfrak{t}^\vee$. We have open subsets $\mathrm{C}_{\mathrm{St}}\subset\overline{\mathrm{C}}_{\mathrm{St}}$ and $\mathrm{C}_{\mathrm{St}'}\subset\overline{\mathrm{C}}_{\mathrm{St}'}$ where the action of $\mathrm{T}^\vee\times\mathrm{T}^\vee$ is free and $\mathrm{St}=\mathrm{C}_{\mathrm{St}}/\mathrm{T}^{\vee 2}$, $\mathrm{St}'=\mathrm{C}_{\mathrm{St}'}/\mathrm{T}^{\vee 2}$.

Notation $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^\vee}_{fr}(\overline{\mathrm{C}}_{\tilde{\mathfrak{g}}^\vee})$, was introduced in Section 4.2.1, tensor categories $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}})$, $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}'})$ etc. are similarly defined as full subcategories in the categories of equivariant coherent sheaves whose objects are obtained from the structure sheaf by tensoring with a representation.

We apply Corollary 18 in the following setting: the group $\mathrm{G}^\vee$ is replaced by $\mathrm{G}^{\vee 2}$ and $\mathcal{C}$ is the category of functors $\hat{\mathrm{D}}\to\hat{\mathrm{D}}$ (respectively $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}\to\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$).

We have two actions of $\mathrm{Rep}(\mathrm{T}^\vee)$ coming from, respectively, left and right convolution with Wakimoto sheaves. We consider the action of $\mathrm{Rep}(\mathrm{G}^{\vee 2})$ obtained as composition of restriction to the diagonal copy of $\mathrm{G}^\vee$ and the action by central functors. The nearby cycles monodromy acting on the cental functor defines a tensor endomorphism E of the $\mathrm{G}^{\vee 2}$ action, while the torus monodromy defines an action of $\mathfrak{t}^{\vee 2}$. It is not hard to see that conditions of Corollary 18 are satisfied. Thus we get an action of $\mathrm{Coh}^{\mathrm{G}^{\vee 2}\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}^2_{\tilde{\mathfrak{g}}^\vee})$ on $\hat{\mathrm{D}}$, $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$.

The fact that the action of $\mathrm{G}^{\vee 2}$ factors through restriction to diagonal is easily seen to imply that the action factors canonically through a uniquely defined action of $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}(\overline{\mathrm{C}}^2_{\tilde{\mathfrak{g}}^\vee})$. Furthermore, since the isomorphism between the two actions of $\mathrm{G}^\vee$ is compatible with the tensor endomorphism E, both actions factor through a uniquely defined action of $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}})$. Finally, since the second (right monodromy) action of $\mathfrak{t}^\vee$ on $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ vanishes, the action of $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}})$ factors through $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}'})$. We denote the two actions by $\Phi_{fr}$, $\Phi'_{fr}$ respectively.

**4.4.2.** *Extending the actions to the perfect derived categories.* — Our next goal is to extend the action described in the previous subsection to complexes. We encounter the standard non-functoriality of cone issue, which we circumvent in the following way.

We use the equivalences $\mathrm{Ho}(\hat{\mathcal{T}})\xrightarrow{\sim}\hat{\mathrm{D}}$, $\mathrm{Ho}(\mathcal{T})\xrightarrow{\sim}\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$.

Assume given a finite complex $\mathcal{F}^\bullet$ of objects in $\mathrm{Coh}^{\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}}(\overline{\mathrm{C}}_{\mathrm{St}})$, where each term $\mathcal{F}^i$ is a trivial vector bundle twisted by a representation $\mathrm{U}^i$ of $\mathrm{G}^\vee\times\mathrm{T}^{\vee 2}$. Pick $\lambda,\mu\in\Lambda$ so that for each character $(\lambda_j,\mu_j)$ of $\mathrm{T}^{\vee 2}$ appearing in one of the representations $\mathrm{U}^i$ we have $\lambda+\lambda_j\in(-\Lambda^+)$, $\mu+\mu_j\in\Lambda^+$.

In view of Corollary 12 and Lemma 4(d) the functor $\Phi_{fr}(\mathcal{F}^i)\circ\mathcal{J}^l_\lambda\circ\mathcal{J}^r_\mu$ sends $\hat{\mathcal{T}}$ to $\hat{\mathcal{P}}$, where $\mathcal{J}^l_\lambda:\mathrm{X}\mapsto\mathcal{J}_\lambda*\mathrm{X}$, $\mathcal{J}^r_\mu:\mathrm{X}\mapsto\mathrm{X}*\mathcal{J}_\mu$; thus one gets a functor $\mathrm{Ho}(\hat{\mathcal{T}})\to$

$\mathrm{Ho}(\hat{\mathcal{P}})$ sending a complex $\mathrm{T}^\bullet$ to the total complex of the bicomplex $\Phi_{fr}(\mathcal{F}^\bullet) \circ \mathcal{J}^l_\lambda \circ \mathcal{J}^r_\mu(\mathrm{T}^\bullet)$.

We now define a functor $\hat{\mathrm{D}} \to \hat{\mathrm{D}}$ as the composition:

$$\hat{\mathrm{D}} \xrightarrow{\mathcal{J}^l_{-\lambda}\circ\mathcal{J}^r_{-\mu}} \hat{\mathrm{D}} \xleftarrow{\sim} \mathrm{Ho}(\hat{\mathcal{T}}) \xrightarrow{\Phi_{fr}(\mathcal{F}^\bullet)\circ\mathcal{J}^l_\lambda\circ\mathcal{J}^r_\mu} \mathrm{Ho}(\hat{\mathcal{P}}) \to \hat{\mathrm{D}}.$$

We claim that different choices of $\lambda, \mu$ produce canonically isomorphic functors. This follows from existence of a canonical up to homotopy quasi-isomorphism $\mathcal{J}_{-\lambda} * \mathrm{T} \to \mathrm{T}'$, $\mathrm{T}' \to \mathrm{T} * \mathcal{J}_\mu$, where $\mathrm{T}$, $\mathrm{T}'$ are finite complexes of objects in $\hat{\mathcal{T}}$ and $\mathrm{T}'$ representing the object in the derived category corresponding to $\mathcal{J}_{-\lambda} * \mathrm{T}$ (respectively, $\mathrm{T} * \mathcal{J}_\mu$), $\lambda, \mu \in \Lambda^+$.

Thus we get a well defined functor $\mathrm{Ho}(\mathrm{Coh}^{\check{G}\times\check{T}^2}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}})) \to \mathrm{End}(\hat{\mathrm{D}})$. It is not hard to see from the definition that the last arrow carries a natural monoidal structure.

Let $\mathrm{Acycl}_{\mathrm{St}} \subset \mathrm{Ho}(\mathrm{Coh}^{\check{G}\times\check{T}^2}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}}))$ be the subcategory of complexes whose restriction to $\mathrm{C}_{\mathrm{St}}$ is acyclic. As in Proposition 19, the fact that the lowest weight arrow $\varpi_\lambda$ extends to a filtration by Wakimoto sheaves compatible with convolution implies that $\mathrm{Acycl}_{\mathrm{St}}$ acts on $\hat{\mathrm{D}}$ by zero. In view of (8) we have

$$\mathrm{Idem}(\mathrm{Ho}(\mathrm{Coh}^{\check{G}\times\check{T}^2}_{fr}(\overline{\mathrm{C}}_{\mathrm{St}}))/\mathrm{Acycl}) \cong \mathrm{D}^{\check{G}}_{perf}(\mathrm{St}).$$

Thus we obtain an action of $\mathrm{D}^{\check{G}}_{perf}(\mathrm{St})$ on $\hat{\mathrm{D}}$. Finally, since the action of the log monodromy endomorphism is pro-nilpotent, we conclude that the action factors through $\mathrm{D}^{\check{G}}_{perf}(\widehat{\mathrm{St}})$.

A parallel argument (with the last sentence omitted) endows $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ with an action of $\mathrm{D}^{\check{G}}_{perf}(\mathrm{St}')$.

**4.4.3.** *Compatibility between the two actions.* — For future reference we record a compatibility between the two actions.

*Lemma* **22.** — *For* $\mathcal{F} \in \mathrm{D}^{\check{G}}_{perf}(\widehat{\mathrm{St}})$, $\mathrm{X} \in \hat{\mathrm{D}}$ *and* $\mathrm{Y} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ *we have canonical isomorphisms*

$$\pi_*(\mathcal{F}(\mathrm{X})) \cong i^*_{\mathrm{St}}(\mathcal{F})(\pi_*(\mathrm{X})),$$
$$\pi^*(i^*_{\mathrm{St}}(\mathcal{F})(\mathrm{Y})) \cong \mathcal{F}(\pi^*(\mathrm{Y})),$$

*where* $i_{\mathrm{St}}$ *denotes the closed embedding* $\mathrm{St}' \to \widehat{\mathrm{St}}$. *The isomorphism is functorial in* $\mathcal{F}$, $\mathrm{X}$, $\mathrm{Y}$ *it is also compatible with the monoidal structure of the action functor.*

*Proof.* — Comparing the procedures of extending the action to the category of complexes for $\hat{\mathrm{D}}$ and $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ and using that $\pi_*$ sends $\hat{\mathcal{T}}$ into $\mathcal{T}$ we see that to get the first isomorphism it suffices to construct a functorial isomorphism for $\mathcal{F} \in \mathrm{Coh}^{\check{G}}_{fr}(\widehat{\mathrm{St}})$. This follows from $\pi_*(\hat{Z}_V) = Z_V$, $\pi_*(\mathcal{J}_\lambda) = J_\lambda$, where the second isomorphism is compatible with the log monodromy endomorphism and the last two isomorphisms are compatible with the lowest weight arrows. The second isomorphism can be deduced using the adjunction

$$\mathrm{Hom}(\mathcal{F}(X), X') \cong \mathrm{Hom}(X, \mathcal{F}^*(X'))$$

which holds for both actions; here $\mathcal{F}^* = R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{O})$. In view of the isomorphism $i^*(\mathcal{F}^*) \cong (i^*(\mathcal{F}))^*$ the second isomorphism follows from the first one. □

## 5. The anti-spherical projector

Set $\hat{\Xi} = \hat{T}_{w_0}$, $\Xi = T_{w_0}$.

Recall that $D_{IW}^{I^0}$ is the derived category of Iwahori-Whittaker sheaves on $\widetilde{\mathcal{F}\ell}$. We have *averaging functors* $\mathrm{Av}^{IW} : D_{I^0I^0} \to D_{IW}^{I^0}$ and $\mathrm{Av}_{IW}^{I^0} : D_{IW}^{I^0} \to D_{I^0I^0}$.

### 5.1. $\hat{\Xi}$ *and Whittaker averaging.*

*Proposition* **23.** — *(a) Right convolution with* $\hat{\Xi}$ *is isomorphic to* $\mathrm{Av}_{IW}^{I^0} \circ \mathrm{Av}^{IW}$.

*(b) Convolution with* $\hat{\Xi}$ *is isomorphic to its left and right adjoint.*

*(c) The full subcategory in* $\hat{\mathcal{T}}$ *consisting of direct sums of copies of* $\hat{\Xi}$ *is a subcategory closed under the convolution product. It is tensor equivalent to the full subcategory in* $\mathrm{Coh}(\widehat{\mathfrak{t}^* \times_{\mathfrak{t}^*/W_f} \mathfrak{t}^*})$ *whose objects are sheaves isomorphic to* $\mathcal{O}^{\oplus N}$ *for some* $N$*; here "hat" stands for completion at zero.*

*(d) Consider the full subcategory* $\mathcal{P}_{I^0I^0}^{fin} \subset \mathcal{P}_{I^0I^0}$ *of sheaves supported on* $G/U \subset \widetilde{\mathcal{F}\ell}$*, and let* $\overline{\mathcal{P}_{I^0I^0}^{fin}}$ *be its Serre quotient by the Serre subcategory generated by all irreducible objects except for* $L_e$*, the irreducible object supported on the closed cell* $\widetilde{\mathcal{F}\ell}_e$*. Then the endofunctor of* $\overline{\mathcal{P}_{I^0I^0}^{fin}}$ *induced by the functor* $\mathcal{F} \mapsto \mathcal{F} * \hat{\Xi}$ *is isomorphic to the functor* $\mathcal{O}(\check{\mathfrak{t}}) \otimes_{\mathcal{O}(\check{\mathfrak{t}})^{W_f}} \mathcal{F}$*, where* $\mathcal{O}(\check{\mathfrak{t}})^{W_f} \subset \mathcal{O}(\check{\mathfrak{t}})$ *acts on* $\overline{\mathcal{P}_{I^0I^0}^{fin}}$ *by log monodromy with respect to the right* $T$ *action.*

*Proof.* — Part (a) follows from [23, Lemma 4.4.11(3)]. To check part (b) we use part (a) and adjunctions in [23, Lemma 4.4.5], which show that the right adjoint to the functor $\mathrm{Av}_{IW}^{I^0} \circ \mathrm{Av}^{IW}$ is isomorphic to $(\mathrm{Av}_{IW}^{I^0})_* \circ \mathrm{Av}^{IW}$; here $(\mathrm{Av}_{IW}^{I^0})_*$ is the right adjoint to the pull-back functor defined using the $*$ direct image. The isomorphism $\mathrm{Av}_{IW}^{I^0} \circ \mathrm{Av}^{IW} \cong (\mathrm{Av}_{IW}^{I^0})_* \circ \mathrm{Av}^{IW}$ follows from the relation between the two convolutions on $D_{I^0I^0}$ explained in footnote 4 and [23, Corollary 5.4.3].

Part (c) is a consequence of [23, Proposition 4.7.3].

Part (d) follows from parts (b,c) since $\hat{\Xi}$ is the projective cover of $L_e$ in $\mathcal{P}_{I^0I^0}^{fin}$. □

*Remark.* — Beilinson-Bernstein Localization Theorem identifies $\mathcal{P}^{fin}_{I^0I^0}$ with category O of modules over the Lie algebra $\mathfrak{g}$ with a regular integral generalized central character $\lambda$. It is easy to see that convolution with $\hat{\Xi}$ is a projective functor isomorphic to the composition of two translation functors: translation from $\lambda$ to the singular central character $-\rho$ followed by translation from $-\rho$ to $\lambda$. Properties (b,d) of this functor are well known and play a central role in Soergel bimodules method.

**5.2.** *Tilting property of* $\hat{\Xi} * Z_\lambda$.

*Proposition* **24.** — *For* $V \in \mathrm{Rep}(G\check{})$ *we have*

(a) $\Xi * Z_V \in \mathcal{T}$.
(b) $\hat{\Xi} * \hat{Z}_V \in \hat{\mathcal{T}}$.

*Proof.* — Recall that $\mathcal{P}_{II} \subset D_{II}$ is the category of perverse sheaves, let ${}^f\mathcal{P}_{II}$ be the Serre quotient of $\mathcal{P}_{II}$ by the Serre subcategory generated by irreducible objects with support $\overline{\mathcal{F}\ell_w}$, where $w$ is not the minimal length element in its coset $W_f w$. Let $\bar{Z}_V$ be the image of $Z_V$ in ${}^f\mathcal{P}_{II}$. It follows from [1, Theorem 7] together with [1, Theorem 2] that $\bar{Z}_V$ admits a standard and a costandard filtration. Here a filtration is called (co)standard if its associated graded is a sum of $\overline{j_{w!}}$, (respectively, $\overline{j_{w*}}$), where $\overline{j_{w!}}, \overline{j_{w*}}$ is the image of $j_{w!}$, respectively $j_{w*}$, under the projection $\mathcal{P}_{II} \to^f \mathcal{P}_{II}$. It is easy to see (either by combining [1, Lemma 4(a)] with a "left–right swap" of Proposition 23(a), or directly) that the functor $\mathcal{P}_{II} \to \mathcal{P}_{I^0I}$, $\mathcal{F} \mapsto \Xi * \mathcal{F}$ factors through ${}^f\mathcal{P}_{II}$. It follows that $\Xi * Z_V$ admits a filtration whose subquotients are of the form $\Xi * j_{w!}$ and another one with subquotients of the form $\Xi * j_{w*}$. It is also easy to see that $\Xi * j_{w!}$ carries a filtration such that $gr(\Xi * j_{w!}) \cong \bigoplus_{v \in W_f} j_{vw!}$ and similarly for $\Xi * j_{w*}$. This proves part (a).

Part (b) follows from Proposition 9, compatibility of central functors with direct image and part (a) of this proposition. □

*Remark.* — The proof of Proposition 24 is the only place in this article where we use the results of [1] directly, without applying the "left–right swap".

*Corollary* **25.** — *For* $T \in \mathcal{T}$, $\hat{T} \in \hat{\mathcal{T}}$ *we have*

$$\mathrm{Ext}^{\neq 0}(\mathcal{J}_\lambda * \Xi * J_\mu * Z_\nu, T) = 0,$$

$$\mathrm{Ext}^{\neq 0}(\mathcal{J}_\lambda * \hat{\Xi} * \mathcal{J}_\mu * \hat{Z}_\nu, \hat{T}) = 0$$

*provided* $(-\lambda), (-\mu) \in \Lambda^+$, *i.e.* $\lambda$, $\mu$ *are anti-dominant.*

*Proof.* — We have

$$\mathrm{Ext}^\bullet(\mathcal{J}_\lambda * \hat{\Xi} * \mathcal{J}_\mu * \hat{Z}_\nu, \hat{T}) \cong \mathrm{Ext}^\bullet(\mathcal{J}_\lambda * \hat{\Xi} * \hat{Z}_\nu, \hat{T} * \mathcal{J}_{-\mu}).$$

Comparing Proposition 24 with Corollary 12 we see that $\mathcal{J}_\lambda * \hat{\Xi} * \hat{Z}_\nu$ admits a free monodromic standard filtration, while $\hat{T} * \mathcal{J}_{-\mu}$ admits a free-monodromic costandard filtration, which implies the second vanishing. The first one is similar. □

**5.3.** *Convolution with $\hat{\Xi}$ and the Springer map.* — We let $p_{\mathrm{Spr}}$ denotes the projection $\tilde{\mathfrak{g}}^\vee \to \mathfrak{g}^\vee$.

*Proposition* **26.** — *The equivalence (6) intertwines the endo-functor $\mathcal{F} \mapsto \hat{\Xi} * \mathcal{F}$ with the endo-functor $p_{\mathrm{Spr}}^* p_{\mathrm{Spr}*}$.*

We start with

*Lemma* **27.** — *Recall that $\widehat{\Phi}_{\mathrm{IW}}$ denotes the equivalence $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \cong \hat{\mathrm{D}}_{\mathrm{IW}}$.*
*(a) The object $\widehat{\Phi}_{\mathrm{IW}}^{-1}(\mathrm{Av}^{\mathrm{IW}}(\hat{\Xi}))$ is canonically isomorphic to $\mathcal{O}(\mathfrak{t}^\vee) \otimes_{\mathcal{O}(\mathfrak{t}^\vee)^{\mathrm{W}_f}} \mathcal{O}$.*
*(b) The composed functor $\mathcal{F} \mapsto \hat{\Xi} * (\widehat{\Phi}_{\mathrm{IW}} \circ p_{\mathrm{Spr}}^*(\mathcal{F}))$ is isomorphic to the functor $\mathcal{F} \mapsto \widehat{\Phi}_{\mathrm{IW}}(\mathcal{O}(\mathfrak{t}^\vee) \otimes_{\mathcal{O}(\mathfrak{t}^\vee/\mathrm{W}_f)} p_{\mathrm{Spr}}^*)$.*
*(c) For $\mathcal{F} \in \hat{\mathrm{D}}_{\mathrm{IW}}$ we have $\hat{\Xi} * \mathcal{F} = 0$ iff $(\widehat{\Phi}_{\mathrm{IW}})^{-1}(\mathcal{F}) \in \mathrm{Ker}(p_{\mathrm{Spr}*})$.*

*Proof of Lemma 27.* — (a) The restriction of the functor $\mathrm{Av}^{\mathrm{IW}}$ to the category $\mathcal{P}^{fin}_{\mathrm{I}^0\mathrm{I}^0}$ factors through the category $\overline{\mathcal{P}^{fin}_{\mathrm{I}^0\mathrm{I}^0}}$ (notations of Proposition 23). Thus Proposition 23(d) shows that $\mathrm{Av}^{\mathrm{IW}}(\hat{\Xi}) \cong \mathcal{O}(\mathfrak{t}^\vee) \otimes_{\mathcal{O}(\mathfrak{t}^\vee)^{\mathrm{W}}} \mathrm{Av}^{\mathrm{IW}}(\Delta_e)$. Since $\mathrm{Av}^{\mathrm{IW}}(\Delta_e) \cong \widehat{\Phi}_{\mathrm{IW}}(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee})$, the claim follows.

(b) The functor $\widehat{\Phi}_{\mathrm{IW}} \circ p_{\mathrm{Spr}}^* : \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\hat{\mathfrak{g}}^\vee)) \to \hat{\mathrm{D}}_{\mathrm{IW}}$ comes from the *central* action of $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\hat{\mathfrak{g}}^\vee))$ on $\hat{\mathrm{D}}$. Since this action commutes with the functor of convolution with $\hat{\Xi}$, (b) follows from (a).

(c) The kernel of $p_{\mathrm{Spr}*}$ is the (right) orthogonal to the objects $\mathcal{O} \otimes \mathrm{V}$, $\mathrm{V} \in \mathrm{Rep}(\mathrm{G}^\vee)$. So we need to show that $\hat{\Xi} * \mathcal{F} = 0 \iff \mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}(\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \mathcal{F}) = 0$ for all $\lambda \in \Lambda^+$. First, if $\hat{\Xi} * \mathcal{F} = 0$ then by self-adjointness of convolution with $\hat{\Xi}$, $\mathrm{Hom}(\hat{\Xi} * \mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \mathcal{F}) = 0$. We have $\hat{\Xi} * \hat{Z}_\lambda \cong \hat{Z}_\lambda * \hat{\Xi}$ and $\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda * \hat{\Xi})$ admits a filtration where each subquotient is isomorphic to $\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda)$. By a standard argument (see e.g. [15, Lemma 5]) it follows that $\mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}(\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \mathcal{F}) = 0$. Conversely, suppose that $\hat{\Xi} * \mathcal{F} \neq 0$. We need to show that $\mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}(\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \mathcal{F}) \neq 0$ for some $\lambda$. Without loss of generality we can assume that $\mathcal{F} \in \hat{\mathcal{P}}_{\mathrm{IW}}$ (recall that convolution with $\hat{\Xi}$ is exact). Then, since $\mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}(\nabla_w^{\mathrm{IW}}, \hat{\Xi} * \mathcal{F})$ depends only on the 2-sided coset $\mathrm{W}_f w \mathrm{W}_f$, we see that $\mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}(\nabla_w^{\mathrm{IW}}, \hat{\Xi} * \mathcal{F}) \neq 0$ for some $w$ which is maximal in its 2-sided $\mathrm{W}_f$-coset. Using the tilting property of $\hat{\Xi} * \hat{Z}_\lambda$ one sees that for such $w$ the object $\nabla_w^{\mathrm{IW}}$ is a quotient of $\hat{\Xi} * \mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda)$ if $\lambda \in \mathrm{W}_f w \mathrm{W}_f$. Thus $\mathrm{Hom}(\hat{\Xi} * \mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \hat{\Xi} * \mathcal{F}) \neq 0$, hence $\mathrm{Hom}(\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda * \hat{\Xi}), \mathcal{F}) \neq 0$ and $\mathrm{Hom}(\mathrm{Av}^{\mathrm{IW}}(\hat{Z}_\lambda), \mathcal{F}) \neq 0$. □

*Proof of Proposition 26.* — Set $\mathrm{F}_{\hat{\Xi}} : \mathcal{F} \to \widehat{\Phi}_{\mathrm{IW}}^{-1}(\hat{\Xi} * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{F}))$. Our goal is to show that $\mathrm{F}_{\hat{\Xi}} \cong \mathrm{F}_{\mathrm{Spr}}$, where $\mathrm{F}_{\mathrm{Spr}} := p_{\mathrm{Spr}}^* p_{\mathrm{Spr}*}$. Notice that both functors are self-adjoint: for $\mathrm{F}_{\hat{\Xi}}$ this is Proposition 23(b) and for $\mathrm{F}_{\mathrm{Spr}}$ this follows from the fact that both $\tilde{\mathfrak{g}}^\vee$ and $\mathfrak{g}^\vee$ have trivial canonical bundle and their dimensions coincide, which yields an isomorphism $p_{\mathrm{Spr}}^* \cong p_{\mathrm{Spr}}^!$.

Lemma 27(b) shows that $\mathrm{F}_{\hat{\Xi}} \circ p_{\mathrm{Spr}}^* \cong \mathrm{F}_{\mathrm{Spr}} \circ p_{\mathrm{Spr}}^*$ (notice that $p_{\mathrm{Spr}*} p_{\mathrm{Spr}}^*(\mathcal{F}) \cong \mathcal{O}(\mathfrak{t}^\vee) \otimes_{\mathcal{O}(\mathfrak{t}^\vee)^{W_f}} \mathcal{F}$ canonically), which implies $\mathrm{F}_{\hat{\Xi}} \circ \mathrm{F}_{\mathrm{Spr}} \cong \mathrm{F}_{\mathrm{Spr}} \circ \mathrm{F}_{\mathrm{Spr}}$. Self-adjointness of $\mathrm{F}_{\mathrm{Spr}}$ yields the adjunction arrow $\mathrm{F}_{\mathrm{Spr}} \circ \mathrm{F}_{\mathrm{Spr}} \to \mathrm{Id}$, thus we get an arrow $\mathrm{F}_{\hat{\Xi}} \circ \mathrm{F}_{\mathrm{Spr}} \to \mathrm{Id}$. Applying self-adjointness of $\mathrm{F}_{\hat{\Xi}}$ we get an arrow $c : \mathrm{F}_{\mathrm{Spr}} \to \mathrm{F}_{\hat{\Xi}}$.

Lemma 27(b) provides an isomorphism $\mathrm{F}_{\hat{\Xi}} \circ p_{\mathrm{Spr}}^* \cong \mathrm{F}_{\mathrm{Spr}} \circ p_{\mathrm{Spr}}^*$, a diagram chase shows that this isomorphism coincides with the arrow induced by $c$. Also, Lemma 27(c) shows that $\mathrm{F}_{\hat{\Xi}}|_{\mathrm{Ker}(\mathrm{F}_{\mathrm{Spr}})} = 0$. Thus $c_{\mathcal{F}} : \mathrm{F}_{\mathrm{Spr}}(\mathcal{F}) \to \mathrm{F}_{\hat{\Xi}}(\mathcal{F})$ is an isomorphism when $\mathcal{F} \in \mathrm{Im}(\mathrm{F}_{\mathrm{Spr}})$ or $\mathcal{F} \in \mathrm{Ker}(\mathrm{F}_{\mathrm{Spr}})$. Again using self-adjointness of $\mathrm{F}_{\mathrm{Spr}}$, $\mathrm{F}_{\hat{\Xi}}$ we see that for any $\mathcal{F}$ the object $\mathrm{Cone}(\mathrm{F}_{\mathrm{Spr}}(\mathcal{F}) \xrightarrow{c} \mathrm{F}_{\hat{\Xi}}(\mathcal{F}))$ lies in the left orthogonal to both $\mathrm{Ker}(\mathrm{F}_{\mathrm{Spr}})$ and $\mathrm{Im}(\mathrm{F}_{\mathrm{Spr}})$. However, ${}^\perp\mathrm{Im}(\mathrm{F}_{\mathrm{Spr}}) = \mathrm{Ker}(\mathrm{F}_{\mathrm{Spr}})$ due to self-adjointness of $\mathrm{F}_{\mathrm{Spr}}$; thus ${}^\perp\mathrm{Ker}(\mathrm{F}_{\mathrm{Spr}}) \cap {}^\perp \mathrm{Im}(\mathrm{F}_{\mathrm{Spr}}) = 0$, which shows that $\mathrm{F}_{\mathrm{Spr}}(\mathcal{F}) \underset{c}{\xrightarrow{\sim}} \mathrm{F}_{\hat{\Xi}}(\mathcal{F})$ for all $\mathcal{F}$. □

## 6. Properties of $\Phi_{perf}$

Recall the actions defined in Section 4.4.2 and objects $\hat{\Xi} = \hat{\mathrm{T}}_{w_0}$, $\Xi = \mathrm{T}_{w_0}$. We define $\widehat{\Phi}_{perf} : \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}}) \to \hat{\mathrm{D}}$, $\widehat{\Phi}_{perf}(\mathcal{F}) = \mathcal{F}(\hat{\Xi})$ and $\Phi_{perf} : \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\mathrm{St}') \to \mathrm{D}_{\mathrm{I}^0\mathrm{I}}$, $\Phi_{perf}(\mathcal{F}) = \mathcal{F}(\Xi)$.

**6.1.** *Compatibility of $\Phi_{perf}$ with projection* $\mathrm{St} \to \tilde{\mathfrak{g}}^\vee$. — We start by recording some of the compatibilities following directly from the definitions.

*Lemma* **28.** — *The following diagrams commute up to a natural isomorphism:*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}}) & \xrightarrow{i^*} & \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\mathrm{St}') \\ \Big\downarrow{\scriptstyle \widehat{\Phi}_{perf}} & & \Big\downarrow{\scriptstyle \Phi_{perf}} \\ \hat{\mathrm{D}} & \xrightarrow{\pi_*} & \mathrm{D} \end{array}$$

$$\begin{array}{ccc} \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) & \xrightarrow{pr^*_{\mathrm{Spr},1}} & \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}}) \\ \Big\downarrow{\scriptstyle \widehat{\Phi}_{\mathrm{IW}}} & & \Big\downarrow{\scriptstyle \widehat{\Phi}_{perf}} \\ \hat{\mathrm{D}}_{\mathrm{IW}} & \xrightarrow{\mathrm{Av}^{right}_{\mathrm{I}^0}} & \hat{\mathrm{D}} \end{array}$$

$$\begin{array}{ccc} D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) & \xrightarrow{(pr'_{\mathrm{Spr},1})^*} & D^{G^\vee}_{perf}(\mathrm{St}')V \\ \downarrow{\scriptstyle \widehat{\Phi}_{\mathrm{IW}}} & & \downarrow{\scriptstyle \Phi_{perf}} \\ \hat{D}_{\mathrm{IW}} & \xrightarrow{\pi_*\circ \mathrm{Av}^{right}_{I^0}} & D \end{array}$$

*Proof.* — Commutativity of the first diagram follows from the corresponding compatibility for action (Lemma 22) and the isomorphism $\pi_*(\hat{\Xi}) \cong \Xi$ (Proposition 11(b)). To see commutativity of the second one observe that the functor $\mathrm{Av}^{right}_{I^0}$ of averaging with respect to the right action of $I^0$ commutes with convolution on the left. For $\mathcal{F} \in D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$ the object $pr^*_{\mathrm{Spr},1}(\mathcal{F}) \in D^{G^\vee}_{perf}(\widehat{\mathrm{St}})$ acts on $\hat{D}$ by the left convolution with $\Phi_{diag}(\mathcal{F})$, thus the required commutativity follows the isomorphism $\mathrm{Av}^{right}_{I^0}(\Delta^{\mathrm{IW}}_e) \cong \hat{\Xi}$, which is a consequence of Proposition 23(a). The third diagram is obtained by concatenation of the first two: the left (respectively, right) vertical arrow in the third diagram coincides with the left (right) arrow in the second (respectively, first) one, the horizontal arrows are compositions of the corresponding horizontal arrows in the first two. □

The goal of this subsection is the following

*Proposition* **29.** — *The functor $\widehat{\Phi}_{perf}$ is compatible with the convolution action of $D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}}))$ on $D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$ and the action of $\hat{D}$ on $\hat{D}_{\mathrm{IW}}$; i.e. for $\mathcal{F} \in D^{G^\vee}_{perf}(\widehat{\mathrm{St}})$, $\mathcal{G} \in D^b(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$ we have an isomorphism*

$$\widehat{\Phi}_{\mathrm{IW}}(\mathcal{F} * \mathcal{G}) \cong \widehat{\Phi}_{perf}(\mathcal{F}) * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G}) \qquad (\mathbf{14})$$

*functorial in $\mathcal{F}, \mathcal{G}$.*

*Proof.* — When $\mathcal{F} \cong \mathcal{O}$, so that $\widehat{\Phi}(\mathcal{F}) \cong \hat{\Xi}$, the isomorphism (for any $\mathcal{G}$) is provided by Proposition 26. Since the functors commute with twist either by a line bundle or by a representation of $G^\vee$ we get an isomorphism for $\mathcal{F}$ of the form $\mathcal{O}(\lambda, \mu) \otimes V$, $V \in \mathrm{Rep}(G^\vee)$, this isomorphism is functorial in $\mathcal{F}, \mathcal{G}$.

By a standard argument[10] (attributed, in particular, to Kontsevich, see also [43], Theorem 2.1 and Example 1.10) any object in $D^{G^\vee}_{perf}(\widehat{\mathrm{St}})$ is a direct summand in one represented by a finite complex of sheaves of the form $\mathcal{O}(\lambda_i, \mu_i) \otimes V_i$, where $\lambda_i, \mu_i$ are antidominant, thus we can assume without loss of generality that $\mathcal{F}$ is of this form.

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[10] More generally, for a reductive group H acting linearly on $\mathbf{A}^{N+1}$ and an H-invariant locally closed subscheme $X \subset \mathbf{P}^N$ every object $\mathcal{F}$ in the perfect equivariant derived category $D^H_{perf}(X)$ is a direct summand in an object represented by a finite complex of equivariant bundles of the form $\oplus V_i \otimes \mathcal{O}_X(n_i)$, $V \in \mathrm{Rep}(H)$. To see this one constructs a bounded above complex $\mathcal{F}^\bullet$ whose terms are finite sums of bundles $V_i \otimes \mathcal{O}_X(n_i)$ representing $\mathcal{F}$, then denoting by $\mathcal{F}_{\geq -N}$ the "stupid" truncation of $\mathcal{F}^\bullet$ we get for $N \gg 0$ a distinguished triangle (15). For large N we have $\mathrm{Ext}^{N+1}(\mathcal{F}, \mathcal{F}_N) = 0$, so the triangle splits.

Pick $\nu \in \Lambda^+$ such that $\mu_i + \nu \in \Lambda^+$ for all $i$. We can choose a finite complex of free-monodromic tilting objects in $\hat{\mathcal{P}}_{\mathrm{IW}}$ representing $\mathcal{J}_{-\nu} * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G})$, then $\widehat{\Phi}(\mathcal{G})$ is represented by a finite complex of objects $\mathcal{J}_\nu * \hat{\mathrm{T}}_j$, where $\hat{\mathrm{T}}_j \in \hat{\mathcal{P}}_{\mathrm{IW}}$ is free-monodromic tilting.

We claim that

$$(\mathcal{J}_{\lambda_i} * \hat{\Xi} * \mathcal{J}_{\mu_i}) * (\mathcal{J}_\nu * \hat{\mathrm{T}}_j) \in \hat{\mathcal{P}}_{\mathrm{IW}},$$

$$\widehat{\Phi}_{\mathrm{IW}}^{-1}((\mathcal{J}_{\lambda_i} * \hat{\Xi} * \mathcal{J}_{\mu_i}) * (\mathcal{J}_\nu * \hat{\mathrm{T}}_j)) \in \mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}).$$

Here the first claim follows from Lemma 4(d) the second one follows from Lemma 30(a) below.

Now (14) follows by comparing Proposition 7(c) to Corollary 47(c) below. □

*Lemma* **30.** — *(a)* $(\widehat{\Phi}_{\mathrm{IW}})^{-1}(\hat{\Xi} * \mathcal{J}_\mu * \hat{\mathrm{T}}) \in \mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})$ *for* $\mu \in \Lambda^+$ *and* $\hat{\mathrm{T}} \in \hat{\mathcal{P}}_{\mathrm{IW}}$ *a free-monodromic tilting object.*

*(b)* $(\Phi^{\mathrm{I}}_{\mathrm{IW}})^{-1}(j^{\mathrm{IW}}_{w*}) \in \mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}})$ *for any* $w \in \mathrm{W}/\mathrm{W}_f$.

*(c)* $(\widehat{\Phi}_{\mathrm{IW}})^{-1}(\nabla^{\mathrm{IW}}_w) \in \mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})$ *for any* $w \in \mathrm{W}/\mathrm{W}_f$.

*(d)* $\Phi_{\mathrm{IW}}^{-1} : \mathcal{P}_{\mathrm{IW}} \to \mathrm{D}^{\geq 0}(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}})) \cap \mathrm{D}^{\leq \dim(\tilde{\mathcal{N}})}(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}}))$;
$\widehat{\Phi}_{\mathrm{IW}}^{-1} : \hat{\mathcal{P}}_{\mathrm{IW}} \to \mathrm{D}^{\geq 0}(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}})) \cap \mathrm{D}^{\leq \dim(\tilde{\mathfrak{g}}^\vee)}(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$.

*Proof.* — (a) An object $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$ lies in the abelian heart iff for large $\lambda$ we have $\mathrm{R}^i\Gamma(\mathcal{F} \otimes \mathcal{O}(\lambda)) = 0$ for $i \neq 0$. Since $\mathrm{R}^i\Gamma(\mathcal{F}) = \mathrm{Hom}^{\mathrm{G}^\vee}_{deeq}(\mathcal{O}, \mathcal{F})$, it suffices to show that $\mathrm{Hom}^i_{\hat{\mathcal{P}}_{\mathrm{IW}}}(\mathcal{J}_{-\nu} * \mathrm{Av}^{\mathrm{IW}}(\hat{\mathrm{Z}}_\lambda), \hat{\Xi} * \mathcal{J}_\mu * \hat{\mathrm{T}}) = 0$ for $i \neq 0$ and $\lambda, \mu, \nu \in \Lambda^+$. Using Propositions 24(b) and 11(c) we see that $\mathcal{J}_{-\nu} * \mathrm{Av}^{\mathrm{IW}}(\hat{\mathrm{Z}}_\lambda)$ has a free-monodromic standard filtration, while $\mathcal{J}_\mu * \hat{\mathrm{T}}$ and hence $\hat{\Xi} * \mathcal{J}_\mu * \hat{\mathrm{T}}$ has a free-monodromic costandard filtration, this implies the desired vanishing.

Similarly, the first statement in (b) follows from $\mathrm{Ext}^i_{\mathrm{D}^{\mathrm{I}}_{\mathrm{IW}}}({}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}(\mathrm{J}_{-\lambda} * \mathrm{Z}_\mu), j^{\mathrm{IW}}_{w*}) = 0$ for $i \neq 0$, $\lambda \in \Lambda^+$. The latter Ext vanishing is clear from the fact that ${}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}(\mathrm{Z}_\mu)$ is tilting in $\mathcal{P}^{\mathrm{I}}_{\mathrm{IW}}$ [1, Theorem 7], hence ${}^{\mathrm{I}}\mathrm{Av}^{\mathrm{IW}}(\mathrm{J}_{-\lambda} * \mathrm{Z}_\mu)$ admits a costandard filtration. The proof of (c) is parallel to that of (b), with (co)standard replaced by free monodromic (co)standard. The inclusion $\Phi_{\mathrm{IW}}^{-1}(\mathcal{P}_{\mathrm{IW}}) \subset \mathrm{D}^{\geq 0}(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}}))$ follows from part (b). To check the other inclusion we use that $\Phi_{\mathrm{IW}}(\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \otimes \mathrm{V}) \in \mathcal{P}_{\mathrm{IW}}$ for all $\lambda \in \Lambda$, $\mathrm{V} \in \mathrm{Rep}(\mathrm{G}^\vee)$ thus for $\mathcal{F} \in \mathcal{P}_{\mathrm{IW}}$ we have $\mathrm{Ext}^{<0}_{\mathrm{Coh}(\tilde{\mathcal{N}})}(\Phi_{\mathrm{IW}}^{-1}(\mathcal{F}), \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = 0$. Applying Grothendieck-Serre duality we conclude that $\mathrm{Ext}^{<0}_{\mathrm{Coh}(\tilde{\mathcal{N}})}(\mathcal{O}(\lambda), \mathrm{S}(\Phi_{\mathrm{IW}}^{-1}(\mathcal{F}))) = 0$, where $\mathrm{S}(\mathcal{G}) = \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{O})$. Thus $\mathrm{S}(\Phi_{\mathrm{IW}}^{-1}(\mathcal{F})) \in \mathrm{D}^{\geq 0}(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}}))$, so $\Phi_{\mathrm{IW}}^{-1}(\mathcal{F}) = \mathrm{S}(\mathrm{S}(\Phi_{\mathrm{IW}}^{-1}(\mathcal{F}))) \in \mathrm{D}^{\leq \dim \tilde{\mathcal{N}}}$. This proves the first formula in (d), the second one is checked in a similar way. □

**6.2.** *The functors* $\Phi_{perf}$, $\widehat{\Phi}_{perf}$ *are fully faithful.* — In this subsection we establish full faithfulness of $\Phi_{perf}$, $\widehat{\Phi}_{perf}$. Since $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\mathrm{St}'}(\mathrm{St}))$ is a full subcategory in $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}}))$, while $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ is a full subcategory in $\hat{\mathrm{D}}$, it is enough to do so for $\widehat{\Phi}_{perf}$ only.

It suffices to show that the map

$$\begin{aligned}&\mathrm{Hom}^\bullet(\mathrm{V}\otimes\mathcal{O}_{\widehat{\mathrm{St}}}(\lambda,\mu),\mathrm{V}'\otimes\mathcal{O}_{\widehat{\mathrm{St}}}(\lambda',\mu'))\\&\quad\to\mathrm{Hom}^\bullet(\hat{\mathrm{Z}}_{\mathrm{V}}*\mathcal{J}_\lambda*\hat{\Xi}*\mathcal{J}_\mu,\hat{\mathrm{Z}}_{\mathrm{V}'}*\mathcal{J}_{\lambda'}*\hat{\Xi}*\mathcal{J}_{\mu'})\end{aligned}$$

induced by $\widehat{\Phi}_{perf}$ is an isomorphism.

The functor $\widehat{\Phi}_{perf}$ sends twisting by a line bundle to convolution by Wakimoto sheaves, and twisting by a representation of $\mathrm{G}^\vee$ to the central functor. Since adjoint to such a twist is twist by the dual representation, and similar adjunction holds for the central functors and convolution by Wakimoto sheaves, we see that it suffices to consider the case when $\lambda=0=\mu'$ and V is trivial.

Then we have:

$$\begin{aligned}&\mathrm{Hom}_{\hat{\mathrm{D}}}(\hat{\Xi}*\mathcal{J}_\mu,\mathcal{J}_{\lambda'}*\hat{\mathrm{Z}}_{\mathrm{V}'}*\hat{\Xi})\\&\quad\cong\mathrm{Hom}_{\hat{\mathrm{D}}_{\mathrm{IW}}}\big(\mathrm{Av}^{\mathrm{IW}}(\hat{\Xi}*\mathcal{J}_\mu),\mathrm{Av}^{\mathrm{IW}}(\mathcal{J}_{\lambda'}*\hat{\mathrm{Z}}_{\mathrm{V}'})\big)\\&\quad\cong\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee))}\big(p^*_{\mathrm{Spr}}p_{\mathrm{Spr}*}\big(\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\mu)\big),\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\lambda')\otimes\mathrm{V}'\big)\\&\quad\cong\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\hat{\tilde{\mathfrak{g}}}^\vee))}\big(p_{\mathrm{Spr},2*}p^*_{\mathrm{Spr},1}\big(\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\mu)\big),\mathcal{O}_{\hat{\tilde{\mathfrak{g}}}^\vee}(\lambda')\otimes\mathrm{V}'\big).\end{aligned}$$

Here the first isomorphism comes from the fact that right convolution with $\hat{\Xi}$ is isomorphic to $\mathrm{Av}^{\mathrm{I}^0}_{\mathrm{IW}}\circ\mathrm{Av}^{\mathrm{IW}}$ (Proposition 23(a)). The second isomorphism uses the "coherent" description of the Iwahori-Whittaker category (6) along with the fact that left convolution with $\hat{\Xi}$ corresponds to $p^*_{\mathrm{Spr}}p_{\mathrm{Spr}*}$ on the coherent side (Proposition 26). Finally, the last isomorphism comes from: $p^*_{\mathrm{Spr}}p_{\mathrm{Spr}*}\cong p_{\mathrm{Spr},2*}p^*_{\mathrm{Spr},1}$, which follows from base change for coherent sheaves and the fact that $\mathrm{Tor}^{\mathcal{O}(\mathfrak{g}^\vee)}_{>0}(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee},\mathcal{O}_{\tilde{\mathfrak{g}}^\vee})=0$.

Using adjointness we get:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathfrak{g}}^\vee))}\big(pr_{2*}pr_1^*\big(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\mu)\big),\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\lambda')\otimes\mathrm{V}'\big)\\&\quad\cong\mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}(\mathrm{St}))}\big(pr_1^*\big(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\mu)\big),pr_2^*\big(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\lambda')\otimes\mathrm{V}'\big)\big),\end{aligned}$$

where we used that $pr_2^*\cong pr_2^!$ since the target of $pr_2$ is smooth, while both its source and target have trivial dualizing complexes (more precisely, in both cases the dualizing complex is isomorphic to $\mathcal{O}[d]$, $d=\dim(\tilde{\mathfrak{g}}^\vee)=\dim(\mathrm{St})$).

Since $\widehat{\Phi}_{perf}:pr_1^*(\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\mu))\mapsto\mathcal{J}_\mu*\hat{\Xi}$, $\widehat{\Phi}_{perf}:\mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\lambda')\otimes\mathrm{V}'\mapsto\mathcal{J}_{\lambda'}*\hat{\mathrm{Z}}_{\mathrm{V}'}*\hat{\Xi}$, we have constructed an isomorphism between the two Hom spaces. A routine diagram chase shows that this isomorphism coincides with the map induced by $\widehat{\Phi}_{perf}$.

## 7. Extending an equivalence from the subcategory of perfect complexes

**7.1.** *A criterion for representability.* — Let algebraic stack X be given by $X = Z/H$ where Z is a quasiprojective scheme over an algebraically closed field of characteristic zero and H is a reductive group. [The results of this section are likely valid in greater generality but we present the setting needed for our applications]. We fix a H-equivariant ample line bundle L on Z, such a bundle exists by Sumihiro embedding Theorem (though in examples considered in this paper Z comes equipped with a supply of such line bundles).

Set $D = D^b(\mathrm{Coh}(X))$ and let $D_{perf}(X) \subset D$ be the subcategory of perfect complexes. Set $D_{perf}^{\leq n} = D^{\leq n}(\mathrm{Coh}(X)) \cap D_{perf}(X)$, and let $D_{perf}^{\geq n} \subset D_{perf}(X)$ be the full subcategory of objects represented by complexes of locally free sheaves placed in degree $n$ and higher, and their direct summands.

*Remark* **31.** — It is obvious that $D_{perf}^{\geq n} \subset D^{\geq n}(\mathrm{Coh}(X)) \cap D_{perf}(X)$. Using [45, Theorem 3.2.6] ("finiteness of finitistic dimension") one can also show that $D_{perf}^{\geq n}(X) \supset D^{\geq n-\dim(Z)} \cap D_{perf}$. This implies that most of the statements below hold with $D_{perf}^{\geq n}$ replaced by $D^{\geq n}(\mathrm{Coh}(X)) \cap D_{perf}(X)$. We neither prove nor use this point.

*Proposition* **32.** — *(a) The natural functor from* $D^b(\mathrm{Coh}(X))$ *to the category of contravariant functors from* $D_{perf}(X)^{op}$ *to vector spaces is fully faithful.*

*(b) A cohomological functor* F *from* $D_{perf}(X)$ *to vector spaces is represented by an object of* $D^b(\mathrm{Coh}(X))$ *if and only if the following conditions hold:*

*(i) For any* $n$ *the functor* $F|_{D_{perf}^{\geq n}}$ *is represented by an object of* $D_{perf}(X)$ *(not necessarily by an object of* $D_{perf}^{\geq n}$*).*

*(ii) There exists* $m$ *such that* $F|_{D_{perf}^{\leq m}} = 0$.

*Proof.* — Fix $\mathcal{F}, \mathcal{G} \in D^b(\mathrm{Coh}(X))$ and let $\phi_{\mathcal{F}}, \phi_{\mathcal{G}}$ be the corresponding functors on $D_{perf}(X)$. Fix a bounded above complex $\mathcal{F}^\bullet$ of locally free sheaves representing $\mathcal{F}$. Let $\mathcal{F}_{\geq -n} = \tau_{\geq -n}^{\text{bête}}(\mathcal{F}^\bullet)$ denote the stupid truncation.

Given a natural transformation $\phi_{\mathcal{F}} \to \phi_{\mathcal{G}}$ we get morphisms $\mathcal{F}_{\geq -n} \to \mathcal{G}$, compatible with the arrows $\mathcal{F}_{\geq -n} \to \mathcal{F}_{\geq -(n+1)}$. Choose $n$ such that $\mathcal{F} \in D^{>-n}(\mathrm{Coh}(X))$. Then for $N > n$ we have a canonical isomorphism $\mathcal{F} \cong \tau_{\geq -n}(\mathcal{F}_{\geq -N})$. Assuming also that $\mathcal{G} \in D^{>-n}(\mathrm{Coh}(X))$, we get an arrow $\mathcal{F} = \tau_{\geq -n}(\mathcal{F}_{\geq -N}) \to \tau_{\geq -n}(\mathcal{G}) = \mathcal{G}$. A standard argument shows that bounded above complexes representing a given $\mathcal{F} \in D^b(\mathrm{Coh}(X))$ form a filtered category (i.e. given two such complexes $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$, there exists a complex $\mathcal{F}_0^\bullet$ with maps of complexes $\mathcal{F}_0^\bullet \to \mathcal{F}_1^\bullet, \mathcal{F}_0^\bullet \to \mathcal{F}_2^\bullet$ inducing identity maps in the derived category). This implies that the arrow $\mathcal{F} \to \mathcal{G}$ does not depend on the choice of $\mathcal{F}^\bullet$.

Thus we have constructed a map $\mathrm{Hom}(\phi_{\mathcal{F}}, \phi_{\mathcal{G}}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. It is clear from the construction that the composition $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\phi_{\mathcal{F}}, \phi_{\mathcal{G}}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is the identity map. It remains to see that the map $\mathrm{Hom}(\phi_{\mathcal{F}}, \phi_{\mathcal{G}}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is injective.

Let $h \in \mathrm{Hom}(\phi_{\mathcal{F}}, \phi_{\mathcal{G}})$ be a nonzero element. Thus for some $\mathcal{P} \in \mathrm{D}_{perf}(\mathrm{X})$ and $\varphi : \mathcal{P} \to \mathcal{F}$ we have $0 \neq h(\varphi) : \mathcal{P} \to \mathcal{G}$. Fix again a complex $\mathcal{F}^\bullet$, for N as above we get a distinguished triangle

$$\mathcal{F}_{\mathrm{N}}[\mathrm{N}] \to \mathcal{F}_{\geq -\mathrm{N}} \to \mathcal{F} \to \mathcal{F}_{\mathrm{N}}[\mathrm{N}+1] \tag{15}$$

for some $\mathcal{F}_{\mathrm{N}} \in \mathrm{Coh}(\mathrm{X})$. For large N we have $\mathrm{Hom}(\mathcal{P}, \mathcal{F}_{\mathrm{N}}[\mathrm{N}+1]) = 0$, thus $\varphi$ factors through an arrow $\mathcal{P} \to \mathcal{F}_{\geq -\mathrm{N}}$. It follows that for $\mathrm{N} \gg 0$ applying $h$ to the tautological map $\mathcal{F}_{\geq -\mathrm{N}} \to \mathcal{F}$ we get a nonzero arrow $\mathcal{F}_{\geq -\mathrm{N}} \to \mathcal{G}$. Since $\mathrm{Hom}(\mathcal{F}_{\mathrm{N}}[\mathrm{N}], \mathcal{G}) = 0 = \mathrm{Hom}(\mathcal{F}_{\mathrm{N}}[\mathrm{N}+1], \mathcal{G})$ for large N, we see that the induced arrow $\mathcal{F} \to \mathcal{G}$ is nonzero. This proves (a).

We now prove (b). We first check the "only if" direction. Condition (ii) is clear, and to check condition (i) let $\mathcal{F}$ be the representing object, and choose a bounded above complex $\mathcal{F}^\bullet$ representing $\mathcal{F}$; we can and will choose $\mathcal{F}^\bullet$ so that its terms are locally free sheaves. Setting again $\mathcal{F}_{\geq \mathrm{N}} = \tau^{\text{bête}}_{\geq \mathrm{N}}(\mathcal{F}^\bullet) \in \mathrm{D}_{perf}(\mathrm{X})$, we claim that $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{G}, \mathcal{F}_{\geq \mathrm{N}})$ when $\mathcal{G} \in \mathrm{D}^{\geq m}_{perf}$, $\mathrm{N} < m - d$, where $d = \dim(\mathrm{Z})$. This follows from the fact that $\mathrm{Ext}^i(\mathcal{E}, \mathcal{K}) = 0$ for $i > d$, where $\mathcal{E}, \mathcal{K} \in \mathrm{Coh}(\mathrm{X})$ and $\mathcal{E}$ is locally free.

To check the "if" direction, given a functor F satisfying the conditions take $n$ in (i) satisfying $n < m - d$ where $m$ is as in (ii) and $d = \dim(\mathrm{Z})$. Let $\mathcal{F}' \in \mathrm{D}_{perf}(\mathrm{X})$ be a representing object for $\mathrm{F}|_{\mathrm{D}^{\geq n}_{perf}}$. We claim that $\mathcal{F} = \tau_{\geq n}(\mathcal{F}')$ represents F.

First observe that

$$\mathcal{F} \in \mathrm{D}^{>m}\big(\mathrm{Coh}(\mathrm{X})\big), \tag{16}$$

to check this we need to see that $\mathrm{H}^i(\mathcal{F}') = 0$ for $i = n, \dots, m$. If $\mathrm{H}^i(\mathcal{F}') \neq 0$ for such an $i$, we can find a locally free sheaf $\mathcal{E}$ such that $\mathrm{Hom}(\mathcal{E}, \mathrm{H}^i(\mathcal{F}')) \neq 0$ and $\mathrm{Ext}^{>0}(\mathcal{E}, \mathrm{H}^j(\mathcal{F}')) = 0$ for all $j$ (in fact, we can take $\mathcal{E} = \mathrm{L}^{\otimes \mathrm{N}} \otimes \mathrm{V}$ where L is an anti-ample H-equivariant line bundle on Z and V is a representation of H). Then we get $\mathrm{Hom}(\mathcal{E}[-i], \mathcal{F}') = \mathrm{F}(\mathcal{E}[-i]) \neq 0$, which contradicts (ii).

We now construct a functorial isomorphism $\mathrm{F}(\mathcal{G}) \cong \mathrm{Hom}(\mathcal{G}, \mathcal{F})$, $\mathcal{G} \in \mathrm{D}_{perf}(\mathrm{X})$. Fix such $\mathcal{G}$, and fix a finite complex $\mathcal{G}^\bullet$ of locally free sheaves representing $\mathcal{G}$. The desired isomorphism is obtained as the following composition:

$$\begin{aligned}\mathrm{Hom}(\mathcal{G}, \mathcal{F}) &\cong \mathrm{Hom}(\tau^{\text{bête}}_{\geq m}(\mathcal{G}^\bullet), \mathcal{F}) \cong \mathrm{Hom}(\tau^{\text{bête}}_{\geq m}(\mathcal{G}^\bullet), \mathcal{F}') \\ &\cong \mathrm{F}(\tau^{\text{bête}}_{\geq m}(\mathcal{G}^\bullet)) \cong \mathrm{F}(\mathcal{G}).\end{aligned}$$

Here the first isomorphism follows from (16), which implies that $\mathrm{Hom}(\tau^{\text{bête}}_{<m}(\mathcal{G}^\bullet), \mathcal{F}) = 0 = \mathrm{Hom}(\tau^{\text{bête}}_{<m}(\mathcal{G}^\bullet)[-1], \mathcal{F})$.

The second isomorphism follows from the distinguished triangle $\tau_{<n}(\mathcal{F}') \to \mathcal{F}' \to \mathcal{F} \to \tau_{<n}(\mathcal{F}')[1]$ and the fact that $\mathrm{Hom}(\mathrm{D}^{\geq m}_{perf}(\mathrm{Coh}(\mathrm{X})), \mathrm{D}^{\leq n}(\mathrm{Coh}(\mathrm{X}))) = 0$, since $m - n > d$ and $\mathrm{Ext}^i(\mathcal{E}, \mathcal{K}) = 0$ for $i > d$, where $\mathcal{E}, \mathcal{K} \in \mathrm{Coh}(\mathrm{X})$ and $\mathcal{E}$ is locally free.

The third isomorphism is the assumption on $\mathcal{F}'$, and the last isomorphism follows from (ii). It is easy to see that the constructed isomorphism is independent on the auxiliary choices and is functorial. □

Let $X = Z/H$ be as in the previous proposition. We assume that Z admits a projective H-equivariant morphism $Z \to Y$ where Y is affine. Let L be an H-equivariant ample line bundle on Z. We have the homogeneous coordinate ring $\hat{\mathcal{O}}(Z) = \bigoplus_{n\geq 0} \Gamma(L^{\otimes n})$. The assumptions on Z imply that $\hat{\mathcal{O}}(Z)$ is Noetherian.

We now assume that $\mathcal{C}$ is a triangulated category with a fixed full triangulated embedding $i : D_{perf}(X) \to \mathcal{C}$.

For $M \in \mathcal{C}$ we can form a module for the homogeneous coordinate ring

$$\tilde{\Psi}(M) = \bigoplus_{n\geq 0,\lambda} \mathrm{Hom}(i(L^{\otimes -n} \otimes \mathcal{O}(H)_\lambda), M),$$

where $\lambda$ runs over the set of dominant weights of H and $\mathcal{O}(H)_\lambda$ denotes the corresponding isotypic component of the translation action of H on $\mathcal{O}(H)$. A section of $L^{\otimes n}$ defines an element in $\mathrm{Hom}_{\mathrm{Coh}(X)}(L^{\otimes m} \otimes \mathcal{O}(H)_\lambda, \bigoplus_\mu L^{\otimes m+n} \otimes \mathcal{O}(H)_\mu)$ for every $\lambda$ and $m$, thus $\tilde{\Psi}(M)$ does carry a natural action of the homogeneous coordinate ring.

Notice that if $\mathcal{C}$ is equipped with a Rep(H) action making $i$ a functor of module categories for Rep(H) then we have: $\tilde{\Psi}(M) = \bigoplus_{n\geq 0} \mathrm{Hom}^{H}_{deeq}(i(L^{\otimes -n}), M)$.

We also set: $\tilde{\Psi}_m(M) = \bigoplus_{n\geq m,\lambda} \mathrm{Hom}^{H}_{deeq}(i(L^{\otimes -n} \otimes \mathcal{O}(H)_\lambda), M)$.

*Proposition* **33.** — *For* $M \in \mathcal{C}$ *the following are equivalent.*

*(a) For any m the functor on* $D^{\geq m}_{perf}(X)$, $\mathcal{F} \mapsto \mathrm{Hom}(i(\mathcal{F}), M)$ *is represented by an object of* $D_{perf}(X)$.

*(b) The module* $\tilde{\Psi}(M[n])$ *is finitely generated for all n and* $\tilde{\Psi}(M[n]) = 0$ *for* $n \gg 0$.

*(c) We have* $\tilde{\Psi}(M[n]) = 0$ *for* $n \gg 0$ *and for any n there exists m, such that* $\tilde{\Psi}_m(M[n])$ *is finitely generated.*

The proof of the proposition is based on the following

*Lemma* **34.** — *If* $\tilde{\Psi}(M[n]) = 0$ *for* $n \geq s$, *then* $\mathrm{Hom}(i(\mathcal{F}), M) = 0$ *for* $\mathcal{F} \in D^{>s+d}_{perf}$, $d = \dim(Z)$.

*Proof.* — We claim that any object in $\mathcal{F} \in D^{>s+d}_{perf}$ is isomorphic to a direct summand in an object represented by a complex placed in degree $s$ and higher, with each term isomorphic to $L^{\otimes i} \otimes V$, $i \leq 0$, $V \in \mathrm{Rep}(H)$. This clearly implies the lemma.

It remains to check that claim. Let $\mathcal{F} \in D^{>s+d}_{perf}$. By a standard argument there exists a bounded above complex $\mathcal{F}^\bullet$ representing $\mathcal{F}$ whose terms are of the form $L^{\otimes n} \otimes V$,

$n \le 0$. Then using the fact that $\mathrm{Ext}^i$ from a locally free sheaf to any sheaf vanishes for $i > d$, we conclude the argument by a standard trick: consider the distinguished triangle $\mathcal{F}_s[s] \to \tau^{\text{bête}}_{\ge -s}(\mathcal{F}^\bullet) \to \mathcal{F}$ and use that $\mathrm{Hom}(\mathcal{F}, \mathcal{F}_s[s+1]) = 0$. □

*Proof of Proposition 33.* — (a) clearly implies (b), while (b) implies (c). We proceed to prove that (c) implies (a).

Assume that (c) holds. In view of the lemma, it suffices to find for every $m$ an object $\mathcal{F}_{\mathrm{M},m} \in \mathrm{D}_{perf}(\mathrm{X})$ and a morphism $c_m : i(\mathcal{F}_{\mathrm{M},m}) \to \mathrm{M}$ so that $\tilde{\Psi}(\mathrm{Cone}(c_m)[l]) = 0$ for $l \ge m$. Moreover, it suffices to do so after possibly replacing the full embedding $i$ by the functor $i' : \mathcal{F} \mapsto i \circ (\mathcal{F} \otimes \mathrm{L}^{\otimes p})$ for some $p \in \mathbf{Z}$ (notice that conclusion of Lemma 34 is not affected by such a substitution).

Let $d_0$ be the largest integer such that $\tilde{\Psi}(\mathrm{M}[d_0]) \ne 0$. We argue by descending induction in $d_0$. Using the finite generation condition we find a locally free sheaf $\mathcal{E} \in \mathrm{Coh}(\mathrm{X})$ and a morphism $i(\mathcal{E})[-d_0] \to \mathrm{M}$, such that the induced map $\tilde{\Psi}_m(i(\mathcal{E})) \to \tilde{\Psi}_m(\mathrm{M}[d_0])$ is surjective for some $m \in \mathbf{Z}$. Fix $m_0 \ge 0$ such that $\mathrm{R}^{>0}\Gamma(\mathrm{L}^{\otimes i} \otimes \mathcal{E}) = 0$ for $i \ge m_0$. We can assume without loss of generality that $m_0 \ge m$. Then upon replacing the embedding $i$ by $i' : \mathcal{F} \mapsto i(\mathcal{F} \otimes \mathrm{L}^{\otimes -m_0})$ we get that $\mathrm{M}' := \mathrm{Cone}(i(\mathcal{E}) \to \mathrm{M})$ satisfies: $\tilde{\Psi}(\mathrm{M}'[i]) = 0$ for $i \ge d_0$. Also it is clear that the finite generation condition is satisfied for $\mathrm{M}', i'$. Thus we can assume that the statement is true for $\mathrm{M}'$ by the induction assumption. Then the statement about M follows from the octahedron axiom. □

**7.2.** *A characterization of* $\mathrm{D}^b(\mathrm{Coh}(\mathrm{X}))$ *as an ambient category of* $\mathrm{D}_{perf}(\mathrm{X})$. — We continue working under the assumption that $i$ is fully faithful. Assume also that equivalent conditions of Proposition 33 hold, thus the condition of Proposition 32(b)(i) is satisfied. Assume also that assumption (b,ii) holds. In view of Proposition 32 we get a functor $\Psi : \mathcal{C} \to \mathrm{D}^b(\mathrm{Coh}(\mathrm{X}))$ sending $\mathrm{M} \in \mathcal{C}$ to $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}(\mathrm{X}))$ representing the functor $\mathcal{G} \mapsto \mathrm{Hom}(i(\mathcal{G}), \mathrm{M})$ on $\mathrm{D}_{perf}(\mathrm{X})$.

It is not hard to see that $\Psi$ is a triangulated functor.

We now assume that $\mathcal{C}$ is equipped with a bounded $t$-structure $\tau$. Consider the following properties of the functor $\Psi$ in relation to the $t$-structures.

(A) The functor $\Psi$ is of bounded amplitude, i.e. there exists $\mathbf{d}$ such that $\Psi : \mathrm{D}^{\tau,\le 0} \to \mathrm{D}^{<\mathbf{d}}(\mathrm{Coh}(\mathrm{X}))$, $\Psi : \mathrm{D}^{\tau,\ge 0} \to \mathrm{D}^{>-\mathbf{d}}(\mathrm{Coh}(\mathrm{X}))$.

(B) There exists $d \in \mathbf{Z}$ such that for $\mathcal{F} \in \mathcal{C}$ we have: $\Psi(\mathcal{F}) \in \mathrm{D}^{\le 0}(\mathrm{Coh}(\mathrm{X})) \Rightarrow \mathcal{F} \in \mathcal{C}^{\tau,\le d}$.

(C) There exists $\mathfrak{d} > 0$ such that for $\mathcal{F} \in \mathcal{C}$ we have: $\mathrm{H}^i(\Psi(\mathcal{F})) = 0$ for $i \in [-\mathfrak{d}, \mathfrak{d}] \Rightarrow \mathrm{H}^{\tau,0}(\mathcal{F}) = 0$. Here $\mathrm{H}^i(\Psi(\mathcal{F})) \in \mathrm{Coh}(\mathrm{X})$ is the cohomology with respect to the standard $t$-structure on $\mathrm{D}^b(\mathrm{Coh}(\mathrm{X}))$.

*Proposition* **35.** — *(a) Property (B) implies that* $\Psi$ *is fully faithful.*
*(b) Properties (A), (C) imply that* $\Psi$ *is an equivalence.*

*Proof.* — (a) To unburden notation we assume without loss of generality that $d = 0$, this can be achieved by replacing the $t$-structure $\tau$ with its shift by $-d$.

Recall that $i$ is assumed to be a full embedding, which implies that $\Psi \circ i \cong \mathrm{Id}_{D_{perf}(X)}$. It follows from the definition of $\Psi$ that for $\mathcal{F} \in D_{perf}(X)$ we have:

$$\mathrm{Hom}(i(\mathcal{F}), M) \cong \mathrm{Hom}(\mathcal{F}, \Psi(M)) \cong \mathrm{Hom}(\Psi i(\mathcal{F}), \Psi(M)).$$

Thus the map $\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(\Psi(M_1), \Psi(M_2))$ is an isomorphism when $M_1 \in \mathrm{Im}(i)$.

Fix $M_1, M_2 \in \mathcal{C}$. Fix $n$ such that $M_2 \in \mathcal{C}^{\tau,>n}$ and $\Psi(M_2) \in D^{>n}(\mathrm{Coh}(X))$. Fix a bounded above complex $\mathcal{F}^\bullet$ of locally free sheaves representing $\mathcal{F} = \Psi(M_1)$, and let $\mathcal{F}_{\geq N} \in D_{perf}(X)$ be the naive truncation as above. We have an exact triangle $\mathcal{F}_N[-N] \to \mathcal{F}_{\geq N} \to \mathcal{F}$ for some $\mathcal{F}_N \in \mathrm{Coh}(X)$.

Assuming $N < n$, we get

$$\mathrm{Hom}(\Psi(M_1), \Psi(M_2)) \cong \mathrm{Hom}(\mathcal{F}_{\geq N}, \Psi(M_2)) \cong \mathrm{Hom}(i(\mathcal{F}_{\geq N}), M_2).$$

We have a morphism $i(\mathcal{F}_{\geq N}) \to M_1$ whose cone lies in $D^{\tau,\leq N-1}$ in view of condition (B). [Notice that $\Psi$ sends this cone to $\mathcal{F}_N[-N+1]$.] Thus $\mathrm{Hom}(M_1, M_2) \cong \mathrm{Hom}(i(\mathcal{F}_{\geq N}), M_2)$, so composing the above isomorphisms we get that $\mathrm{Hom}(M_1, M_2) \cong \mathrm{Hom}(\Psi(M_1), \Psi(M_2))$. It is easy to see that this map coincides with the map induced by $\Psi$, so (a) is proved.

(b) Property (C) implies (B), thus $\Psi$ is fully faithful by (a), it remains to show that it is essentially surjective. Fix $\mathcal{F} \in D^b(\mathrm{Coh}(X))$ and a bounded above complex of locally free sheaves $\mathcal{F}^\bullet$ representing $\mathcal{F}$. Let $n$ be such that $\mathcal{F} \in D^{\geq n}(\mathrm{Coh}(X))$. Fix $N < n$ and let $A = i(\mathcal{F}_{\geq N})$. We assume as we may that $N < n - 2\mathfrak{d}$, then condition (C) implies that $H^{\tau,m}(A) = 0$ for $m \in [N + \mathfrak{d} + 1, n - \mathfrak{d} - 1]$. Pick such an $m$ and set $B = \tau_{\geq m}(A)$. We have an exact triangle $A \to B \to C$ where $A \in \mathcal{C}^{\tau,\geq n-\mathfrak{d}}$ and $C \in \mathcal{C}^{\tau,<N+\mathfrak{d}}$. Thus applying condition (A) we get an exact triangle $\Psi(A) \to \Psi(B) \to \Psi(C)$, where $\Psi(B) \cong \mathcal{F}_{\geq N}$, $\Psi(A) \in D^{\geq n-\mathfrak{d}-\mathbf{d}}(\mathrm{Coh}(X))$ and $\Psi(C) \in D^{<N+\mathfrak{d}+\mathbf{d}}(\mathrm{Coh}(X))$. Assuming as we may that $N < n - 2(\mathfrak{d} + \mathbf{d})$, we see that $\Psi(A) \cong \mathcal{F}$ which proves that $\Psi$ is essentially surjective. □

## 8. Compatibility between the *t*-structures and construction of the functor from constructible to coherent category

**8.1.** *Almost exactness of* $\Phi_{perf}$.

*Proposition* **36.** — *For some* $d > 0$ *the following holds. If* $\mathcal{F} \in D$ *is such that* $\mathrm{Hom}^i(\Phi_{perf}(\mathcal{O}(\lambda, \mu) \otimes V), \mathcal{F}) = 0$ *for all* $\lambda, \mu \in \Lambda$, $V \in \mathrm{Rep}(G\check{})$ *and* $i \in [-d, d]$, *then* $H^{p,0}(\mathcal{F}) = 0 \in \mathcal{P}$.

The proof of proposition is preceded by some auxiliary results.

*Lemma* **37.** — *For* $X \in D_{I^0 I}$ *there exists a finite subset* $S \subset W$, *such that for* $\lambda \in \Lambda^+$ *we have*

$$j_w^!(X * J_\lambda) \neq 0 \Rightarrow w \in S \cdot (\lambda) \subset W. \tag{17}$$

*Proof.* — By the !-support of an object $X \in D$ we mean the set of points $i_x : \{x\} \hookrightarrow \mathcal{F}\ell$ such that $i_x^!(X) \neq 0$. Proper base change shows that the !-support of $X * J_\lambda$ lies in the convolution of sets supp(X) and $\mathcal{F}\ell_\lambda$. This implies (17). □

*Lemma* **38.** — *Let* $\mathcal{F}$ *be as in Proposition 36.*
*For large* $\lambda$ *and* $n \in [-d + 2\dim(\tilde{\mathfrak{g}}^\vee), d - 2\dim(\tilde{\mathfrak{g}}^\vee)]$ *we have*

$$\mathrm{Ext}^n(j_{w!}, \mathcal{F} * J_\lambda) = 0 \tag{18}$$

*for all* $w$.

*Proof.* — According to Lemma 37 there exists a finite set $S \subset W$ such that for large $\lambda$ the left hand side of (18) vanishes for all $n$ unless $w \in S \cdot (\lambda)$. Also for large $\lambda$ we have $S \cdot (\lambda) \subset W_f \cdot (\Lambda^+)$ and each element in this set is the minimal length representative of its right $W_f$ coset. Hence for all $w \in W$ we have

$$\begin{aligned} &\mathrm{Ext}^p_D(j_{w!}, \mathcal{F} * J_\lambda) \cong \mathrm{Ext}^p_D(\Delta_w * \Xi, \mathcal{F} * J_\lambda), \\ &\qquad\qquad \text{or } \mathrm{Ext}^p_D(j_{w!}, \mathcal{F} * J_\lambda) = 0, \end{aligned} \tag{19}$$

which follows from the fact that $\Delta_w * \Xi$ admits a filtration with associated graded $\bigoplus_{w_f \in W_f} j_{ww_f!}$, and for $w_f \neq e$ we have $\mathrm{Ext}^\bullet(j_{ww_f!}, \mathcal{F} * J_\lambda) = 0$ provided that $\mathrm{Ext}^\bullet(j_{w!}, \mathcal{F} * J_\lambda) \neq 0$. We can rewrite the right hand side of (19) as

$$\mathrm{Ext}^p_{\hat{D}}(\Delta_w * \hat{\Xi}, \pi^*(F * J_\lambda)[r]) \cong \mathrm{Ext}^p(\mathrm{Av}_{IW}(\Delta_w), \mathrm{Av}_{IW}(\pi^*(F * J_\lambda)[r])),$$

$r = \mathrm{rank}(G)$, where we used Proposition 23(a) and isomorphisms $\pi_*(\hat{\Xi}) \cong \pi_!(\hat{\Xi})[r] \cong \Xi$, $\pi^! \cong \pi^*[2r]$.

Now $\Phi_{IW}^{-1}(\mathrm{Av}_{IW}(\Delta_w)) \in D^{\geq 0}(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \cap D^{\leq \dim \tilde{\mathfrak{g}}^\vee}(\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$ by Lemma 30(d), while the condition of Proposition 36 implies that $\Phi_{IW}^{-1}(\pi^*(\mathcal{F} * J_\lambda))[r]$ is concentrated in homological degrees less than $-d$ and greater than $d$. Since $\mathrm{Coh}^{G^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})$ has homological dimension $\dim(\tilde{\mathfrak{g}}^\vee)$, we get the desired vanishing. □

*Proof of Proposition 36.* — In the assumptions of part (a) Lemma 38(a) implies that for large $\lambda$ the object $\mathcal{F} * J_\lambda$ is concentrated in homological degrees less than $-d + 2\dim(\mathfrak{g}^\vee)$ and greater than $d - 2\dim(\mathfrak{g}^\vee)$.

We finish the proof by invoking a result of Lusztig [36] saying that Lusztig's $a$-function for the affine Weyl group is bounded by $\dim(G/B)$, thus convolution of two object in $\mathcal{P}_{II}$ lies in perverse degrees from $-\dim(G/B)$ to $\dim(G/B)$. Thus $\mathcal{F} = (\mathcal{F} * J_\lambda) * J_{-\lambda}$ has no cohomology in perverse degree zero provided that $d > 2\dim(\mathfrak{g}^\vee) + \dim(G/B)$. □

**8.2.** *The functor from constructible to coherent category.* — Applying the general construction of Section 7.1 (see notation introduced prior to Proposition 33) in the present situation: $X = St'/G^\vee$, $\mathcal{C} = D_{I^0 I}$, $L = \mathcal{O}(\lambda, \mu)$ for strictly dominant weights $\lambda, \mu$, we get a functor $\tilde{\Psi}$ from $D_{I^0 I}$ to $G^\vee$-equivariant modules over the homogeneous coordinate ring of $St'$.

*Proposition* **39.** — *For* $\mathcal{F} \in Perv_N(G/B) \subset \mathcal{P}$ *we have* $\tilde{\Psi}(\mathcal{F} * J_\rho[n]) = 0$ *for* $n \neq 0$.

*Proof.* — It suffices to check that for $\mathcal{F} = j_{w!}, j_{w*}$, $w \in W_f$ we have $\tilde{\Psi}(\mathcal{F} * J_\rho[n]) = 0$ for $n \neq 0$. This reduces to showing that for dominant $\lambda, \mu, \nu$ with $\mu$ strictly dominant we have $Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * \mathcal{J}_{-\mu}, \mathcal{F}) = 0$ for $i \neq 0$. We have $\ell(w\mu) = \ell(\mu) - \ell(w)$, $\ell(\lambda w) = \ell(\lambda) + \ell(w)$ for $w \in W_f$. Thus for such $w$ we have

$$\begin{aligned} Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * J_{-\mu}, j_{w!}) &= Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu, j_{w!} j_{\mu*}) \\ &= Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu, j_{w\mu*}), \\ Ext^i(\mathcal{J}_{-\lambda} * \Xi * Z_\nu * J_{-\mu}, j_{w*}) &= Ext^i(\Xi * Z_\nu * J_{-\mu}, j_{\lambda*} j_{w*}) \\ &= Ext^i(\Xi * Z_\nu * J_{-\mu}, j_{\lambda w*}). \end{aligned}$$

Since $\Xi * Z_\nu$ is tilting, $\mathcal{J}_{-\lambda} * \Xi * Z_\nu$ admits a standard filtration, which shows that the first Ext group vanishes for $i \neq 0$. Likewise, $\Xi * Z_\nu * J_{-\mu}$ admits a standard filtration which shows vanishing of the second Ext group for $i \neq 0$. □

*Proposition* **40.** — *The module* $\tilde{\Psi}(\mathcal{F})$ *is finitely generated for any* $\mathcal{F} \in D$.

*Proof.* — For $\mathcal{F}$ in the image of $\Phi_{perf}$ this is clear from the fact that $\Phi_{perf}$ is a full embedding. Every irreducible object in $Perv_N(G/B)$ is a subquotient of $\Xi$. Then it follows from the previous proposition that if $\mathcal{L}$ is such an irreducible object, $\tilde{\Psi}(\mathcal{L} * J_\rho)$ is a subquotient of $\tilde{\Psi}(\Xi * J_\rho)$, hence it is finitely generated (since the homogeneous coordinate ring of Steinberg variety is Noetherian), while $\tilde{\Psi}(\mathcal{L} * J_\rho[n]) = 0$ for $n \neq 0$. It follows that the same is true for any $\mathcal{L} \in Perv_N(G/B)$. Now it follows from Proposition 33 that $\tilde{\Psi}(\mathcal{J}_\lambda * \mathcal{F} * J_\mu[n])$ is finitely generated for $\mathcal{F} \in Perv_N(G/B)$ and any $\lambda, \mu \in \Lambda$, $n \in \mathbf{Z}$. Such objects generate D, so the claim follows. □

*Proposition* **41.** — *There exists* $\mathbf{d}$, *such that for all* $\mathcal{F} \in \mathcal{P}_{I^0 I^0}$ *we have*

$$Hom_{deeq}^{G^\vee \times T^{\vee 2}}(\hat{\Xi}, \mathcal{F}[i]) = 0$$

*for* $i \notin [-\mathbf{d}, \mathbf{d}]$.

*Proof.* — We need to check that for some $\mathbf{d} \in \mathbf{Z}$ we have

$$Ext^i_{\hat{D}}(\mathcal{J}_{-\lambda} * \hat{Z}_\nu * \hat{\Xi} * \mathcal{J}_{-\mu}, \mathcal{F}) = 0,$$

for $i \notin [-\mathbf{d}, \mathbf{d}]$, $\mathcal{F} \in \mathcal{P}$. According to a result of Lusztig [36], Lusztig's $a$-function for an affine Weyl group is bounded by $\dim(G/B)$, which implies that the convolution of any two objects in $\mathcal{P}_{II}$ is concentrated in perverse degrees from $-\dim(G/B)$ to $\dim(G/B)$. It follows that $\mathcal{J}_\lambda * \mathcal{F} * \mathcal{J}_\mu \in D^{\geq -2\dim(G/B)}(\mathcal{P}_{I^0I^0}) \cap D^{\leq 2\dim(G/B)}(\mathcal{P}_{I^0I^0})$.

Thus it suffices to show that for some $\mathbf{d} > 0$ we have

$$\mathrm{Ext}^i_{D}(\mathcal{Z}_\nu * \hat{\Xi}, \mathcal{F}) = 0 \quad \text{for } i \notin [-\mathbf{d}, \mathbf{d}],\ \mathcal{F} \in \mathcal{P}_{I^0I^0}.$$

Using Proposition 23(a) we can rewrite the right hand side[11] as

$$\mathrm{Ext}^i_{\mathrm{Coh}^{G^\vee}(\hat{\tilde{\mathfrak{g}}})}(\mathcal{O} \otimes V_\nu, \widehat{\Phi}^{-1}_{IW}(\mathcal{F})).$$

Now the statement follows from Lemma 30(d). □

## 9. The equivalences

**9.1.** *Equivalence (3).* — We use the criterion of Proposition 32(b) to show that for $\mathcal{F} \in D_{I^0I}$ the functor $M \mapsto \mathrm{Hom}(\Phi_{perf}(\mathcal{F}), M)$ is represented by an object of $D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}'))$; this object is then defined uniquely up to a unique isomorphism in view of Proposition 32(a) and we obtain a functor $\Psi' : D_{I^0I} \to D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}'))$ sending $M \in D_{I^0I}$ to the corresponding representing object.

We need to check that conditions of Proposition 32(b) are satisfied. Condition 32(b)(i) (representability of the restriction to $D^{\geq n}_{perf}$ for all $n$) follows from Propositions 40 and 41 (finite generation and bounded amplitude) in view of Proposition 33. Condition 32(b)(ii) (vanishing on $D^{\leq m}_{perf}$ for $m \ll 0$) follows from Proposition 41.

Functor $\Psi'$ is now defined.

Proposition 35(b) shows it is an equivalence, conditions (A) and (C) are provided respectively by Proposition 41 and Proposition 36.

For future reference we record another favorable property of $\Psi'$ in relation to the standard $t$-structures on the triangulated categories involved.

*Corollary* **42.** — *(a) For $\mathcal{F} \in \mathrm{Perv}_N(G/B) \subset \mathcal{P}$ we have $\Psi'(\mathcal{F}) \in \mathrm{Coh}^{G^\vee}(\mathrm{St}')$.*

*(b) $\Psi'(j_{w*}) \in \mathrm{Coh}^{G^\vee}(\mathrm{St}')$ for $w \in W^f$ and $\Psi'(j_{w!}) \in \mathrm{Coh}^{G^\vee}(\mathrm{St}')$ when $w \in W_f\nu$, $\nu \in -\Lambda^+$.*

*Proof.* — (a) follows from the Proposition 39.

(b) follows from (a) since $w \in W^f$ can be written as $w = w'\lambda$, $\lambda \in \Lambda^+$, $w' \in W_f$, so that $\ell(w) = \ell(w') + \ell(\lambda)$. Then we get $\Psi'(j_{w*}) = \Psi'(j_{w'*} * j_{\lambda*}) = \Psi'(j_{w'*}) \otimes \mathcal{O}(0, \lambda)$. Similarly, if $w = w'\nu$, $\nu \in -\Lambda^+$, then $\ell(w) = \ell(w') + \ell(\nu)$. □

[11] In fact, the main result of [12] (see [12, Theorem 2], cf. a related statement Theorem 54(b) below) shows that for an irreducible $\mathcal{F}$ this Ext group is an isotypic component in the cohomology of a coherent IC sheaf on the nilpotent cone $\mathcal{N}$; in particular, it shows that in this case required vanishing holds with $\mathbf{d} = \frac{1}{2}\dim\mathcal{N}$.

**9.2.** *Equivalence (2).* — We again use the criterion of Proposition 32(b) to show that the functor $\mathcal{F} \mapsto \mathrm{Hom}(\widehat{\Phi}_{perf}(\widehat{\mathcal{F}}), \mathrm{M})$ is represented by an object of $\mathrm{D}^b(\mathrm{Coh}_{\mathcal{N}}^{\check{G}}(\mathrm{St}))$, here $\widehat{\mathcal{F}}$ denotes pull back of $\mathcal{F}$ under the morphism $\widehat{\mathrm{St}} \to \mathrm{St}$. The representing object is then defined uniquely up to a unique isomorphism in view of Proposition 32(a) and we obtain a functor $\Psi : \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0} \to \mathrm{D}^b(\mathrm{Coh}_{\mathcal{N}}^{\check{G}}(\mathrm{St}))$ sending $\mathrm{M} \in \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ to the corresponding representing object.

We need to check that conditions of Proposition 32(b) are satisfied. In view of Proposition 33 condition 32(b)(i) (representability of the restriction to $\mathrm{D}_{perf}^{\geq n}$ for all $n$) follows from Propositions 40 and 41 which provide respectively finite generation and bounded amplitude properties (Proposition 40 states a similar property for an object of D, the case of $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ follows).

Condition 32(b)(ii) (vanishing on $\mathrm{D}_{perf}^{\leq m}$ for $m \ll 0$) follows from Proposition 41. Since the log monodromy endomorphisms act on objects of $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0}$ nilpotently, this object is set theoretically supported on the preimage of $\mathcal{N}$ in St. Thus we get the functor $\Psi : \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0} \to \mathrm{D}^b(\mathrm{Coh}_{\mathcal{N}}^{\check{G}}(\mathrm{St}))$.

It also induces a functor between the subcategories in the categories of pro-objects: $\widehat{\Psi} : \hat{\mathrm{D}} \to \mathrm{D}^b(\mathrm{Coh}^{\check{G}}(\widehat{\mathrm{St}}))$. Recall that $i_{\mathrm{St}}$ denotes for the embedding $\mathrm{St}' \to \mathrm{St}$.

*Lemma* **43.** — *(a) The following diagrams commute:*

$$\begin{array}{ccc} \hat{\mathrm{D}} & \xrightarrow{\widehat{\Psi}} & \mathrm{D}^b(\mathrm{Coh}^{\check{G}}(\widehat{\mathrm{St}})) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle i_{\mathrm{St}}^*} \\ \mathrm{D}_{\mathrm{I}^0\mathrm{I}} & \xrightarrow{\Psi'} & \mathrm{D}^b(\mathrm{Coh}^{\check{G}}(\mathrm{St}')) \end{array}$$

$$\begin{array}{ccc} \mathrm{D}_{\mathrm{I}^0\mathrm{I}} & \xrightarrow{\Psi'} & \mathrm{D}^b(\mathrm{Coh}^{\check{G}}(\mathrm{St}')) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle i_{\mathrm{St}*}} \\ \mathrm{D}_{\mathrm{I}^0\mathrm{I}^0} & \xrightarrow{\Psi} & \mathrm{D}^b(\mathrm{Coh}_{\mathcal{N}}^{\check{G}}(\mathrm{St})) \end{array}$$

*(b) We have $\widehat{\Psi} \circ \Phi_{diag} \cong \delta_*$, where $\delta : \widehat{\tilde{\mathfrak{g}}}^{\vee} \to \widehat{\mathrm{St}}$ is the diagonal embedding.*

*Proof.* — Lemma 22 implies that both compositions in the first diagram are compatible with the action of $(\mathrm{D}_{perf}^{\check{G}}(\widehat{\mathrm{St}}), \otimes)$, i.e. if $\mathrm{F}_1 = i_{\mathrm{St}}^* \circ \Psi$, $\mathrm{F}_2 = \Psi' \circ \pi_*$, then $\mathrm{F}_i(\mathcal{F}(\mathrm{X})) \cong i^*(\mathcal{F}) \otimes \mathrm{F}_i(\mathrm{X})$ canonically for $\mathcal{F} \in \mathrm{D}_{perf}^{\check{G}}(\widehat{\mathrm{St}})$, $\mathrm{X} \in \hat{\mathrm{D}}$. We also have $\mathrm{F}_1(\hat{\Xi}) \cong \mathcal{O} \cong \mathrm{F}_2(\hat{\Xi})$, thus we get a functorial isomorphism $\mathrm{F}_1(\mathrm{X}) \cong \mathrm{F}_2(\mathrm{X})$ for X in the image of $\widehat{\Phi}_{perf}$.

Now given any $\mathrm{X} \in \hat{\mathrm{D}}$, choose a bounded above complex of equivariant locally free sheaves $\mathcal{F}^\bullet$ representing the object $\widehat{\Psi}(\mathrm{X})$; then using Proposition 41 and the fact that $\pi_*$, $i^*_{\mathrm{St}}$ have bounded homological dimension we get for $N' \ll N \ll 0$:

$$F_1(X) \cong \tau^{\text{bête}}_{\geq N} \widehat{\Phi}_{perf}(\tau^{\text{bête}}_{\geq N'}(\mathcal{F}^\bullet)) \cong F_2(X).$$

It is easy to check that the resulting isomorphism does not depend on the choice of $\mathcal{F}^\bullet$ and is functorial in X, thus commutativity of the first diagram is established.

The proof for the second diagram is similar, this proves part (a).

The same observation that all the functors involved commute with the action of $(D^{G^\vee}_{perf}(\widehat{\mathrm{St}}), \otimes_{\mathcal{O}})$ reduce (b) to checking that

$$\widehat{\Psi}(\Delta_e) \cong \delta_*(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}).$$

To this end it suffices to construct an isomorphism of $\mathcal{O}(\overline{C}_{\mathrm{St}})$-modules:

$$\bigoplus_{\lambda,\mu\in\Lambda^+} \mathrm{Hom}^\bullet_{deeq}(\hat{\Xi}, \mathcal{J}_\lambda * \Delta_e * \mathcal{J}_\mu) \cong \bigoplus_{\lambda,\mu\in\Lambda^+} R\Gamma\big(\widehat{\tilde{\mathfrak{g}}}^\vee, \mathcal{O}(\lambda+\mu)\big) \tag{20}$$

compatible with the $\mathcal{O}(\overline{C}_{\mathrm{St}})$ action; here subscript $_{deeq}$ refers to the $G^\vee$-deequivariantization (see Section 4.2.1).

Using 23(a,b) we can rewrite the left hand side of (20) as

$$\bigoplus_{\lambda,\mu\in\Lambda^*} \mathrm{Hom}^\bullet_{deeq}(\mathrm{Av}^{\mathrm{IW}}(\Delta_e), \mathrm{Av}^{\mathrm{IW}}(\mathcal{J}_{\lambda+\mu})).$$

Since $\mathrm{Av}^{\mathrm{IW}}(\Delta_e) = \widehat{\Phi}_{\mathrm{IW}}(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee})$, $\mathrm{Av}^{\mathrm{IW}}(\mathcal{J}_{\lambda+\mu}) = \widehat{\Phi}_{\mathrm{IW}}(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}(\lambda+\mu))$, we see that the displayed expression is canonically isomorphic to $\bigoplus\limits_{\lambda,\mu\in\Lambda^+} \mathrm{Hom}^\bullet_{\mathrm{Coh}(\widehat{\tilde{\mathfrak{g}}}^\vee)}(\mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}, \mathcal{O}_{\widehat{\tilde{\mathfrak{g}}}^\vee}(\lambda+\mu))$, which yields (20); compatibility with the $\mathcal{O}_{\overline{C}_{\widehat{\mathrm{St}}}}$ action is clear from the construction. □

We are now ready to prove that $\Psi$, and hence $\widehat{\Psi}$ is an equivalence. Since we know that $\Psi'$ is an equivalence and the essential image of $i_* : D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}')) \to D^b(\mathrm{Coh}^{G^\vee}_{\mathcal{N}}(\mathrm{St}))$ generates the target category, Lemma 43(a) shows that the essential image of $\Psi$ generates the target category. Thus it suffices to check that $\Psi$ is fully faithful. It is enough to see that

$$\mathrm{Hom}(A, B) \xrightarrow{\ \Psi\ } \mathrm{Hom}(\Psi(A), \Psi(B))$$

is an isomorphism when B is obtained from an object $B' \in D_{I^0 I}$ by forgetting the equivariance. This follows from the corresponding statement for $\Psi'$ and Lemma 43(a).

**9.3.** *Equivalence (4).*

**9.3.1.** *Passing from monodromic to equivariant category by killing monodromy.* — Let X be a scheme with an action of an algebraic torus A. Let $\mathcal{P}_{mon}$ be the category of unipotently monodromic perverse sheaves on X.

We have an action of $\mathfrak{a} = \mathrm{Lie}(A)$ on $\mathcal{P}_{mon}$ by log monodromy. Let $\mathbf{K}_\mathfrak{a}$ be the Koszul complex of the vector space $\mathfrak{a}$; in other words, $\mathbf{K}_\mathfrak{a}$ is the standard complex for homology of the abelian algebra $\mathfrak{a}$ with coefficients in the free module $U\mathfrak{a} = \mathrm{Sym}(\mathfrak{a})$. Thus $\mathbf{K}_\mathfrak{a}$ is a graded commutative DG-algebra with $\mathfrak{a} \oplus \mathfrak{a}[1]$ as the space of generators and differential sending $\mathfrak{a}[1]$ to $\mathfrak{a}$ by the identity map. It is clear that $\mathbf{K}_\mathfrak{a}$ is quasi-isomorphic to the base field $k$ and its degree zero part is the enveloping algebra $U\mathfrak{a}$.

We define a DG-category $\mathcal{P}_{eq}$ as the category of complexes of objects in $\mathcal{P}_{mon}$ equipped with an action of $\mathbf{K}_\mathfrak{a}$, such that the action of $\mathfrak{a} \subset \mathbf{K}_\mathfrak{a}^0$ coincides with the log monodromy action. Let $D(\mathcal{P}_{eq}) = \mathrm{Ho}(\mathcal{P}_{eq})/\mathrm{Ho}_{acycl}(\mathcal{P}_{eq})$ be the quotient of the homotopy category by the subcategory of acyclic complexes.

We will also write $D(X/A)$ for the A-equivariant derived category of constructible sheaves on X (equivalently, constructible derived category of the stack X/A).

*Lemma* **44.** — *(a) We have a natural equivalence* $D(\mathcal{P}_{eq}) \cong D(X/A)$ *(the equivalence will be denoted by* $real_{eq}$*).*

*(b) Consider the functors* $\mathrm{Forg} : \mathcal{P}_{eq} \to \mathrm{Com}(\mathcal{P}_{mon})$ *and* $\mathrm{Ind}_{U(\mathfrak{a})}^{\mathbf{K}_\mathfrak{a}} : \mathrm{Com}(\mathcal{P}_{mon}) \to \mathcal{P}_{eq}$*, where the first one is the functor of forgetting the* $\mathbf{K}_\mathfrak{a}$ *action and the second one is the functor of induction from* $U(\mathfrak{a})$ *which acts by log monodromy to* $\mathbf{K}_\mathfrak{a}$*.*

*The induced functors on the derived categories fit into the following diagrams which commute up to a natural isomorphism:*

$$\begin{array}{ccc} D(\mathcal{P}_{eq}) & \xrightarrow{\mathrm{Forg}} & D^b(\mathcal{P}_{mon}) \\ \downarrow{\scriptstyle real_{eq}} & & \downarrow{\scriptstyle real} \\ D(X/A) & \xrightarrow{pr^*} & D(X) \end{array}$$

$$\begin{array}{ccc} D(\mathcal{P}_{eq}) & \xleftarrow{\mathrm{Ind}_{U(\mathfrak{a})}^{\mathbf{K}_\mathfrak{a}}[-d]} & D^b(\mathcal{P}_{mon}) \\ \downarrow{\scriptstyle real_{eq}} & & \downarrow{\scriptstyle real} \\ D(X/A) & \xleftarrow{pr_*} & D(X) \end{array}$$

*where* $pr$ *denotes the projection* $X \to X/A$*, real denotes Beilinson's realization functor [6] and* $d = \dim(\mathfrak{a})$.

*(c) Suppose that* $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{eq}$ *are such that* $\mathrm{Ext}^{>0}_{\mathrm{D(X)}}(\mathcal{F}^i, \mathcal{G}^j) = 0$ *for all* $i, j$. *Then*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{P}_{eq})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D(X/A)}}(real_{eq}(\mathcal{F}), real_{eq}(\mathcal{G})).$$

*Proof*. — (a) Assume first that the action of A on X is free and the quotient $Y = X/A$ is represented by a scheme. The abelian category Perv(Y) of perverse sheaves on Y admits a full embedding into the category $\mathrm{Perv}_{mon}(X)$ of unipotently monodromic perverse sheaves on X, and the essential image of the embedding consists of sheaves with zero action of log monodromy. Thus we have a natural embedding $\mathrm{Com}(\mathrm{Perv}(Y)) \to \mathrm{Com}(\mathcal{P}_{eq})$ sending a complex of equivariant sheaves to the same complex equipped with zero action of $\mathfrak{a}$ and $\mathfrak{a}[1]$. We claim that the induced functor $\mathrm{D}^b(\mathrm{Perv}(Y)) \to \mathrm{D}(\mathcal{P}_{eq})$ is an equivalence.

This claim is readily seen to be local on Y, i.e. it suffices to check it assuming that $X = A \times Y$ where A acts on the first factor by translations. In the latter case the category $\mathrm{Perv}_{mon}(X)$ is readily identified with the tensor product of the abelian category Perv(Y) and the abelian category of unipotently monodromic local systems of A, the latter is equivalent to the category of modules over the symmetric algebra $U(\mathfrak{a}) \cong \mathrm{Sym}(\mathfrak{a})$ set-theoretically supported at zero (see [26, §5] for the notion of tensor product of abelian categories). Thus the claim is clear in this case.

Let now X be general. Then an object of D(X/A) is by definition (see [11]) a collection of objects in $\mathrm{D}(\tilde{Y})$ given for every A-equivariant smooth map $\tilde{X} \to X$ where the action of A on $\tilde{X}$ is free and $\tilde{Y} = \tilde{X}/A$, subject to certain compatibilities. We have the pull back functor $\mathcal{P}_{eq}(X) \to \mathcal{P}_{eq}(\tilde{X})$, composing it with the functor $\mathcal{P}_{eq}(\tilde{X}) \to \mathrm{D}^b(\mathrm{Perv}(\tilde{Y})) \cong \mathrm{D}(\tilde{Y})$ we get the desired system of objects, the compatibilities are easy to see.

(b) Commutativity of the first diagram is clear from the proof of (a) and commutativity of the second one follows by passing to adjoint functors (notice that in view of self-duality of Koszul complex the functor $\mathrm{Ind}^{\mathbf{K}_\mathfrak{a}}_{U(\mathfrak{a})}[-d]$ is *right* adjoint to the forgetful functor Forg).

(c) By a standard argument the condition in (c) implies that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{P}_{mon})}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathrm{Forg}(\mathcal{F}), \mathrm{Forg}(\mathcal{G})).$$

We have adjoint pairs of functors compatible with the natural functor from the homotopy category to the derived category:

$$\mathrm{Ho}(\mathcal{P}_{eq}) \xrightarrow{\mathrm{Forg}} \mathrm{Ho}(\mathcal{P}_{mon}) \xrightarrow{\mathrm{Ind}} \mathrm{Ho}(\mathcal{P}_{eq}),$$

$$\mathrm{D}(\mathcal{P}_{eq}) \xrightarrow{\mathrm{Forg}} \mathrm{D}^b(\mathcal{P}_{mon}) \xrightarrow{\mathrm{Ind}} \mathrm{D}(\mathcal{P}_{eq}).$$

The composition in each case admits a filtration with associated graded $\mathrm{Id} \otimes \Lambda(\mathfrak{a}[1])$, i.e. for $\mathcal{F} \in \mathrm{Ho}(\mathcal{P}_{eq})$ or $\mathcal{F} \in \mathrm{D}(\mathcal{P}_{eq})$ we have

$$\mathrm{Ind}\circ\mathrm{Forg}(\mathcal{F})\in\{\Lambda^d(\mathfrak{a})\otimes\mathcal{F}[d]\}*\{\Lambda^{d-1}(\mathfrak{a})\otimes\mathcal{F}[d-1]\}$$
$$*\cdots*\{\mathfrak{a}\otimes\mathcal{F}[1]\}*\mathcal{F},$$

where we used the notation of [8]: $X*Y$ is the set of objects $z$ such that there exists a distinguished triangle $x\to z\to y$, $x\in X, y\in Y$. Since $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{P}_{eq})}(\mathrm{Ind}\circ\mathrm{Forg}(\mathcal{F}),\mathcal{G})\xrightarrow{\sim}\mathrm{Hom}_{D(X/A)}(\mathrm{Ind}\circ\mathrm{Forg}(\mathcal{F}),\mathcal{G})$, it follows by induction in $n$ that $\mathrm{Hom}^n_{\mathrm{Ho}(\mathcal{P}_{eq})}(\mathcal{F},\mathcal{G})\xrightarrow{\sim}\mathrm{Hom}^n_{D(X/A)}(\mathcal{F},\mathcal{G})$. ☐

*Corollary* **45.** — *Let* $\hat{\mathcal{T}}_{\mathrm{II}}$ *denote the DG category whose objects are finite complexes of objects in* $\hat{\mathcal{T}}$ *equipped with an action of* $\mathbf{K}_{\mathfrak{t}^2}$ *such that the action of* $\mathbf{K}^0_{\mathfrak{t}^2}=\mathrm{U}(\mathfrak{t}^2)$ *coincides with the action induced by the torus monodromy. Then the homotopy category* $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ *is naturally equivalent to* $D_{\mathrm{II}}$.

*Proof.* — Lemma 44(c) yields a fully faithful functor $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})\to D_{\mathrm{II}}$. To see that this functor is essentially surjective, notice that Lemma 44(b) implies that the composition of the natural functors $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})\to\mathrm{Ho}(\hat{\mathcal{T}})\to\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ contains identity functor as a direct summand (more precisely, this composition is isomorphic to tensoring with $H^*(T^2)\in D^b(\mathrm{Vect})$). Thus every object of $D_{\mathrm{II}}$ is a direct summand in an object which belongs to the essential image of the full embedding $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$. Thus we will be done if we check that $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ is Karoubian (idempotent complete).

Since a direct summand of a free-monodromic tilting object is again free-monodromic tilting, the category $\hat{\mathcal{T}}_{\mathrm{II}}$ is idempotent complete. For $T^\bullet\in\hat{\mathcal{T}}_{\mathrm{II}}$ the space of closed endomorphisms of the complex commuting with the $\mathbf{K}_{\mathfrak{t}^2}$ action is a pro-finite dimensional ring whose quotient by its pro-nilpotent radical is finite dimensional. The subspace of endomorphisms homotopic to zero is a two-sided ideal in this ring. By elementary algebra an idempotent in a quotient of a finite dimensional algebra by a two-sided ideal can be lifted to an idempotent in the original ring; thus we see that every idempotent endomorphism of an object in $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ lifts to an idempotent in the ring of endomorphisms of the corresponding object in $\hat{\mathcal{T}}_{\mathrm{II}}$, this shows that $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ is idempotent complete. ☐

We are now ready to establish (4).

Consider the category of finite complexes of objects in $\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})$ equipped with an action of $\mathbf{K}_{\mathfrak{t}^2}$ extending the action of $\mathfrak{t}^2=(\mathfrak{t}^{\vee*})^2$ coming from the action of linear functions on $\mathfrak{t}^{\vee 2}$ pulled back under the natural map $\mathrm{St}\to\mathfrak{t}^{\vee 2}$. (It is easy to see that replacing $\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})$ in the previous sentence by $\mathrm{Coh}^{G^\vee}(\mathrm{St})$ one gets definition of an equivalent category.) Let $\mathrm{Coh}^{G^\vee}_{\mathbf{K}_{\mathfrak{t}^2}}(\mathrm{St})$ denote this category and $\mathrm{Ho}(\mathrm{Coh}^{G^\vee}_{\mathbf{K}_{\mathfrak{t}^2}}(\mathrm{St}))$ be the corresponding homotopy category.

It follows from the definition of the derived coherent category of a DG-scheme that there exists a natural functor

$$real_{coh}:\mathrm{Ho}(\mathrm{Coh}^{G^\vee}_{\mathbf{K}_{\mathfrak{t}^2}}(\mathrm{St}))\to\mathrm{DGCoh}^{G^\vee}(\mathrm{St}\overset{L}{\times}_{\mathfrak{t}^{\vee 2}}\{0\})=\mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}}\overset{L}{\times}_{\mathfrak{g}^\vee}\tilde{\mathcal{N}}).$$

Moreover, given two complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Coh}^{G^\vee}_{\mathbf{K}_{t^2}}(\mathrm{St})$ such that $\mathrm{Ext}^{>0}_{\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})}(\mathcal{F}^i, \mathcal{G}^j) = 0$ we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Coh}^{G^\vee}_{\mathbf{K}_{t^2}}(\mathrm{St}))}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{Hom}(real_{coh}(\mathcal{F}^\bullet), real_{coh}(\mathcal{G}^\bullet)).$$

Corollary 25 implies that the functor $\widehat{\Psi}$ sends free-monodromic tilting sheaves to coherent sheaves; thus equivalence $\Phi_{I^0 I^0}$ and Corollary 45 yield a fully faithful functor $\Psi_{II} : D_{II} \to \mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$. The essential image of $\Phi_{II}$ contains the essential image of the functor $\mathrm{Ind}^{\mathbf{K}_{t^2}}_{\mathcal{O}(t^{\vee 2})} : D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})) \to \mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$, since the diagram

$$\begin{array}{ccc} \hat{D} & \xrightarrow{\widehat{\Psi}} & D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})) \\ \Big\downarrow {\scriptstyle \mathrm{Av}^{I}_{I^0} \circ \pi_*} & & \Big\downarrow {\scriptstyle \mathrm{Ind}^{\mathbf{K}_{t^2}}_{\mathcal{O}(t^{\vee 2})}} \\ D_{II} & \xrightarrow{\Psi_{II}} & \mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}) \end{array}$$

is commutative by Lemma 44(b). Since every object of $\mathrm{DGCoh}^{G^\vee}(\tilde{\mathcal{N}} \overset{L}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$ is a direct summand in an object which lies in the image of $\mathrm{Ind}^{\mathbf{K}_{t^2}}_{\mathcal{O}(t^{\vee 2})}$ and $D_{II}$ has been shown to be idempotent complete, the functor $\Psi_{II}$ is an equivalence.

## 10. Monoidal structure

**10.1.** *A* DG*-model for convolution of coherent sheaves.*

*Lemma* **46.** — *Let* X, Y *be two algebraic stacks and* $F = F_{\mathcal{K}} : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$ *be a functor coming from an object* $\mathcal{K} \in D^b(\mathrm{Coh}(X \times Y))$, *i.e.* $F : \mathcal{F} \mapsto pr_{2*}(\mathcal{K} \otimes^L pr_1^*(\mathcal{F}))$. *Let* $M \in D^b(\mathrm{Coh}(X))$ *be represented by a complex of sheaves* $M^\bullet$ *such that* $F(M^i) \in \mathrm{Coh}(Y)$. *Then* $F(M)$ *is canonically isomorphic to the object represented by* $F(M^\bullet)$.

*Proof.* — A functor as above lifts to a functor between filtered derived categories $F^{fil} : DF(\mathrm{Coh}(X)) \to DF(\mathrm{Coh}(Y))$. Recall that DF contains the category of bounded complexes in Coh(X) as a full subcategory, the canonical functor from the filtered derived category to the derived category restricted to this subcategory coincides with the canonical functor from the category of complexes to the derived category. The conditions of the lemma show that $F^{fil}$ sends the object corresponding to the complex $M^\bullet$ to the object corresponding to $F(M^\bullet)$, which yields the desired statement. □

Recall from Section 1.2.2 and [21] that for a proper map $X \to Y$ of smooth varieties convolution yields a monoidal structure on the derived coherent category of the

DG-scheme $X\overset{L}{\times}_Y X$. If $X \to Y$ is semi-small then $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) = 0$ for $i > 0$ thus we get a convolution monoidal structure on $D^b(\mathrm{Coh}(X \times_Y X))$ and on $D^b(\mathrm{Coh}^H(X \times_Y X))$ for an algebraic group H acting compatibly on X, Y. These monoidal categories act on module categories $D^b(\mathrm{Coh}(X))$, $D^b(\mathrm{Coh}^H(X))$ respectively, the action functor is also denoted by $*$.

*Corollary* **47.** — *Let $X \to Y$ be a proper semi-small morphism of smooth quasiprojective varieties equipped with an action of a reductive algebraic group* H.

*(a) Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be finite complexes of* H*-equivariant coherent sheaves on $X \times_Y X$ such that the convolution $\mathcal{F}^i * \mathcal{G}^j$ lies in $\mathrm{Coh}^H(X \times_Y X)$ for all $i, j$. Let $\mathcal{F}, \mathcal{G}$ be the corresponding objects in the derived category. Then $\mathcal{F} * \mathcal{G}$ is canonically isomorphic to the object represented by the total complex of the bicomplex $\mathcal{F}^i * \mathcal{G}^j$.*

*(b) Assume that three complexes $\mathcal{F}_1^\bullet$, $\mathcal{F}_2^\bullet$, $\mathcal{G}^\bullet$ of* H*-equivariant coherent sheaves on $X \times_Y X$ are such that $\mathcal{F}_1^i * \mathcal{F}_2^j$, $\mathcal{F}_2^j * \mathcal{G}^l$ and $\mathcal{F}_1^i * \mathcal{F}_2^j * \mathcal{G}^l$ lie in $\mathrm{Coh}^H(X \times_Y X)$ for all $i, j, l$. Then the two isomorphisms provided by part (a) between $\mathcal{F}_1 * \mathcal{F}_2 * \mathcal{G} \in D^b(\mathrm{Coh}^H(X \times_Y X))$ and the object represented by the complex $\mathcal{C}^d = \bigoplus_{i+j+l=d} \mathcal{F}_1^i * \mathcal{F}_2^j * \mathcal{G}^l$ coincide.*

*(c) Let $\mathcal{F}^\bullet$ be as in (a) and $\mathcal{G}^\bullet$ be a finite complex of* H*-equivariant coherent sheaves on* X*. Then $\mathcal{F} * \mathcal{G}$ is canonically isomorphic to the object represented by the total complex of the bicomplex $\mathcal{F}^i * \mathcal{G}^j$ provided that $\mathcal{F}^i * \mathcal{G}^j \in \mathrm{Coh}^H(X)$.*

*(d) Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$ be as in (b) and $\mathcal{G}$ as in (c). Assume that $\mathcal{F}_1^i * \mathcal{F}_2^j \in \mathrm{Coh}^H(X \times_Y X)$, while $\mathcal{F}_2^j * \mathcal{G}^l, \mathcal{F}_1^i * \mathcal{F}_2^j * \mathcal{G}^l \in \mathrm{Coh}^H(X)$ for all $i, j, l$. Then the isomorphism between $\mathcal{F}_1 * \mathcal{F}_2 * \mathcal{G} \in D^b(\mathrm{Coh}^H(X))$ and the object represented by the complex $\mathcal{C}^d = \bigoplus_{i+j+l=d} \mathcal{F}_1^i * \mathcal{F}_2^j * \mathcal{G}^l$ obtained by applying part (c) twice coincides with the isomorphism obtained by applying part (a) and part (c).*

*Proof.* — The convolution product comes from a functor

$$F : D^b\big(\mathrm{Coh}^H\big((X \times_Y X)^2\big)\big) \to D^b\big(\mathrm{Coh}^H(X \times_Y X)\big)$$

of the type considered in Lemma 46, namely we have $F = F_{\mathcal{K}}$, where $\mathcal{K} \in D^b(\mathrm{Coh}^H(X \times_Y X)^3)$ is given by $\mathcal{K} = \upsilon^* \delta_*(\mathcal{O}_{X^3})$; here $\upsilon$ stands for the embedding $(X \times_Y X)^3 \to (X \times X)^3 = X^6$ and $\delta : X^3 \to X^6$ is given by $(x_1, x_2, x_3) \mapsto (x_3, x_1, x_1, x_2, x_2, x_3)$. Thus statement (a) follows from Lemma 46. Part (b) follows by considering the functor between filtered derived categories $DF(\mathrm{Coh}^H(X \times_Y X)^3) \to DF(\mathrm{Coh}^H(X \times_Y X))$ corresponding to the triple convolution. The proof of (c,d) is similar to the proof of (a,b) respectively. □

*Lemma* **48.** — *(a) Let $X \to Y$ be a semi-small proper morphism of smooth quasi-projective varieties equipped with an action of a reductive algebraic group* H. *For $\mathcal{F} \in D^b(\mathrm{Coh}^H(X \times_Y X))$ let $a(\mathcal{F})$ denote the corresponding functor $D^b(\mathrm{Coh}^H(X)) \to D^b(\mathrm{Coh}^H(X))$.*

*For $\mathcal{F} \in \mathrm{Coh}^H(X \times_Y X)$, $\mathcal{F}' \in D^b(\mathrm{Coh}^H(X \times_Y X))$ any isomorphism of functors $a(\mathcal{F}) \cong a(\mathcal{F}')$ comes from a unique isomorphism $\mathcal{F} \cong \mathcal{F}'$.*

*(b) Given an* $\mathrm{H}$*-invariant closed subvariety* $\mathrm{Z} \subset \mathrm{Y}$*, the statement in (a) remains true for* $\mathcal{F}, \mathcal{F}' \in \mathrm{Coh}^{\mathrm{H}}(\widehat{\mathrm{X} \times_{\mathrm{Y}} \mathrm{X}})$*,* $a(\mathcal{F}), a(\mathcal{F}') \in \mathrm{End}(\mathrm{D}^b(\mathrm{Coh}^{\mathrm{H}}(\widehat{\mathrm{X}})))$*, where* $\widehat{\mathrm{X}}$*,* $\widehat{\mathrm{X} \times_{\mathrm{Y}} \mathrm{X}}$ *denote formal completions at the preimage of* $\mathrm{Z}$*.*

*Proof.* — In the setting of either part (a) or part (b), an equivariant coherent sheaf $\mathcal{F}$ can be reconstructed from the corresponding module $\mathrm{M}(\mathcal{F})$ over the homogeneous coordinate ring, $\mathrm{M}(\mathcal{F}) = \bigoplus_{n,m\geq 0} \Gamma(\mathcal{F} \otimes pr_1^*(\mathrm{L}^n) \otimes pr_2^*(\mathrm{L}^m))$, where $\mathrm{L}$ is an equivariant ample line bundle on $\mathrm{X}$. Thus lemma follows from the following expression for $\mathrm{M}(\mathcal{F})$ in terms of the functor of convolution by $\mathcal{F}$: $\mathrm{M}(\mathcal{F}) = \bigoplus_{m,n} \mathrm{Hom}_{deeq}(\mathrm{L}^{-n}, \mathcal{F} * \mathrm{L}^m)$. □

**10.2.** *Monoidal structure on* $\Phi_{\mathrm{I}^0\mathrm{I}^0}$.

*Lemma* **49.** — *The equivalence* $\widehat{\Phi}$ *is compatible with the action on* $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee})) \cong \hat{\mathrm{D}}_{\mathrm{IW}}$ *via the equivalence* $\widehat{\Phi}_{\mathrm{IW}}$*, i.e. we have a functorial isomorphism*

$$\widehat{\Phi}_{\mathrm{IW}}(\mathcal{F} * \mathcal{G}) \cong \widehat{\Phi}(\mathrm{F}) * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G})$$

*where* $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}}))$*,* $\mathcal{G} \in \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$*.*

*Proof.* — For $\mathcal{F} \in \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}})$ this is Proposition 29.

Let now $\mathcal{F}$ be general. For any sufficiently large $\mathrm{N}$ we can find $\mathcal{F}' \in \mathrm{D}^{\mathrm{G}^\vee}_{perf}(\widehat{\mathrm{St}})$ such that $\mathcal{F} = \tau_{\geq -\mathrm{N}}(\mathcal{F}')$. The functor $\mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}})) \to \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\tilde{\mathfrak{g}}^\vee}))$, $\mathcal{F} \mapsto \mathcal{F} * \mathcal{G}$ has bounded homological amplitude; the functor $\hat{\mathrm{D}} \to \hat{\mathrm{D}}_{\mathrm{IW}}$ $\mathrm{X} \mapsto \mathrm{X} * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G})$ has bounded homological amplitude and by Proposition 36 the functor $\widehat{\Phi}$ has homological amplitude bounded above, i.e. it sends $\mathrm{D}^{\leq 0}(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}}))$ to $\mathrm{D}^{\leq n}(\hat{\mathcal{P}})$ for some $n$. It follows that for $\mathrm{N} \gg m \gg 0$ and $\mathcal{F}'$ as above we have

$$\begin{aligned}\widehat{\Phi}_{\mathrm{IW}}(\mathcal{F} * \mathcal{G}) &\cong \widehat{\Phi}_{\mathrm{IW}}\big(\tau_{\geq -m}\big(\mathcal{F}' * \mathcal{G}\big)\big) \cong \tau_{\geq -m}\widehat{\Phi}_{\mathrm{IW}}\big(\mathcal{F}' * \mathcal{G}\big)\\ &\cong \tau_{\geq -m}\big(\widehat{\Phi}(\mathcal{F}) * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G})\big) \cong \widehat{\Phi}(\mathcal{F}) * \widehat{\Phi}_{\mathrm{IW}}(\mathcal{G}),\end{aligned}$$

which proves the lemma. □

We are now ready to equip $\widehat{\Phi}$ with a monoidal structure. We work with the inverse equivalence $\widehat{\Psi}$. We need to construct an isomorphism

$$\widehat{\Psi}(\mathcal{F} * \mathcal{G}) \cong \widehat{\Psi}(\mathcal{F}) * \widehat{\Psi}(\mathcal{G}) \tag{21}$$

compatible with the associativity isomorphism.

Given $\mathcal{F}, \mathcal{G} \in \hat{\mathrm{D}}$ and $\mathcal{M} \in \hat{\mathrm{D}}_{\mathrm{IW}}$, Lemma 49 provides isomorphisms

$$\widehat{\Psi}(\mathcal{F} * \mathcal{G}) * \widehat{\Psi}_{\mathrm{IW}}(\mathcal{M}) \cong \widehat{\Psi}_{\mathrm{IW}}(\mathcal{F} * \mathcal{G} * \mathcal{M}) \cong \widehat{\Psi}(\mathcal{F}) * \widehat{\Psi}_{\mathrm{IW}}(\mathcal{G} * \mathcal{M})$$
$$\cong \widehat{\Psi}(\mathcal{F}) * \widehat{\Psi}(\mathcal{G}) * \widehat{\Psi}_{\mathrm{IW}}(\mathcal{M}),$$

where $\widehat{\Psi}_{\mathrm{IW}}$ is the equivalence inverse to $\widehat{\Phi}_{\mathrm{IW}}$.

Thus we get an isomorphism

$$a(\widehat{\Psi}(\mathcal{F} * \mathcal{G})) \cong a(\widehat{\Psi}(\mathcal{F})) \circ a(\widehat{\Psi}(\mathcal{G})) \cong a(\widehat{\Psi}(\mathcal{F}) * \widehat{\Psi}(\mathcal{G})),$$

where we used notations of Lemma 48. This isomorphism is compatible with the associativity constraint, since the equivalence $\widehat{\Phi}_{\mathrm{IW}}$ sends the corresponding equality to an equality which holds since monoidal category $\hat{\mathrm{D}}$ acts on $\hat{\mathrm{D}}_{\mathrm{IW}}$.

Since $\widehat{\Psi} : \hat{\mathcal{T}} \to \mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})$, Lemma 48(b) yields (21) in the case when $\mathcal{F}, \mathcal{G} \in \hat{\mathcal{T}}$, which is compatible with the associativity isomorphism for three objects in $\hat{\mathcal{T}}$. Now Corollary 47(a) compared to Proposition 7(b) yields (21) in general, while Corollary 47(b) together with Proposition 7(b) shows that the constructed isomorphism is compatible with associativity constraint.

### 10.3. *Monoidal structure on* $\Phi_{\mathrm{II}}$.

**10.3.1.** *A monoidal structure on* $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$. — In order to equip $\Phi_{\mathrm{II}}$ with a monoidal structure we describe the monoidal structure on $\mathrm{D}_{\mathrm{II}}$ in terms of the DG-model $\hat{\mathcal{T}}_{\mathrm{II}}$ (see Corollary 45).

Let $\hat{\mathcal{T}}_{\mathrm{II}}^{(2)}$ denote the category of finite complexes of objects in $\hat{\mathcal{T}}$ equipped with an action of $\mathbf{K}_{\mathfrak{t}^2} \otimes \Lambda(\mathfrak{t}[1])$, and $\hat{\mathcal{T}}_{\mathrm{II}}^{(3)}$ be the category of finite complexes of objects in $\hat{\mathcal{T}}$ equipped with an action of $\mathbf{K}_{\mathfrak{t}^2} \otimes \Lambda(\mathfrak{t}^2[1])$. In both cases we require that $\mathfrak{t}^2 \subset \mathbf{K}_{\mathfrak{t}^2}$ acts by logarithm of monodromy.

We have a functor $\hat{\mathcal{T}}_{\mathrm{II}} \times \hat{\mathcal{T}}_{\mathrm{II}} \xrightarrow{\star} \hat{\mathcal{T}}_{\mathrm{II}}^{(2)}$ sending $(\mathrm{T}_1, \mathrm{T}_2)$ to the convolution $\mathrm{T}_1 * \mathrm{T}_2$; the latter complex is equipped with two actions of $\mathbf{K}_{\mathfrak{t}}$ coming respectively from the left action on $\mathrm{T}_1$ and the right action on $\mathrm{T}_2$. To define the action of $\Lambda(\mathfrak{t}[1])$ observe that the right monodromy action on $\mathrm{T}_1$ and the left monodromy action on $\mathrm{T}_2$ induce the same action on $\mathrm{T}_1 * \mathrm{T}_2$, the diagonal action of $\mathbf{K}_{\mathfrak{t}}$ kills the augmentation ideal of $\mathbf{K}_{\mathfrak{t}}^0 = \mathrm{Sym}(\mathfrak{t})$, thus it factors through an action of $\Lambda(\mathfrak{t}[1])$.

Similarly, we have a functor $\hat{\mathcal{T}}_{\mathrm{II}} \times \hat{\mathcal{T}}_{\mathrm{II}} \times \hat{\mathcal{T}}_{\mathrm{II}} \xrightarrow{\star_2} \hat{\mathcal{T}}_{\mathrm{II}}^{(3)}$ sending $(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3)$ to $\mathrm{T}_1 * \mathrm{T}_2 * \mathrm{T}_3$ where the two actions of $\mathbf{K}_{\mathfrak{t}}$ come respectively from the left action on $\mathrm{T}_1$ and the right action on $\mathrm{T}_3$, and the two actions of $\Lambda(\mathfrak{t}[1])$ come from the diagonal action of $\mathbf{K}_{\mathfrak{t}}$ on the first and the second factor, and the diagonal action of $\mathbf{K}_{\mathfrak{t}}$ on the second and the third factor respectively. We use the same notation $\star$, $\star_2$ for the corresponding functors on the homotopy categories.

Furthermore, we have functors $\mu : \mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}}^{(2)}) \to \mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$, $\mu : \mathrm{M} \mapsto \mathcal{M} \otimes^{\mathrm{L}}_{\Lambda(\mathfrak{t}[1])} k$ and $\mu^{(2)} : \mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}}^{(3)}) \to \mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$, $\mu^{(2)} : \mathrm{M} \mapsto \mathrm{M} \otimes^{\mathrm{L}}_{\Lambda(\mathfrak{t}^2[1])} k$.

The following proposition obviously yields a monoidal structure on the equivalence (4).

*Proposition* **50.** — *(a) The product* $(\mathrm{M}_1, \mathrm{M}_2) \mapsto \mu(\mathrm{M}_1 \star \mathrm{M}_2)$ *makes* $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}})$ *into a monoidal category, where the associativity constraint comes from the natural isomorphisms*

$$(\mathrm{M}_1 \otimes \mathrm{M}_2) \otimes \mathrm{M}_3 \cong \mu^{(2)} \star_2 (\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3) \cong \mathrm{M}_1 \otimes (\mathrm{M}_2 \otimes \mathrm{M}_3). \tag{22}$$

*(b) The equivalence* $real_{eq} : \mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}}) \cong \mathrm{D}_{\mathrm{II}}$ *is naturally enhanced to a monoidal functor.*

*(c) The equivalence* $\mathrm{Ho}(\hat{\mathcal{T}}_{\mathrm{II}}) \cong \mathrm{DGCoh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$ *is naturally enhanced to a monoidal functor.*

*Proof.* — To check (a) and (b) it suffices to provide a bi-functorial isomorphism

$$real_{eq}(\mathrm{M}_1 \star \mathrm{M}_2) \cong real_{eq}(\mathrm{M}_1) * real_{eq}(\mathrm{M}_2)$$

sending the isomorphism (22) to the associativity constraint in $\mathrm{D}_{\mathrm{II}}$. This follows from the next Lemma 51.

(c) follows from the definition of convolution in $\mathrm{DGCoh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}})$. □

In order to state the next lemma we return to the setting of Section 9.3.1. Let X be an algebraic variety equipped with an action of an algebraic torus A and let $f : \mathrm{X}/\mathrm{A} \to \mathrm{Y}$ be a map where Y is an algebraic variety and X/A is the stack quotient. Let $pr : \mathrm{X} \to \mathrm{X}/\mathrm{A}$ be the projection and set $\tilde{f} = f \circ pr : \mathrm{X} \to \mathrm{Y}$.

*Lemma* **51.** — *(a) Let* $\mathrm{M}^\bullet \in \mathcal{P}_{eq}$ *be a complex of monodromic perverse sheaves on* X *equipped with a* $\mathbf{K}_{\mathfrak{a}}$ *action and let* $\bar{\mathrm{M}}$ *be the corresponding object in* $\mathrm{D}(\mathrm{X}/\mathrm{A})$ *(see Lemma 44).*

*Assume that* $\tilde{f}_*(\mathrm{M}^i)$ *is a perverse sheaf for all* $i$.

*We then have a canonical isomorphism*

$$f_*(\bar{\mathrm{M}}) \cong real_{eq}(\tilde{f}_*(\mathrm{M}^\bullet) \overset{\mathrm{L}}{\underset{\Lambda(\mathfrak{a}[1])}{\otimes}} k[\dim(\mathrm{A})]).$$

*(b) Assume also that a torus* $\mathrm{A}'$ *acts on* X, Y *so that* $f$ *is* $\mathrm{A}'$*-equivariant and the action on* X *commutes with* A. *Let* $\bar{f}$ *be the morphism* $\mathrm{X}/(\mathrm{A} \times \mathrm{A}') \to \mathrm{Y}/\mathrm{A}'$.

*Let* $\mathrm{M}^\bullet \in \mathcal{P}_{eq}$ *be a complex of monodromic perverse sheaves on* X *equipped with a* $\mathbf{K}_{\mathfrak{a}\oplus\mathfrak{a}'}$ *action and let* $\bar{\mathrm{M}}$ *be the corresponding object in* $\mathrm{D}(\mathrm{X}/(\mathrm{A} \times \mathrm{A}'))$.

*Assume that* $\tilde{f}_*(\mathrm{M}^i)$ *is a perverse sheaf for all* $i$.

*We then have a canonical isomorphism of objects in* $\mathrm{D}(\mathrm{Y}/\mathrm{A}')$*:*

$$\bar{f}_*(\bar{\mathrm{M}}) \cong real_{eq}(\tilde{f}_*(\mathrm{M}^\bullet) \overset{\mathrm{L}}{\underset{\Lambda(\mathfrak{a}[1])}{\otimes}} k[\dim(\mathrm{A})]).$$

*Proof.* — (a) is a particular case of (b), which we will presently deduce from the following two statements:

(I) The equivalence of Lemma 44(a) satisfies the following functoriality. Consider an A-equivariant map of schemes $f : \mathrm{X} \to \mathrm{Y}$ and use Lemma 44(a) to identify $\mathrm{D}_{\mathrm{A}}(\mathrm{X}) \cong \mathrm{D}(\mathcal{P}_{eq}(\mathrm{X}))$, $\mathrm{D}_{\mathrm{A}}(\mathrm{Y}) \cong \mathrm{D}(\mathcal{P}_{eq}(\mathrm{Y}))$. Then for $\mathcal{F}^\bullet \in \mathcal{P}_{eq}(\mathrm{X})$ such that $f_*(\mathcal{F}^i) \in \mathrm{Perv}(\mathrm{Y})$ the object of $\mathcal{P}_{eq}(\mathrm{Y})$ obtained from $\mathcal{F}^\bullet$ by term-wise application of $f_*$ corresponds to the object $f_*(\mathcal{F}) \in \mathrm{D}_{\mathrm{A}}(\mathrm{Y})$.

The special case of this functoriality where the group A is trivial is checked in [6], the general case is similar.

(II) For a subtorus A′ of A the functor $\mathrm{Res}^{\mathbf{K}_{\mathfrak{a}}}_{\mathbf{K}_{\mathfrak{a}'}} : \mathcal{P}^{\mathrm{A}}_{eq}(\mathrm{X}) \to \mathcal{P}^{\mathrm{A}'}_{eq}(\mathrm{X})$ corresponds under the equivalence of Lemma 44(a) to the restriction of equivariance functor $\mathrm{Res}^{\mathrm{A}}_{\mathrm{A}'} : \mathrm{D}_{\mathrm{A}}(\mathrm{X}) \to \mathrm{D}_{\mathrm{A}'}(\mathrm{X})$, while the functor $\mathrm{Ind}^{\mathbf{K}_{\mathfrak{a}}}_{\mathbf{K}_{\mathfrak{a}'}}[\dim \mathrm{A}' - \dim \mathrm{A}] : \mathcal{P}^{\mathrm{A}'}_{eq}(\mathrm{X}) \to \mathcal{P}^{\mathrm{A}}_{eq}(\mathrm{X})$ corresponds to the functor of direct image under the morphism of stacks $\mathrm{X}/\mathrm{A}' \to \mathrm{X}/\mathrm{A}$.

This is a straightforward generalization of Lemma 44(b).

Now (I) applied to the torus $\mathrm{A} \times \mathrm{A}'$ acting compatibly on X, Y yields a description of an object in $\mathcal{P}^{\mathrm{A}\times\mathrm{A}'}_{eq}(\mathrm{Y})$ representing $\bar{f}_*(\bar{\mathrm{M}})$: we have $\bar{f}_*(\bar{\mathrm{M}}) \cong real_{eq}(\tilde{f}_*(\mathrm{M}^\bullet))$, where $\tilde{f}_*(\mathrm{M}^\bullet)$ is equipped with the action of $\mathbf{K}_{\mathfrak{a}\oplus\mathfrak{a}'}$ inherited from the action on $\mathrm{M}^\bullet$. However, $\bar{f}_*(\bar{\mathrm{M}})$ can also be rewritten as the direct image of $f_*(\bar{\mathrm{M}})$ under the morphism $\mathrm{Y}/\mathrm{A}' \to \mathrm{Y}/(\mathrm{A} \times \mathrm{A}')$. Using (II) we get an isomorphism

$$\Lambda(\mathfrak{a}[1]) \otimes_k (real^{\mathrm{A}'}_{eq})^{-1}(f_*(\bar{\mathrm{M}}))[-\dim(\mathrm{A})] \cong (real^{\mathrm{A}\times\mathrm{A}'}_{eq})^{-1}(\bar{f}_*(\bar{\mathrm{M}})),$$

where we have adorned $real_{eq}$ with an additional superscript making clear in which equivariant category it lands. Applying the functor $-\otimes_{\Lambda(\mathfrak{a}[1])} k[\dim(\mathrm{A})]$ to both sides of the last isomorphism we get the lemma. □

**10.4.** *Compatibility of (3) with the action of categories from (2), (4).* — To finish the proof of Theorem 1 it remains to establish compatibility of equivalence (3) with the structure of a module category over the monoidal categories appearing in (2) and (4).

To check compatibility with the action of $\mathrm{D}_{\mathrm{I}^0\mathrm{I}^0} \cong \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\mathcal{N}}(\mathrm{St}))$ we pass to the pro-completions and check compatibility of (3) with the action of $\hat{\mathrm{D}} \cong \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}}))$. We have an action of the monoidal category of free monodromic tilting complexes $\hat{\mathcal{T}}$ on the category of tilting objects $\mathcal{T} \subset \mathcal{P}$ which induces a structure of a module category for $\mathrm{Ho}(\hat{\mathcal{T}})$ on $\mathrm{Ho}(\mathcal{T})$. In view of Proposition 7(c) this module structure is compatible with one arising from the equivalences $\mathrm{Ho}(\mathcal{T}) \cong \mathrm{D}$, $\mathrm{Ho}(\hat{\mathcal{T}}) \cong \hat{\mathrm{D}}$. On the other hand, using Corollary 47(c,d) we see that the equivalence $\mathrm{Ho}(\mathcal{T}) \cong \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\mathrm{St}'))$ is compatible with the action of $\mathrm{Ho}(\hat{\mathcal{T}}) \cong \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}(\widehat{\mathrm{St}}))$. This yields compatibility with the action of categories in (2).

To check compatibility with the action of $\mathrm{D}_{\mathrm{II}} \cong \mathrm{DGCoh}^{\mathrm{G}^\vee}(\tilde{\mathcal{N}} \overset{\mathrm{L}}{\times}_{\mathcal{N}} \tilde{\mathcal{N}})$ we use Lemma 44 to identify $\mathrm{D}_{\mathrm{I}^0\mathrm{I}}$ with the homotopy category of complexes in $\hat{\mathcal{T}}$ equipped with an action of $\mathbf{K}_{\mathfrak{t}}$ compatible with the right log monodromy action. This category of complexes carries a natural action of the monoidal DG-category of complexes in $\hat{\mathcal{T}}$ with a compatible action of $\mathbf{K}_{\mathfrak{t}^2}$. The resulting triangulated module category is module equivalent to both $D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}'))$ and $D_{I^0I}$ by an argument parallel to that of Section 10.3.

This establishes the compatibilities thereby completing the proof of Theorem 1.

**10.5.** *Compatibility with projections.* — We finish the section by recording another useful compatibility between equivalences of Theorems 1 and 2.

*Proposition* **52.** — *The following diagrams commute*

$$\begin{array}{ccc} D^b(\mathrm{Coh}^{G^\vee}(\widehat{\mathrm{St}})) & \xrightarrow{\widehat{\Phi}} & \hat{D} \\ \downarrow {\scriptstyle pr_{\mathrm{Spr},1*}} & & \downarrow {\scriptstyle \mathrm{Av}_{\mathrm{IW}}} \\ D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee)) & \xrightarrow{\widehat{\Phi}_{\mathrm{IW}}} & \hat{D}_{\mathrm{IW}} \end{array}$$

$$\begin{array}{ccc} D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}')) & \xrightarrow{\widehat{\Phi}} & D_{I^0I} \\ \downarrow {\scriptstyle pr'_{\mathrm{Spr},1*}} & & \downarrow {\scriptstyle \mathrm{Av}_{\mathrm{IW}}} \\ D^b(\mathrm{Coh}^{G^\vee}(\tilde{\mathcal{N}})) & \xrightarrow{\Phi_{\mathrm{IW}}} & D^{I}_{\mathrm{IW}} \end{array}$$

*Proof.* — These diagrams are obtained from the last two diagrams in Lemma 28 by passing to adjoint functors. Alternatively, the first diagram above follows from compatibility of the equivalences (9) and $\Phi^{I^0}_{\mathrm{IW}}$ with the action of the monoidal category on the module category, since $pr_{\mathrm{Spr},1*}(\mathcal{F}) = \mathcal{F} * \mathcal{O}_{\tilde{\mathfrak{g}}^\vee}$ and $\mathrm{Av}_{\mathrm{IW}}(\mathcal{G}) = \mathcal{G} * \Delta_i^{\mathrm{IW}}$ for $\mathcal{F} \in D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}))$, $\mathcal{G} \in \hat{D}$. □

## 11. Further properties

In this section we mention further properties and possible generalizations of the constructed equivalences.

**11.1.** *Frobenius compatibility.* — As pointed out in the Introduction, our main result is inspired by different geometric realizations of the affine Hecke algebra. However, the Grothendieck group of the categories in Theorem 1 is isomorphic to a less interesting ring $\mathbf{Z}[W]$. A possible "upgrade" of the theorem involving categories whose Grothendieck group is related to the affine Hecke algebra is an equivalence between $D^b(\mathrm{Coh}_{\mathcal{N}}^{G^\vee \times \mathbf{G}_m}(\mathrm{St}))$

and an appropriately defined mixed version of $D_{I^0 I^0}$. However, for many applications (cf. [20], [14]) the following simpler version is sufficient.

Fix a finite field $\mathbf{F}_q$ and assume that the base field $k = \overline{\mathbf{F}_q}$. Then the categories in the left hand side of (2)–(6) carry an automorphism coming from the Frobenius automorphism of $k$.

Let $\mathbf{q} : \mathrm{St} \to \mathrm{St}$ be the map given by $\mathbf{q} : (x, \mathfrak{b}_1, \mathfrak{b}_2) \mapsto (qx, \mathfrak{b}_1, \mathfrak{b}_2)$. We use the same letter to denote the induced automorphisms of $\mathrm{St}'$, $\widehat{\mathrm{St}}$ etc.

*Proposition* **53.** — *The equivalences in Theorems 1, 2 intertwine Frobenius automorphism with the functor* $\mathbf{q}^*$ *acting on the derived categories of coherent sheaves.*

The proof is parallel to the proof of [1, Proposition 1].

**11.2.** *Category $\mathcal{P}$ and the noncommutative Springer resolution.* — Recall that the main result of [20][12] is a construction of a certain noncommutative $\mathcal{O}(\mathfrak{g}^\vee)$ algebra A and its quotient $A^0$ with derived equivalences $D^b(A - mod_{fg}) \cong D^b(\mathrm{Coh}(\tilde{\mathfrak{g}}^\vee))$, $D^b(A^0 - mod_{fg}) \cong D^b(\mathrm{Coh}(\tilde{\mathcal{N}}))$, see [20, §1.5.3; Theorem 1.5.1(b)]. The algebras come equipped with a natural $G^\vee$ action and equivalences admit an equivariant version. Furthermore, applying a version of [20, Theorem 1.5.1(b)] to the group $G^\vee \times G^\vee$ one gets an equivalence

$$D^b(\mathrm{Coh}^{G^\vee}(\mathrm{St}')) \cong D^b(A \otimes_{\mathcal{O}(\mathfrak{g}^\vee)} A^0 - mod_{fg}^{G^\vee}).$$

Composing it with equivalence (3) and recalling that $D_{I^0 I} \cong D^b(\mathcal{P})$ we get an equivalence

$$D^b(\mathcal{P}) \cong D^b\big(A \otimes_{\mathcal{O}(\mathfrak{g}^\vee)} A^0 - mod_{fg}^{G^\vee}\big). \tag{23}$$

We now describe the relation between the natural $t$-structures on the two sides of (23).

For a nilpotent orbit $O \subset \mathcal{N}$ consider the full subcategory of complexes such that each cohomology module considered as a module over the center $\mathcal{O}(\mathcal{N}) \subset A \otimes_{\mathcal{O}(\mathfrak{g}^\vee)} A^0$ is set theoretically supported on the closure of O. These subcategories define a filtration by thick subcategories on the triangulated category $D^b(A \otimes_{\mathcal{O}(\mathfrak{g}^\vee)} A^0 - mod_{fg}^{G^\vee})$ indexed by the partially ordered set of nilpotent orbits. We will refer to this filtration as the support filtration.

For an orbit $O \subset \mathcal{N}$ we let $j^O : O \to \mathcal{N}$ be the embedding, $d_O = \frac{\dim O}{2}$, $d = \frac{\dim \mathcal{N}}{2}$. Recall the *perverse t-structure of middle perversity* on $D^b(\mathrm{Coh}^{G^\vee}(\mathcal{N}))$, [4, Example 4.15] and the minimal extension functor $j^O_{!*}$ from equivariant perverse coherent sheaves on O to those on $\mathcal{N}$, [4, §4]. A straightforward generalization of *loc. cit.* produces a functor on (a subcategory of) the derived category of $G^\vee$-equivariant modules over a finite $\mathcal{O}(\mathcal{N})$ algebra equipped with a compatible $G^\vee$-action.

[12] Note the difference of notation: the group denoted here by $G^\vee$ is denoted by G in [20].

*Theorem* **54.** — *(a) The support filtration is compatible with the image of the tautological t-structure on* $D^b(\mathcal{P})$ *under the equivalence* (23). *The induced t-structure on the associated graded category corresponding to the nilpotent orbit* $O$ *coincides with t-structure coming from the tautological t-structure on* $D^b(A \otimes_{\mathcal{O}(\mathfrak{g}^{\check{}})} A^0 - mod_{fg}^{G^{\check{}}})$ *shifted by* $d - d_O$.

*(b) Let* $\mathcal{F} \in D_{I^0 I} \cong D^b(\mathcal{P})$ *be an object and* $\mathcal{M} \in D^b(A \otimes_{\mathcal{O}(\mathfrak{g}^{\check{}})} A^0 - mod_{fg}^{G^{\check{}}})$ *be its image under* (23). *Then the following are equivalent:*

(1) $\mathcal{F} \in \mathcal{P}$.
(2) $\mathrm{Forg}(\mathcal{M})[-d]$ *is a perverse coherent sheaf for the middle perversity. Here* $\mathrm{Forg} : A \otimes_{\mathcal{O}(\mathfrak{g}^{\check{}})} A^0 - mod_{fg}^{G^{\check{}}} \to \mathrm{Coh}^{G^{\check{}}}(\mathcal{N})$ *is the forgetful functor.*

*(c) For* $\mathcal{F}$, $\mathcal{M}$ *as in (b) the following are equivalent:*

(1) $\mathcal{F}$ *is an irreducible object in* $\mathcal{P}$.
(2) *There exists an orbit* $O \subset \mathcal{N}$ *and an irreducible object* $\mathcal{L}$ *in the category of* $G^{\check{}}$*-equivariant* $A \otimes A^0|_O$*-modules, such that* $\mathcal{M}[d] = j_{!*}^O(\mathcal{L}[d_O])$.

*Proof.* — Part (c) follows from (b) by a straightforward generalization of [4, Proposition 4.11], while (b) follows from part (a) by comparing it with the definition of the perverse coherent *t*-structure in [4]. The proof of part (a) is parallel to the proof of [20, Theorem 6.2.1] which asserts the similar property of equivalence (5). □

Similar properties hold for the rest of the equivalences (2)–(6).

**11.3.** *Lusztig's cells.* — In order to simplify the statement in this subsection we assume that G is simply-connected, thus W is a Coxeter group. Recall the notion of a *two sided cell* in W. These are certain subsets in W. In [37] Lusztig has established a bijection between 2-sided cells in W and the set $\mathcal{N}/G^{\check{}}$ of nilpotent conjugacy classes in $\mathfrak{g}^{\check{}}$. The set of two sided cells is equipped with a partial order. It has been conjectured by Lusztig and proved in [12] that this order matches the adjacency order on the set of nilpotent orbits under the bijection between two-sided cells and $\mathcal{N}/G^{\check{}}$. We now present a stronger statement relating the 2-sided cells to the support filtration introduced in Section 11.2.

*Theorem* **55.** — *Let* $\underline{c}$ *be a two sided cell in* W *and* $O_{\underline{c}} \subset \mathcal{N}$ *be the corresponding nilpotent orbit.*

*Let* $D_{\leq \underline{c}} \subset D_{I^0 I}$ *be the thick subcategory generated by irreducible objects* $IC_w \in \mathcal{P}$, $w \in \underline{c}' \leq \underline{c}$.

*Let* $D^b(\mathrm{Coh}_{O_{\underline{c}}}^{G^{\check{}}}(\mathrm{St}'))$ *be the full subcategory in* $D^b(\mathrm{Coh}^{G^{\check{}}}(\mathrm{St}'))$ *consisting of complexes whose cohomology is set-theoretically supported on the preimage of the closure of* $O_{\underline{c}}$.

*Then* $D^b(\mathrm{Coh}_{O_{\underline{c}}}^{G^{\check{}}}(\mathrm{St}'))$ *is the image of* $D_{\leq \underline{c}}$ *under the equivalence* $\Psi_{I^0 I}$.

*Proof.* — Fix an orbit O and let $W_O$ be the set of all $w \in W$ such that $IC_w \in \Phi(D^b(\mathrm{Coh}_O^{G^{\check{}}}(\mathrm{St}')))$. Theorem 54(c) implies that $\Phi(D^b(\mathrm{Coh}_O^{G^{\check{}}}(\mathrm{St}')))$ is generated as a triangulated category by $IC_w$, $w \in W_O$.

Thus the theorem follows once we check that $W_{O_{\underline{c}}} = \bigcup_{\underline{c}' \leq \underline{c}} \underline{c}'$, equivalently that for $w \in \underline{c}$ the closure of the orbit $\overline{O_{\underline{c}}}$ coincides with the image in $\mathcal{N}$ of the support of $\Psi_{I^0 I}(IC_w)$. Let $S_w$ denote that image.

We deduce this from [12, Theorem 4(a)], which provides the similar statement for the equivalence (5) of Theorem 2. (In fact, *loc. cit.* deals with the category ${}^f\mathcal{P}$, a quotient category of the category of Iwahori equivariant perverse sheaves on $\mathcal{F}\ell$; however, ${}^f\mathcal{P}$ is equivalent to the category in the left hand side of (5) by [1, Theorem 2].) Thus we see that

$$\textbf{(24)} \qquad \Phi^I_{IW} : D^b\big(Coh^{G^\vee}_{O_{\underline{c}}}(\tilde{\mathcal{N}})\big) \xrightarrow{\sim} \big(D^I_{IW}\big)_{\leq \underline{c}},$$

where $(D^I_{IW})_{\leq \underline{c}}$ is the image of $D_{\leq \underline{c}}$ under the functor the Whittaker averaging functor $Av^{IW} : D_{I^0 I} \to D^{IW}_I$ (the same notation was used above for the Whittaker averaging functor $D_{I^0 I^0} \to D^{IW}_{I^0}$).

The second commutative diagram in Proposition 52 yields

$$\textbf{(25)} \qquad \begin{aligned} &(\Psi^I_{IW})^{-1}(IC^{IW}_w) \cong p'_{Spr,2*}(\Psi_{I^0 I}(IC_w)) \quad \text{for } w \in W^f, \\ &p'_{Spr,2*}(\Psi_{I^0 I}(IC_w)) = 0 \quad \text{for } w \notin W^f; \end{aligned}$$

here $IC_w$, $w \in W$ and $IC^{IW}_w$, $w \in W^f$ are irreducible objects in $\mathcal{P}$ and $\mathcal{P}^{IW}_I$ respectively.

It is clear that

$$\textbf{(26)} \qquad S_w = \bigcup_{\lambda,\mu} supp\big(p_{Spr*}\Phi^{-1}_{I^0 I}(IC_w)(\lambda,\mu)\big),$$

$$\textbf{(27)} \qquad S_w \supset supp\big(\Phi^{-1}_{I^0 I}(IC_w)(M * IC_w * N)\big).$$

Since $D_{\leq \underline{c}}$ is invariant under both left and right convolution, we see that (26) combined with (24), (25) shows that $S_w \subseteq \overline{O_{\underline{c}}}$. Also, for any $w_1, w_2 \in \underline{c}$ the object $IC_{w_1}$ is a direct summand in the convolution $X * IC_{w_2} * Y$ for some X, Y; thus (27) shows that $S_w \supseteq \overline{O_{\underline{c}}}$. □

**11.4.** *Exactness and Hodge* D*-modules.* — Recall that in view of Corollary 42(a), the restriction of the functors $\Psi_{I^0 I}$, $\Psi_{I^0 I^0}$ to the subcategory of sheaves supported on the finite dimensional flag variety $G/B \subset \mathcal{F}\ell$ is $t$-exact, i.e. it sends a perverse sheaf to a coherent sheaf.

On the other hand, a well known result in representation theory asserts that the category O for Langlands dual Lie algebras are equivalent, i.e. we have an equivalence of abelian categories

$$\Upsilon : Perv_{U^\vee}(G^\vee/B^\vee) \xrightarrow{\sim} Perv_U(G/B) = Perv_{I^0}(G/B).$$

This allows to state a relation between the restriction of our equivalence $\Phi_{I^0 I}$ to $Perv_{I^0}(G/B) \subset \mathcal{P}_{I^0 I}$ and Hodge D-module theory.

Notice that the stack $\mathrm{St}'/\check{G}$ can be interpreted as the *cotangent* to the stack $\check{U}\backslash\check{G}/\check{B}$. Thus for a $\check{U}$-equivariant D-module M on $\check{G}/\check{B}$ equipped with a $\check{U}$-equivariant good filtration we get $gr(\mathrm{M}) \in \mathrm{Coh}^{\check{G}}(\mathrm{St}')$.

Let $\mathcal{MH}_{\check{U}}(\check{G}/\check{B})$ be the category of mixed Hodge modules on $\check{G}/\check{B}$ equivariant with respect to $\check{U}$. We have the forgetful functor $\mathrm{Forg} : \mathcal{MH}_{\check{U}}(\check{G}/\check{B}) \to \mathrm{D}-mod_{\check{U}}(\check{G}/\check{B}) \cong \mathrm{Perv}_{\check{U}}(\check{G}/\check{B})$ where the second equivalence is the Riemann-Hilbert functor. Recall that a part of the data of a mixed Hodge structure on a D-module is a good filtration, i.e. for $\tilde{\mathrm{M}} \in \mathcal{MH}_{\check{U}}(\check{G}/\check{B})$ the D-module $\mathrm{M} = \mathrm{Forg}(\tilde{\mathrm{M}})$ is equipped with a canonical good filtration. Thus we get a functor $gr : \mathcal{MH}_{\check{U}}(\check{G}/\check{B}) \to \mathrm{Coh}^{\check{G}}(\mathrm{St}')$.

*Conjecture* **56.** — *For* $\tilde{\mathrm{M}} \in \mathcal{MH}_{\check{U}}(\check{G}/\check{B})$ *we have a canonical isomorphism*

$$gr(\tilde{\mathrm{M}}) \otimes \mathcal{O}(-\rho) \cong \Psi_{\mathrm{I}^0\mathrm{I}}(\Upsilon(\mathrm{M})).$$

This conjecture can be compared to the results of Ben-Zvi and Nadler [10].

*Example* **57.** — Recall that the finite Weyl group $\mathrm{W}_f$ acts on the open subvariety $\tilde{\check{\mathfrak{g}}}^{reg} \subset \tilde{\check{\mathfrak{g}}}$.

For $w \in \mathrm{W}_f$ let $\Gamma_w \subset \mathrm{St}$ be the closure of the graph of $w$. Let $\Gamma'_w$ be the scheme theoretic intersection $\Gamma_w \cap \mathrm{St}'$. One can show that:

$$\Psi_{\mathrm{I}^0\mathrm{I}} : \Xi \mapsto \mathcal{O}_{\mathrm{St}'},$$
$$\Psi_{\mathrm{I}^0\mathrm{I}} : j_{w*} \mapsto \mathcal{O}_{\Gamma'_w},$$
$$\Psi_{\mathrm{I}^0\mathrm{I}} : j_{w!} \mapsto \Omega_{\Gamma'_w},$$

where $\Omega_{\Gamma'_w}$ is the dualizing sheaf for the Cohen-Macaulay variety $\Gamma'_w$ (the Cohen-Macaulay property is proven in [21]). Parallel results for associated graded of Hodge D-modules will be shown in [22].

We finish by sketching some generalizations of the equivalences described in the paper. We expect they can be obtained by similar methods.

**11.5.** *Nonunipotent monodromy.* — Consider the category of $\mathbf{I}^2$ monodromic sheaves on $\widetilde{\mathcal{F}\ell}$ with a fixed generalized eigenvalues of monodromy. The latter corresponds to a tame rank one local system on $\check{T}^2$, such local systems are in bijection with elements of $\check{T}^2$ (or a subset of that in the $l$-adic setting). For $\theta_1, \theta_2 \in \check{T}$ let $\mathrm{D}_{\theta_1,\theta_2}$ be the category of monodromic sheaves on $\widetilde{\mathcal{F}\ell}$ with corresponding generalized eigenvalues of monodromy.

Let $\widetilde{\check{G}} \subset \check{G} \times \check{G}/\check{B}$ be the closed subvariety given by $\widetilde{\check{G}} = \{(g, x) \mid g(x) = x\}$. We have a projection $\widetilde{\check{G}} \to \check{T}$. Set $\mathrm{St}_{grp} = \widetilde{\check{G}} \times_{\check{G}} \widetilde{\check{G}}$, and for $t_1, t_2 \in \check{T}$ let $\mathrm{St}_{grp}^{t_1,t_2}$ be the preimage of $(t_1, t_2)$ under the projection $\mathrm{St}_{grp} \to \check{T} \times \check{T}$.

*Conjecture* **58.** — *We have a canonical equivalence of triangulated categories:*

$$\mathrm{D}_{\theta_1,\theta_2} \cong \mathrm{D}^b(\mathrm{Coh}^{\mathrm{G}^\vee}_{\mathrm{St}^{t_1,t_2}_{grp}}(\mathrm{St}_{grp})).$$

An equivariant isomorphism between the variety of unipotent elements in $\mathrm{G}^\vee$ and $\mathcal{N}$ and its extension to the formal neighborhoods in $\mathrm{G}^\vee$ (respectively, $\mathfrak{g}^\vee$) can be used to identify the category $\mathrm{Coh}^{\mathrm{G}^\vee}_{\mathrm{St}^{1,1}_{grp}}(\mathrm{St}_{grp})$ with $\mathrm{Coh}^{\mathrm{G}^\vee}_{\mathcal{N}}(\mathrm{St})$, thus in the special case $t_1 = t_2 = 1$ the conjecture amounts to equivalence (2). One can also state similar generalizations of (3), (4). Appearance of a group rather than a Lie algebra element here agrees with Langlands duality where the element is interpreted as the image of a topological generator of the tame ramification subquotient of the Galois group of the local field. On the other hand, working with the Lie algebra as we did in the present article, makes it easier to describe a graded version of the category, Koszul duality etc.

**11.6.** *Parabolic-Whittaker categories.* — Let P be a parabolic subgroup in G, and let $\mathbf{I}_\mathrm{P} \subset \mathbf{G_O}$ be the parahoric subgroup which is the preimage of P under the projection $\mathbf{G_O} \to \mathrm{G}$. Let $\mathcal{F}\ell_\mathrm{P} = \mathbf{G_F}/\mathbf{I}_\mathrm{P}$ be the corresponding partial affine flag variety.

Let Q be another parabolic subgroup and let $\psi_\mathrm{Q}$ be an additive character of $\mathbf{I}^0$ vanishing on the finite simple roots which are not in the Levi subgroup of Q as well as on the affine roots and not vanishing on the simple roots in the Levi of Q. Let $\mathrm{D}_{\mathrm{IW_Q}}(\mathcal{F}\ell_\mathrm{P})$ be the corresponding category of partial Whittaker sheaves.

Let $\mathrm{Q}^\vee$, $\mathrm{P}^\vee$ be the corresponding parabolic subgroups in $\mathrm{G}^\vee$. Define $\tilde{\mathcal{N}}_{\mathrm{Q}^\vee} \subset \tilde{\mathcal{N}}'_{\mathrm{Q}^\vee} \subset \tilde{\mathfrak{g}}^\vee_{\mathrm{Q}^\vee} \subset \mathrm{G}^\vee/\mathrm{Q}^\vee \times \mathfrak{g}^\vee$ and $\tilde{\mathcal{N}}_{\mathrm{P}^\vee} \subset \mathrm{G}^\vee/\mathrm{P}^\vee \times \mathfrak{g}^\vee$ by: $\tilde{\mathfrak{g}}^\vee_{\mathrm{Q}^\vee} = \{(\mathfrak{q}, x) \mid x \in \mathfrak{q}\}$, $\tilde{\mathcal{N}}_{\mathrm{P}^\vee} = \{(\mathfrak{p}, x) \mid x \in rad(\mathfrak{p})\}$, $\tilde{\mathcal{N}}'_{\mathrm{Q}^\vee} = \{(\mathfrak{q}, x) \mid x \in rad(\mathfrak{q}) + \mathfrak{z}(\mathfrak{q}/rad(\mathfrak{q}))\}$, where we used the identification between $\mathrm{G}^\vee/\mathrm{Q}^\vee$, respectively $\mathrm{G}^\vee/\mathrm{P}^\vee$ and the corresponding conjugacy class of parabolic subalgebras, *rad* stands for the nilpotent radical and $\mathfrak{z}$ denotes the center.

*Conjecture* **59.** — *We have canonical equivalences*

$$\mathrm{D}_{\mathrm{IW_Q}}(\mathcal{F}\ell_\mathrm{P}) \cong \mathrm{D}^b\big(\mathrm{Coh}^{\mathrm{G}^\vee}(\tilde{\mathfrak{g}}^\vee_{\mathrm{Q}^\vee} \times_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}_{\mathrm{P}^\vee})\big), \tag{28}$$

$$\mathrm{D}_{\mathbf{I}_\mathrm{Q}}(\mathcal{F}\ell_\mathrm{P}) \cong \mathrm{DGCoh}^{\mathrm{G}^\vee}\big(\tilde{\mathcal{N}}_{\mathrm{Q}^\vee} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}_{\mathrm{P}^\vee}\big), \tag{29}$$

$$\mathrm{D}_{\mathbf{I}'_\mathrm{Q}}(\mathcal{F}\ell_\mathrm{P}) \cong \mathrm{DGCoh}^{\mathrm{G}^\vee}\big(\tilde{\mathcal{N}}'_{\mathrm{Q}^\vee} \overset{\mathrm{L}}{\times}_{\mathfrak{g}^\vee} \tilde{\mathcal{N}}_{\mathrm{P}^\vee}\big), \tag{30}$$

*where* $\mathbf{I}'_\mathrm{Q}$ *is the derived subgroup of* $\mathbf{I}_\mathrm{Q}$.

There are natural pull-back, push-forward and Iwahori-Whittaker averaging functors between the categories of constructible sheaves which should correspond to the functors between the derived categories of coherent sheaves given by the natural correspondences, Proposition 52 is an example of such a compatibility.

*Example* **60.** — Some special cases of Conjecture 59 follow from results found in the literature.

Let $P = Q = G$. Then the right hand side of (28) is $D^b(\mathrm{Coh}^{G^\vee}(\{0\})) = D^b(\mathrm{Rep}(G^\vee))$. In this case (28) is essentially equivalent to the so-called geometric Casselman-Shalika formula established in [27].

The right hand side of (29) for $P = Q = G$ is $\mathrm{DGCoh}^{G^\vee}(\{0\} \overset{L}{\times}_{\mathfrak{g}^\vee} \{0\})$. In view of Koszul duality, this special case of (29) follows from the second equivalence in [17, Theorem 5], it is discussed in detail (along with equivalences for various Ind-completions of the two categories) in [5, §12].

For $Q = B$ and $P = G$ the left hand side in (30) is the derived category of $\mathbf{I}^0$ equivariant sheaves on the affine Grassmannian, while the right hand is $\mathrm{DGCoh}^{G^\vee}(\tilde{\mathfrak{g}}^\vee \overset{L}{\times}_{\mathfrak{g}^\vee} \{0\})$. This special case of (30) amounts to one of the main results of [2]; again one needs to apply linear Koszul duality [41] to pass from the coherent side of the equivalence in *loc. cit.* to the right hand side of (30); see also [18, §2.4].

Finally, let us mention the *Koszul duality* functors which give equivalences between the graded version of $D_{\mathrm{IW}_Q}(\mathcal{F}\ell_P)$ and $D_{\mathrm{IW}_P}(\mathcal{F}\ell_Q)$, see [23]. Under the first equivalence of Conjecture 59 these should correspond to *linear Koszul duality* [41]. In the special case when $P = Q = B$ is a Borel subgroup, this would provide a categorification of the main result of [42].

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