THE HYPERBOLIC AX-LINDEMANN-WEIERSTRASS CONJECTURE

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1. Introduction

1.1. *Bi-algebraic geometry and the Ax-Lindemann-Weierstraß property.* — Let X and S be complex algebraic varieties and suppose $\pi : \mathrm{X}^{\mathrm{an}} \longrightarrow \mathrm{S}^{\mathrm{an}}$ is a complex analytic, *non-algebraic*, morphism between the associated complex analytic spaces. In this situation the image $\pi(\mathrm{Y})$ of a generic algebraic subvariety $\mathrm{Y} \subset \mathrm{X}$ is usually highly transcendental and the pairs $(\mathrm{Y} \subset \mathrm{X}, \mathrm{V} \subset \mathrm{S})$ of irreducible algebraic subvarieties such that $\pi(\mathrm{Y}) = \mathrm{V}$ are rare and of particular geometric significance. We will say that an irreducible subvariety $\mathrm{Y} \subset \mathrm{X}$ (resp. $\mathrm{V} \subset \mathrm{S}$) is *bi-algebraic* if $\pi(\mathrm{Y})$ is an algebraic subvariety of S (resp. any analytic irreducible component of $\pi^{-1}(\mathrm{V})$ is an irreducible algebraic subvariety of X). Notice that $\mathrm{V} \subset \mathrm{S}$ is bi-algebraic if and only if any analytic irreducible component of $\pi^{-1}(\mathrm{V})$ is bi-algebraic.

Example **1.1.** — Let $\pi := (\exp(2\pi i\cdot), \ldots, \exp(2\pi i\cdot)) : \mathbf{C}^n \longrightarrow (\mathbf{C}^*)^n$. One easily shows that an irreducible algebraic subvariety $\mathrm{Y} \subset \mathbf{C}^n$ (resp. $\mathrm{V} \subset (\mathbf{C}^*)^n$) is bi-algebraic if and only if Y is a translate of a rational linear subspace of $\mathbf{C}^n = \mathbf{Q}^n \otimes_{\mathbf{Q}} \mathbf{C}$ (resp. V is a translate of a subtorus of $(\mathbf{C}^*)^n$).

Example **1.2.** — Let $\pi : \mathbf{C}^n \longrightarrow \mathrm{A}$ be the uniformising map of a complex Abelian variety A of dimension n. One checks that an irreducible algebraic subvariety $\mathrm{V} \subset \mathrm{A}$ is bi-algebraic if and only if V is the translate of an Abelian subvariety of A (cf. [32, prop. 5.1] for example).

More generally, given $\mathrm{Y} \subset \mathrm{X}$ an algebraic subvariety, one may ask for a description of the Zariski-closure $\overline{\pi(\mathrm{Y})}^{\mathrm{Zar}}$ of its image $\pi(\mathrm{Z})$. We will say that $\pi : \mathrm{X} \longrightarrow \mathrm{S}$ satisfy the Ax-Lindemann-Weierstraß property if for any such $\mathrm{Y} \subset \mathrm{X}$ the irreducible components of $\overline{\pi(\mathrm{Y})}^{\mathrm{Zar}}$ are bi-algebraic. One checks that the Ax-Lindemann-Weierstraß property is equivalent to the following: for any algebraic subvariety $\mathrm{V} \subset \mathrm{S}$, any irreducible algebraic subvariety Y of X contained in $\pi^{-1}(\mathrm{V})$ and maximal for this property is bi-algebraic.

Example **1.3.** — In the situations of Examples 1.1 and 1.2 Ax [2] showed that $\pi : \mathrm{X} \longrightarrow \mathrm{S}$ has the Ax-Lindemann-Weierstraß property. Namely:

- if $\pi := (\exp(2\pi i\cdot), \ldots, \exp(2\pi i\cdot)) : \mathbf{C}^n \longrightarrow (\mathbf{C}^*)^n$ and $\mathrm{Y} \subset \mathbf{C}^n$ is an algebraic subvariety then any irreducible component of $\overline{\pi(\mathrm{Y})}^{\mathrm{Zar}}$ is the translate of a subtorus of $(\mathbf{C}^*)^n$.

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- if $\pi : \mathbf{C}^n \longrightarrow A$ is the uniformising map of a complex abelian variety A of dimension n and $Y \subset \mathbf{C}^n$ is an algebraic subvariety then any irreducible component of $\overline{\pi(Y)}^{\mathrm{Zar}}$ is the translate of an Abelian subvariety of A.

Remark **1.4.** — Notice that Ax's theorem for $\pi := (\exp(2\pi i\cdot), \dots, \exp(2\pi i\cdot)) : \mathbf{C}^n \longrightarrow (\mathbf{C}^*)^n$ is the functional analog of the classical Lindemann-Weierstraß transcendence theorem ([13], [36]) stating that if $\alpha_1, \dots, \alpha_n$ are $\mathbf{Q}$-linearly independent algebraic numbers then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbf{Q}$. This explain our terminology.

1.2. *The hyperbolic Ax-Lindemann-Weierstraß conjecture.* — The main result of this paper is the proof of the Ax-Lindemann-Weierstraß property for the uniformising map $\pi : X \longrightarrow S := \Gamma\backslash X$ of any *arithmetic variety* S. Here X denotes a Hermitian symmetric domain and Γ is any *arithmetic subgroup* of the real adjoint Lie group G of biholomorphisms of X. This means that there exists a semisimple $\mathbf{Q}$-algebraic group $\mathbf{G}$ and a surjective morphism with compact kernel $p : \mathbf{G}(\mathbf{R}) \longrightarrow G$ such that Γ is commensurable with the projection $p(\mathbf{G}(\mathbf{Z}))$ (cf. Section 2 for the definition of $\mathbf{G}(\mathbf{Z})$ and [14] for a general reference on arithmetic lattices).

The Ax-Lindemann-Weierstraß property does not make sense directly for π: the arithmetic variety S admits a natural structure of complex quasi-projective variety via the Baily-Borel embedding [3] but the Hermitian symmetric domain X is not a complex algebraic variety. However X admits a canonical realisation as a bounded symmetric domain $\mathcal{D} \subset \mathbf{C}^N$ (with $N = \dim_{\mathbf{C}} X$) (cf. [28, §II.4]).

Definition **1.5.** — *We will say that a subset* $Y \subset \mathcal{D}$ *is an* irreducible algebraic subvariety *of* $\mathcal{D}$ *if* Y *is an irreducible component of the analytic set* $\mathcal{D} \cap \widetilde{Y}$ *where* $\widetilde{Y}$ *is an algebraic subset of* $\mathbf{C}^N$. *An algebraic subvariety of* $\mathcal{D}$ *is then defined as a finite union of irreducible algebraic subvarieties.*

With these definitions the morphism π is far from algebraic (in the simplest case where $\mathcal{D}$ is the Poincaré disk and S is the modular curve, the map $\pi : \mathcal{D} \longrightarrow S$ is the usual j-invariant seen on the disk) and it makes sense to study the bi-algebraic subvarieties for π. In [32] Ullmo and Yafaev proved that the bi-algebraic subvarieties of S for π are the *weakly special* ones, namely the irreducible complex algebraic subvarieties of S whose smooth locus is totally geodesic in S endowed with its canonical Hermitian metric.

Our main result is the proof of the Ax-Lindemann-Weierstraß property in this context:

Theorem **1.6** *(The hyperbolic Ax-Lindemann-Weierstraß conjecture).* — *Let* $S = \Gamma\backslash\mathcal{D}$ *be an arithmetic variety with uniformising map* $\pi : \mathcal{D} \longrightarrow S$. *Let* $Y \subset \mathcal{D}$ *be an algebraic subvariety. Then any irreducible component of the Zariski-closure* $\overline{\pi(Y)}^{\mathrm{Zar}}$ *of* $\pi(Y)$ *is weakly special.*

Equivalently, let V *be an algebraic subvariety of* S. *Irreducible algebraic subvarieties of* $\mathcal{D}$ *contained in* π^{-1}V *and maximal for this property are precisely the irreducible components of the preimages of maximal weakly special subvarieties contained in* V.

Remarks **1.7.**

(a) The Ax-Lindemann-Weierstraß property in an hyperbolic context was first proven by Pila in the case where S is a product of modular curves: cf. [23, section 1.4 and theor. 6.8]. It is a crucial ingredient in Pila's proof of the André-Oort conjecture for product of modular curves. The hyperbolic Ax-Lindemann-Weierstraß conjecture for the uniformising map of a general connected Shimura variety S is stated in [30, conjecture 1.2], where Ullmo explains its role in the proof of the André-Oort conjecture. In [34] Ullmo and Yafaev prove Theorem 1.6 in the special case where S is compact. In [26], in part inspired by [34] and relying on [20], Pila and Tsimerman proved Theorem 1.6 in the special case $S = \mathcal{A}_g$, the moduli space of principally polarised Abelian varieties of dimension g.

(b) Mok has a nice, entirely complex-analytic, approach to the hyperbolic Ax-Lindemann-Weierstraß conjecture. In the rank 1 case his approach should extend some of the results of this text to the case where Γ is a non-arithmetic lattice. We refer to [16], [17] for partial results.

(c) We defined algebraic subvarieties of X using the Harish-Chandra realisation $\mathcal{D}$ of X but we could have used as well any other *realisation* of X in the sense of [30, section 2.1]. Indeed morphisms of realisations are necessarily semi-algebraic, thus X admits a canonical semi-algebraic structure and a canonical notion of algebraic subvarieties (cf. Appendix B for details). Hence one can replace $\mathcal{D}$ in Theorem 1.6 by any other realisation of X, for example the Borel realisation (cf. [15, p. 52]).

1.3. *Motivation: the André-Oort conjecture.* — Let $(\mathbf{G}, X_{\mathbf{G}})$ be a Shimura datum. Let X be a connected component of $X_{\mathbf{G}}$ (hence X is a Hermitian symmetric domain). We denote by $\mathbf{G}(\mathbf{Q})_+$ the stabiliser of X in $\mathbf{G}(\mathbf{Q})$. Let K_f be a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$, where $\mathbf{A}_f$ denotes the finite adèles of $\mathbf{Q}$ and let $\Gamma := \mathbf{G}(\mathbf{Q})_+ \cap K_f$ be the corresponding congruence arithmetic lattice of $\mathbf{G}(\mathbf{Q})$.

Then the arithmetic variety $S := \Gamma\backslash X$ is a component of the complex quasi-projective Shimura variety

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbf{Q})_+\backslash X \times \mathbf{G}(\mathbf{A}_f)/K_f.$$

The variety S contains the so-called special points and special subvarieties (these are the weakly special subvarieties of S containing one special point, we refer to [6] or [18] for the detailed definitions). One of the main motivations for studying the Ax-Lindemann-Weierstraß conjecture is the André-Oort conjecture predicting that irreducible subvarieties of S containing Zariski dense sets of special points are precisely the special subvarieties. The André-Oort conjecture has been proved under the assumption of the Generalised Riemann Hypothesis (GRH) by the authors of this paper ([31], [12]), relying on ideas of Edixhoven [9]. Recently Pila and Zannier [27] came up with a new proof of the Manin-Mumford conjecture for abelian varieties using the flat Ax-Lindemann-Weierstraß theorem. This gave hope to prove the André-Oort conjecture unconditionally with the same strategy. In [23] Pila succeeded in applying this strategy to the case where S is a product of modular curves (and more generally, in the context of mixed Shimura varieties, when S is a product of modular curves, of elliptic curves defined over $\mathbf{Q}$ and of an algebraic torus $\mathbf{G}_{\mathrm{m}}^{l}$). Roughly speaking, the strategy of [23] consists of two main ingredients: the first is the problem of bounding below the sizes of Galois orbits of special points and the second is the hyperbolic Ax-Lindemann-Weierstraß conjecture. We refer to [30] for details on how the general hyperbolic Ax-Lindemann-Weierstraß conjecture and a good lower bound on the sizes of Galois orbits of special points imply the full André-Oort conjecture. As a direct corollary of Theorem 1.6 and the proof of [30, theor. 5.1] one obtains:

Corollary **1.8.** — *The André-Oort conjecture holds for $\mathcal{A}_6^n$ for any positive integer n.*

Notice also that (as explained in [30]) a new proof of the André-Oort conjecture under the GRH, alternative to [31] and [12], is a consequence of three ingredients: Theorem 1.6, a lower bound under GRH for the size of Galois orbits of special points (provided by Tsimerman [35] in the case of $\mathcal{A}_g$ and by Ullmo-Yafaev [33] in general) and an upper bound for the height of special points in Siegel sets. This upper-bound has been announced by C. Daw and M. Orr [5].

1.4. *Strategy of the proof of Theorem 1.6.* — Our general strategy for proving Theorem 1.6, which originates in [23], is also the one used in [34] and [26]: it ultimately relies on estimations of rational points in transcendental real-analytic varieties or more generally in spaces definable in an o-minimal structure (cf. [29] for a survey). Let us describe roughly this strategy and emphasize the new ideas involved.

(i) Let $\mathrm{S} := \Gamma\backslash\mathrm{X}$ and $\pi : \mathrm{X} \longrightarrow \mathrm{S}$ be the uniformising map. Even though the map π is transcendental, it still enables us to relate the semi-algebraic structures on X and S through a larger o-minimal structure. We refer to [7], [8], [34, section 3] for details on o-minimal structures. Recall that a fundamental set for the action of Γ on X is a connected open subset $\mathcal{F}$ of X such that $\Gamma\overline{\mathcal{F}} = \mathrm{X}$ and such that the set $\{\gamma \in \Gamma \mid \gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite. Our first result of independent interest is the following:

Theorem **1.9.** — *There exists a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ on* X *such that the restriction $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow \mathrm{S}$ is definable in the o-minimal structure $\mathbf{R}_{\mathrm{an,exp}}$.*

Remarks **1.10.**

(a) The special case of Theorem 1.9 when S is compact is easy and was proven in [34, Prop. 4.2]. In this case, the map $\pi_{|\mathcal{F}}$ is even definable in $\mathbf{R}_{\mathrm{an}}$. Theorem 1.9 in the case where $\mathrm{X} = \mathcal{H}_g$ is the Siegel upper half plane of genus g was proven by Peterzil and Starchenko (see [20] and [21]): in this case they use an explicit description for π in terms of θ-function and delicate computations with these. Their result is a crucial ingredient in [26]. Notice moreover that this particular case implies Theorem 1.9 for any special subvariety S of $\mathcal{A}_g$ (see Proposition 2.5 of [30]).

(b) On the other hand Peterzil and Starchenko's method does not generalize to general arithmetic varieties, where an explicit description of π is not available. Moreover, while the definability of π restricted to $\mathcal{F}$ is of geometric essence, the geometric meaning of computations with θ-functions is difficult to follow. On the contrary our general proof of Theorem 1.9 is completely geometric: it relies on the general theory of toroidal compactifications of arithmetic varieties (cf. [1]). In particular it does not use [20] or [21].

(ii) Choose a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ as in Theorem 1.9 above. The choice of a reasonable representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(\mathrm{E})$ (cf. Section 2) allows us to define a *height function* $\mathrm{H} : \Gamma \longrightarrow \mathbf{R}$ (cf. Definition 5.1). In Section 5 we show the following result, which is the most original part of the proof (it mixes the geometry of toroidal compactifications and various arguments from hyperbolic geometry, like Theorem 5.7 of Hwang-To):

Theorem **1.11.** — *Let* Y *be a positive dimensional irreducible algebraic subvariety of* X. *Define*

$$\mathrm{N}_{\mathrm{Y}}(\mathrm{T}) = \left|\left\{\gamma \in \Gamma : \mathrm{H}(\gamma) \leq \mathrm{T},\ \mathrm{Y} \cap \gamma\mathcal{F} \neq \emptyset\right\}\right|.$$

Then there exists a positive constant c_1 *such that for all positive real number* T *large enough:*

$$\mathrm{N}_{\mathrm{Y}}(\mathrm{T}) \geq \mathrm{T}^{c_1}.$$

Remark **1.12.** — When S is compact Ullmo and Yafaev proved in [34, theor. 2.7] a more refined result. Indeed let $\mathrm{F} := \{\gamma \in \mathcal{F},\ \gamma\overline{\mathcal{F}} \cap \overline{\mathcal{F}} \neq 0\}$ be a finite symmetric set of generators for Γ and let $l : \Gamma \longrightarrow \mathbf{N}$ be the word length function on Γ associated to F. Then Ullmo and Yafaev show that the function $\mathrm{N}_{\mathrm{Y}}(n) := |\{\gamma \in \Gamma,\ \dim(\gamma\mathcal{F} \cap \mathrm{Y}) = \dim \mathrm{Y} \text{ and } l(\gamma) \leq n\}|$ grows exponentially with $n \in \mathbf{N}$ and Theorem 1.11 follows in this case. We were not able to obtain such a result in the general case.

(iii) In Section 6, applying the counting result above and some strong form of Pila-Wilkie's theorem [24], we prove:

Theorem **1.13.** — *Let* V *be an algebraic subvariety of* S *and* Y *a maximal irreducible algebraic subvariety of* π^{-1}V. *Let* Θ_Y *denotes the stabiliser of* Y *in* $\mathbf{G}(\mathbf{R})$ *and define* $\mathbf{H}_Y$ *as the connected component of the identity of the Zariski closure of* $\mathbf{G}(\mathbf{Z}) \cap \Theta_Y$. *Then* $\mathbf{H}_Y$ *is a non-trivial* $\mathbf{Q}$*-subgroup of* $\mathbf{G}$, *such that* $\mathbf{H}_Y(\mathbf{R})$ *is non-compact.*

(iv) Without loss of generality one can assume that V is the smallest algebraic subvariety of S containing $\pi(Y)$. With this assumption we show in Section 7 that $\widetilde{V}$ is invariant under $\mathbf{H}_Y(\mathbf{Q})$, where $\widetilde{V}$ is an analytic irreducible component of $\pi^{-1}V$ containing Y, and then conclude that $\pi(Y) = V$ is weakly special using monodromy arguments.

2. Notations

In the rest of the text:

- X denotes a Hermitian symmetric domain (not necessarily irreducible).
- G is the adjoint semi-simple real algebraic group, whose set of real points, also denoted by G, is the group of biholomorphisms of X; hence $X = G/K$ where K is a maximal compact subgroup of G.
- $\Gamma \subset G$ is an arithmetic lattice. This means (cf. [14]) that there exists a semi-simple linear algebraic group $\mathbf{G}$ over $\mathbf{Q}$ and $p : \mathbf{G}(\mathbf{R}) \longrightarrow G$ a surjective morphism with compact kernel such that Γ is commensurable with $p(\mathbf{G}(\mathbf{Z}))$. Here we recall that two subgroups of a group are commensurable if their intersection is of finite index in both of them; moreover $\mathbf{G}(\mathbf{Z})$ denotes $\mathbf{G}(\mathbf{Q}) \cap \rho^{-1}(\mathbf{GL}(E_{\mathbf{Z}}))$ for some faithful representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(E)$, where E is a finite-dimensional $\mathbf{Q}$-vector space and $E_{\mathbf{Z}}$ is a $\mathbf{Z}$-lattice in E; the commensurability of Γ and $p(\mathbf{G}(\mathbf{Z}))$ is independant of the choice of ρ and $E_{\mathbf{Z}}$.
- We denote by n the dimension of E as a $\mathbf{Q}$-vector space.
- One easily checks that Theorem 1.6 holds for Γ if and only if it holds for any Γ' commensurable with Γ. In particular without loss of generality one can and will assume that the group $\mathbf{G}(\mathbf{Z})$ is neat (meaning that for any $\gamma \in \mathbf{G}(\mathbf{Z})$ the group generated by the eigenvalues of $\rho(\gamma)$ is torsion-free) and the group Γ coincides with $p(\mathbf{G}(\mathbf{Z}))$ (hence is torsion-free).
- Without loss of generality we can and will assume that *the group* $\mathbf{G}$ *is of adjoint type*. Indeed let $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ denotes the natural algebraic morphism to the adjoint group $\mathbf{G}^{\mathrm{ad}}$ of $\mathbf{G}$ (quotient by the centre). As the Lie group G is adjoint the morphism $p : \mathbf{G}(\mathbf{R}) \longrightarrow G$ factorises through

$$\begin{tikzcd} \mathbf{G}(\mathbf{R}) \arrow[r, "\lambda"] \arrow[dr, "p"'] & \mathbf{G}^{\mathrm{ad}}(\mathbf{R}) \arrow[d, "p^{\mathrm{ad}}"] \\ & G \end{tikzcd}$$

and Γ is commensurable with $p^{\mathrm{ad}}(\mathbf{G}^{\mathrm{ad}}(\mathbf{Z}))$.

- Without loss of generality we can and will assume that *each* $\mathbf{Q}$*-simple factor of* $\mathbf{G}$ *is* $\mathbf{R}$*-isotropic*. Indeed let $\mathbf{H}$ be the quotient of $\mathbf{G}$ by its $\mathbf{R}$-anisotropic $\mathbf{Q}$-factors. Again, the morphism $p : \mathbf{G}(\mathbf{R}) \longrightarrow \mathrm{G}$ factorises through $\mathbf{H}(\mathbf{R})$ and Γ is commensurable with the projection of $\mathbf{H}(\mathbf{Z})$.
- The group $\mathrm{K}_\infty := p^{-1}\mathrm{K}$ is a maximal compact subgroup of $\mathbf{G}(\mathbf{R})$. Hence $\mathrm{X} = \mathbf{G}(\mathbf{R})/\mathrm{K}_\infty$. We denote by x_0 the base-point $e\mathrm{K}_\infty$ of X.
- The quotient $\mathrm{S} := \Gamma\backslash\mathrm{X}$ is a smooth complex quasi-projective variety. We denote by $\pi : \mathrm{X} \longrightarrow \mathrm{S}$ the uniformisation map.
- We choose $\|\cdot\|_\infty : \mathrm{E}_{\mathbf{R}} \longrightarrow \mathbf{R}$ a Euclidean norm which is $\rho(\mathrm{K}_\infty)$-invariant.
- We denote by $\mathcal{X}$ any realisation of X (cf. Appendix B).

3. Compactification of arithmetic varieties

3.1. *Siegel sets.* — First we recall the definition of Siegel sets for Γ. We refer to [4, §12] for details. We follow Borel's conventions, except that for us the group G acts on X on the left.

Let $\mathbf{P}$ be a minimal $\mathbf{Q}$-parabolic subgroup of $\mathbf{G}$ such that $\mathrm{K}_\infty \cap \mathbf{P}(\mathbf{R})$ is a maximal compact subgroup of $\mathbf{P}(\mathbf{R})$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$ and let $\mathbf{A}$ be a maximal split torus of $\mathbf{P}$. We denote by $\mathbf{S}$ a maximal split torus of $\mathbf{GL}(\mathrm{E})$ containing $\rho(\mathbf{A})$. We denote by $\mathbf{M}$ the maximal anisotropic subgroup of the connected centralizer $\mathbf{Z}(\mathbf{A})^0$ of $\mathbf{A}$ in $\mathbf{P}$ and by Δ the set of positive simple roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and $\mathbf{P}$. We denote by $\mathrm{A} \subset \mathbf{S}(\mathbf{R})$ the real torus $\mathbf{A}(\mathbf{R})$. For any real number $t > 0$ we let

$$\mathrm{A}_t := \left\{ a \in \mathrm{A} \;\middle|\; a^\alpha \geq t \text{ for any } \alpha \in \Delta \right\}.$$

A Siegel set for $\mathbf{G}(\mathbf{R})$ for the data $(\mathrm{K}_\infty, \mathbf{P}, \mathbf{A})$ is a product:

$$\Sigma'_{t,\Omega} := \Omega \cdot \mathrm{A}_t \cdot \mathrm{K}_\infty \subset \mathbf{G}(\mathbf{R})$$

where Ω is a compact neighborhood of e in $\mathbf{M}^0(\mathbf{R}) \cdot \mathbf{U}(\mathbf{R})$.

The image

$$\Sigma_{t,\Omega} := \Omega \cdot \mathrm{A}_t \cdot x_o \subset \mathcal{X}$$

of $\Sigma'_{t,\Omega}$ in $\mathcal{X}$ is called a Siegel set in $\mathcal{X}$.

Theorem **3.1** *[4, theor. 13.1]. — Let* X, G, $\mathbf{G}$, Γ, $\mathbf{P}$, $\mathbf{A}$, K_∞, *and* $\mathcal{X}$ *be as above. Then for any Siegel set* $\Sigma_{t,\Omega}$*, the set* $\{\gamma \in \Gamma \mid \gamma\Sigma_{t,\Omega} \cap \Sigma_{t,\Omega} \neq \emptyset\}$ *is finite. There exist a Siegel set (called a Siegel set for* Γ*)* $\Sigma_{t_0,\Omega}$ *and a finite subset* J *of* $\mathbf{G}(\mathbf{Q})$ *such that* $\mathcal{F} := \mathrm{J} \cdot \Sigma_{t_0,\Omega}$ *is a fundamental set for the action of* Γ *on* $\mathcal{X}$.

When Ω is chosen to be semi-algebraic the Siegel set $\Sigma_{t,\Omega}$ and the fundamental set $\mathcal{F}$ are semi-algebraic as by definition of a complex realisation (cf. Appendix B) the action of $\mathbf{G}(\mathbf{R})$ on $\mathcal{X}$ is semi-algebraic and the subset $\Omega \cdot \mathrm{A}_t$ of $\mathbf{G}(\mathbf{R})$ is semi-algebraic.

We will only consider semi-algebraic Siegel sets in the rest of the text.

3.2. *Boundary components.* — General references for this section and the next one are [19] and [1].

Let $\mathcal{D} \hookrightarrow \mathbf{C}^{\mathrm{N}}$ be the Harish-Chandra realisation of X as a bounded symmetric domain. The action of G extends to the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $\mathbf{C}^{\mathrm{N}}$. The boundary $\partial\mathcal{D} := \overline{\mathcal{D}}\backslash\mathcal{D}$ is a smooth manifold which decomposes into a (continuous) union of *boundary components*, which are defined as maximal complex analytic submanifolds of $\partial\mathcal{D}$ (or alternatively as holomorphic path components of $\partial\mathcal{D}$). Explicitly, let us say that a real affine hyperplane $\mathrm{H} \subset \mathbf{C}^{\mathrm{N}}$ is a supporting hyperplane if $\mathrm{H} \cap \overline{\mathcal{D}}$ is nonempty but $\mathrm{H} \cap \mathcal{D}$ is empty. Let H be a supporting hyperplane and let $\overline{\mathrm{F}} = \mathrm{H} \cap \overline{\mathcal{D}} = \mathrm{H} \cap \partial\mathcal{D}$. Let L be the smallest affine subspace of $\mathbf{C}^{\mathrm{N}}$ which contains $\overline{\mathrm{F}}$. Then $\overline{\mathrm{F}}$ is the closure of a nonempty open subset $\mathrm{F} \subset \mathrm{L}$ which is then a single boundary component of $\mathcal{D}$ (cf. [28, §III.8.11]). The boundary component F turns out to be a bounded symmetric domain in L.

Fix a boundary component F. The normaliser $\mathrm{N}(\mathrm{F}) := \{g \in \mathrm{G} \mid g\mathrm{F} = \mathrm{F}\}$ turns out to be a proper parabolic subgroup of G. The Levi decomposition $\mathrm{N}(\mathrm{F}) = \mathrm{R}(\mathrm{F}) \cdot \mathrm{W}(\mathrm{F})$ (where W(F) denotes the unipotent radical of N(F) and R(F) is the unique reductive Levi factor stable under the Cartan involution corresponding to K) can be refined into

$$\mathrm{N}(\mathrm{F}) = \big(\mathrm{G}_h(\mathrm{F}) \cdot \mathrm{G}_l(\mathrm{F}) \cdot \mathrm{M}(\mathrm{F})\big) \cdot \mathrm{V}(\mathrm{F}) \cdot \mathrm{U}(\mathrm{F}), \tag{3.1}$$

where:

- U(F) is the centre of W(F). It is a real vector space;

- $\mathrm{V}(\mathrm{F}) = \mathrm{W}(\mathrm{F})/\mathrm{U}(\mathrm{F})$ turns out to be abelian. It is a real vector space of even dimension $2l$, and we get a decomposition $\mathrm{W}(\mathrm{F}) = \mathrm{V}(\mathrm{F}) \cdot \mathrm{U}(\mathrm{F})$ using "exp";

- $\mathrm{G}_l(\mathrm{F}) \cdot \mathrm{M}(\mathrm{F}) \cdot \mathrm{V}(\mathrm{F}) \cdot \mathrm{U}(\mathrm{F})$ acts trivially on F and $\mathrm{G}_h(\mathrm{F})$ modulo a finite center is $\mathrm{Aut}^0(\mathrm{F})$;

- $\mathrm{G}_h(\mathrm{F}) \cdot \mathrm{M}(\mathrm{F}) \cdot \mathrm{V}(\mathrm{F}) \cdot \mathrm{U}(\mathrm{F})$ commutes with U(F) and $\mathrm{G}_l(\mathrm{F})$ modulo a finite central group acts faithfully on U(F) by inner automorphisms;

- M(F) is compact.

The boundary component F is said to be *rational* if $\Gamma_{\mathrm{F}} := \Gamma \cap \mathrm{N}(\mathrm{F})$ is an arithmetic subgroup of N(F). There are only finitely many Γ-orbits of rational boundary components, we choose representatives $\mathrm{F}_1, \ldots, \mathrm{F}_r$ for these Γ-orbits. Then the Baily-Borel compactification of S is

$$\overline{\mathrm{S}}^{\mathrm{BB}} = \mathrm{S} \cup \bigcup_{i=1}^{r} (\Gamma_{\mathrm{F}_i}\backslash \mathrm{F}_i)$$

with a suitable analytic structure.

3.3. *Toroidal compactifications and local coordinates.* — Let $X^\vee$ be the compact dual of X and $\mathcal{D} \hookrightarrow X^\vee$ be the Borel embedding. Recall that $X^\vee$ has an algebraic action by $G_{\mathbf{C}}$. Given a boundary component F of $\mathcal{D}$ we define, following [19, section 3], an open subset $\mathcal{D}_F$ of $X^\vee$ containing $\mathcal{D}$ as follows:

$$\mathcal{D}_F = \bigcup_{g \in U(F)_{\mathbf{C}}} g \cdot \mathcal{D}.$$

The embedding of $\mathcal{D}$ in $\mathcal{D}_F$ is Piatetskii-Shapiro's realisation of $\mathcal{D}$ as Siegel Domain of the third kind. In fact there is a canonical holomorphic isomorphism (we refer to the proof of Lemma 4.2 for a precise description of this isomorphism):

$$\mathcal{D}_F \overset{j}{\simeq} U(F)_{\mathbf{C}} \times \mathbf{C}^l \times F.$$

This biholomorphism defines complex coordinates (x, y, t) on $\mathcal{D}_F$, such that

$$\mathcal{D} \overset{j}{\simeq} \left\{(x, y, t) \in U(F)_{\mathbf{C}} \times \mathbf{C}^l \times F \,\middle|\, \mathrm{Im}(x) + l_t(y, y) \in C(F)\right\} \subset \mathcal{D}_F$$

where $\mathrm{Im}(x)$ is the imaginary part of x, $C(F) \subset U(F)$ is a self-adjoint convex cone homogeneous under the $G_l(F)$-action on $U(F)$ and $l_t \colon \mathbf{C}^l \times \mathbf{C}^l \longrightarrow U(F)$ is a symmetric $\mathbf{R}$-bilinear form varying real-analytically with $t \in F$. The group $U(F)_{\mathbf{C}}$ acts on $\mathcal{D}_F$ and in these coordinates the action of $a \in U(F)(\mathbf{C})$ is given by:

$$(x, y, t) \longrightarrow (x + a, y, t).$$

From now on we fix a Γ-admissible collection of polyhedra $\boldsymbol{\sigma} = (\sigma_\alpha)$ (cf. [1, definition 5.1]) such that the associated toroidal compactification $\overline{S} = \overline{S}_{\boldsymbol{\sigma}}$ constructed in [1] is smooth projective and the complement $\overline{S} \setminus S$ is a divisor with normal crossings. We refer to [1] for details and we just recall what is needed for our purposes.

The compactification $\overline{S}$ is covered by a finite set of coordinates charts constructed as follows (cf. [19, p. 255–256]):

(a) Take a rational boundary component F of $\mathcal{D}$;

(b) We may choose some complex coordinates $x = (x_1, \ldots, x_k)$ on $U(F)_{\mathbf{C}}$ (depending on the choice of $\boldsymbol{\sigma}$) such that the following diagram commutes:

$$(3.2)\qquad \begin{array}{ccccc}
\mathcal{D} & \hookrightarrow & \mathcal{D}_F \overset{j}{\simeq} U(F)_{\mathbf{C}} \times \mathbf{C}^l \times F & & \\
\downarrow & & \downarrow{\scriptstyle \exp_F} & & \\
\exp_F(\mathcal{D}) = \Gamma \cap U_F \backslash \mathcal{D} & \hookrightarrow & \mathbf{C}^{*k} \times \mathbf{C}^l \times F & \hookrightarrow & \mathbf{C}^k \times \mathbf{C}^l \times F \\
{\scriptstyle \pi_F}\downarrow & & & & \\
S & & & &
\end{array}$$

where $\exp_{\mathrm{F}} : \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^{l} \times \mathrm{F} \to \mathbf{C}^{*k} \times \mathbf{C}^{l} \times \mathrm{F}$ is given by

$$(\mathbf{3.3}) \qquad (x, y, t) \mapsto \big(\exp(2i\pi x), y, t\big), \text{ where } \exp(2i\pi x) = \big(\exp(2i\pi x_1), \ldots, \exp(2i\pi x_k)\big).$$

(c) Define the "partial compactification of $\exp_{\mathrm{F}}(\mathcal{D})$ in the direction F" to be the set $\exp_{\mathrm{F}}(\mathcal{D})^{\vee}$ of points P in $\mathbf{C}^{k} \times \mathbf{C}^{l} \times \mathrm{F}$ having a neighborhood Θ such that

$$\Theta \cap \mathbf{C}^{*k} \times \mathbf{C}^{l} \times \mathrm{F} \subset \exp_{\mathrm{F}}(\mathcal{D}).$$

Then there exists an integer m, $1 \leq m \leq k$, such that $\exp_{\mathrm{F}}(\mathcal{D})^{\vee}$ contains

$$\mathrm{S}(\mathrm{F}, \boldsymbol{\sigma}) = \bigcup_{i=1}^{m} \big\{(z, y, t) \bigm| z = (z_1, \ldots, z_k), z_i = 0\big\}.$$

(d) The basic property of $\overline{\mathrm{S}}$ is that the covering map $\pi_{\mathrm{F}} : \exp_{\mathrm{F}}(\mathcal{D}) \to \mathrm{S}$ extends to a local homeomorphism $\overline{\pi_{\mathrm{F}}} : \exp_{\mathrm{F}}(\mathcal{D})^{\vee} \to \overline{\mathrm{S}}$ making the diagram

$$(\mathbf{3.4}) \qquad \begin{tikzcd}
\mathcal{D} \arrow[d, "\exp_{\mathrm{F}}"] \arrow[dd, bend right=60, "\pi"'] & \\
\exp_{\mathrm{F}}(\mathcal{D}) \arrow[r, hook] \arrow[d, "\pi_{\mathrm{F}}"] & \exp_{\mathrm{F}}(\mathcal{D})^{\vee} \arrow[d, "\overline{\pi}_{\mathrm{F}}"] \\
\mathrm{S} \arrow[r, hook] & \overline{\mathrm{S}}
\end{tikzcd}$$

commutative. Moreover every point P of $\overline{\mathrm{S}} - \mathrm{S}$ is of the form $\overline{\pi}_{\mathrm{F}}((z, y, t))$ with $z_i = 0$ for some $i \leq m$, for some F.

The following proposition summarizes what we will need:

Proposition **3.2.** — *Let* $\Sigma = \Sigma_{t,\Omega} \subset \mathcal{D}$ *be a Siegel set for the action of* Γ. *Then* Σ *is covered by a finite number of open subsets* Θ *having the following properties. For each* Θ *there is a rational boundary component* F, *a simplicial cone* $\sigma \in \boldsymbol{\sigma}$ *with* $\sigma \subset \overline{\mathrm{C}(\mathrm{F})}$, *a point* $a \in \mathrm{C}(\mathrm{F})$, *relatively compact subsets* U', Y' *and* F' *of* $\mathrm{U}(\mathrm{F})$, $\mathbf{C}^{l}$ *and* F *respectively such that the set* Θ *is of the form*

$$\begin{aligned}\Theta \overset{j}{\simeq} \big\{&(x, y, t) \in \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^{l} \times \mathrm{F},\ \mathrm{Re}(x) \in \mathrm{U}',\ y \in \mathrm{Y}',\ t \in \mathrm{F}' \bigm| \\ &\mathrm{Im}(x) + l_t(y, y) \in \sigma + a\big\} \\ &\subset \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^{l} \times \mathrm{F} \overset{j^{-1}}{\simeq} \mathcal{D}_{\mathrm{F}}.\end{aligned}$$

Proof. — Let us provide a proof of this proposition, essentially stated without proof in [19, p. 259]. Let $\mathcal{D} \overset{\Psi}{\simeq} \mathrm{W}(\mathrm{F}) \times \mathrm{C}(\mathrm{F}) \times \mathrm{F}$ be the real-analytic isomorphism deduced from the group-theoretic isomorphism (3.1) constructed in [1, p. 233]. Following [1, p. 266, corollary of proof], the Siegel set Σ is covered by a finite number of sets Θ of the form

$$\Theta \overset{\Psi}{\simeq} \omega_{\mathrm{F}} \times \left(\mathrm{C}_0 \cap \sigma_\alpha^{\mathrm{F}}\right) \times \mathrm{E},$$

where $\mathrm{E} \subset \mathrm{F}$ and $\omega_{\mathrm{W}} \subset \mathrm{W(F)}$ are compact, $\mathrm{C}_0 \subset \mathrm{C(F)}$ is a rational core and $\sigma_\alpha^{\mathrm{F}}$ is one of the polyhedra in our decomposition of $\mathrm{C(F)}$.

Considering $\mathrm{C(F)}$ as a cone in $\sqrt{-1} \cdot \mathrm{U(F)}$ and decomposing $\mathrm{W(F)}$ as $\mathrm{U(F)} \cdot \mathrm{V(F)}$, the isomorphism Ψ extends to the real-analytic isomorphism $\mathcal{D}_{\mathrm{F}} \overset{\Psi}{\simeq} \mathrm{U(F)}_{\mathbf{C}} \times \mathrm{V(F)} \times \mathrm{F}$ constructed in [1, p. 235]. Hence the Siegel set Σ is covered by a finite number of sets Θ of the form

$$\Theta \overset{\Psi}{\simeq} \Psi(\mathcal{D}) \cap \left\{(x, s, t) \in \mathrm{U(F)}_{\mathbf{C}} \times \mathrm{V(F)} \times \mathrm{F} \,\middle|\, \mathrm{Re}(x) \in \mathrm{U}', s \in \mathrm{S}', t \in \mathrm{F}'\right\} \tag{3.5}$$

where $\mathrm{F}' \subset \mathrm{F}$, $\mathrm{U}' \subset \mathrm{U(F)}$ and $\mathrm{S}' \subset \mathrm{V(F)}$ are relatively compact.

Using the definition of j given in [37, §7] and recalled in the proof of Lemma 4.2 below, it follows, as stated in [1, p. 238], that the diffeomorphism $j \circ \Psi^{-1} : \mathrm{U(F)}_{\mathbf{C}} \times \mathrm{V(F)} \times \mathrm{F} \simeq \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}$ is a change of trivialisation of the real-analytic bundle

$$\begin{tikzcd}
\mathcal{D}_{\mathrm{F}} \arrow[d, "\pi'_{\mathrm{F}}"] \arrow[dd, bend right=60, "\pi_{\mathrm{F}}"'] \\
\mathcal{D}'_{\mathrm{F}} \arrow[d, "p_{\mathrm{F}}"] \\
\mathrm{F}
\end{tikzcd}$$

studied in [1, p. 237]. Here the map π'_{F} is a $\mathrm{U(F)}_{\mathbf{C}}$-principal homogeneous space, the map p_{F} is a $\mathrm{V(F)}$-principal homogeneous space, and the map $j \circ \Psi^{-1}$ is $\mathrm{U(F)}_{\mathbf{C}}$-equivariant and respects the fibrations over F. These two properties ensure that $j \circ \Psi^{-1}$ identifies the set $\Psi(\Theta)$ of (3.5) to a set of the required form

$$\begin{aligned}\Theta \overset{j}{\simeq} \Big\{&(x, y, t) \in \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F},\ \mathrm{Re}(x) \in \mathrm{U}',\ y \in \mathrm{Y}',\ t \in \mathrm{F}' \,\Big|\\ &\mathrm{Im}(x) + l_t(y, y) \in \sigma + a\Big\}\\ \subset\ &\mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}.\end{aligned}$$

□

4. Definability of the uniformisation map: proof of Theorem 1.9

First notice that, although the variety S does not canonically embed into some $\mathbf{R}^n$, the statement of Theorem 1.9 makes sense as S has a canonical structure of real algebraic manifold, hence of $\mathbf{R}_{\mathrm{an,exp}}$-manifold: cf. Appendix A.

By Theorem 3.1 there exist a semi-algebraic Siegel set Σ and a finite subset J of $\mathbf{G}(\mathbf{Q})$ such that $\mathcal{F} := \mathrm{J} \cdot \Sigma$ is a (semi-algebraic) fundamental set for the action of Γ on $\mathcal{D}$. Hence Theorem 1.9 follows from the following more precise result.

Theorem **4.1.** — *The restriction* $\pi_{|\Sigma} : \Sigma \longrightarrow \mathrm{S}$ *of the uniformising map* $\pi : \mathcal{D} \longrightarrow \mathrm{S}$ *is definable in* $\mathbf{R}_{\mathrm{an,exp}}$.

Proof. — By Proposition 3.2 we know that Σ is covered by a finite union of open subsets Θ with the following properties. For each Θ there is a rational boundary component F, a simplicial cone $\sigma \in \boldsymbol{\sigma}$ with $\sigma \subset \overline{\mathrm{C(F)}}$, a point $a \in \mathrm{C(F)}$, relatively compact subsets U′, Y′ and F′ of U(F), $\mathbf{C}^l$ and F respectively such that the set Θ is of the form

$$\begin{aligned}(\mathbf{4.1}) \qquad \Theta \overset{j}{\simeq} \big\{(x, y, t) \in \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F},\ \mathrm{Re}(x) \in \mathrm{U}',\ y \in \mathrm{Y}',\ t \in \mathrm{F}' \ \big| \\ \mathrm{Im}(x) + l_t(y, y) \in \sigma + a\big\} \\ \subset \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}.\end{aligned}$$

We first prove that the holomorphic coordinates we introduced on $\mathcal{D}_{\mathrm{F}}$ are definable:

Lemma **4.2.** — *The canonical isomorphism* $j : \mathcal{D}_{\mathrm{F}} \simeq \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}$ *is semi-algebraic.*

Proof. — The isomorphism j was studied in [22] and in full generality in [37, §7] (cf. [3, §1.6] for a survey). To keep the amount of definitions at a reasonable level we follow in this proof (and this proof only) the notations of Wolf and Koranyi in [37]. For example our X, resp. $\mathrm{X}^\vee$ is denoted by M, resp. M^*.

Let $\xi : \mathfrak{p}^- = \mathbf{C}^{\mathrm{N}} \longrightarrow \mathrm{M}^*$ be the Harish-Chandra morphism defined by $\xi(\mathrm{E}) = \exp(\mathrm{E}) \cdot x$ (cf. [37, p. 901]; in the notations of Wolf and Koranyi x is the base point of M^*). This is a holomorphic embedding onto a dense open subset of M^*. Notice that the map ξ is real algebraic: indeed $\mathfrak{p}^-$ is a nilpotent sub-algebra of $\mathfrak{g}^{\mathbf{C}}$ hence the exponential is polynomial in restriction to $\mathfrak{p}^-$. The bounded symmetric domain $\mathcal{D}$ is $\xi^{-1}(\mathrm{G}^0(x))$.

Let Δ be a maximal set of strongly orthogonal positive non-compact roots of $\mathfrak{g}^{\mathbf{C}}$ as in [37, p. 901]. For any $\alpha \in \Delta$ let $c_\alpha \in \mathrm{G}$ be the partial Cayley transform of M associated to α (cf. [37, p. 902], recall that with the notations of Wolf and Koranyi G is the compact form of the complexified group $\mathbf{G}^{\mathbf{C}}$!). For a subset $\theta \subset \Delta$ we denote by $c_\theta := \prod_{\alpha \in \theta} c_\alpha$ the partial Cayley transform associated with θ (cf. [37, §4.1]).

Following [37, theor. 4.8] there exists a unique subset $\theta \subset \Delta$ such that $\mathrm{F} = \xi^{-1} c_{\Delta-\theta} \mathrm{M}_\theta$, where $\mathrm{M}_\theta = \mathrm{G}^0_\theta(x)$ is defined in [37, p. 912]. Let $\mathfrak{p}_\theta^{-1} \subset \mathfrak{p}^-$ be defined as in [37, p. 912], let $\mathfrak{p}^-_{\Delta-\theta,1}$ be the $(+1)$-eigenspace of $\mathrm{ad}(c^4_{\Delta-\theta})$ on $\mathfrak{p}^-_{\Delta-\theta}$ and $\mathfrak{p}_2^{\theta,-}$ be the (-1)-eigenspace of $\mathrm{ad}(c^4_{\Delta-\theta})$ on $\mathfrak{p}^-$. One has a canonical decomposition (cf. [37, p. 933]):

$$(\mathbf{4.2}) \qquad \mathfrak{p}^- = \mathfrak{p}^-_{\Delta-\theta,1} \oplus \mathfrak{p}_2^{\theta,-} \oplus \mathfrak{p}^-_\theta.$$

The decomposition (3.1) of the normaliser $\mathrm{N}(\mathrm{F}) = \mathrm{B}^{\theta}$ (cf. [37, remark 3 p. 932]) is proven in [37, theorem 6.8]. In particular it follows that $\exp_{\Delta-\theta} := \exp \circ \mathrm{ad} c_{\Delta-\theta} : \mathfrak{p}^-_{\Delta-\theta,1} \longrightarrow \mathrm{U}(\mathrm{F})_{\mathbf{C}}$ and $\exp : \mathfrak{p}_2^{\theta,-} \longrightarrow \mathbf{C}^l$ are polynomial isomorphisms, while $\mathrm{F} \subset \mathfrak{p}^-$ is a bounded symmetric domain of $\mathfrak{p}^-_{\theta}$.

Following [37, §7.6 and §7.7] the map $j : \mathcal{D} \longrightarrow \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F} \subset \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^l \times \mathfrak{p}^-_{\theta}$ is the composition of the semi-algebraic holomorphic maps

$$\mathcal{D} \xrightarrow{\xi^{-1} c_{\Delta-\theta} \xi} \mathfrak{p}^- = \mathfrak{p}^-_{\Delta-\theta,1} \oplus \mathfrak{p}_2^{\theta,-} \oplus \mathfrak{p}^-_{\theta} \xrightarrow{(\exp_{\Delta-\theta}, \exp, \mathrm{Id})} \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^l \times \mathfrak{p}^-_{\theta}$$

which finishes the proof of Lemma 4.2. □

The previous lemma enables us to forget about the definable biholomorphism j. From now on and for simplicity of notations we simply write $\mathcal{D}_{\mathrm{F}} = \mathrm{U}(\mathrm{F})_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}$.

In the description (4.1) we may and do assume that U', Y' and F' are semi-algebraic subsets respectively of $\mathrm{U}(\mathrm{F})_{\mathbf{C}}$, $\mathbf{C}^l$ and F. Then the set Θ is definable in $\mathbf{R}_{\mathrm{an}}$ because:

- the function $\psi : \mathrm{Y}' \times \mathrm{F}' \to \mathrm{U}(\mathrm{F})$ defined by $\psi(y, t) = l_t(y, y)$ is analytic and defined on a compact semi-algebraic set.
- the cone σ is polyhedral, hence semi-algebraic.

Hence the restriction $\pi_{|\Sigma} : \Sigma \longrightarrow \mathrm{S}$ is definable in $\mathbf{R}_{\mathrm{an,exp}}$ if and only if the restriction $\pi_{|\Theta} : \Theta \longrightarrow \mathrm{S}$ to any set Θ appearing in Proposition 3.2 is definable in $\mathbf{R}_{\mathrm{an,exp}}$.

Fix such a set

$$\Theta = \left\{ (x, y, t), y \in \mathrm{Y}', t \in \mathrm{F}', \mathrm{Re}(x) \in \mathrm{U}' \;\middle|\; \mathrm{Im}(x) + l_t(y, y) \in \sigma + a \right\}$$

associated to a rational boundary component $\mathrm{F} \in \{\mathrm{F}_1, \ldots, \mathrm{F}_r\}$.

Consider the left-hand side of the diagram (3.4):

$$\begin{tikzcd}
\mathcal{D} \arrow[d, "\exp_{\mathrm{F}}"] \arrow[dd, bend right=60, "\pi"'] \\
\exp_{\mathrm{F}}(\mathcal{D}) \arrow[d, "\pi_{\mathrm{F}}"] \\
\mathrm{S}
\end{tikzcd}$$

Recall that $\exp_{\mathrm{F}} : \mathcal{D}_{\mathrm{F}} \to \mathbf{C}^{*k} \times \mathbf{C}^l \times \mathrm{F}$ is given by

$$(x, y, t) \mapsto \big(\exp(2i\pi x), y, t\big),$$
$$\text{where } \exp(2i\pi x) = \big(\exp(2i\pi x_1), \ldots, \exp(2i\pi x_k)\big).$$

The function $\mathrm{Re}(x_i)$, $1 \leq i \leq k$, is bounded on Θ hence the restriction to Θ of the map $x \mapsto \exp(2i\pi \mathrm{Re}(x))$ is definable in $\mathbf{R}_{\mathrm{an}}$. On the other hand the restriction to Θ of the function $x \mapsto \exp(-2\pi \mathrm{Im}(x))$ is definable in $\mathbf{R}_{\exp}$ by definition of $\mathbf{R}_{\exp}$. Thus the restriction to Θ of the map $\exp_{\mathrm{F}}$ is definable in $\mathbf{R}_{\mathrm{an},\exp}$ and we are reduced to showing that $\pi_{\mathrm{F}} : \exp_{\mathrm{F}}(\Theta) \longrightarrow \mathrm{S}$ is definable in $\mathbf{R}_{\mathrm{an},\exp}$.

Consider the lower part of the diagram (3.4):

$$\begin{tikzcd}
\exp_{\mathrm{F}}(\mathcal{D}) \arrow[r, hook] \arrow[d, "\pi_{\mathrm{F}}"] & \exp_{\mathrm{F}}(\mathcal{D})^{\vee} \arrow[d, "\overline{\pi}_{\mathrm{F}}"] \\
\mathrm{S} \arrow[r, hook] & \overline{\mathrm{S}}.
\end{tikzcd}$$

As U', V', F' are relatively compact and the imaginary part of x has a lower bound on Θ, the closure $\overline{\exp_{\mathrm{F}}(\Theta)}$ of $\exp_{\mathrm{F}}(\Theta)$ is compact in $\exp_{\mathrm{F}}(\mathcal{D})^{\vee}$. Hence $\pi_{\mathrm{F}} : \exp_{\mathrm{F}}(\Theta) \longrightarrow \mathrm{S}$, which is the restriction of the analytic map $\overline{\pi}_{\mathrm{F}} : \exp_{\mathrm{F}}(\mathcal{D})^{\vee} \longrightarrow \overline{\mathrm{S}}$ to the relatively compact subset $\exp_{\mathrm{F}}(\Theta)$ of $\exp_{\mathrm{F}}(\mathcal{D})^{\vee}$, is definable in $\mathbf{R}_{\mathrm{an}}$. □

5. Proof of Theorem 1.11

5.1. *Distance, norm, height.*

5.1.1. *Distance.* — Let $*$ be the adjunction on $\mathrm{E}_{\mathbf{R}}$ associated to the Hilbert structure $\|\cdot\|_\infty$ on $\mathrm{E}_{\mathbf{R}}$. The restriction of the bilinear form $(u, v) \mapsto \mathrm{tr}(u^* v)$ to the Lie algebra $\mathrm{Lie}(\mathbf{G}(\mathbf{R}))$ defines a $\mathbf{G}(\mathbf{R})$-invariant Kähler metric g_{X} on X. We denote by $d : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbf{R}$ the associated distance and by ω the associated Kähler form.

5.1.2. *Norm.* — We still denote by $\|\cdot\|_\infty : \mathrm{End}\, \mathrm{E}_{\mathbf{R}} \longrightarrow \mathbf{R}$ the operator norm associated to the norm $\|\cdot\|_\infty$ on $\mathrm{E}_{\mathbf{R}}$. By restriction we also denote by $\|\cdot\|_\infty : \mathbf{G}(\mathbf{R}) \longrightarrow \mathbf{R}$ the function $\|\cdot\|_\infty \circ \rho$. As K_∞ preserves the norm $\|\cdot\|_\infty$ on $\mathrm{E}_{\mathbf{R}}$, the function $\|\cdot\|_\infty : \mathbf{G}(\mathbf{R}) \longrightarrow \mathbf{R}$ is K_∞-bi-invariant, in particular descends to a K_∞-invariant function $\|\cdot\|_\infty : \mathrm{X} \longrightarrow \mathbf{R}$.

Choose $(e_1, \ldots, e_n)$ a basis of $\mathrm{E}_{\mathbf{Z}}$ in which $\mathbf{A}$ diagonalises. It will be useful to compare the norm $\|\cdot\|_\infty$ with the norm $|\cdot|_\infty : \mathrm{End}\, \mathrm{E}_{\mathbf{R}} \longrightarrow \mathbf{R}$ defined by

$$(\mathbf{5.1}) \qquad \forall \varphi \in \mathrm{End}\, \mathrm{E}_{\mathbf{R}}, \quad |\varphi|_\infty = \max_{i,j} |\varphi_{ij}|,$$

where (φ_{ij}) is the matrix of φ in the basis $(e_1, \ldots, e_n)$ of $\mathrm{E}_{\mathbf{R}}$.

5.1.3. *Height.*

Definition **5.1.** — *We define the (multiplicative) height function* $\mathrm{H} : \mathrm{End}\,\mathrm{E}_{\mathbf{Z}} \longrightarrow \mathbf{R}$ *as*

$$\forall \varphi \in \mathrm{End}\,\mathrm{E}_{\mathbf{Z}}, \quad \mathrm{H}(\varphi) = \max\big(1, \|\varphi\|_\infty\big).$$

Remark **5.2.** — When $\dim_{\mathbf{Q}} \mathrm{E} = 1$, this height function coincides with the classical multiplicative height function on rational numbers.

By restriction, we also denote by $\mathrm{H} : \mathbf{G}(\mathbf{Z}) \longrightarrow \mathbf{R}$ the function $\mathrm{H} \circ \rho$. Notice that for $\varphi \in \mathrm{End}\,\mathrm{E}_{\mathbf{R}}$, $\|\varphi\|_\infty$ is the square root of the largest eigenvalue of the positive definite matrix $\varphi^*\varphi$. If $\varphi \in \mathrm{End}\,\mathrm{E}_{\mathbf{Z}}$ it follows that $\|\varphi\|_\infty$ is at least 1, hence

$$\forall \varphi \in \mathbf{G}(\mathbf{Z}), \quad \mathrm{H}(\varphi) = \|\varphi\|_\infty \geq 1.$$

We also define $\mathrm{H}_{\mathrm{class}}$ the classical multiplicative height on $\mathrm{End}\,\mathrm{E}$ using the basis $(e_i^* \otimes e_j)_{i,j}$. In particular if $\varphi \in \mathrm{End}\,\mathrm{E}_{\mathbf{Z}}$ then $\mathrm{H}_{\mathrm{class}}(\varphi) = |\varphi|_\infty$. As the norms $\|\cdot\|_\infty$ and $|\cdot|_\infty$ are equivalent on $\mathrm{End}\,\mathrm{E}_{\mathbf{R}}$ we obtain the following:

Lemma **5.3.** — *There exist a positive number* C *such that*

$$\forall \varphi \in \mathrm{End}\,\mathrm{E}_{\mathbf{Z}}, \quad \frac{1}{\mathrm{C}} \cdot \mathrm{H}_{\mathrm{class}}(\varphi) \leq \mathrm{H}(\varphi) \leq \mathrm{C} \cdot \mathrm{H}_{\mathrm{class}}(\varphi).$$

5.2. *Comparing norm and distance.*

Lemma **5.4.** — *For any* $g \in \mathbf{G}(\mathbf{R})$ *the following inequality holds:*

$$\log \|g\|_\infty \leq d(g \cdot x_0, x_0).$$

Proof. — Let $\mathbf{G}(\mathbf{R}) = \mathrm{K}_\infty \cdot \mathrm{A}_\infty \cdot \mathrm{K}_\infty$ be a Cartan decomposition of $\mathbf{G}(\mathbf{R})$ associated to K_∞, where A_∞ is a maximal split real torus of G containing A. Let $g \in \mathbf{G}(\mathbf{R})$ and write $g = k_1 \cdot a \cdot k_2$ its Cartan decomposition, with k_1, k_2 in K_∞ and $a \in \mathrm{A}_\infty$. As $\|\cdot\|_\infty$ is K_∞-bi-invariant and d is $\mathbf{G}(\mathbf{R})$-equivariant the equalities $\log \|g\|_\infty = \log \|a\|_\infty$ and $d(g \cdot x_0, x_0) = d(a \cdot x_0, x_0)$ do hold.

The torus A_∞ is diagonalisable in an orthonormal basis $(f_1, \ldots, f_n)$ of $\mathrm{E}_{\mathbf{R}}$. Write $a = \mathrm{diag}(a_1, \ldots, a_n)$ in this basis, then:

$$\log \|a\|_\infty = \max_i \log |a_i| \quad \text{and} \quad d(a \cdot x_0, x_0) = \sqrt{\sum_{i=1}^{n} (\log |a_i|)^2}$$

hence the result. $\square$

5.3. *Comparing height and norms.* — The main result of this section is the following:

Lemma **5.5.** — *Let $\mathcal{F} \subset X$ be the fundamental domain described in Theorem 3.1. There exists a positive number* B *such that:*

$$\forall \gamma \in \mathbf{G}(\mathbf{Z}),\ \forall u \in \gamma\mathcal{F}, \quad H(\gamma) \leq B \cdot \|u\|_\infty^n. \tag{5.2}$$

Proof. — Write $u = \gamma \cdot j \cdot x$ with $j \in J$ and $x = \omega \cdot a \cdot k \in \Sigma'_{t_0,\Omega} = \Omega \cdot A_{t_0} \cdot K_\infty$. Thus:

$$u = j \cdot \left(j^{-1}\gamma j\right) \cdot a \cdot \left(a^{-1}\omega a\right) \cdot k. \tag{5.3}$$

Notice that for each $j \in \mathbf{G}(\mathbf{Q})$ the groups $\mathbf{G}(\mathbf{Z})$ and $j^{-1}\mathbf{G}(\mathbf{Z})j$ are commensurable (i.e. their intersection is of finite index in both of them). As the subset $J \subset \mathbf{G}(\mathbf{Q})$ is finite, it follows that the subgroup $\mathbf{G}(\mathbf{Z})_J := \mathbf{G}(\mathbf{Z}) \bigcap (\bigcap_{j\in J} j^{-1}\mathbf{G}(\mathbf{Z})j)$ is of finite index in $j^{-1}\mathbf{G}(\mathbf{Z})j$, $j \in J$. Choose a finite set S of representatives in $\mathbf{G}(\mathbf{Q})$ for the cosets $j^{-1}\mathbf{G}(\mathbf{Z})j/\mathbf{G}(\mathbf{Z})_J$, $j \in \{1\} \cup J$. Hence there exists a unique $s \in S$ and $\gamma' \in \mathbf{G}(\mathbf{Z})_J \subset \mathbf{G}(\mathbf{Z})$ such that $j^{-1}\gamma j = s \cdot \gamma'$. We deduce from (5.3):

$$u = js \cdot \left(\gamma' \cdot a\right) \cdot \left(a^{-1}\omega a\right) \cdot k. \tag{5.4}$$

The set $J \cdot S$ is finite. The group K_∞ is compact. Moreover the set $\bigcup_{a\in A_{t_0}} a^{-1}\Omega a$ is relatively compact in G by [4, Lemma 12.1]. As $\|\cdot\|_\infty$ is sub-multiplicative, it follows from (5.4) that there exists a positive number b, depending only on Ω and t_0, such that

$$\|u\|_\infty \geq b\left\|\gamma' \cdot a\right\|_\infty. \tag{5.5}$$

As $j^{-1}\gamma j = s \cdot \gamma'$ and J and S are finite sets, there exists a positive number b', depending only on Ω and t_0, such that

$$\left\|\gamma'\right\|_\infty \geq b'\|\gamma\|_\infty. \tag{5.6}$$

Thus Lemma 5.5 follows the equality $H(\gamma) = \|\gamma\|_\infty$, inequalities (5.5) and (5.6) and Sublemma 5.6 below. □

Sublemma **5.6.** — *There exists a positive number* B *depending only on* Ω *and* t_0 *such that for all* $\gamma \in \mathbf{G}(\mathbf{Z})$ *and* $a \in A_{t_0}$ *the following inequality holds:*

$$\|\gamma\|_\infty \leq B \cdot \|\gamma \cdot a\|_\infty^n. \tag{5.7}$$

Proof. — As the norm $\|\cdot\|_\infty$ on $\operatorname{End} E_{\mathbf{R}}$ is equivalent to the norm $|\cdot|_\infty$, it is enough to show that $|\gamma|_\infty \leq |\gamma \cdot a|_\infty^n$.

Let $\gamma = (\gamma_{k,l})$ be the matrix of γ in the basis $(e_1, \dots, e_n)$ of $E_{\mathbf{Z}}$. As the torus $\mathbf{A}$ is diagonalisable in the basis $(e_1, \dots, e_n)$, we write $a = \operatorname{diag}(a_1, \dots, a_n)$, with $a_i \in \mathbf{R}^{>0}$. It follows that:

$$\forall k, l \in \{1, \dots, n\}, \quad (\gamma \cdot a)_{kl} = \gamma_{kl} \cdot a_l. \tag{5.8}$$

As γ is invertible, there exists for each $s \in \{1, \dots, n\}$ an index $r_s \in \{1, \dots, n\}$ such that $\gamma_{r_s,s} \neq 0$. It follows from Equation (5.8) that:

$$\forall k, l \in \{1, \dots, n\}, \quad (\gamma \cdot a)_{k,l} \cdot \prod_{s \neq l} (\gamma \cdot a)_{r_s,s} = \gamma_{k,l} \cdot \prod_{s \neq l} \gamma_{r_s,s} \cdot \prod_{s=1}^{n} a_s = \gamma_{k,l} \cdot \prod_{s \neq l} \gamma_{r_s,s}, \tag{5.9}$$

where we used that $\prod_{l=1}^{n} a_i = 1$ as $\rho(\mathbf{G}) \subset \mathbf{SL}(\mathrm{E})$.

Notice that $\Gamma = \mathbf{G}(\mathbf{Z})$ hence each $\gamma_{k,l}$ is an integer. It follows from Equation (5.9) that:

$$\forall k, l \in \{1, \dots, n\},$$
$$|\gamma_{k,l}| \leq \left| \gamma_{k,l} \cdot \prod_{s \neq l} \gamma_{r_s,s} \right| = \left| (\gamma \cdot a)_{k,l} \cdot \prod_{s \neq l} (\gamma \cdot a)_{r_s,s} \right| \leq \left(\max_{r,s} \left| (\gamma \cdot a)_{r,s} \right| \right)^n.$$

In other words: $|\gamma|_\infty \leq |\gamma \cdot a|_\infty^n$. Hence the inequality (5.7) follows. □

5.4. *Lower bound for the volume of an algebraic curve.* — In [11, Corollary 3 p. 1227], Hwang and To prove the following lower bound for the area of any complex analytic curve in $\mathcal{D}$:

Theorem **5.7** *(Hwang and To). — Let* C *be a complex analytic curve in* $\mathcal{D}$*. For any point* $x_0 \in \mathrm{C}$ *there exist positive constants* a_1, b_1 *such that for any positive real number* R *one has:*

$$\mathrm{Vol}_{\mathrm{C}}\big(\mathrm{C} \cap \mathrm{B}(x_0, \mathrm{R})\big) \geq a_1 \exp(b_1 \cdot \mathrm{R}). \tag{5.10}$$

Here $\mathrm{Vol}_{\mathrm{C}}$ denotes the area for the Riemanian metric on C restriction of the metric g_{X} on $\mathcal{D}$ and $\mathrm{B}(x_0, \mathrm{R})$ denotes the geodesic ball of $\mathcal{D}$ with center x_0 and radius R.

5.5. *Upper bound for the volume of algebraic curves on Siegel sets.*

Lemma **5.8.**

(i) *There exists a constant* $\mathrm{A}_0 > 0$ *such that for any algebraic curve* $\mathrm{C} \subset \mathcal{D}$ *of degree* d *we have the bound*

$$\mathrm{Vol}_{\mathrm{C}}(\mathrm{C} \cap \Sigma) \leq \mathrm{A}_0 \cdot d.$$

(ii) *There exists a constant* $\mathrm{A} > 0$ *such that for any algebraic curve* $\mathrm{C} \subset \mathcal{D}$ *of degree* d *we have the bound*

$$\mathrm{Vol}_{\mathrm{C}}(\mathrm{C} \cap \mathcal{F}) \leq \mathrm{A} \cdot d.$$

Proof. — We first prove (i). Recall that Σ is covered by a finite union of open subsets Θ described in Proposition 3.2: there is a rational boundary component F, a simplicial cone $\sigma \in \Sigma$ with $\sigma \subset \overline{\mathrm{C(F)}}$, a point $a \in \mathrm{C(F)}$, relatively compact subsets U', Y' and F' of $\mathrm{U(F)}$, $\mathbf{C}^l$ and F respectively such that the set Θ is of the form

$$\begin{aligned}\Theta &= \left\{(x, y, t) \in \mathcal{D}_{\mathcal{F}}, y \in \mathrm{Y}', t \in \mathrm{F}', \mathrm{Re}(x) \in \mathrm{U}' \;\middle|\; \mathrm{Im}(x) + l_t(y, y) \in \sigma + a\right\}\\ &\subset \mathcal{D}_{\mathrm{F}} = \mathrm{U(F)}_{\mathbf{C}} \times \mathbf{C}^l \times \mathrm{F}.\end{aligned}$$

Recall that ω denotes the natural Kähler form on X. As $\mathrm{C} \subset \mathrm{X}$ is a complex analytic curve, one has:

$$\mathrm{Vol}_{\mathrm{C}}(\mathrm{C} \cap \Theta) = \int_{\mathrm{C}\cap\Theta} \omega.$$

On the other hand let $\omega_{\mathcal{D}_{\mathrm{F}}}$ be the Poincaré metric on $\mathcal{D}_{\mathrm{F}}$ defined in the Siegel coordinates by:

$$\omega_{\mathcal{D}_{\mathrm{F}}} = \sum \frac{dx_i \wedge d\overline{x}_i}{\mathrm{Im}(x_i)^2} + \sum dy_j \wedge d\overline{y}_j + \sum df_k \wedge d\overline{f}_k.$$

Mumford [19, Theor. 3.1] proved that there exists a positive constant c such that on $\mathcal{D}$:

$$\omega \leq c \cdot \omega_{\mathcal{D}_{\mathrm{F}}}.$$

Hence:

$$\mathrm{Vol}_{\mathrm{C}}(\mathrm{C} \cap \Theta) \leq c \int_{\mathrm{C}\cap\Theta} \omega_{\mathcal{D}_{\mathrm{F}}}.$$

Let p_{x_i}, p_{y_j} and p_{f_k} be the projections on $\mathcal{D}_{\mathrm{F}}$ to the coordinates x_i, y_j and f_k.

As the curve C has degree d the restriction of these maps to $\mathrm{C} \cap \Theta$ are either constant or at most d to 1, hence

$$\begin{aligned}\mathrm{Vol}_{\mathrm{C}}(\mathrm{C} \cap \Theta) \leq c \cdot d \cdot \Bigg(\sum \int_{p_{x_i}(\Theta)} \frac{dx_i \wedge d\overline{x}_i}{\mathrm{Im}(x_i)^2} + \sum \int_{p_{y_j}(\Theta)} dy_j \wedge d\overline{y}_j\\ + \sum \int_{p_{f_k}(\Theta)} df_k \wedge d\overline{f}_k\Bigg).\end{aligned}$$

Let i be such that the map p_{x_i} is not constant. In view of the description of Θ the projection $p_{x_i}(\Theta)$ is contained in a usual fundamental set of the upper-half plane, of finite hyperbolic area.

Let w be a coordinate y_j, f_k and p_w be the associated projection on the w axis. By the definition of Θ the projection $p_w(\Theta)$ is a relatively compact open set of the plane, hence of finite Euclidean area.

This finishes the proof of (i).

Let us prove (*ii*). As $C \cap \mathcal{F} = C \cap J \cdot \Sigma$, one has the inequality:

$$\mathrm{Vol}_C(C \cap \mathcal{F}) \leq \sum_{j \in J} \mathrm{Vol}_C(C \cap j \cdot \Sigma) = \sum_{j \in J} \mathrm{Vol}_{j^{-1}C}\big(j^{-1}C \cap \Sigma\big) \leq |J| \cdot A_0 \cdot d$$

where we used part (*i*) applied to the algebraic curves $j^{-1}C$ of $\mathcal{D}$, $j \in J$, which are of degree d.

This finishes the proof of Lemma 5.8. □

5.6. *Proof of Theorem 1.11*. — Choose $C \subset Y$ an irreducible algebraic curve. To prove Theorem 1.11 for Y it is enough to prove it for C.

Consider the set

$$C(T) := \big\{z \in C \text{ and } \|z\|_\infty \leq T\big\}.$$

As $\mathcal{F}$ is a fundamental domain for the action of Γ one has on the one hand:

$$\begin{aligned} C(T) &= \bigcup_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset}} \big\{u \in \gamma\mathcal{F} \cap C \text{ and } \|u\|_\infty \leq T\big\} \\ &\subset \bigcup_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T^n}} \{u \in \gamma\mathcal{F} \cap C\} \quad \text{by Lemma 5.5.} \end{aligned}$$

Taking volumes:

$$\mathrm{Vol}_C\big(C(T)\big) \leq \sum_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T^n}} \mathrm{Vol}_C\big(\mathcal{F} \cap \gamma^{-1}C\big)$$

hence

$$\mathrm{Vol}_C\big(C(T)\big) \leq (A \cdot d) \cdot N_C\big(B \cdot T^n\big) \tag{5.11}$$

where we applied Lemma 5.8(ii) to the algebraic curves $\gamma^{-1}C$, $\gamma \in \Gamma$, which are all of degree d.

On the other hand if follows from Lemma 5.4 that

$$C \cap B(x_0, \log T) \subset C(T),$$

hence

$$\mathrm{Vol}_C\big(C \cap B(x_0, \log T)\big) \leq \mathrm{Vol}_C\big(C(T)\big). \tag{5.12}$$

Finally:

$$\begin{aligned}(\mathrm{A}\cdot d)\cdot \mathrm{N}_{\mathrm{C}}\big(\mathrm{B}\cdot \mathrm{T}^{n}\big) &\geq \mathrm{Vol}_{\mathrm{C}}\big(\mathrm{C}(\mathrm{T})\big) \quad \text{by inequality (5.11)}\\ &\geq \mathrm{Vol}_{\mathrm{C}}\big(\mathrm{C}\cap \mathrm{B}(x_0,\log \mathrm{T})\big) \quad \text{by inequality (5.12)}\\ &\geq a_1 \exp(b_1 \log \mathrm{T}) \quad \text{by Theorem 5.7.}\end{aligned}$$

Hence the result. □

6. Stabilisers of a maximal algebraic subset: proof of Theorem 1.13

6.1. *Pila-Wilkie theorem.*

Definition **6.1.** — *The classical height* $\mathrm{H}_{\mathrm{class}}(x)$ *of a point* $x=(x_1,\ldots,x_m)\in \mathbf{Q}^m$ *is defined as*

$$\mathrm{H}_{\mathrm{class}}(x)=\max\big(\mathrm{H}(x_1),\ldots,\mathrm{H}(x_m)\big)$$

where H *is the usual multiplicative height of a rational number.*

Let $\mathrm{Z}\subset \mathbf{R}^m$ be a subset and $\mathrm{T}\geq 0$ a real number, we define:

$$\Psi_{\mathrm{class}}(\mathrm{Z},\mathrm{T}) := \big\{x\in \mathrm{Z}\cap \mathbf{Q}^m : \mathrm{H}_{\mathrm{class}}(x)\leq \mathrm{T}\big\}$$

and

$$\mathrm{N}_{\mathrm{class}}(\mathrm{Z},\mathrm{T}) := \big|\Psi_{\mathrm{class}}(\mathrm{Z},\mathrm{T})\big|.$$

For $\mathrm{Z}\subset \mathbf{R}^m$ a definable set in a o-minimal structure we define the algebraic part $\mathrm{Z}^{\mathrm{alg}}$ of Z to be the union of all positive dimensional semi-algebraic subsets of Z.

Recall (cf. definition 3.3 of [34]), that a semi-algebraic block of dimension w in $\mathbf{R}^m$ is a connected definable set $\mathrm{W}\subset \mathbf{R}^m$ of dimension w, regular at every point, such that there exists a semi-algebraic set $\mathrm{A}\subset \mathbf{R}^m$ of dimension w, regular at every point with $\mathrm{W}\subset \mathrm{A}$.

The following result is a strong form, proven by Pila [23, theor. 3.6], of the original theorem of Pila and Wilkie [24]:

Theorem **6.2** *(Pila-Wilkie). — Let* $\mathrm{Z}\subset \mathbf{R}^m$ *be a definable set in a o-minimal structure. For every* $\epsilon>0$*, there exists a constant* $\mathrm{C}_\epsilon>0$ *such that*

$$\mathrm{N}_{\mathrm{class}}\big(\mathrm{Z}\backslash \mathrm{Z}^{\mathrm{alg}},\mathrm{T}\big) < \mathrm{C}_\epsilon \mathrm{T}^\epsilon$$

and the set $\Psi_{\mathrm{class}}(\mathrm{Z},\mathrm{T})$ *is contained in the union of at most* $\mathrm{C}_\epsilon \mathrm{T}^\epsilon$ *semi-algebraic blocks.*

As a corollary of Theorem 6.2 and Lemma 5.3 one obtains:

Corollary **6.3.** — *Let* $Z \subset \mathrm{End}\, E_{\mathbf{R}}$ *be a definable set in a o-minimal structure. Define* $\Psi(Z, T) := \{x \in Z \cap \mathrm{End}\, E_{\mathbf{Z}} : H(x) \leq T\}$ *and* $N(Z, T) := |\Psi(Z, T)|$. *For every* $\epsilon > 0$, *there exists a constant* $C_\epsilon > 0$ *such that*

$$N\big(Z \backslash Z^{\mathrm{alg}}, T\big) < C_\epsilon T^\epsilon$$

and the set $\Psi(Z, T)$ *is contained in the union of at most* $C_\epsilon T^\epsilon$ *semi-algebraic blocks.*

6.2. *Proof of Theorem 1.13.* — Let V be an algebraic subvariety of S and Y a maximal irreducible algebraic subvariety of $\pi^{-1}V$. Let Θ_Y be the stabiliser of Y in $\mathbf{G}(\mathbf{R})$ and $\mathbf{H}_Y$ be the neutral component of the Zariski-closure of $\mathbf{G}(\mathbf{Z}) \cap \Theta_Y$ in $\mathbf{G}$. We want to show that $\mathbf{H}_Y$ is a non-trivial subgroup of $\mathbf{G}$, acting non-trivially on X.

Via $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(E)$, we view $\mathbf{G}(\mathbf{R})$ as a semi-algebraic (and hence definable) subset of $\mathrm{End}\, E_{\mathbf{R}}$. As $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S$ is definable by Theorem 1.9, lemmas 5.1 and 5.2 of [34] show the following:

Proposition **6.4.** — *Let us define*

$$\Sigma(Y) = \big\{g \in \mathbf{G}(\mathbf{R}) : \dim\big(gY \cap \pi^{-1}V \cap \mathcal{F}\big) = \dim(Y)\big\} \quad \textit{and}$$
$$\Sigma'(Y) = \big\{g \in \mathbf{G}(\mathbf{R}) : g^{-1}\mathcal{F} \cap Y \neq \emptyset\big\}.$$

The following properties hold:

(1) *The set* $\Sigma(Y)$ *is definable and for all* $g \in \Sigma(Y)$, $gY \subset \pi^{-1}V$.
(2) *For all* $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbf{Z})$, γY *is a maximal algebraic subset of* $\pi^{-1}V$.
(3) *The following equality holds:*

$$\Sigma(Y) \cap \mathbf{G}(\mathbf{Z}) = \Sigma'(Y) \cap \mathbf{G}(\mathbf{Z}).$$

It follows that the number $N_Y(T)$ defined in Theorem 1.11 coincide with $|\Theta(Y, T)|$, where

$$\Theta(Y, T) := \mathbf{G}(\mathbf{Z}) \cap \Psi\big(\Sigma(Y), T\big).$$

We can now finish the proof of Theorem 1.13 in exactly the same way as the proof of theorem 5.4 of [34]. For the sake of completeness, we reproduce it here. As $\Theta(Y, T) \subset \Psi(\Sigma(Y), T)$ it follows from Corollary 6.3 that for T large enough, the set $\Theta(Y, T^{\frac{1}{2n}})$ is contained in at most $T^{\frac{c_1}{4n}}$ semi-algebraic blocks. As $|\Theta(Y, T^{\frac{1}{2n}})| = N_Y(T^{\frac{1}{2n}}) \geq T^{\frac{c_1}{2n}}$ by Theorem 1.11, we see that there is a semi-algebraic block W in $\Sigma(Y)$ containing at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbf{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Using lemma 5.5 of [31] which applies verbatim in our case, we see that there exists an element σ in $\Sigma(Y)$ such that $\sigma\Theta_Y$ contains at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbf{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Let γ_1 and γ_2 be two elements of $\sigma\Theta_Y \cap \mathbf{G}(\mathbf{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Let $\gamma := \gamma_2^{-1}\gamma_1 \in \mathbf{G}(\mathbf{Z}) \cap \Theta_Y$. Using elementary properties of heights, we see that $H(\gamma) \leq c_n T^{1/2}$ where c_n is a constant depending on n only. It follows that for all T large enough, Θ_Y contains at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \mathbf{G}(\mathbf{Z})$ with $H(\gamma) \leq T$. Hence the connected component of the identity $\mathbf{H}_Y$ of the Zariski closure of $\mathbf{G}(\mathbf{Z}) \cap \Theta_Y$ in $\mathbf{G}$ is a positive dimensional algebraic subgroup of $\mathbf{G}$ contained in Θ_Y. This finishes the proof of Theorem 1.13.

7. End of the proof of Theorem 1.6

Let V be an algebraic subvariety of S. Our aim is to show that maximal irreducible algebraic subvarieties Y of $\pi^{-1}V$ are precisely the irreducible components of the preimages of maximal weakly special subvarieties contained in V.

Using Deligne's interpretation of Hermitian symmetric spaces in terms of Hodge theory the representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(E)$ defines a polarised $\mathbf{Z}$-variation of Hodge structure on S. We refer to [18, section 2] for the definition of the Hodge locus of X and S. Recall that an irreducible analytic subvariety M of X or S is said to be Hodge generic if it is not contained in the Hodge locus. If M is not irreducible we say that M is Hodge generic if all the irreducible components of M are Hodge generic.

Let $V' \subset V$ be the Zariski closure of $\pi(Y)$, as Y is analytically irreducible it easily follows that V' is irreducible. Replacing V by V' we can without loss of generality assume that $\pi(Y)$ is not contained in a proper algebraic subvariety of V. We now have to show that $\pi(Y) = V$ and V is an arithmetic subvariety of S.

Since the group $\mathbf{G}$ is adjoint, it is a direct product

$$\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$$

where the $\mathbf{G}_i$'s are the $\mathbf{Q}$-simple factors of $\mathbf{G}$. This induces decompositions

$$G = \prod_{i=1}^{r} G_i, \qquad X = \prod_{i=1}^{r} X_i, \qquad \mathbf{G}(\mathbf{Z}) = \prod_{i=1}^{r} \mathbf{G}_i(\mathbf{Z}),$$

$$\Gamma = \prod_{i=1}^{r} \Gamma_i, \qquad S = \prod_{i=1}^{r} S_i,$$

where G_i is a group of Hermitian type, X_i its associated Hermitian symmetric domain, Γ_i is an arithmetic lattice in G_i, $S_i := \Gamma_i \backslash X_i$ is the associated arithmetic variety and $\pi_i : X_i \longrightarrow S_i$ the associated uniformisation map.

Our main Theorem 1.6 is then a consequence of the following:

Theorem **7.1.** — *Let* $\widetilde{\mathrm{V}}$ *be the an analytic irreducible component of* $\pi^{-1}\mathrm{V}$ *containing* Y. *In the situation described above, after, if necessary, reordering the factors, one has*

$$\widetilde{\mathrm{V}} = \mathrm{X}_1 \times \widetilde{\mathrm{V}_{>1}}$$

where $\widetilde{\mathrm{V}_{>1}}$ *is an analytic subvariety of* $\mathrm{X}_2 \times \cdots \times \mathrm{X}_r$ *(in particular if* $r = 1$ *then* $\widetilde{\mathrm{V}} = \mathrm{X}_1 = \mathrm{X}$*).*

We first show:

Proposition **7.2.** — *Theorem 7.1 implies the main Theorem 1.6.*

Proof. — Let t, $1 \le t \le r$, be the largest integer such that, after reordering the factors if necessary, we have:

$$\widetilde{\mathrm{V}} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_t \times \widetilde{\mathrm{V}_{>t}}$$

with $\widetilde{\mathrm{V}_{>t}}$ an analytic irreducible subvariety of $\mathrm{X}_{t+1} \times \cdots \times \mathrm{X}_r$ which does not (after reordering the factors if necessary) decompose into a product $\mathrm{X}_{t+1} \times \mathrm{V}_{>t+1}$.

In this case necessarily one has:

$$\mathrm{Y} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_t \times \mathrm{Y}_{>t}$$

where $\mathrm{Y}_{>t}$ is a maximal algebraic subset of $\widetilde{\mathrm{V}_{>t}}$.

Suppose that $\dim_{\mathbf{C}}(\widetilde{\mathrm{V}_{>t}}) > 0$. Let $x_{\le t}$ be a special point on $\mathrm{X}_1 \times \cdots \times \mathrm{X}_t$ and $x_{>t}$ be a Hodge generic point of $\mathrm{Y}_{>t}$. Let $\mathbf{H} \subset \mathbf{G}$ be the Mumford-Tate group of the point $(x_{\le t}, x_{>t})$ of X and let $\mathrm{X}_{\mathrm{H}} \subset \mathrm{X}$ be the $\mathbf{H}(\mathbf{R})$-orbit of x. Replace G by H the group of biholomorphisms of X_{H}, X by X_{H}, $\mathbf{G}$ by $\mathbf{H}^{\mathrm{ad}}$, Γ by Γ_{H} the projection of $\mathbf{H}(\mathbf{Z})$ on H, S by $\mathrm{S}_{\mathrm{H}} := \Gamma_{\mathrm{H}} \backslash \mathrm{X}_{\mathrm{H}}$, $\pi : \mathrm{X} \longrightarrow \mathrm{S}$ by $\pi_{\mathrm{H}} : \mathrm{X}_{\mathrm{H}} \longrightarrow \mathrm{S}_{\mathrm{H}}$, V by $\mathrm{V}_{\mathrm{H}} := \pi_{\mathrm{H}}(x_{\le t} \times \widetilde{\mathrm{V}_{>t}})$ and Y by $x_{\le t} \times \mathrm{Y}_{>t}$ and apply Theorem 7.1 for these new data: this shows that there exists $t' > t+1$ such that $\widetilde{\mathrm{V}_{>t}} = \mathrm{X}_{t+1} \times \cdots \times \mathrm{X}_{t'} \times \widetilde{\mathrm{V}_{>t'}}$. This contradicts the maximality of t.

Hence $\widetilde{\mathrm{V}_{>t}}$ is a point $(x_{t+1}, \ldots, x_r)$. Thus

$$\widetilde{\mathrm{V}} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_t \times (x_{t+1}, \ldots, x_r)$$

is weakly special, in particular algebraic, hence by maximality

$$\mathrm{Y} = \widetilde{\mathrm{V}} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_t \times (x_{t+1}, \ldots, x_r)$$

and Y is weakly special. □

Let us prove Theorem 7.1. Let $\mathbf{H}_{\mathrm{Y}}$ be the maximal connected $\mathbf{Q}$-subgroup in the stabiliser of Y in $\mathbf{G}(\mathbf{R})$. By Theorem 1.13 the group $\mathbf{H}_{\mathrm{Y}}$ is a non-trivial algebraic subgroup of $\mathbf{G}$.

Lemma **7.3.** — *The group* $\mathbf{H}_{\mathrm{Y}}(\mathbf{Q})$ *stabilises* $\widetilde{\mathrm{V}}$.

Proof. — Suppose there exists $h \in \mathbf{H}_{\mathrm{Y}}(\mathbf{Q})$ such that

$$\widetilde{\mathrm{V}} \neq h\widetilde{\mathrm{V}}.$$

As Y is contained in $\widetilde{\mathrm{V}} \cap h\widetilde{\mathrm{V}}$ and Y is irreducible, we can choose an analytic irreducible component $\widetilde{\mathrm{V}}'$ of $\widetilde{\mathrm{V}} \cap h\widetilde{\mathrm{V}}$ containing Y. Notice that $\pi(\widetilde{\mathrm{V}}')$ is an irreducible component, say V', of $\mathrm{V} \cap \mathrm{T}_h(\mathrm{V})$. As $\dim_{\mathbf{C}}(\widetilde{\mathrm{V}}') < \dim_{\mathbf{C}}(\widetilde{\mathrm{V}})$, we have that $\dim_{\mathbf{C}}(\mathrm{V}') < \dim_{\mathbf{C}}(\mathrm{V})$.

As $\pi(\mathrm{Y}) \subset \mathrm{V}'$, this contradicts the assumption that $\pi(\mathrm{Y})$ is Zariski dense in V. □

Choose a Hodge generic point z of V^{sm} (smooth locus of V) and a point $\widetilde{z}$ of $\widetilde{\mathrm{V}}$ lying over z. Let

$$\rho^{\mathrm{mon}} : \pi_1(\mathrm{V}^{\mathrm{sm}}, z) \longrightarrow \mathbf{GL}(\mathrm{E}_{\mathbf{Z}})$$

be the corresponding monodromy representation. We let $\Gamma_{\mathrm{V}} \subset \mathbf{G}(\mathbf{Z})$ be the image of ρ. By usual topological Galois theory the group Γ_{V} is the subgroup of $\mathbf{G}(\mathbf{Z})$ stabilising $\widetilde{\mathrm{V}}$ (cf. section 3 of [18]), in particular Γ_{V} contains $\mathbf{H}_{\mathrm{Y}}(\mathbf{Z})$.

By Deligne's monodromy theorem (see Theorem 1.4 of [18]), the connected component of the identity $\mathbf{H}^{\mathrm{mon}}$ of the Zariski closure $\overline{\Gamma_{\mathrm{V}}}^{\mathrm{Zar},\mathbf{Q}}$ of Γ_{V} in $\mathbf{G}$ is a normal subgroup of $\mathbf{G}$. As $\mathbf{G}$ is semi-simple of adjoint type, after reordering the factors we may assume that $\mathbf{H}^{\mathrm{mon}}$ coincides with $\mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$ for some integer $t \geq 1$. In particular $\mathbf{H}_{\mathrm{Y}} \subset \mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$.

We claim that Γ_{V} normalises $\mathbf{H}_{\mathrm{Y}}$. Let $\gamma \in \Gamma_{\mathrm{V}}$. Consider the $\mathbf{Q}$-algebraic group $\mathbf{F}$ generated by $\mathbf{H}_{\mathrm{Y}}$ and $\gamma \mathbf{H}_{\mathrm{Y}} \gamma^{-1}$. Then $\mathbf{F}(\mathbf{R})^{+} \cdot \widetilde{\mathrm{V}} = \widetilde{\mathrm{V}}$, where $\mathbf{F}(\mathbf{R})^{+}$ denotes the connected component of the identity of $\mathbf{F}(\mathbf{R})$. Hence $\mathbf{F}(\mathbf{R})^{+} \cdot \mathrm{Y} \subset \widetilde{\mathrm{V}}$. By Lemma B.3 there exists an irreducible (complex) algebraic subvariety $\tilde{\mathrm{Y}}$ of $\tilde{\mathrm{V}}$ containing U, hence Y. By maximality of Y one has $\tilde{\mathrm{Y}} = \mathrm{Y}$ hence

$$\mathbf{F}(\mathbf{R})^{+} \cdot \mathrm{Y} = \mathrm{Y}.$$

By maximality of $\mathbf{H}_{\mathrm{Y}}$, we have $\mathbf{F} = \mathbf{H}_{\mathrm{Y}}$. This proves the claim.

As $\mathbf{H}_{\mathrm{Y}}$ is normalised by Γ_{V}, it is normalised by $\mathbf{H}^{\mathrm{mon}} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$. It follows that (after possibly reordering factors) $\mathbf{H}_{\mathrm{Y}}$ contains $\mathbf{G}_1 \times \{1\}$.

The fact that $\mathbf{H}_{\mathrm{Y}}(\mathbf{R})$ stabilises $\widetilde{\mathrm{V}}$ shows (by taking the $\mathbf{H}_{\mathrm{Y}}(\mathbf{R})$-orbit of any point of $\widetilde{\mathrm{V}}$) that $\widetilde{\mathrm{V}} = \mathrm{X}_1 \times \widetilde{\mathrm{V}}_{>1}$. This concludes the proof of Theorem 7.1 and hence of Theorem 1.6.

Appendix A: Definability

A.1 *About Theorem 1.9*. — Let $\mathcal{R}$ be any fixed o-minimal expansion of $\mathbf{R}$ (in our case $\mathcal{R} = \mathbf{R}_{\mathrm{an,exp}}$). Recall [7, Chap. 10] that a *definable manifold* of dimension n is an equivalence class (for the usual relation) of triple $(\mathrm{X}, \mathrm{X}_i, \phi_i)_{i \in \mathrm{I}}$ where $\{\mathrm{X}_i : i \in \mathrm{I}\}$ is a finite cover of the set X and for each $i \in \mathrm{I}$:

(i) we have injective maps $\phi_i : X_i \longrightarrow \mathbf{R}^n$ such that $\phi_i(X_i)$ is an open, definably connected, definable set.
(ii) each $\phi(X_i \cap X_j)$ is an open definable subset of $\phi_i(X_i)$.
(iii) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} = \phi_j \cap \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

We say that a subset $Z \subset X$ is definable (resp. open or closed) if $\phi_i(Z \cap X_i)$ is a definable (resp. open or closed) subset of $\phi_i(X_i)$ for all $i \in I$. A definable map between abstract definable manifolds is a map whose graph is a definable subset of the definable product manifold.

Notice in particular that $X = \mathbf{P}^n\mathbf{C}$ has a canonical structure of a definable manifold (for any $\mathcal{R}$): take $X_i = \mathbf{C}^n = \{[z_o, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n] \in \mathbf{P}^n\mathbf{C}\}$, $0 \leq i \leq n$ where we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. As a corollary any complex quasi-projective variety is canonically a definable manifold. This apply in particular to S. In particular the statement of Theorem 1.9 has an intrinsic meaning.

Appendix B: Algebraic subvarieties of X

Recall from [30, section 2.1] that a realisation $\mathcal{X}$ of X for $\mathbf{G}$ is any analytic subset of a complex quasi-projective variety $\widetilde{\mathcal{X}}$, with a transitive holomorphic action of $\mathbf{G}(\mathbf{R})$ on $\mathcal{X}$ such that for any $x_0 \in \mathcal{X}$ the orbit map $\psi_{x_0} : \mathbf{G}(\mathbf{R}) \longrightarrow \mathcal{X}$ mapping g to $g \cdot x_0$ is semi-algebraic and identifies $\mathbf{G}(\mathbf{R})/K_\infty$ with X. A morphism of realisations is a $\mathbf{G}(\mathbf{R})$-equivariant biholomorphism. By [30, lemma 2.1] any realisation of X has a canonical semi-algebraic structure and any morphism of realisations is semi-algebraic. Hence X has a canonical semi-algebraic structure.

Let $\mathcal{X}$ be a realisation of X for $\mathbf{G}$. A subset $Y \subset \mathcal{X}$ is called an *irreducible algebraic subvariety* of $\mathcal{X}$ if Y is an irreducible component of the analytic set $\mathcal{X} \cap \widetilde{Y}$ where $\widetilde{Y}$ is an algebraic subset of $\widetilde{\mathcal{X}}$. By [10, section 2] the set Y has only finitely many analytic irreducible components and these components are semi-algebraic. An *algebraic subvariety* of $\mathcal{X}$ is defined to be a finite union of irreducible algebraic subvarieties of $\mathcal{X}$.

Lemma **B.1.** — *A subset* Y *of* $\mathcal{X}$ *is algebraic if and only if* Y *is a closed complex analytic subvariety of* $\mathcal{X}$ *and semi-algebraic in* $\mathcal{X}$.

Proof. — Let $Y \subset X$ be a closed complex analytic subvariety of $\mathcal{X}$, semi-algebraic in $\mathcal{X}$. Without loss of generality we can assume that Y is irreducible as an analytic subvariety, of dimension d. Consider the real Zariski-closure $\widetilde{Y}$ of Y in the real algebraic variety $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\widetilde{\mathcal{X}}$, where $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}$ denotes the Weil restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$. Let us show that $\widetilde{Y}_{\mathbf{R}}$ has a canonical structure of a complex subvariety of $\widetilde{\mathcal{X}}$. Choose an affine open cover $(\widetilde{\mathcal{X}}_i)_{i \in I} \subset \mathbf{A}^{n_i}$ of $\widetilde{\mathcal{X}}$ and denote by $\widetilde{Y}_i$ the intersection $\widetilde{Y} \cap \widetilde{\mathcal{X}}_i$. Let $i \in I$ such that $\widetilde{Y}_i$ is non-empty. As Y is semi-algebraic, Y is open in $\widetilde{Y}$ for the Hausdorff topology, hence $\mathrm{Y}_i := \mathrm{Y} \cap \widetilde{\mathcal{X}}_i$ is non-empty and open in $\widetilde{\mathrm{Y}}_i$ for the Hausdorff topology. Consider the Gauss map φ_i from the smooth part $\widetilde{\mathrm{Y}}_i^{\mathrm{sm}}$ of $\widetilde{\mathrm{Y}}_i$ to the real Grassmannian $\mathbf{Gr}^{2d,2n_i}$ of real $2d$-planes of $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{A}^{n_i}$ associating to a point its tangent space. The map φ_i is real analytic and its restriction to the open subset $\mathrm{Y}_i^{\mathrm{sm}}$ of $\widetilde{\mathrm{Y}}_i^{\mathrm{sm}}$ takes values in the closed real analytic subvariety $\mathbf{Gr}_{\mathbf{C}}^{d,n_i} \subset \mathbf{Gr}^{2d,2n_i}$ of complex d-planes of $\mathbf{A}_{\mathbf{C}}^{n_i}$. By analytic continuation φ_i takes values in $\mathbf{Gr}_{\mathbf{C}}^{d,n_i}$. Hence $\widetilde{\mathrm{Y}}_i$ is a complex algebraic subvariety of $\mathbf{A}^{n_i}$. As this is true for all $i \in \mathrm{I}$, $\widetilde{\mathrm{Y}}$ is a complex algebraic subvariety of $\widetilde{\mathcal{X}}$. As $\mathrm{Y} \subset \widetilde{\mathrm{Y}}$ is open and Y is closed analytically irreducible in $\mathcal{X}$, it follows that Y is an irreducible component of $\mathcal{X} \cap \widetilde{\mathrm{Y}}$, hence algebraic.

The other implication is clear. □

As any morphism of realisations is an analytic biholomorphism and semi-algebraic the previous lemma implies immediately:

Corollary **B.2.** — *Let* $\varphi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ *be a morphism of realisations of* X. *A subset* Y_1 *of* $\mathcal{X}_1$ *is algebraic if and only if its image* $\mathrm{Y}_2 := \varphi(\mathrm{Y}_1) \subset \mathcal{X}_2$ *is algebraic.*

This defines the notion of algebraic subsets of X.

Lemma **B.3.** — *Let* $\mathcal{X}$ *be a realisation of a Hermitian symmetric domain* X. *Let* $\mathrm{Z} \subset \mathcal{X} \subset \mathbf{C}^n$ *be a complex analytic subvariety and* $\mathrm{W} \subset \mathrm{Z}$ *a semi-algebraic set. There exists an irreducible complex algebraic subvariety* $\mathrm{Y} \subset \mathbf{C}^n$ *such that*

$$\mathrm{W} \subset \mathrm{Y} \cap \mathrm{X} \subset \mathrm{Z}$$

Proof. — This is a consequence of the proof of [25, lemma 4.1]. □

REFERENCES

1. A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth Compactification of Locally Symmetric Varieties*, Lie Groups: History Frontiers and Applications, vol. 4, Math. Sci. Press, Brookline, 1975.
2. J. Ax, On Schanuel's conjecture, *Ann. Math.*, **93** (1971), 1–24.
3. W. L. Baily and A. Borel, Compactification of arithmetic quotients of bounded symmetric domains, *Ann. Math.*, **84** (1966), 442–528.
4. A. Borel, *Introduction aux Groupes Arithmétiques*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, vol. 1341, Hermann, Paris, 1969.
5. C. Daw and M. Orr, Heights of pre-special points of Shimura varieties, *Math. Ann.*, to appear, arXiv:1502.00822.
6. P. Deligne, Variétés de Shimura: interprétation modulaire et techniques de construction de modèles canoniques, in A. Borel and W. Casselman (eds.) *Automorphic Forms, Representations, and* L*-Functions, Part. 2*, Proc. of Symp. in Pure Math., vol. 33, pp. 247–290, Am. Math. Soc., Providence, 1979.
7. L. van den Dries, *Tame Topology and o-Minimal Structures*, LMS Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.
8. L. van den Dries and C. Miller, On the real exponential field with restricted analytic functions, *Isr. J. Math.*, **85** (1994), 19–56.
9. B. Edixhoven and A. Yafaev, Subvarieties of Shimura varieties, *Ann. Math.*, **157** (2003), 621–645.
10. E. Fortuna and S. Lojasiewicz, Sur l'algébricité des ensembles analytiques complexes, *J. Reine Angew. Math.*, **329** (1981), 215–220.

11. J. M. Hwang and W. K. To, Volumes of complex analytic subvarieties of Hermitian symmetric spaces, *Am. J. Math.*, **124** (2002), 1221–1246.
12. B. Klingler and A. Yafaev, The André-Oort conjecture, *Ann. Math.*, **180** (2014), 867–925.
13. F. Lindemann, Über die Zahl π, *Math. Ann.*, **20** (1882), 213–225.
14. G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grengebiete, vol. 17, Springer, Berlin, 1991.
15. N. Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Math., vol. 6, World Scientific, Singapore, 1989.
16. N. Mok, On the Zariski closure of a germ of totally geodesic complex submanifold on a subvariety of a complex hyperbolic space form of finite volume, in *Complex Analysis*, Trends Math., vol. 2-79-300, Springer, Berlin, 2010.
17. N. Mok, Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric, *J. Eur. Math. Soc.*, **14** (2012), 1617–1656.
18. B. Moonen, Linearity properties of Shimura varieties. I, *J. Algeb. Geom.*, **7** (1998), 539–567.
19. D. Mumford, Hirzebruch's proportionality theorem in the non-cocompact case, *Invent. Math.*, **42** (1979), 239–272.
20. Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of Abelian varieties, *Duke Math. J.*, **162** (2013), 731–765.
21. Y. Peterzil and S. Starchenko, Tame complex analysis and o-minimality, in *Proceedings of the ICM*, *Hyderabad*, 2010. Available on first author's web-page.
22. I. I. Pyateskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains*, Mathematics and Its Applications, vol. 8, Gordon & Breach, New York, 1969, translated from the Russian.
23. J. Pila, O-minimality and the Andre-Oort conjecture for $\mathbf{C}^n$, *Ann. Math.*, **173** (2011), 1779–1840.
24. J. Pila and A. Wilkie, The rational points on a definable set, *Duke Math. J.*, **133** (2006), 591–616.
25. J. Pila and J. Tsimerman, The André-Oort conjecture for the moduli space of abelian surfaces, *Compos. Math.*, **149** (2013), 204–216.
26. J. Pila and J. Tsimerman, Ax-Lindemann for $\mathcal{A}_g$, *Ann. Math.*, **179** (2014), 659–681.
27. J. Pila and U. Zannier, Rational points in periodic analytic sets and the Manin-Mumford conjecture, *Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl.*, **19** (2008), 149–162.
28. I. Satake, *Algebraic Structures of Symmetric Domains*, Kanu Memorial Lectures, vol. 4, Iwanami Shoten and Princeton University Press, Tokyo and Princeton, 1980.
29. T. Scanlon, O-minimality as an approach to the André-Oort conjecture, preprint (2012). Available on author's web-page.
30. E. Ullmo, Applications du théorème d'Ax-Lindemann hyperbolique, *Compos. Math.*, **150** (2014), 175–190.
31. E. Ullmo and A. Yafaev, Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture, *Ann. Math.*, **180** (2014), 823–865.
32. E. Ullmo and A. Yafaev, A characterisation of special subvarieties, *Mathematika*, **57** (2011), 263–273.
33. E. Ullmo and A. Yafaev, Nombre de classes des tores de multiplication complexe et bornes inférieures pour orbites Galoisiennes de points spéciaux, *Bull. Soc. Math. Fr.*, **143** (2015), 197–228.
34. E. Ullmo and A. Yafaev, Hyperbolic Ax-Lindemann theorem in the cocompact case, *Duke Math. J.*, **163** (2014), 433–463.
35. J. Tsimermann, Brauer-Siegel theorem for tori and lower bounds for Galois orbits of special points, *J. Am. Math. Soc.*, **25** (2012), 1091–1117.
36. K. Weierstrass, Zu Lindemanns Abhandlung: "Über die Ludolph'sche Zahl", *Berl. Ber.* (1885), 1067–1086.
37. J. A. Wolf and A. Korányi, Generalized Cailey transformations of bounded symmetric domains, *Am. J. Math.*, **87** (1965), 899–939.

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