DIFFEOMORPHISMS WITH POSITIVE METRIC ENTROPY

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To the memory of Jean-Christophe Yoccoz

ABSTRACT

We obtain a dichotomy for C^1-generic, volume-preserving diffeomorphisms: either all the Lyapunov exponents of almost every point vanish or the volume is ergodic and non-uniformly Anosov (i.e. nonuniformly hyperbolic and the splitting into stable and unstable spaces is dominated). This completes a program first put forth by Ricardo Mañé.

Introduction

From a probabilistic perspective, ergodicity is the most basic irreducibility property of a dynamical system. A measurable map $f: M \to M$ is *ergodic* with respect to an invariant probability measure μ if every f-invariant subset of M is μ-trivial: $f^{-1}(A) = A$ implies $\mu(A) = 0$ or 1, for every measurable $A \subset M$. In the context of this paper, where M is a compact manifold, f is a homeomorphism, and $\mu = m$ is a normalized volume, ergodicity is equivalent to equidistribution of almost every orbit: for m-almost every $x \in M$ and every continuous $\phi: M \to \mathbf{R}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \phi\big(f^j(x)\big) = \int_M \phi\, dm.$$

Is ergodicity with respect to volume a typical property? The question was first addressed by Oxtoby and Ulam in the 1930's [OU], who proved that the *generic* volume-preserving homeomorphism is ergodic; that is, the set of ergodic maps in the space $\mathrm{Homeo}^+_{\mathrm{vol}}(M)$ of volume-preserving homeomorphisms contains a countable intersection of open and dense sets in the uniform topology. A natural question, still open in general, is whether such a result extends to the space of volume-preserving *diffeomorphisms*.

If one looks at the other extreme of regularity, C^∞ diffeomorphisms, ergodicity is not a typical property at all: KAM theory guarantees on any manifold of dimension at least 2 an *open* set of diffeomorphisms in $\mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ that are not ergodic. This paper focuses on the lowest class of differentiability, C^1 diffeomorphisms, where the question is still open: *is ergodicity a generic property in the space* $\mathrm{Diff}^1_{\mathrm{vol}}(M)$ *of* C^1 *volume-preserving diffeomorphisms of a compact manifold* M*?*

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As a first approach to this question, one should ask whether the techniques of the Oxtoby–Ulam proof can be extended to the C^1 setting. There is an immediate obstruction: metric entropy. The same technique (namely periodic approximation) that proves genericity of ergodicity in [OU] also proves that the metric entropy $h_m(f)$ of a generic $f \in \mathrm{Homeo}^+_{\mathrm{vol}}(M)$ is 0. The corresponding statement is false for $\mathrm{Diff}^1_{\mathrm{vol}}(M)$, as we explain below: there are *open* sets of diffeomorphisms $f \in \mathrm{Diff}^1_{\mathrm{vol}}(M)$ with $h_m(f) > 0$. Thus the Oxtoby–Ulam technique cannot be naïvely extended from the C^0-category to prove general results about C^1-generic diffeomorphisms.

This phenomenon of robustly positive entropy is most clearly demonstrated by the Anosov maps, in which every direction in the tangent bundle to M sees expansion or contraction under iteration of the derivative Df^n. Interestingly, this *uniformly hyperbolic* behavior that gives rise to positive metric entropy in Anosov systems is also the source of a powerful mechanism for ergodicity, known as the Hopf argument [An1], which is of a very different nature than the Oxtoby–Ulam mechanism. Here we show for generic diffeomorphisms in $\mathrm{Diff}^1_{\mathrm{vol}}(M)$, positive metric entropy is associated with a strong type of non-uniformly hyperbolic behavior, which we call *non-uniformly Anosov*. Harnessing this hyperbolicity, we prove:

Theorem **A.** — C^1*-generically, a volume-preserving diffeomorphism* $f : M \to M$ *of a compact manifold* M *with positive entropy is ergodic.*

Our proof of this theorem completes a program first put forth by Ricardo Mañé to understand the Lyapunov exponents of volume-preserving diffeomorphisms from a C^1-generic perspective. In his 1983 ICM address [M], Mañé announced the following remarkable result, whose proof was later completed by Bochi [Boc1].

Theorem (Mañé–Bochi). — C^1*-generically, an area preserving diffeomorphism* f *of a compact connected surface* M *is either Anosov (and ergodic) or satisfies*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|D_x f^n v\| = 0,$$

for a.e. $x \in M$ *and every* $0 \neq v \in T_x M$.

Our main result gives the optimal generalization to higher dimensions:

Theorem **B.** — C^1*-generically, a volume-preserving diffeomorphism* f *of a compact connected manifold* M *is either nonuniformly Anosov and ergodic or satisfies*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|D_x f^n v\| = 0$$

for a.e. $x \in M$ *and every* $0 \neq v \in T_x M$.

Theorem B was conjectured in its present form by Avila–Bochi [AB] where it was shown that generic diffeomorphisms in $\mathrm{Diff}^1_{\mathrm{vol}}(M)$ with only non-zero Lyapunov exponents almost everywhere are ergodic and non-uniformly Anosov. In dimension three, Theorem B was proved by M.A. Rodríguez-Hertz [R] by reducing to an analysis of dominated splittings admitting some uniformly hyperbolic subbundles, which have been thoroughly described for 3-manifolds. Our proof of Theorem B in the general case follows a very different route, focused on the elimination of zero Lyapunov exponents throughout large parts of the phase space.

In another paper [ACW], we will use Theorem B above in order to prove a C^1-version of a conjecture by Pugh and Shub: among smooth partially hyperbolic volume-preserving diffeomorphisms, the stably ergodic ones are C^1-dense.

Before exploring further consequences of Theorems A and B, we put it in context and explain the terminology. Throughout, M will denote a closed connected Riemannian manifold with dimension d, and $\mathrm{Diff}^r(M)$ will denote the set of C^r diffeomorphisms of M endowed with the C^r-topology. The volume induces, after normalization, a Borel probability measure m and we denote by $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ the set of $f \in \mathrm{Diff}^r(M)$ preserving m. Both $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^r_{\mathrm{vol}}(M)$ are Baire spaces. We say that a property of (volume-preserving) diffeomorphisms is C^r *generic* if it holds on a dense G_δ (i.e., a countable intersection of open-dense sets) in $\mathrm{Diff}^r(M)$ (respectively $\mathrm{Diff}^r_{\mathrm{vol}}(M)$).

A measure of chaoticity for volume-preserving diffeomorphisms is given by the notion of Lyapunov exponents. A real number χ is a *Lyapunov exponent* of f at $x \in M$ if there exists a nonzero vector $v \in T_xM$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\left\| Df^n(v) \right\| = \chi. \tag{1}$$

Oseledets's ergodic theorem implies that there is a set $\Omega \subset M$ of total measure—i.e., $\mu(\Omega) = 1$, for every invariant Borel probability measure μ—with the following property: for any $x \in \Omega$ there exists $\ell(x) \geq 1$ and a Df-invariant splitting

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{\ell(x)}(x), \tag{2}$$

depending measurably on x such that the limit $\chi = \chi(x, v)$ in (1) exists for every $v \in E_i(x) \setminus \{0\}$. The value $\chi(x, v)$ is constant in $E_i(x) \setminus \{0\}$ so that $\chi(x, \cdot)$ can assume at most $\dim(M)$ distinct values $\chi_1(x), \ldots, \chi_{\ell(x)}(x)$. If f preserves the volume m, then the sum of the Lyapunov exponents is zero on a set of total measure.

Lyapunov exponents can be used to control a more familiar barometer of chaos, namely the metric (or measure-theoretic) entropy. Entropy and Lyapunov exponents of C^1 diffeomorphisms are related by Ruelle's inequality, which states that for $f \in \mathrm{Diff}^1(M)$ preserving a Borel probability μ,

$$h_\mu(f) \leq \int_M \sum_{\chi_i(x)\geq 0} \dim\big(E_i(x)\big)\, \chi_i(x)\, d\mu(x).$$

For $\mu = m$, the reverse equality was proved by Pesin for all $f \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathrm{M})$ and generically in $\mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ by Tahzibi [T1] and Sun–Tian [ST]. In particular for generic $f \in \mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$, the metric entropy vanishes exactly when the second case of Theorem B occurs. Hence Theorem B implies Theorem A.

In his 1983 address mentioned above, Mañé proposed to study how the "Oseledets splitting" (2) varies as a function of the diffeomorphism f, in the C^1 topology. A diffeomorphism f of a compact manifold M is Anosov if there exists a continuous $\mathrm{D}f$-invariant splitting

$$\mathrm{TM} = \mathrm{E}^u \oplus \mathrm{E}^s \tag{3}$$

and $0 < \lambda < 1$, $n_0 \in \mathbf{N}$, such that $\|\mathrm{D}f^n|\mathrm{E}^u\| \leq \lambda^n$ and $\|(\mathrm{D}f^n|\mathrm{E}^s)^{-1}\| \leq \lambda^n$ for every $n \geq n_0$. In this case, the (measurable) Oseledets splitting (2) refines the (continuous) Anosov splitting (3) and the Lyapunov exponents are nonzero (either smaller than $-|\log(\lambda)|$ or larger than $+|\log(\lambda)|$). This property is extremely rigid in low dimension (and conjecturally rigid in all dimensions): in particular, if f is an Anosov diffeomorphism of a surface, then M is a torus, and f is topologically conjugate to a hyperbolic linear automorphism. Thus the Mañé–Bochi theorem implies that if M is not a torus, then the C^1-generic area-preserving diffeomorphism of M has metric entropy 0.

However, in higher dimensions *uniform* hyperbolicity is too much to aim for: any volume-preserving diffeomorphism admitting a *dominated splitting* must have robustly positive metric entropy. A diffeomorphism $f \in \mathrm{Diff}^1(\mathrm{M})$ is said to admit a (global) dominated splitting if there exists a continuous non-trivial decomposition $\mathrm{TM} = \mathrm{E}_1 \oplus \mathrm{E}_2$ that is $\mathrm{D}f$-invariant and satisfies

$$\left\|\left(\mathrm{D}f^{\mathrm{N}}\big|\mathrm{E}_1\right)^{-1}\right\| \left\|\mathrm{D}f^{\mathrm{N}}\big|\mathrm{E}_2\right\| < 1,$$

for some $\mathrm{N} \in \mathbf{N}$. Thus f is an Anosov map if and only if it admits a uniformly hyperbolic dominated splitting.

While in dimension 2 a dominated splitting for an area-preserving diffeomorphism is always Anosov, already in dimension 3 there are manifolds that do not support Anosov dynamics, but which are compatible with a dominated splitting.[1]

In the presence of robust obstructions to uniform hyperbolicity, the best one can hope for is to obtain a dominated splitting $\mathrm{TM} = \mathrm{E}^+ \oplus \mathrm{E}^-$ that is *non-uniformly hyperbolic*, in the sense that there exists $\chi_0 > 0$ such that for m-a.e. $x \in \mathrm{M}$, each Lyapunov exponent is either smaller than χ_0 or larger than χ_0. This leads to:

Definition. — *A diffeomorphism* $f \in \mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ *admitting a non-uniformly hyperbolic dominated splitting will be called* non-uniformly Anosov. *Equivalently, f is non-uniformly Anosov if it*

[1] On the unit tangent bundle of a hyperbolic surface, the geodesic flow is Anosov; hence its time-one map is a diffeomorphism preserving a dominated splitting. However this manifold does not support any Anosov diffeomorphism, since, in dimension 3, only the torus has this property (Franks–Newhouse theorem [F, N]).

possesses a dominated splitting $TM = E^+ \oplus E^-$ *and if there exists* $0 < \lambda < 1$ *such that for m-almost every* $x \in M$, *there exists* $n_0(x) \in \mathbf{N}$ *such that* $\|Df^n(x)|E^-(x)\| \leq \lambda^n$ *an* $\|Df^{-n}(x)|E^+(x)\| \leq \lambda^n$ *for every* $n \geq n_0(x)$.

The class of non-uniformly Anosov diffeomorphisms is strictly larger than the Anosov class; Shub and Wilkinson [SW] constructed an open set of non-uniformly Anosov diffeomorphisms in $\mathrm{Diff}^2_{\mathrm{vol}}(\mathbf{T}^3)$ that are not Anosov. (Their construction is at the root of one of the arguments used in this paper; see Section 1.)

The existence of a dominated splitting is a robust dynamical property (i.e., stable under perturbations in $\mathrm{Diff}^1(M)$), as is uniform hyperbolicity. A striking consequence of Theorem B (proved in Section 4.1) is thus:

Corollary **1.** — *A map* $f \in \mathrm{Diff}^1_{\mathrm{vol}}(M)$ *has robust positive metric entropy if and only if it admits a dominated splitting.*

These results highlight the unique features of the C^1 topology. At least conjecturally, sufficiently regular volume-preserving diffeomorphisms are expected to be compatible with a quite different phenomenon: the coexistence of quasiperiodic behavior (where Lyapunov exponents vanish) with chaotic, non-uniformly hyperbolic behavior (inducing positive metric entropy). Even on surfaces, this problem remains open.

Discussion and questions. — We return briefly to the question posed at the beginning of the paper: *Is ergodicity a generic property in* $\mathrm{Diff}^1_{\mathrm{vol}}(M)$*?* Some partial results are known. Bonatti and Crovisier proved [BC] that transitivity (i.e., existence of a dense orbit) is a generic property in $\mathrm{Diff}^1_{\mathrm{vol}}(M)$ (the topological mixing also holds [AC]). A property in between transitivity and ergodicity with respect to volume is *metric transitivity*, where almost every orbit is dense. A weaker question is thus:

Question **1.** — *Is metric transitivity generic in* $\mathrm{Diff}^1_{\mathrm{vol}}(M)$*?*

The next question relates to the Oxtoby–Ulam technique [OU]. If f has entropy 0, then the results in [BDP], [BC] and [Av] show that it can be perturbed to have a dense set of periodic balls.

Question **2.** — *Can every* $f \in \mathrm{Diff}^1_m(M)$ *of entropy* 0 *be* C^1 *approximated by an almost everywhere periodic diffeomorphism (i.e. a diffeomorphism whose periodic points have full measure)?*

In the case of C^1-generic diffeomorphisms with positive entropy, a next goal would be to describe better their measurable dynamics. Some additional argument gives the following corollary of Theorem B, which is proved in Section 4.2:

Corollary **2.** — *The generic* $f \in \mathrm{Diff}^1_{\mathrm{vol}}(M)$ *with positive metric entropy is weakly mixing.*

Due to the lack of regularity in $\mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ we cannot use Pesin theory to get the Bernoulli property to be generic.

Question **3.** — *Are generic nonuniformly Anosov diffeomorphisms in* $\mathrm{Diff}^1_m(\mathrm{M})$ *Bernoulli? or at least strongly mixing?*

Even among the class of C^1 Anosov diffeomorphisms, genericity of the mixing condition is an open question.

When M is endowed with a symplectic form ω, one may also consider the space of diffeomorphisms $\mathrm{Diff}^1_\omega(\mathrm{M})$ that preserve ω. For "technical reasons," Mañé focuses on this case in [M]: the symplectic rigidity imposes some symmetry in the Oseledets splitting. The argument developed in the present paper (Theorem C below) can not be transposed in this setting. Some partial results have been obtained for C^1-generic symplectomorphisms: for instance [ABW] proves that if there exists an invariant global dominated splitting, then the volume is ergodic (but it is not non-uniformly hyperbolic, unless the diffeomorphism is Anosov). In an upcoming work we will prove the symplectic version of Theorem A, using different (and simpler!) methods that are special to the symplectic setting.

1. The main technique: localized, pointwise perturbations of central Lyapunov exponents

From the development of Pesin Theory and the gradual taming of (sufficiently regular) non-uniformly hyperbolic dynamics which followed ([P], [Ka]), it has been a central problem to understand how often such systems arise. While it is understood that the "opposite" behavior, the vanishing of all Lyapunov exponents, does appear robustly (through the KAM mechanism), it has been proposed by Shub and Wilkinson ([SW], Question 1a) that for typical orbits of a generic C^r conservative dynamical system, the presence of some non-zero Lyapunov exponent implies in fact that all Lyapunov exponents are non-zero. Such an optimistic picture was motivated by an argument, introduced in the same paper, which allows one to leverage (in a particularly controlled setting) the non-zero Lyapunov exponents to "perturb away" the zero Lyapunov exponents.

The specific situation considered by Shub and Wilkinson consisted of a trivial circle extension of a linear Anosov map. This is a partially hyperbolic dynamical system with a one-dimensional central direction along which the Lyapunov exponent vanishes everywhere. Through a carefully designed perturbation, the central bundle borrows some of the hyperbolicity from the uniformly expanding bundle, so the *average* central Lyapunov exponent becomes positive. In order to show that the actual Lyapunov exponent along the center is non-zero almost everywhere, they observe that the system can be, at the same time, made ergodic by a separate argument (based on the Pugh–Shub ergodicity mechanism).

This argument has been pursued further, in low regularity, by Baraviera and Bonatti [BaBo]. They consider conservative diffeomorphisms admitting a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k$ and show that the average of the sum of the Lyapunov exponents along any subbundle can be made non-zero by a C^1 perturbation. This result was used by Bochi, Fayad and Pujals in [BFP] to show that stably ergodic diffeomorphisms, which admit a dominated splitting by [BDP], can be made non-uniformly hyperbolic by perturbation.

In a sense, here we do just the opposite of [BFP]: we show the generic absence of zero Lyapunov exponents almost everywhere (under the positive entropy assumption) is a means to conclude ergodicity (via [AB]). In order to do this, we must develop a perturbation argument that can affect directly the actual Lyapunov exponents of certain orbits inside an invariant region, and not just their averages over the whole manifold. Without an assumption of ergodicity, these can be different. This is obtained through the following local, pointwise version of Baraviera–Bonatti's argument [BaBo] (see also [SW]). Even for a diffeomorphism that preserves a globally partially hyperbolic structure, this is a new result. A more precise statement will be given in Section 3.

If μ and ν are finite Borel measures on M, the notation $\mu \le \nu$ means that $\mu(A) \le \nu(A)$ for all measurable sets A. For $f \in \mathrm{Diff}^1_m(M)$, $x \in M$, and a nontrivial subspace $F \subset T_xM$, we denote by $\mathrm{Jac}_F(f, x)$ the Jacobian of Df restricted to F, i.e., the product of the singular values of $Df(x)|F$.

Theorem **C.** — *Let* $f \in \mathrm{Diff}^1_m(M)$, *and let* $K \subset M$ *be an invariant compact set such that:*

- K *admits a dominated splitting* $T_KM = E_1 \oplus E_2 \oplus E_3$ *into three non-trivial subbundles;*
- *for almost every point* $x \in K$ *one has*

$$\limsup_{n \to \pm\infty} \frac{1}{n} \log \mathrm{Jac}_{E_2(x)}\big(f^n, x\big) \le 0.$$

Then for every $\varepsilon > 0$ *and every small neighborhood* Q *of* K, *there exists a diffeomorphism* g *arbitrarily close to* f *in* $\mathrm{Diff}^1_m(M)$ *such that for every* g*-invariant measure* ν *such that* $\nu \le m|Q$ *and* $\nu(M) \ge \varepsilon$, *one has* $\int \log \mathrm{Jac}_{E_2(g,x)}(g, x)\,d\nu(x) < 0$.

In the previous statement the fibers of the bundles E_1, E_2, E_3 do not necessarily have constant dimension, but one can easily reduce the theorem to this case by decomposing the compact set K. The expression $E_1(g) \oplus E_2(g) \oplus E_3(g)$ denotes the *continuation of the dominated splitting* for the diffeomorphism g on any g-invariant set contained in a neighborhood of Q. (See Section 2.1.2.)

We remark that the existence of a global dominated splitting, which is a starting point in [SW] and [BaBo], is here also obtained as a consequence of non-uniform hyperbolicity (again, via [AB]). The hypothesis of positive entropy (and hence the existence of some non-zero Lyapunov exponents) is however enough to obtain local dominated splittings, thanks to a result of Bochi and Viana [BV2] who showed that, for almost every orbit of generic conservative diffeomorphisms, the Oseledets splitting extends continuously to a dominated splitting on its closure.

Our basic technique is the following. First, we may assume that the initial (generic) diffeomorphism has a positive measure set K of orbits having some, but not all, non-zero Lyapunov exponents (otherwise [AB] yields the conclusion at once). Consider a sufficiently long segment of a typical orbit that admits a dominated splitting $E_1 \oplus E_2 \oplus E_3$, where E_2 corresponds to zero Lyapunov exponents. If this orbit segment is long enough, then it "sees" the Lyapunov exponents of the orbit. We can then reproduce the perturbation technique of [SW] and [BaBo] along the orbit: since this technique concerns average exponents, we first thicken the initial point to a small positive measure set, and conclude that the average of the sum of the Lyapunov exponents along the central bundle can be decreased. In order to produce a pointwise estimate, we use a randomization technique introduced by Bochi in [Boc2], which allows us to apply the Law of Large Numbers to promote the averaged estimate to a pointwise one. Using a standard towers technique, this argument can be carried out simultaneously a large set of the orbits remaining within the domain of definition U of the local dominated splitting.

Naturally, the perturbation changes the dynamics, so in principle the decrease of the sum of Lyapunov exponents could be cancelled later. In fact the dynamics could change so much that many orbits escape U and we lose all control, but this "loss of mass" is an irreversible event and thus relatively harmless. As for possible cancellations, we simply assume away the problem by restricting attention to the case where the Lyapunov exponents along E_2 are non-positive for almost every orbit that remains within U. Remarkably, this seemingly very strong hypothesis can be in fact verified along the steps of a carefully designed inductive argument. In any case, with this assumption we can conclude directly that for most orbits remaining in U the number of zero Lyapunov exponents is strictly less than the dimension of E_2, after perturbation.

Iterating this argument, we eventually succeed in either eliminating all non-zero Lyapunov exponents, or in obtaining vanishing Lyapunov exponents almost everywhere (this happens when we keep running into the situation where orbits escape the domains of definition of local dominated splittings).

2. A dichotomy for conservative diffeomorphisms

In this section we prove Theorem B assuming Theorem C.

2.1. *Dominated splittings and center Jacobians.* — Let $f \in \mathrm{Diff}^1(M)$. We recall well-known properties of dominated splittings. As before $d = \dim(M)$.

2.1.1. Given an f-invariant compact set K_f, we say that $f|K_f$ admits a *dominated splitting of type* (d_1, d_2, d_3) (where $d_1, d_2, d_3 \geq 0$ and $d_1 + d_2 + d_3 = d$) if there is an f-invariant splitting $T_xM = E_1(x) \oplus E_2(x) \oplus E_3(x)$ defined over K, where $E_*(x) = E_*(f, x)$ are subspaces of dimension d_*, $* = 1, 2, 3$, and there is an $n \in \mathbf{N}$ such that for each $x \in K$ one has

$$\left\| \left(Df^n(x)\middle|E_1(x)\right)^{-1} \right\| < \left\|Df^n(x)\middle|E_2(x)\right\|^{-1},$$
$$\left\| \left(Df^n(x)\middle|E_2(x)\right)^{-1} \right\| < \left\|Df^n(x)\middle|E_3(x)\right\|^{-1}.$$

In other words, the smallest contraction along E_1 and the largest expansion along E_3 dominate the behavior along E_2. For a fixed d_1, d_2, d_3, the $E_*(x)$ are uniquely defined in this way and depend continuously on x.

2.1.2. This dominated splitting is robust in the following sense. Consider an arbitrary continuous extension of the $E_*(x)$ to a neighborhood of K_f and consider arbitrary metrics on the Grassmanian manifolds of M. Then for every $\alpha > 0$, there are neighborhoods $\mathcal{V} \subset \mathrm{Diff}^1(M)$ of f and $V \subset M$ of K_f such that if $g \in \mathcal{V}$ and $K_g \subset V$ is a compact invariant set, then $g|K_g$ admits a dominated splitting of type (d_1, d_2, d_3), and moreover the spaces $E_*(g, x)$ are α-close to (the extension of) $E_*(f, x)$ for every $x \in K_g$.

2.1.3. Given a compact set $Q \subset M$, we let $K(f, Q) = \bigcap_{n\in\mathbf{Z}} f^n(Q)$ be its maximal f-invariant subset. Notice that $K(g, Q) \subset V$ for every neighborhood V of $K(f, Q)$ and every $g \in \mathrm{Diff}^1(M)$ close to f in the C^0 topology.

The previous paragraph thus implies that *the set of all* $g \in \mathrm{Diff}^1(M)$ *such that* $g|K(g, Q)$ *admits a dominated splitting of type* (d_1, d_2, d_3) *is open*. This includes the diffeomorphisms g such that $K(g, Q)$ is empty.

2.1.4. Let $X_f \subset M$ be the set of *Oseledets regular points* x of f, i.e. which have well-defined Oseledets splitting and Lyapunov exponents

$$\lambda_1(f, x) \geq \lambda_2(f, x) \geq \cdots \geq \lambda_d(f, x).$$

By Oseledets's theorem, X_f is a measurable f-invariant set of total measure. Moreover, the Lyapunov exponents define d functions $\lambda_1, \ldots, \lambda_d \in L^1(\mu)$.

For any regular point x, by summing all the directions associated to the positive, zero, or negative Lyapunov exponents, we obtain a splitting:

$$T_xM = E^+(x) \oplus E^0(x) \oplus E^-(x).$$

The dimensions $\dim(E^+(x))$, $\dim(E^-(x))$ are called *unstable* and *stable dimensions* of x.

An invariant probability measure is *hyperbolic* if for almost every point the Lyapunov exponents are all different from zero.

2.1.5. For $x \in \mathrm{M}$ and a subspace $\mathrm{F} \subset \mathrm{T}_x\mathrm{M}$, we let

$$\Delta_{\mathrm{F}}(f, x) = \lim_{n \to \pm\infty} \frac{1}{n} \log \mathrm{Jac}_{\mathrm{F}}\big(f^n, x\big),$$

which is well-defined on a set of x of total measure. If x is Oseledets regular, and F is a sum of Oseledets subspaces, then $\frac{1}{n} \log \mathrm{Jac}_{\mathrm{F}}(f^n, x)$ converges to the sum of the Lyapunov exponents of f along F. Moreover if ν is an f-invariant finite Borel measure, and $\mathrm{F}(x) \subset \mathrm{T}_x\mathrm{M}$ is a measurable f-invariant distribution of subspaces defined ν-almost everywhere, then for every $n \geq 1$ we have

$$\int \Delta_{\mathrm{F}(x)}(f, x)\, d\nu(x) = \frac{1}{n} \int \log \mathrm{Jac}_{\mathrm{F}(x)}\big(f^n, x\big)\, d\nu(x).$$

2.1.6. Recall that if μ and ν are finite Borel measures, the notation $\mu \leq \nu$ means that $\mu(\mathrm{A}) \leq \nu(\mathrm{A})$ for all measurable sets A. This property is equivalent to the two conditions: μ is absolutely continuous with respect to ν and the Radon–Nikodym derivative $d\mu/d\nu$ is essentially bounded above by 1. When ν is fixed, the set of measures μ satisfying $\mu \leq \nu$ is clearly compact in the weak-$*$ topology.

2.1.7. Recall that m is a smooth volume on M. For $\varepsilon > 0$ and $\mathrm{Q} \subset \mathrm{M}$ compact, we denote by $\mathcal{G}_\varepsilon(\mathrm{Q}, d_1, d_2, d_3)$ the set of all $g \in \mathrm{Diff}^1(\mathrm{M})$ such that

- $g|\mathrm{K}(g, \mathrm{Q})$ admits a dominated splitting of type (d_1, d_2, d_3) (including the case where $\mathrm{K}(g, \mathrm{Q}) = \emptyset$),
- for every g-invariant measure $\nu \leq m|\mathrm{Q}$ satisfying $\nu(\mathrm{M}) \geq \varepsilon$, one has

$$\int \mathrm{Jac}_{\mathrm{E}_2(g,x)}(g, x)\, d\nu(x) < 0.$$

The compactness of the set of ν satisfying $\nu \leq m|\mathrm{Q}$ and the openness of the dominated splitting condition give:

Lemma **2.1.** — *For every* $\varepsilon > 0$, *the set* $\mathcal{G}_\varepsilon(\mathrm{Q}, d_1, d_2, d_3)$ *is open in* $\mathrm{Diff}^1(\mathrm{M})$.

Proof. — Consider (g_n) converging to g in $\mathrm{Diff}^1(\mathrm{M})$ and assume $g_n \notin \mathcal{G}_\varepsilon(\mathrm{Q}, d_1, d_2, d_3)$. We have to prove that $g \notin \mathcal{G}_\varepsilon(\mathrm{Q}, d_1, d_2, d_3)$. For the sake of contradiction, by Section 2.1.2 it suffices to assume that the $g_n|\mathrm{K}(g_n, \mathrm{Q})$ admit dominated splittings of type (d_1, d_2, d_3). Let $\nu_n \leq m|\mathrm{Q}$ be a sequence of g_n-invariant measures satisfying $\nu_n(\mathrm{M}) \geq \varepsilon$ and $\int \mathrm{Jac}_{\mathrm{E}_2(g_n,x)}(g_n, x)\, d\nu_n(x) \geq 0$. Let ν be a weak-$*$ limit of ν_n. Then $\nu \leq m|\mathrm{Q}$ is g-invariant and satisfies $\nu(\mathrm{M}) \geq \varepsilon$ and $\int \mathrm{Jac}_{\mathrm{E}_2(g,x)}\, d\nu(x) \geq 0$. Hence $g \notin \mathcal{G}_\varepsilon(\mathrm{Q}, d_1, d_2, d_3)$. □

2.2. *Oseledets blocks.* — For $f \in \mathrm{Diff}^1_m(\mathrm{M})$, the set of regular points X_f splits into f-invariant measurable subsets $\mathrm{X}_f(d_1, d_2, d_3)$, $d_1 + d_2 + d_3 = d$ and $d_* \geq 0$, defined as the set of points admitting d_1 positive, d_2 zero and d_3 negative Lyapunov exponents (counted with multiplicity). Note that:

- $\mathrm{X}_f(0, d, 0)$ is the set of points whose Lyapunov exponents are all zero;
- the set of *non-uniformly hyperbolic points*, denoted by Nuh_f is the union of the sets $\mathrm{X}(d_1, 0, d_3)$, with $d_1, d_3 > 0$;
- by volume preservation, the other non-empty sets satisfy $d_1, d_2, d_3 > 0$.

2.2.1. *Domination.* — Oseledets and dominated splittings coincide generically.

Theorem **2.2** *(Bochi–Viana [BV2]). — For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1_m(\mathrm{M})$ *and for any $\varepsilon > 0$, for each Oseledets block $\mathrm{X}_f(d_1, d_2, d_3)$ there exists an f-invariant compact set* K *satisfying:*

- *$f|\mathrm{K}$ admits a dominated splitting of type (d_1, d_2, d_3),*
- *$m(\mathrm{X}_f(d_1, d_2, d_3) \setminus \mathrm{K}) \leq \varepsilon$.*

In the previous theorem, the set K is not necessarily contained in $\mathrm{X}_f(d_1, d_2, d_3)$.

2.2.2. *The non-uniformly hyperbolic set.* — Generically the non-uniformly hyperbolic set Nuh_f coincides m-almost everywhere with a single Oseledets block.

Theorem **2.3** *(Avila–Bochi [AB], Theorem A). — For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1_m(\mathrm{M})$*, either $m(\mathrm{Nuh}_f) = 0$ or Nuh_f is dense in* M *and the restriction $m|\mathrm{Nuh}_f$ is ergodic.*

2.2.3. *The set where all exponents vanish.* — As a consequence we get (see also [AB], Corollary 1.1):

Corollary **2.4.** — *For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1_m(\mathrm{M})$*, if $m(\mathrm{Nuh}_f) > 0$, then there exists a global dominated splitting $\mathrm{TM} = \mathrm{E} \oplus \mathrm{F}$ on* M *such that for m-almost every point $x \in \mathrm{Nuh}_f$,*

$$v \in \mathrm{E}(x) \setminus \{0\} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n} \log \left\| \mathrm{D}_x f^n(v) \right\| > 0,$$

and

$$v \in \mathrm{F}(x) \setminus \{0\} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n} \log \left\| \mathrm{D}_x f^n(v) \right\| < 0.$$

In particular, $\mathrm{X}_f(0, d, 0) = \emptyset$.

Proof. — Theorem 2.3 implies that, C^1-generically, if Nuh_f has positive volume, then it is dense in M, the restriction of m is ergodic and it coincides with a set $X(d_1, 0, d_3)$. Suppose then that $m(\mathrm{Nuh}_f) > 0$, and let $\varepsilon = m(\mathrm{Nuh}_f)/2$. By Theorem 2.2, there exists an invariant compact set K with $m(\mathrm{Nuh}_f \setminus K) < \varepsilon$ that admits a non-trivial dominated splitting. In particular, $m(\mathrm{Nuh}_f \cap K) > 0$; since $m|\,\mathrm{Nuh}_f$ is ergodic, this implies that $m(\mathrm{Nuh}_f \setminus K) = 0$. This proves that the compact set K contains m-almost every point of Nuh_f, and hence coincides with M, since Nuh_f is dense in M. We have thus proved that M has a non-trivial dominated splitting, and so the set $X_f(0, d, 0)$ is empty. □

2.2.4. *The other Oseledets blocks.* — Using Theorem C we get:

Corollary **2.5.** — *For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1_m(M)$*, the Oseledets blocks* $X_f(d_1, d_2, d_3)$ *with* $d_1, d_2, d_3 > 0$ *have volume zero.*

Proof. — Let $\mathcal{K}$ be a countable family of compact sets of M such that for any $K \subset U \subset M$, with K compact and U open, there exists $Q \in \mathcal{K}$ satisfying $K \subset Q \subset U$. By Lemma 2.1, one can assume that for any $Q \in \mathcal{K}$, any $\varepsilon > 0$ such that $1/\varepsilon \in \mathbf{N}$, and any type (d_1, d_2, d_3), the diffeomorphism f either belongs to $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$ or to $\mathrm{Diff}^1_{\mathrm{vol}}(M) \setminus \overline{\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)}$.

Case 1. The case Nuh_f *has zero volume.* — We prove by increasing induction on $d_2 + d_3$ that $X_f(d_1, d_2, d_3)$ has volume zero, for each triple (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = d$ and $d_1, d_2, d_3 > 0$. We thus fix (d_1, d_2, d_3) and assume that $m(X_f(d_1', d_2', d_3')) = 0$ for each triple (d_1', d_2', d_3') such that $d_2' + d_3' < d_2 + d_3$ and $d_1', d_2', d_3' > 0$.

Claim. — *For any set* $X_f(d_1', d_2', d_3')$ *with positive volume, one has* $d_2' + d_3' \geq d_2 + d_3$.

Proof. — We consider separately the three possible cases:

- $(d_1', d_2', d_3') = (0, d, 0)$: the claim holds trivially,
- d_1', d_2', d_3' are all nonzero: our inductive assumption implies the claim,
- $d_2' = 0$: this does not occur since Nuh_f has zero volume. □

We fix $\varepsilon > 0$ with $1/\varepsilon \in \mathbf{N}$. By Theorem 2.2 there exists an invariant compact set K (possibly empty) such that $m(X_f(d_1, d_2, d_3) \setminus K)$ is smaller than ε and such that $f|K$ admits a dominated splitting $E_1 \oplus E_2 \oplus E_3$ of type (d_1, d_2, d_3).

Almost every point $x \in K$ belongs to a set $X_f(d_1', d_2', d_3')$ with positive volume. By the claim above, $d_2' + d_3' \geq d_2 + d_3$. As a consequence $E_2(f, x)$ is contained in the sum of the central and the stable spaces of the Oseledets decomposition at x. This implies $\Delta_{E_2(f,x)}(f, x) \leq 0$.

We have proved that the assumptions of Theorem C are satisfied. We choose a small neighborhood $Q \in \mathcal{K}$ of K. There exists g arbitrarily close to f in $\mathrm{Diff}^1_{\mathrm{vol}}(M)$

such that for every invariant measure $\nu \leq m|Q$ such that $\nu(M) \geq \varepsilon$, one has $\int_X \log \mathrm{Jac}_{E_2(g,x)} \, d\nu(x) < 0$. In particular g belongs to $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$, and hence f does as well (recall that f belongs to the union of the open sets $\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)$ and $\mathrm{Diff}^1_{\mathrm{vol}}(M) \setminus \overline{\mathcal{G}_\varepsilon(Q, d_1, d_2, d_3)}$). It follows that $X_f(d_1, d_2, d_3) \cap K$ has volume smaller than ε. With our choice of K, this proves $m(X_f(d_1, d_2, d_3)) \leq 2\varepsilon$. Since $\varepsilon > 0$ has been arbitrarily chosen we get $m(X_f(d_1, d_2, d_3)) = 0$, as desired. The induction on $d_2 + d_3$ in $\{1, \dots, d-1\}$ concludes the proof in this case.

Case 2. The case Nuh_f *has positive volume.* — In the case Nuh_f has positive volume, we modify the previous argument. By Theorem 2.3, there exists d_+, d_- such that Nuh_f and $X_f(d_+, 0, d_-)$ coincide up to a set of volume zero and by Corollary 2.4 there exists a global domination $TM = E \oplus F$ with $\dim(E) = d_+$.

Claim. — *If* $d_2 + d_3 \leq d_-$, *then for any set* $X_f(d_1', d_2', d_3')$ *with positive volume, one has* $d_2' + d_3' \geq d_2 + d_3$.

Proof. — One considers the three possible case:

- $(d_1', d_2', d_3') = (0, d, 0)$: the claim holds trivially,
- d_1', d_2', d_3' are all nonzero: our inductive assumption implies the claim,
- $d_2' = 0$: this implies $X_f(d_1', d_2', d_3') = \mathrm{Nuh}_f$; hence $d_2' + d_3' = d_- \geq d_2 + d_3$. □

The induction of case 1 can thus be repeated while the condition $d_2 + d_3 \leq d_-$ of the claim holds. This proves that the Oseledets blocks $X(d_1, d_2, d_3)$ with $d_1, d_2, d_3 > 0$ and $d_2 + d_3 \leq d_-$ have measure zero.

Replacing f by f^{-1}, one gets the same conclusion for the blocks $X(d_1, d_2, d_3)$ with $d_1, d_2, d_3 > 0$ and $d_1 + d_2 \leq d_+$, i.e. such that $d_- \leq d_3$. This completes the proof in this second case. □

2.3. *Proof of Theorem B.* — Theorem 2.3 and Corollaries 2.4 and 2.5 now imply Theorem B.

3. Local perturbations of center exponents

This section is devoted to the proof of the following, which implies Theorem C.

Theorem **C'.** — *Let* $f \in \mathrm{Diff}^1_{\mathrm{vol}}(M)$, *and let* K *be an* f*-invariant compact set admitting a dominated splitting* $T_K M = E_1 \oplus E_2 \oplus E_3$ *into three non-trivial subbundles. Then for any* $\alpha > 0$ *small and for any neighborhood* $\mathcal{U} \subset \mathrm{Diff}^1_{\mathrm{vol}}(M)$ *of the identity, there exists* $\delta > 0$ *such that for any* $\eta > 0$, *there exists* $n_0 \geq 1$ *satisfying the following property.*

For any $n \geq n_0$, *any compact neighborhood* Q *of* K *and any* $\chi > 0$, *there exist a smooth diffeomorphism* $\varphi \in \mathcal{U}$, *and a measurable subset* $\Lambda \subset Q$ *such that:*

- *φ is supported on Q and is χ-close to the identity in the C^0 topology,*
- *$m(K \setminus \Lambda) < \eta$,*
- *the diffeomorphism $g = f \circ \varphi$ satisfies*

$$\frac{1}{n}\log \mathrm{Jac}_F\big(g^n, y\big) \le \frac{1}{n}\log \mathrm{Jac}_{E_2(f,y)}\big(f^n, y\big) - \delta, \tag{4}$$

for every $y \in \Lambda$ such that $y, g^n(y) \in K$, and every subspace $F \subset T_yM$ such that F is α-close to $E_2(f, y)$ and $Dg^n(y) \cdot F$ is α-close to $E_2(f, g^n(y))$.

Proof of Theorem C from Theorem C'. — Consider f, K, ε as in the statement of Theorem C and small neighborhoods $\mathcal{V} \subset \mathrm{Diff}^1_{\mathrm{vol}}(M)$ of f and $Q \subset M$ of K such that the maximal invariant set $K(g, Q)$ for any $g \in \mathcal{V}$ still has a dominated splitting that extends the splitting $T_KM = E_1 \oplus E_2 \oplus E_3$ on K. We construct g satisfying the conclusion of the Theorem C.

Let C_0 be an upper bound for $d \log \|Dg(x)\|$, where $x \in M$, $g \in \mathcal{V}$. Fix $\alpha > 0$ small. Reducing $\mathcal{V}$, Q if necessary, for any point $x \in K(g, Q) \cap K$ the spaces $E_2(f, x)$ and $E_2(g, x)$ are α-close. Theorem C' applied to α, $\mathcal{V}$, gives δ. One then chooses $\eta > 0$ smaller than $\min(\varepsilon/10, \delta\varepsilon/100C_0)$ and Theorem C' gives n_0. We also take $\kappa > 0$ smaller than $\min(\varepsilon/10, \delta\varepsilon/100C_0)$.

We choose $n \ge n_0$ and define the compact set

$$\Omega = \left\{x \in K,\ \frac{1}{n}\log \mathrm{Jac}_{E_2(f,x)}\big(f^n, x\big) \le \delta/2\right\}.$$

If n is large enough, $K \setminus \Omega$ has measure less than κ. For $\chi > 0$ sufficiently small, shrinking if necessary the neighborhood Q, for any g such that $g \circ f^{-1}$ is χ-close to the identity in the C^0 topology, we have:

$$m\big(K \setminus g^{-n}(K)\big) \le \kappa, \qquad m\big(K(g, Q) \setminus K\big) \le \kappa.$$

Theorem C' provides us with a diffeomorphism $g \in \mathcal{V}$ and a set Λ such that for every $x \in K(g, Q) \cap K \cap \Lambda \cap \Omega \cap g^{-n}(K)$ one has

$$\frac{1}{n}\log \mathrm{Jac}_{E_2(g,x)}\big(g^n, x\big) \le \frac{1}{n}\log \mathrm{Jac}_{E_2(f,x)}\big(f^n, x\big) - \delta \le -\delta/2.$$

Moreover the complement of the set $Z := K(g, Q) \cap K \cap \Lambda \cap \Omega \cap g^{-n}(K)$ in $K(g, Q)$ has volume smaller than $3\kappa + \eta$.

If $\nu \le m|Q$ is a g-invariant measure with $\nu(M) \ge \varepsilon$, then $\nu(Z) \ge \varepsilon - 3\kappa - \eta \ge \varepsilon/2$. Thus

$$\int \log \mathrm{Jac}_{E_2(g,x)}(g, x)\, d\nu(x) = \int \frac{1}{n}\log \mathrm{Jac}_{E_2(g,x)}\big(g^n, x\big)\, d\nu(x)$$

$$\leq C_0\nu(M \setminus Z) - \frac{\delta}{2}\nu(Z)$$

$$< C_0(3\kappa + \eta) - \frac{\delta\varepsilon}{4} < 0.$$

The result follows. □

The construction of the perturbation in Theorem C' follows three natural steps, and will occupy the remainder of this section.

3.1. *Infinitesimal.* — Let $\mathbf{R}^d = E^+ \oplus E^0 \oplus E^-$ be an orthogonal decomposition, and set $d_0 = \dim(E^0)$. Let $G \subset \mathbf{R}^d$ be a two-dimensional subspace that intersects both E^0 and E^- in one-dimensional subspaces, endowed with an arbitrary orientation. For a subspace $F \subset \mathbf{R}^d$, we let $F^\perp$ denote its orthogonal complement, and we let $P_F : \mathbf{R}^d \to F$ be the projection with kernel $F^\perp$. For $\theta \in \mathbf{R}$, let $R_\theta : \mathbf{R}^d \to \mathbf{R}^d$ be the orthogonal operator that is the identity on $G^\perp$ and that restricted to G is a rotation of angle $2\pi\theta$ (measured according to the chosen orientation).

Elementary perturbation. — We introduce a diffeomorphism ψ^ε which will be used at different places for the perturbation. Let $\alpha : \mathbf{R}^d \to \mathbf{R}$ be a smooth function with the following properties:

- $\alpha(x) = 0$ for x in the complement of the unit ball $B := \{x, \|x\| \geq 1\}$,
- $\alpha(x) = 1$ for $\|x\| \leq 1/2$,
- $\|\alpha\|_{C^0} \leq 1$,
- $\alpha(R_\theta \cdot x) = \alpha(x)$ for every $\theta \in \mathbf{R}$ and $x \in \mathbf{R}^d$.

Given $\varepsilon > 0$, let $\psi^\varepsilon : \mathbf{R}^d \to \mathbf{R}^d$ be defined by $\psi^\varepsilon(x) = R_{\varepsilon\alpha(x)} \cdot x$. It is a smooth, volume-preserving diffeomorphism of $\mathbf{R}^d$ and is the identity outside the unit ball. See Figure 1. We have $\|\psi^\varepsilon - \mathrm{Id}\|_{C^1} \leq \kappa\varepsilon$ for some constant $\kappa > 0$.

Let μ_ε be a probability measure in $SL(d, \mathbf{R})$ given by the push-forward under $x \mapsto D\psi^\varepsilon(x)$ of normalized Lebesgue measure m on the unit ball. Note that for every $A \in \operatorname{supp}\mu_\varepsilon$, we have $A \cdot (E^0 + G) = (E^0 + G)$. We set

$$c(\varepsilon) = -\int \log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A)\, d\mu_\varepsilon(A). \tag{5}$$

Taking $\varepsilon > 0$ small enough, the $A \in \operatorname{supp}\mu_\varepsilon$ are close enough to the identity so that the $\log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A)$ are uniformly bounded. Consequently, $c(\varepsilon)$ is finite.

We describe the effect of an elementary perturbation averaged on the unit ball.

Lemma **3.1.** — *For every $\varepsilon > 0$ sufficiently small, we have $c(\varepsilon) > 0$.*

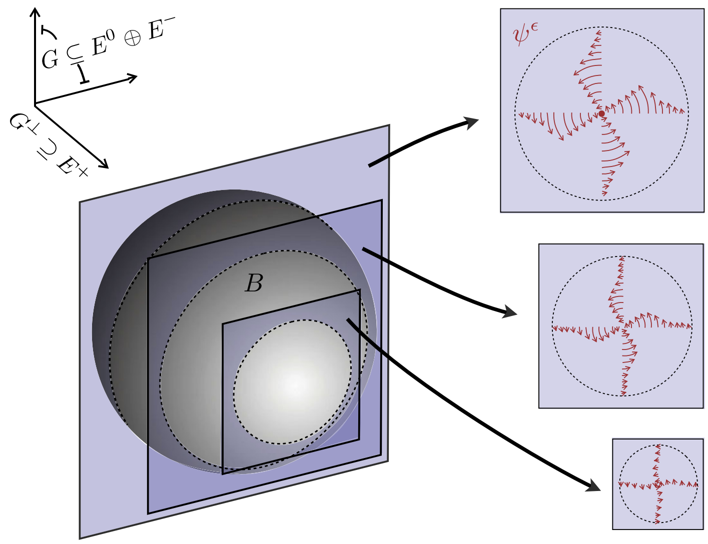

FIG. 1. — The map ψ^ε

Proof. — Observe that for any $x_0 \in G^\perp$, $x \mapsto P_{E^0} \cdot \psi^\varepsilon(x_0 + x)$ defines a diffeomorphism of G that is the identity outside the ball of radius $\max(0, (1 - |x_0|^2)^{1/2})$. In particular, Fubini's theorem implies

$$\text{(6)} \qquad \int_{SL(d,\mathbf{R})} \mathrm{Jac}_{E^0}(P_{E^0} \cdot A) d\mu_\varepsilon(A) = \int_B \mathrm{Jac}_{E^0}\big(P_{E^0} \cdot D\psi^\varepsilon(z)\big) dm(z)$$
$$= \int_{G^\perp \int_G \mathrm{Jac}_{E^0}(P_{E^0} \cdot D\psi^\varepsilon(x_0+x))\, dx\, dx_0 = 1.}$$

Observe also that for $|x| < 1/2$ we have $\mathrm{Jac}_{E^0}(P_{E^0} \cdot D\psi^\varepsilon(x)) = \cos(2\pi\varepsilon) < 1$. Thus $c(\varepsilon) > 0$ follows from Jensen's inequality:

$$-\int \log \mathrm{Jac}_{E^0}(P_{E^0} \cdot A) d\mu_\varepsilon(A) > -\log\Big(\int_{SL(d,\mathbf{R})} \mathrm{Jac}_{E^0}(P_{E^0} \cdot A) d\mu_\varepsilon(A)\Big)$$
$$= 0. \qquad \square$$

Random composition of elementary perturbations. — By the Law of Large Numbers, the effect of an elementary perturbation composed along most random sequences of points of the unit ball is the same as the average effect of a single elementary perturbation.

Proposition **3.2.** — *If* $\varepsilon > 0$ *is small, there exists* $\lambda \in (0, 1/4)$ *such that for every* $\theta > 0$ *there exist* $R_0 \in \mathbf{N}$ *and for each* $R \geq R_0$ *a compact set* $W_R \subset SL(d, \mathbf{R})^R$ *with* $\mu_\varepsilon^{\otimes R}(SL(d, \mathbf{R})^R \setminus W_R) < \theta$ *with the following property. Let* $R \geq R_0$ *and let* $L_j : \mathbf{R}^d \to \mathbf{R}^d$, $0 \leq j \leq R-1$, *be invertible linear* *operators preserving* E^+, E^0 *and* E^- *such that*

$$\|L_j|E^0\| \cdot \|L_j^{-1}|E^+\| \leq \lambda \quad \textit{and} \quad \|L_j|E^-\| \cdot \|L_j^{-1}|E^0\| \leq \lambda.$$

Then

$$\log \operatorname{Jac}_F\big((L_{R-1} \cdot A_{R-1}) \cdots (L_1 \cdot A_1) \cdot (L_0 \cdot A_0)\big) < \sum_{j=0}^{R-1} \log \operatorname{Jac}_{E^0}(L_j) - \frac{c(\varepsilon)}{2} R,$$

for every $(A_0, \dots A_{R-1}) \in W_R$ *and for every* d_0*-dimensional subspace* F *such that* $\|P_{E^-}|F\| \leq 1/2$ *and* $\|P_{E^+}|(L_{R-1} \cdot A_{R-1} \cdots L_0 \cdot A_0) \cdot F\| \leq 1/2$.

The proof will use the following lemma about dominated splittings.

Lemma **3.3.** — *There exists* $C > 0$ *such that if* $\varepsilon > 0$ *is sufficiently small, then the following holds. Let* $L : \mathbf{R}^d \to \mathbf{R}^d$ *be an invertible linear operator that preserves each of* E^+, E^0 *and* E^-, *and assume that for some* $\lambda \in (0, 1/4)$ *we have*

$$\|L|E^0\| \cdot \|L^{-1}|E^+\| \leq \lambda \quad \textit{and} \quad \|L|E^-\| \cdot \|L^{-1}|E^0\| \leq \lambda. \tag{7}$$

Let $A \in \operatorname{supp} \mu_\varepsilon$ *and let* $F \subset \mathbf{R}^d$ *be a* d_0*-dimensional subspace. Then* (7) *implies:*

1. *if* $\|P_{E^-}|F\| \leq 1/2$ *then* $\|P_{E^-}|(L \cdot A) \cdot F\| \leq \lambda$*;*
2. *if* $\|P_{E^+}|(L \cdot A) \cdot F\| \leq 1/2$ *then* $\|P_{E^+}|F\| \leq \lambda$*; and*
3. *if* $\|P_{E^-}|F\|, \|P_{E^+}|(L \cdot A) \cdot F\| \leq \gamma$, *for some* $\gamma \in (0, 1/2)$, *then*

$$\log \operatorname{Jac}_F(L \cdot A) < \log \operatorname{Jac}_{E^0}(L) + \log \operatorname{Jac}_{E^0}(P_{E^0} \cdot A) + C(\lambda + \gamma).$$

Proof. — If $v \in \mathbf{R}^d$ is a unit vector with $\|P_{E^-} \cdot v\|^2 \leq 1/2$, then $\|P_{E^-} \cdot v\| \leq \|P_{E^+ \oplus E^-} \cdot v\|$. With (7) this gives

$$\begin{aligned}\|(P_{E^-} \cdot L) \cdot v\| = \|(L \cdot P_{E^-}) \cdot v\| &\leq \lambda \, \|(L \cdot P_{E^+ \oplus E^0}) \cdot v\| \\ &= \lambda \, \|(P_{E^+ \oplus E^0} \cdot L) \cdot v\|.\end{aligned}$$

Since $\varepsilon > 0$ is small, $\|P_{E^-}|F\| \leq 1/2$ implies $\|P_{E^-}|A \cdot F\|^2 \leq 1/2$. The first estimate follows.

Symmetrically if $v \in \mathbf{R}^d$ is a unit vector with $\|P_{E^+} \cdot v\|^2 \leq 1/2$, then

$$\begin{aligned}\|\big(P_{E^+} \cdot L^{-1}\big) \cdot v\| = \|\big(L^{-1} \cdot P_{E^+}\big) \cdot v\| &\leq \lambda \, \|\big(L^{-1} \cdot P_{E^0 \oplus E^-}\big) \cdot v\| \\ &= \lambda \, \|(P_{E^0 \oplus E^-} \cdot L) \cdot v\|.\end{aligned}$$

Since $\varepsilon > 0$ is small, $\|P_{E^+}|(L \cdot A) \cdot F\| \leq 1/2$ implies $\|P_{E^+}|L \cdot F\|^2 \leq 1/2$. The second estimate follows.

For any unit vector $v \in \mathbf{R}^d$ such that $\|P_{E^-} \cdot v\|^2 \leq 1/2$ and $\|P_{E^+} \cdot L \cdot v\| \leq \gamma \|L \cdot v\|$,

$$\begin{aligned}\big\|L \cdot v - (P_{E^0} \cdot L) \cdot v\big\| &\leq \big\|(P_{E^+} \cdot L) \cdot v\big\| + \big\|(P_{E^-} \cdot L) \cdot v\big\| \\ &\leq \gamma \|L \cdot v\| + \lambda \big\|(P_{E^+ \oplus E^0} \cdot L) \cdot v\big\| \\ &\leq (\gamma + \lambda)\|L \cdot v\|.\end{aligned}$$

Thus if $F \subset \mathbf{R}^d$ satisfies $\|P_{E^-}|F\| \leq 1/2$ (and hence $\|P_{E^-}|A \cdot F\|^2 \leq 1/2$) and $\|P_{E^+}|(L \cdot A) \cdot F\| \leq \gamma$, we can write $L|A \cdot F$ as $S_F \cdot L \cdot (P_{E^0}|A \cdot F)$, where $S_F : E^0 \to \mathbf{R}^d$ is a linear map with $\|S_F\| \leq (1 - \gamma - \lambda)^{-1}$. We conclude that

$$\log \operatorname{Jac}_F(L \cdot A) \leq -d_0 \log(1 - \gamma - \lambda) + \log \operatorname{Jac}_{E^0}(L) + \log \operatorname{Jac}_F(P_{E^0} \cdot A).$$

On the other hand, the function $\log \operatorname{Jac}_F(P_{E^0} \cdot A)$ is uniformly (on $A \in \operatorname{supp} \mu_\varepsilon$) Lipschitz as a function of those F satisfying $\|P_{E^+ \oplus E^-}|A \cdot F\| \leq 1/2$. Thus

$$\big|\log \operatorname{Jac}_F(P_{E^0} \cdot A) - \log \operatorname{Jac}_{E^0}(P_{E^0} \cdot A)\big| \leq C_0 \|P_{E^+ \oplus E^-}|F\|,$$

for some $C_0 > 0$. Since $\|P_{E^+ \oplus E^-}|F\| \leq \|P_{E^-}|F\| + \|P_{E^+}|F\| \leq \gamma + \lambda$, the third estimate follows. □

Proof of Proposition 3.2. — Define F_j, $0 \leq j \leq R$ by $F_0 = F$, $F_{j+1} = L_j \cdot A_j \cdot F$. First notice $\|P_{E^+}|F_R\| \leq 1/2$ and $\|P_{E^-}|F_0\| \leq 1/2$ imply, by iterated application of estimates (1–2) in the previous lemma, that $\|P_{E^+}|F_j\| \leq \lambda$ for $0 \leq j \leq R - 1$, while $\|P_{E^-}|F_j\| \leq \lambda$ for $1 \leq j \leq R$. By item (3) in Lemma 3.3 we get that $\log \operatorname{Jac}_{F_j}(L_j \cdot A_j) - (\log \operatorname{Jac}_{E^0}(L_j) + \log \operatorname{Jac}_{E^0}(P_{E^0} \cdot A_j))$ is at most $2C\lambda$ if $1 \leq j \leq R - 2$, and at most $C\lambda + \frac{C}{2}$ for $j = 0$ or $j = R - 1$. It follows that

$$\begin{aligned}&\log \operatorname{Jac}_F\big((L_{R-1} \cdot A_{R-1}) \cdots (L_0 \cdot A_0)\big) \\ &\quad \leq \sum_{j=0}^{R-1} \log \operatorname{Jac}_{E^0}(L_j) + \sum_{j=0}^{R-1} \operatorname{Jac}_{E^0}(P_{E^0} \cdot A_j) + 2CR\lambda + C.\end{aligned}$$

If $0 < \lambda \leq (10C)^{-1} c(\varepsilon)$ and $R \geq 10C c(\varepsilon)^{-1}$, this gives

$$\begin{aligned}&\log \operatorname{Jac}_F\big((L_{R-1} \cdot A_{R-1}) \cdots (L_0 \cdot A_0)\big) \\ &\quad \leq \sum_{j=0}^{R-1} \log \operatorname{Jac}_{E^0}(L_j) + \sum_{j=0}^{R-1} \operatorname{Jac}_{E^0}(P_{E^0} \cdot A_j) + \frac{3c(\varepsilon)}{10} R.\end{aligned}$$

Recalling the definition (5) of $c(\varepsilon)$, the Law of Large Numbers implies that for every $\theta > 0$, if R is sufficiently large, the probability, with respect to $\mu_\varepsilon^{\otimes R}$, that

$$\frac{1}{R} \sum_{j=0}^{R-1} \operatorname{Jac}_{E^0}(P_{E^0} \cdot A_j) \geq -\frac{4c(\varepsilon)}{5}$$

is less than θ. The result follows. □

3.2. *Local.* — In the second step, we explain how to perturb along an orbit.

Proposition **3.4.** — *If* $\varepsilon > 0$ *is small, there exists* $\lambda \in (0, 1/4)$ *such that for every* $\theta > 0$ *there exists* $R_0 \in \mathbf{N}$ *with the following property. Let* $R \geq R_0$, $N \geq R$, *and let* $f_j : (\mathbf{R}^d, 0) \to (\mathbf{R}^d, 0)$, $0 \leq j \leq N-1$, *be germs of volume-preserving diffeomorphisms such that the* $L_j = Df_j(0)$ *preserve* E^+, E^0 *and* E^-, *and such that*

$$\|L_j|E^0\| \cdot \|L_j^{-1}|E^+\| \leq \lambda \quad \textit{and} \quad \|L_j|E^-\| \cdot \|L_j^{-1}|E^0\| \leq \lambda.$$

Then for every small neighborhood U *of* $0 \in \mathbf{R}^d$, *and* $0 \leq j \leq N-1$, *there exist measurable subsets* Z_j *of* $U_j := f_{j-1} \circ \cdots \circ f_0(U)$, *smooth volume-preserving diffeomorphisms* $\varphi_j : \mathbf{R}^d \to \mathbf{R}^d$ *and perturbations* $\tilde{f}_j := f_j \circ \varphi_j$ *such that:*

- $m(Z_j) \geq (1-2\theta)m(U_j)$,
- φ_j *coincides with* Id *outside* U_j *and* $D\varphi_j(x) \in \operatorname{supp} \mu_\varepsilon$ *for every* $x \in \mathbf{R}^d$,
- *for any* $0 \leq j \leq N-R$, *any* $y \in Z_j$ *and any* d_0*-dimensional space* F *satisfying* $\|P_{E^-}|F\| \leq 1/3$ *and* $\|P_{E^+}|D(\tilde{f}_{j+R-1} \circ \cdots \circ \tilde{f}_j)(y) \cdot F\| \leq 1/3$, *we have:*

$$\log \operatorname{Jac}_F(\tilde{f}_{j+R-1} \circ \cdots \circ \tilde{f}_j, y) \leq \operatorname{Jac}_{E^0}(L_{j+R-1} \circ \cdots \circ L_j) - \frac{c(\varepsilon)}{3}R.$$

The proof of Proposition 3.4 uses the following lemma, which allows us to construct a sequence of perturbations along an orbit that act like random perturbations.

Lemma **3.5.** — *Consider a sequence* $f_j : U_j \to U_{j+1}$, $0 \leq j \leq N-1$, *of* C^1 *volume-preserving diffeomorphisms between bounded open sets of* $\mathbf{R}^d$ *and* $f^j = f_{j-1} \circ \cdots \circ f_0$. *Let* ψ_j *be volume-preserving diffeomorphisms of* $\mathbf{R}^d$ *supported on the unit ball* B. *Let* μ_j *be the push-forward of normalized Lebesgue measure* m *on* B *under the map*

$$B \ni x \mapsto D\psi_j(x) \in SL(d, \mathbf{R}).$$

Then for any $\chi > 0$ *there exist orientation- and volume-preserving diffeomorphisms* φ_j *of* $\mathbf{R}^d$ *such that, setting* $\tilde{f}_j = f_j \circ \varphi_j$ *and* $\tilde{f}^j = \tilde{f}_{j-1} \circ \cdots \circ \tilde{f}_0$, *we have:*

1. *for* $0 \leq j \leq N-1$, *the diffeomorphism* φ_j *is* χ*-close to the identity in the* C^0*-distance, equals* Id *outside* U_j, *and satisfies* $D\varphi_j(x) \in \operatorname{supp} \mu_j$ *for each* $x \in \mathbf{R}^d$;
2. *the push-forward of normalized Lebesgue measure* m *on* U_0 *under the map*

$$U_0 \ni x \mapsto \big(D\varphi_j\big(\tilde{f}^j(x)\big)\big)_{j=0}^{N-1} \in SL(d, \mathbf{R})^N$$

is arbitrarily close to $\mu_0 \otimes \cdots \otimes \mu_{N-1}$.

Proof. — The proof is by induction on N. For $N = 0$ there is nothing to do. Assume it holds for $N-1$, and apply the result for the sequence $(f_j)_{0 \leq j \leq N-2}$, yielding the sequence

$(\varphi_j)_{0\le j\le N-2}$. Define $\tilde{f}_j$ and $\tilde{f}^j$ as before, and let ν_{N-1} be the push-forward of normalized Lebesgue measure on U_0 under the map

$$H_{N-1} : U_0 \ni x \mapsto \left(D\varphi_j\left(\tilde{f}^j(x)\right)\right)_{j=0}^{N-2} \in SL(d,\mathbf{R})^{N-1},$$

so that ν_{N-1} is arbitrarily close to $\mu_0 \otimes \cdots \otimes \mu_{N-2}$.

For $n \in \mathbf{N}$, let $\{D_\ell^n\}_\ell$ be a finite family of disjoint closed balls in U_{N-1} chosen using the Vitali lemma such that:

- $\operatorname{diam}(D_\ell^n) < n^{-1}$;
- defining $\hat{D}_\ell^n \subset U_0$ by $D_\ell^n = \tilde{f}^{N-1}(\hat{D}_\ell^n)$, we have: $\sum_\ell m(\hat{D}_\ell^n) \ge (1-n^{-1})\, m(U_0)$;
- if $x, y \in \hat{D}_\ell^n$ then $\|H_{N-1}(x) - H_{N-1}(y)\| \le n^{-1}$.

Let $\xi_{n,\ell}$ be the conformal affine dilation that sends B into D_ℓ^n. Define $\varphi_{N-1,n}$ to be the identity outside $\bigcup_\ell D_\ell^n$ and by

$$\varphi_{N-1,n}(x) = \xi_{n,\ell}\psi_{N-1}\left(\xi_{n,\ell}^{-1}x\right), \qquad x \in D_\ell^n.$$

Let $\nu_{N,n}$ be the push-forward of normalized Lebesgue measure on U_0 under

$$H_{N,n} : U_0 \ni x \mapsto \left(H_{N-1}(x), D\varphi_{N-1,n}\left(\tilde{f}^{N-1}(x)\right)\right) \in SL(d,\mathbf{R})^N.$$

The properties of the first item are immediate. For instance $\operatorname{diam}(D_\ell^n) < n^{-1}$ above implies that, for n large enough, $\varphi_{N-1,n}$ is C^0-close to the identity.

Since ν_{N-1} is close to $\mu_0 \otimes \cdots \otimes \mu_{N-2}$, it is enough to show that $\lim_{n\to\infty} \nu_{N,n} = \nu_{N-1} \otimes \mu_{N-1}$ to establish the second item. Equivalently, we must show that for a dense subset of compactly supported, continuous functions $\rho : SL(d,\mathbf{R})^N \to \mathbf{R}$, we have

$$\lim_{n\to\infty} \int \rho \, d\nu_{N,n} = \int \rho \, d\nu_{N-1} \otimes d\mu_{N-1}. \tag{8}$$

Take ρ to be Lipschitz with constant C_ρ. Since $\operatorname{diam}(H_{N-1}(\hat{D}_\ell^n)) \le n^{-1}$, the quantities

$$\frac{1}{m(\hat{D}_\ell^n)} \int_{\hat{D}_\ell^n} \rho\left(H_{N,n}(x)\right) dx,$$

and

$$\frac{1}{m(\hat{D}_\ell^n)^2} \int_{\hat{D}_\ell^n} \int_{\hat{D}_\ell^n} \rho\left(H_{N-1}(x), D\varphi_{N-1,n}\left(\tilde{f}^{N-1}(y)\right)\right) dx\, dy$$

differ by at most $C_\rho n^{-1}$. By construction, for any $x \in \hat{D}_\ell^n$ we have

$$\frac{1}{m(\hat{D}_\ell^n)} \int_{\hat{D}_\ell^n} \rho\left(H_{N-1}(x), D\varphi_{N-1}\left(\tilde{f}^{N-1}(y)\right)\right) dy$$

$$= \int_{\mathrm{SL}(d,\mathbf{R})} \rho\big(\mathrm{H}_{\mathrm{N}-1}(x), z\big)\, d\mu_{\mathrm{N}-1}(z),$$

so that

$$\left|\int_{\bigcup_\ell \hat{\mathrm{D}}_\ell^n} \rho\big(\mathrm{H}_{\mathrm{N},n}(x)\big)\, dx - \int\int_{\bigcup_\ell \hat{\mathrm{D}}_\ell^n} \rho\big(\mathrm{H}_{\mathrm{N}-1}(x), z\big)\, dx\, d\mu_{\mathrm{N}-1}(z)\right| \leq \mathrm{C}_\rho m\Big(\bigcup_\ell \hat{\mathrm{D}}_\ell^n\Big) n^{-1}. \tag{9}$$

Clearly

$$\left|\int \rho\, d\nu_{\mathrm{N},n} - \frac{1}{m(\mathrm{U}_0)}\int_{\bigcup_\ell \hat{\mathrm{D}}_\ell^n} \rho\big(\mathrm{H}_{\mathrm{N},n}(x)\big)\, dx\right| \leq \|\rho\|_\infty n^{-1} \quad \text{and}$$

$$\left|\int \rho\, d\nu_{\mathrm{N}-1}\otimes d\mu_{\mathrm{N}-1} - \frac{1}{m(\mathrm{U}_0)}\int\int_{\bigcup_l \hat{\mathrm{D}}_\ell^n} \rho\big(\mathrm{H}_{\mathrm{N}-1}(x), z\big)\, dx\, d\mu_{\mathrm{N}-1}(z)\right| \leq \|\rho\|_\infty n^{-1},$$

so that (9) implies (8). □

Proof of Proposition 3.4. — Use Proposition 3.2 to select λ, R_0 and compact sets W_R. Lemma 3.5 applied with $\psi_j = \psi^\varepsilon$ gives the φ_j. In particular, for every $0 \leq j \leq \mathrm{N}-\mathrm{R}$, there exists $\mathrm{Z}_j \subset \mathrm{U}_j$ with $m(\mathrm{Z}_j) > (1-2\theta)m(\mathrm{U}_j)$ such that the image under

$$\mathrm{U}_j \ni x \mapsto \big(\mathrm{D}\varphi_n\big(\tilde{f}^{n-j}(x)\big)\big)_{n=j}^{j+\mathrm{R}-1} \in \mathrm{SL}(d,\mathbf{R})^\mathrm{R}$$

of the set Z_j is arbitrarily close to W_R. It follows that if y is a point in Z_j and if F is a d_0-dimensional space satisfying $\|\mathrm{P}_{\mathrm{E}^-}|\mathrm{F}\| \leq 1/3$ and $\|\mathrm{P}_{\mathrm{E}^+}|\mathrm{F}'\| \leq 1/3$ for $\mathrm{F}' = (\mathrm{L}_{j+\mathrm{R}-1}\cdot \mathrm{A}_{j+\mathrm{R}-1})\cdots(\mathrm{L}_j\cdot \mathrm{A}_j)\cdot \mathrm{F}$, then

$$\log \mathrm{Jac}_\mathrm{F}\big((\mathrm{L}_{j+\mathrm{R}-1}\cdot \mathrm{A}_{j+\mathrm{R}-1})\cdots(\mathrm{L}_j\cdot \mathrm{A}_j)\big) \leq \log \mathrm{Jac}_{\mathrm{E}^0}(\mathrm{L}_{j+\mathrm{R}-1}\cdots \mathrm{L}_j) - \frac{2c(\varepsilon)}{5}\mathrm{R},$$

where we denote $\mathrm{A}_{j+i} = \mathrm{D}\varphi_{j+i}(\tilde{f}_{j+i-1}\circ\cdots\circ\tilde{f}_j(y_j))$.

Since the f_i are diffeomorphisms, if the neighborhood U is small enough,

$$\log \mathrm{Jac}_\mathrm{F}\big(\mathrm{D}(\tilde{f}_{j+\mathrm{R}-1}\circ\cdots\circ\tilde{f}_j)(y_j)\big) \leq \log \mathrm{Jac}_\mathrm{F}\big((\mathrm{L}_{j+\mathrm{R}-1}\cdot \mathrm{A}_{j+\mathrm{R}-1})\cdots(\mathrm{L}_j\cdot \mathrm{A}_j)\big) + \frac{c(\varepsilon)}{20}\mathrm{R}.$$

The result follows. □

3.3. *Global: proof of Theorem C'.* — Using the local perturbation technique along orbits, we define in this third step the global perturbation by building towers.

Proof of Theorem C'. — Let $B_\xi \subset \mathbf{R}^d$ be the ball centered at the origin of radius $\xi > 0$ small. Fix a precompact family of volume-preserving smooth embeddings $\Psi_x : B_\xi \to M$, $x \in K$, such that $\Psi_x(0) = x$ and $D\Psi_x(0)$ sends E^+, E^0, E^- to $E_1(x)$, $E_2(x)$ and $E_3(x)$, respectively.

Let $\alpha > 0$ be small enough so that (from the dominated splitting $T_K M = E_1 \oplus E_2 \oplus E_3$) for all $x \in K$, if F is α-close to $E_2(x)$ then for each $j \geq 0$ the image $Df^j(x) \cdot F$ is close to a subspace of $E_1(f^j(x)) \oplus E_2(f^j(x))$ and $Df^{-j}(x) \cdot F$ is close to a subspace of $E_2(f^{-j}(x)) \oplus E_3(f^{-j}(x))$. In particular for every $j \geq 0$,

$$\left\| P_{E^+} \middle| \left(D\Psi_{f^{-j}(x)}(0)^{-1} \cdot Df^{-j}(x)\right) \cdot F \right\|, \ \left\| P_{E^-} \middle| \left(D\Psi_{f^j(x)}(0)^{-1} \cdot Df^j(x)\right) \cdot F \right\| \leq 1/5.$$

If $\mathcal{U}$ is small in the C^1-topology, for any $g \in \mathcal{U}$ and $j \geq 0$ we still have:

- if $g(x), g^2(x), \dots, g^j(x)$ are close enough to $f(x), f^2(x), \dots, f^j(x)$, then
$$\left\| P_{E^-} \middle| \left(D\Psi_{g^j(x)}(0)^{-1} \cdot Dg^j(x)\right) \cdot F \right\| \leq 1/4,$$
- if $g^{-1}(x), \dots, g^{-j}(x)$ are close enough to $f^{-1}(x), \dots, f^{-j}(x)$, then
$$\left\| P_{E^+} \middle| \left(D\Psi_{g^{-j}(x)}(0)^{-1} \cdot Dg^{-j}(x)\right) \cdot F \right\| \leq 1/4.$$

We choose $\varepsilon > 0$ small (this choice depends on the neighborhood $\mathcal{U}$, see below) and apply Proposition 3.4 to get λ. The dominated splitting gives $J_0 \in \mathbf{N}$ such that for $x \in K$, the map $L_x = D\Psi_{f^{J_0}(x)}(0)^{-1} Df^{J_0}(x) D\Psi_x(0)$ satisfies

$$\|L_x|E^0\| \cdot \|L_x^{-1}|E^+\| \leq \lambda \quad \text{and} \|L_x|E^-\| \cdot \|L_x^{-1}|E^0\| \leq \lambda.$$

We then fix $\delta < c(\varepsilon)/(3J_0)$. Now take $\theta \in (0, \eta/10)$ and apply Proposition 3.4 to get R_0. Next, fix R much larger than R_0 (see the choice below) and set $r = R \cdot J_0$.

Since K has a dominated splitting, any periodic point $p \in K$ with period k satisfies $Df^k(p) \neq \mathrm{Id}$. The Implicit Function Theorem implies that the periodic points for f in K have measure 0. This implies that there exists a Rokhlin tower, i.e. a measurable set $Z \subset K$ and a large integer $n_0 \geq 1$ such that the iterates $Z, f(Z), \dots, f^{n_0-1}(Z)$ are pairwise disjoint and $\bigcup_{k=0}^{n_0-1} f^k(Z)$ has measure larger than $m(K) - \theta/2$. Fix such a tower. Since n_0 is large, one can introduce $n := N \cdot J_0$ with $N := [n_0/J_0]$, and by regularity of the measure, one can replace Z by a compact subset Y, so that

$$m\left(K \setminus \bigcup_{k=0}^{n-1} f^k(Y)\right) < \theta.$$

For each $x \in \mathrm{Y}$, considers the sequence of diffeomorphisms

$$f_{j,x} := \Psi^{-1}_{f^{(j+1)\mathrm{J}_0}(x)} \circ f^{\mathrm{J}_0} \circ \Psi_{f^{j\mathrm{J}_0}(x)}, \quad 0 \le j \le \mathrm{N}-1,$$

and a neighborhood D_x (which is the image $\Psi_x(\mathrm{U}_x)$ of some small neighborhood U_x of 0). By compactness, one can find finitely many such points $x_s \in \mathrm{Y}$, $s \in \mathrm{S}$, and reduce the associated neighborhoods $\mathrm{D}_s := \mathrm{D}_{x_s}$, so that the $f^k(\mathrm{D}_s)$, $s \in \mathrm{S}$, $0 \le k < n$ are pairwise disjoint, and

$$m\Bigg(\mathrm{K} \setminus \bigcup_{s\in\mathrm{S}} \bigcup_{0\le k<n} f^k(\mathrm{D}_s)\Bigg) < 2\theta.$$

The domains D_s may be chosen with small diameter so that for each point $z \in \mathrm{K}$ in an iterate $f^{j\mathrm{J}_0}(\mathrm{D}_s)$, $0 \le j \le \mathrm{N}-1$, and for any d_0-dimensional affine subspace $\mathrm{F} \subset \mathbf{R}^d$,

$$(\mathbf{10}) \qquad \|\mathrm{P}_{\mathrm{E}^-}|\mathrm{F}\| \le 1/4 \quad \Rightarrow \quad \left\|\mathrm{P}_{\mathrm{E}^-}\left|\mathrm{D}\Psi_{f^{j\mathrm{J}_0}(x_s)}(0)^{-1} \cdot \mathrm{D}\Psi_z(0) \cdot \mathrm{F}\right\|\right. \le 1/3,$$

and

$$\|\mathrm{P}_{\mathrm{E}^+}|\mathrm{F}\| \le 1/4 \quad \Rightarrow \quad \left\|\mathrm{P}_{\mathrm{E}^+}\left|\mathrm{D}\Psi_{f^{(j+\mathrm{R})\mathrm{J}_0}(x_s)}(0)^{-1} \cdot \mathrm{D}\Psi_{f^{\mathrm{R}\mathrm{J}_0}(z)}(0) \cdot \mathrm{F}\right\|\right. \le 1/3.$$

Proposition 3.4 applied to x_s and to R, N gives a sequence of diffeomorphisms $\varphi_{j,s}$, and a sequence of sets $\mathrm{Z}_{j,s} \subset f^{j\mathrm{J}_0}(\mathrm{D}_s)$ such that $m(\mathrm{Z}_{j,s}) \ge (1-2\theta)m(\mathrm{D}_s)$. Define the diffeomorphism φ in each $f^{j\mathrm{J}_0}(\mathrm{D}_s)$, $0 \le j \le \mathrm{N}-1$ by

$$\varphi = \Psi_{f^{j\mathrm{J}_0}(x_s)} \circ \varphi_{j,s} \circ \Psi^{-1}_{f^{j\mathrm{J}_0}(x_s)},$$

and let $\varphi = \mathrm{Id}$ otherwise. It is clear that if the neighborhoods D_s are chosen small enough, then φ is arbitrarily close to the identity in the C^0 topology. Also, if ε is small enough then φ is close to the identity in the C^1 topology. We set $g = f \circ \varphi$.

Define the set Λ to be the set of all points y belonging to some $f^k(\mathrm{D}_s)$, with $0 \le k \le (\mathrm{N}-1)\mathrm{J}_0 - r$, such that $f^{j\mathrm{J}_0-k}(y) \in \mathrm{Z}_{j,s}$, where $j = [k/\mathrm{J}_0] + 1$. Hence

$$k \le j\mathrm{J}_0 \le (j+\mathrm{R})\mathrm{J}_0 \le k + r.$$

Clearly, if n is large and since $10\theta < \eta$, we have $m(\mathrm{K} \setminus \Lambda) < \eta$.

Now consider $y \in \Lambda \cap \mathrm{K} \cap g^{-r}(\mathrm{K})$ and a d_0-dimensional subspace $\mathrm{F} \subset \mathrm{T}_y\mathrm{M}$ that is α-close to $\mathrm{E}_2(f, y)$ and whose image $\mathrm{D}g^r \cdot \mathrm{F}$ is α-close to $\mathrm{E}_2(f, g^r(y))$. We also introduce j, k, x_s as defined above such that $f^{j\mathrm{J}_0-k}(y)$ belongs to $\mathrm{Z}_{j,s}$. Since $k - j\mathrm{J}_0$ and $(j+\mathrm{R})\mathrm{J}_0 - (k+r)$ are bounded (by $2\mathrm{J}_0$) and g can be chosen arbitrarily close to f in the C^1-topology, by the choice of α we have

$$\left\|\mathrm{P}_{\mathrm{E}^-}\left|\mathrm{D}\Psi_{f^{j\mathrm{J}_0-k}(y)}(0)^{-1} \cdot \mathrm{D}^{j\mathrm{J}_0-k}g(y) \cdot \mathrm{F}\right\|\right. \le 1/4,$$
$$\left\|\mathrm{P}_{\mathrm{E}^+}\left|\mathrm{D}\Psi_{f^{(j+\mathrm{R})\mathrm{J}_0-k}(g^r(y))}(0)^{-1} \cdot \mathrm{D}g^{(j+\mathrm{R})\mathrm{J}_0-k}(y) \cdot \mathrm{F}\right\|\right. \le 1/4.$$

By (10), this gives:

$$\left\|P_{E^-}\left|D\Psi_{f^{jJ_0}(x_s)}(0)^{-1}\cdot D^{jJ_0-k}g(y)\cdot F\right\|\leq 1/3,$$
$$\left\|P_{E^+}\left|D\Psi_{f^{(j+R)J_0}(x_s)}(0)^{-1}\cdot Dg^{(j+R)J_0-k}(y)\cdot F\right\|\leq 1/3.$$

Let $F'=D^{jJ_0-k}g(y)\cdot F$. Since $f^{jJ_0-k}(y)$ belongs to $Z_{j,s}$, by applying Proposition 3.4 we obtain:

$$\log \mathrm{Jac}_{F'}\left(g^{RJ_0},g^{jJ_0-k}(y)\right)\leq \log \mathrm{Jac}_{E_2(f,f^{jJ_0}(x_s))}\left(f^{RJ_0},f^{jJ_0}(x_s)\right)-\frac{c(\varepsilon)}{3}R+4C_0,$$

where C_0 bounds $|\log \mathrm{Jac}_H(D\Psi_x)|$ for any $x\in K$ and any d_0-dimensional space H.

If g is sufficiently C^0-close to f, and if the sets D_s have small diameter, then the orbits $(f^{-k}(y),\dots,f^{2n-k}(y))$ and $(x_s,\dots,f^{2n}(x_s))$ are arbitrarily close. It follows that there exists a constant $C_1>0$, which depends on J_0 but not on R, such that:

$$\log \mathrm{Jac}_F\left(g^r,y\right)\leq \log \mathrm{Jac}_{E_2(f,y)}\left(f^r,y\right)-\frac{c(\varepsilon)}{3J_0}r+4C_0+C_1.$$

If r (and R) has been chosen large enough, one gets (4) by our choice of δ. This ends the proof of Theorem C'. □

4. Proof of the corollaries

4.1. *Robust positive metric entropy.* — We prove here Corollary 1.

For m-almost every point x, we denote the Lyapunov exponents by

$$\lambda_1(x)\geq\cdots\geq\lambda_{\dim M}(x).$$

If f has a (non-trivial) dominated splitting $TM=E\oplus F$, then by the Pesin-type inequality for C^1 diffeomorphisms with a dominated splitting proved in [ST], we have:

$$h_m(f)\geq\int\left(\lambda_1(x)+\cdots+\lambda_{\dim E}(x)\right)dm(x).$$

The dominated splitting also gives that there exists $a>0$ such that $\lambda_{\dim E}(x)>\lambda_{\dim E+1}(x)+a$ for almost every point x. In particular,

$$a+\frac{1}{\dim F}\int(\lambda_{\dim E+1}+\cdots+\lambda_{\dim M})\,dm<\frac{1}{\dim E}\int(\lambda_1+\cdots+\lambda_{\dim E})\,dm.$$

Since f is conservative,

$$\int(\lambda_1+\cdots+\lambda_{\dim E})\,dm+\int(\lambda_{\dim E+1}+\cdots+\lambda_{\dim M})\,dm=0.$$

All these estimates together imply that the metric entropy is positive:

$$h_m(f) \geq \int (\lambda_1 + \cdots + \lambda_{\dim \mathrm{E}})\, dm > \frac{a \dim \mathrm{E} \dim \mathrm{F}}{\dim \mathrm{M}} > 0.$$

To prove the converse, assume that f has no dominated splitting on M. Then Theorem B implies that the generic diffeomorphism g in the open set $\mathcal{U}$ provided by the lemma below has zero metric entropy. In particular f is the limit of diffeomorphisms with zero metric entropy.

Lemma **4.1.** — *If f has no dominated splitting on* M, *then there exists an open set $\mathcal{U} \subset \mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ of diffeomorphisms with no dominated splitting such that f belongs to the closure of $\mathcal{U}$.*

Proof. — Fix $\varepsilon > 0$. There exists [BC] an arbitrarily C^1-small perturbation f_1 with a sequence of periodic orbits O_n converging to M in the Hausdorff topology. Since f_1 is arbitrarily close to f, the dominated splittings that may exist on O_n, for n large, are weak: by [BoBo] and the Franks lemma, for each $1 \leq i < \dim \mathrm{M}$, one can, after a $\varepsilon/2$-perturbation f_2 (with respect to the C^1-distance), ensure that O_n has simple eigenvalues and that the i^{th} and the $(i+1)^{\text{st}}$ eigenvalues are complex and conjugate. In particular, any diffeomorphism g that is C^1-close to f_2 has no dominated splitting $\mathrm{E} \oplus \mathrm{F}$, with $\dim(\mathrm{E}) = i$. This last perturbation is supported on a small neighborhood of O_n. Considering different periodic orbits, one can perform independently $\dim \mathrm{M} - 1$ such perturbations and obtain a diffeomorphism which robustly has no dominated splitting, as required. □

4.2. *Weak mixing.* — We now prove Corollary 2.

Consider a diffeomorphism $f \in \mathrm{Diff}^r_{\mathrm{vol}}(\mathrm{M})$ with $r > 1$. For m-almost every point x we have introduced in Section 2.1.4 the splitting $T_x\mathrm{M} = \mathrm{E}^+(x) \oplus \mathrm{E}^0(x) \oplus \mathrm{E}^-(x)$ induced by the Oseledets decomposition. The Pesin stable manifold theorem asserts that if $x \in \mathrm{M}$ is a regular point and $\varepsilon > 0$ is small, then

$$\mathrm{W}^-(x) := \left\{ z : \limsup_{n \to +\infty} \frac{1}{n} \log d\big(f^n(x), f^n(z)\big) \leq -\varepsilon \right\}$$

is an injectively immersed submanifold tangent to $\mathrm{E}^-(x)$. Symmetrically, one obtains an injectively immersed submanifold $\mathrm{W}^+(x)$ tangent to $\mathrm{E}^+(x)$.

If O is a hyperbolic periodic orbit, we define the *Pesin homoclinic class*:

$$\mathrm{H}^s_{\mathrm{Pes}}(\mathrm{O}) = \{x \text{ Oseledets regular } : \mathrm{W}^-(x) \pitchfork \mathrm{W}^u(\mathrm{O}) \neq \emptyset\},$$
$$\mathrm{H}^u_{\mathrm{Pes}}(\mathrm{O}) = \{x \text{ Oseledets regular } : \mathrm{W}^+(x) \pitchfork \mathrm{W}^s(\mathrm{O}) \neq \emptyset\},$$

where $\mathrm{W}_1 \pitchfork \mathrm{W}_2$ denotes the set of transverse intersections between manifolds W_1, W_2, i.e. the set of points x such that $T_x\mathrm{W}_1 + T_x\mathrm{W}_2 = T_x\mathrm{M}$. The Pesin homoclinic class is $\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}) := \mathrm{H}^s_{\mathrm{Pes}}(\mathrm{O}) \cap \mathrm{H}^u_{\mathrm{Pes}}(\mathrm{O})$. See Figure 2. We stress the fact that $\mathrm{H}^s_{\mathrm{Pes}}(\mathrm{O})$ can contain

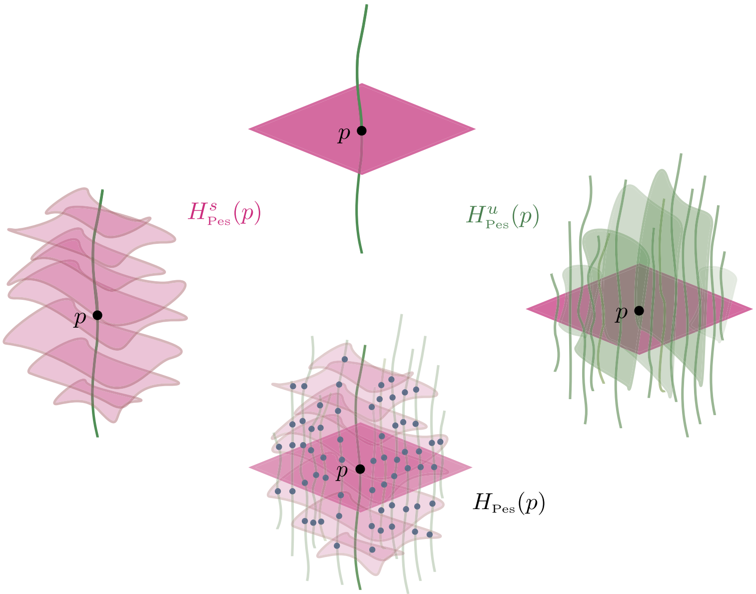

FIG. 2. — The Pesin homoclinic class

points x whose stable dimension $\dim(E^-(x))$ is strictly larger than the stable dimension of O. However the set $H_{\text{Pes}}(O)$ only contains non-uniformly hyperbolic points whose stable/unstable dimensions are the same as O.

An improvement of Hopf argument gives:

Theorem **4.2** *(Rodriguez-Hertz–Rodriguez-Hertz–Tahzibi–Ures [RRTU]). — Let* $f \in \text{Diff}^r_m(M)$ *with* $r > 1$ *and let* O *be a hyperbolic periodic orbit such that* $m(H^s_{\text{Pes}}(O))$ *and* $m(H^u_{\text{Pes}}(O))$ *are positive. Then* $H^s_{\text{Pes}}(O)$, $H^u_{\text{Pes}}(O)$, $H_{\text{Pes}}(O)$ *coincide m-almost everywhere and* $m|H_{\text{Pes}}(O)$ *is ergodic.*

Recall that $f \in \text{Diff}^1_{\text{vol}}(M)$ is weakly mixing if and only if $f \times f$ is ergodic with respect to $m \times m$.

Given a continuous function $\phi : M \times M \to \mathbf{R}$ and $\epsilon > 0$, we denote by $\mathcal{U}(\phi, \epsilon)$ the set of all $f \in \text{Diff}^1_{\text{vol}}(M)$ such that, for some $n \geq 1$, the set of all $(x, y) \in M \times M$ satisfying

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \phi\big(f^k(x), f^k(y)\big) - \int \phi(x, y)\, dm(x)\, dm(y) \right| < \epsilon$$

has measure strictly larger than $1 - \epsilon$. Note that $\mathcal{U}(\phi, \epsilon)$ is open, and that for any dense subset $\Omega \subset C^0(M \times M, \mathbf{R})$, $f \times f$ is ergodic if and only if f belongs to $\bigcap_{\phi \in \Omega} \bigcap_{\epsilon > 0} \mathcal{U}(\phi, \epsilon)$.

We say that f is *ϵ-weak mixing* if it admits an invariant subset X of measure strictly larger than $1 - \epsilon$, such that $f|X$ is weak mixing. Notice that if f is ϵ-weak mixing then $f \in \mathcal{U}(\phi, 3\epsilon \|\phi\|_{L^\infty})$ for every $\phi \in C^0(M \times M, \mathbf{R})$. Thus to prove the genericity statement of

Corollary 2, it is enough to prove that ϵ-weak mixing is dense among the diffeomorphisms in $\mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ with positive metric entropy.

Let $f \in \mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$ be a C^1-generic diffeomorphism given by Theorem B and let us assume that it has positive metric entropy. We may also assume that f has the following additional C^1-generic properties:

(1) f is topologically transitive, by [BC, Théorème 1.3],
(2) for any hyperbolic periodic point p, we have $\mathrm{W}^u(p) \cap \mathrm{W}^s(f(p)) \neq \emptyset$ and the intersection is transverse, by [AC, Theorems 3 and 4], and
(3) there exist hyperbolic periodic points $p_1, \ldots, p_k$ such that for every $\epsilon > 0$ and every $g \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathrm{M})$ sufficiently C^1-close to f, there exists a p_i with the following property: the Pesin homoclinic class $\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}(p_i(g)))$ of the orbit $\mathrm{O}(p_i)$ of p_i has m-measure $> 1 - \epsilon$ and the restriction of the volume is ergodic, non-uniformly hyperbolic and its Oseledets splitting is dominated, by [AB, Lemma 5.1].

Let $p_1, \ldots, p_k$ be given by item (3) and let $\epsilon > 0$. By [Av], there exists $g \in \mathrm{Diff}^2_{\mathrm{vol}}(\mathrm{M})$ arbitrarily C^1-close to f. Then, by item (2) for each $i = 1, \ldots, k$, there still exists a transverse intersection point between $\mathrm{W}^u(p_i(g))$ and $\mathrm{W}^s(g(p_i(g)))$ associated to the hyperbolic continuation $p_i(g)$. By item (3) there exists $i \in 1, \ldots, k$ such that the Pesin homoclinic class $\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}(p_i(g)))$ has m-measure $> 1 - \epsilon$ and the restriction of the volume is ergodic, non-uniformly hyperbolic and its Oseledets splitting is dominated.

It follows from [P] that $\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}(p_i(g)))$ decomposes as a disjoint union of measurable sets $\mathrm{A} \cup g(\mathrm{A}) \cup \cdots \cup g^{\ell-1}(\mathrm{A})$ and that the restriction $m|\mathrm{A}$ is Bernoulli for g^ℓ. On the other hand, since $\mathrm{W}^u(p_i(g)) \cap \mathrm{W}^s(g(p_i(g))) \neq \emptyset$, the Pesin homoclinic class of the orbits of $p_i(g)$ for g and g^ℓ coincide, implying by Theorem 4.2 that $m|\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}(p_i(g)))$ is ergodic for g^ℓ. This shows that $\ell = 1$, and that g is Bernoulli, and in particular weakly mixing, on $\mathrm{H}_{\mathrm{Pes}}(\mathrm{O}(p_i(g)))$. Thus g is ϵ-weakly mixing, and so ϵ-weak mixing is dense in $\mathrm{Diff}^1_{\mathrm{vol}}(\mathrm{M})$. This completes the proof of Corollary 2. □

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REFERENCES

[AC] F. Abdenur and S. Crovisier, Transitivity and topological mixing for C^1 diffeomorphisms, in *Essays in Mathematics and Its Applications*, pp. 1–16, Springer, Heidelberg, 2012.

[An1] D. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, *Tr. Mat. Inst. Steklova*, **90** (1967).

[Av] A. Avila, On the regularization of conservative maps, *Acta Math.*, **205** (2010), 5–18.

[AB] A. Avila and J. Bochi, Nonuniform hyperbolicity, global dominated splittings and generic properties of volume-preserving diffeomorphisms, *Trans. Am. Math. Soc.*, **364** (2012), 2883–2907.

[ABW] A. Avila, J. Bochi and A. Wilkinson, Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms, *Ann. Sci. Éc. Norm. Supér.*, **42** (2009), 931–979.

[ACW] A. Avila, S. Crovisier and A. Wilkinson, C^1-density of stable ergodicity, in preparation, arXiv:1408.4252v3.

[BaBo] A. Baraviera and C. Bonatti, Removing zero Lyapunov exponents, *Ergod. Theory Dyn. Syst.*, **23** (2003), 1655–1670.

[Boc1] J. Bochi, Genericity of zero Lyapunov exponents, *Ergod. Theory Dyn. Syst.*, **22** (2002), 1667–1696.

[Boc2] J. Bochi, C^1-Generic symplectic diffeomorphisms: partial hyperbolicity and zero centre Lyapunov exponents, *J. Inst. Math. Jussieu*, **9** (2010), 49–93.

[BoBo] J. Bochi and C. Bonatti, Perturbation of the Lyapunov spectra of periodic orbits, *Proc. Lond. Math. Soc.*, **105** (2012), 1–48.

[BV2] J. Bochi and M. Viana, The Lyapunov exponents of generic volume-preserving and symplectic maps, *Ann. Math.*, **161** (2005), 1423–1485.

[BFP] J. Bochi, B. Fayad and E. Pujals, A remark on conservative diffeomorphisms, *C. R. Math. Acad. Sci. Paris*, **342** (2006), 763–766.

[BC] C. Bonatti and S. Crovisier, Récurrence et généricité, *Invent. Math.*, **158** (2004), 33–104.

[BDP] C. Bonatti, L. Diaz and E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, *Ann. Math.*, **158** (2003), 355–418.

[F] J. Franks, Anosov diffeomorphisms. Global analysis I, *Proc. Symp. Pure Math.*, **14** (1970), 61–93.

[Ka] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math. Inst. Hautes Études Sci.*, **51** (1980), 137–173.

[M] R. Mañé, Oseledec's theorem from the generic viewpoint, in *Proc. Int. Congress of Mathematicians (Warszawa, 1983)*, vol. 2, pp. 1259–1276.

[N] S. Newhouse, On codimension one Anosov diffeomorphisms, *Am. J. Math.*, **92** (1970), 761–770.

[OU] J. Oxtoby and S. Ulam, Measure-preserving homeomorphisms and metrical transitivity, *Ann. Math. (2)*, **42** (1941), 874–920.

[P] Y. Pesin, Characteristic Lyapunov exponents, and smooth ergodic theory, *Usp. Mat. Nauk*, **32** (1977), 55–112, 287.

[R] M. A. Rodriguez-Hertz, Genericity of nonuniform hyperbolicity in dimension 3, *J. Mod. Dyn.*, **6** (2012), 121–138.

[RRTU] F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, A. Tahzibi and R. Ures, New criteria for ergodicity and nonuniform hyperbolicity, *Duke Math. J.*, **160** (2011), 599–629.

[SW] M. Shub and A. Wilkinson, Pathological foliations and removable zero exponents, *Invent. Math.*, **139** (2000), 495–508.

[ST] W. Sun and X. Tian, Dominated splitting and Pesin's entropy formula, *Discrete Contin. Dyn. Syst.*, **32** (2012), 1421–1434.

[T1] A. Tahzibi, C^1-Generic Pesin's entropy formula, *C. R. Acad. Sci. Paris*, **335** (2002), 1057–1062.

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