MEROMORPHIC TENSOR EQUIVALENCE FOR YANGIANS AND QUANTUM LOOP ALGEBRAS

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ABSTRACT

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $Y_\hbar(\mathfrak{g})$, $U_q(L\mathfrak{g})$ the corresponding Yangian and quantum loop algebra, with deformation parameters related by $q = e^{\pi\iota\hbar}$. When $\hbar$ is not a rational number, we constructed in Gautam and Toledano Laredo (J. Am. Math. Soc. 29:775, 2016) a faithful functor Γ from the category of finite-dimensional representations of $Y_\hbar(\mathfrak{g})$ to those of $U_q(L\mathfrak{g})$. The functor Γ is governed by the additive difference equations defined by the commuting fields of the Yangian, and restricts to an equivalence on a subcategory of $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ defined by choosing a branch of the logarithm. In this paper, we construct a tensor structure on Γ and show that, if $|q| \neq 1$, it yields an equivalence of meromorphic braided tensor categories, when $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ are endowed with the deformed Drinfeld coproducts and the commutative part of their universal R-matrices. This proves in particular the Kohno–Drinfeld theorem for the abelian qKZ equations defined by $Y_\hbar(\mathfrak{g})$. The tensor structure arises from the abelian qKZ equations defined by an appropriate regularisation of the commutative part of the R-matrix of $Y_\hbar(\mathfrak{g})$.

CONTENTS

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ the Yangian and quantum loop algebra of $\mathfrak{g}$. Recall that the latter are deformations of the enveloping algebras of the current Lie algebra $\mathfrak{g}[s]$ of $\mathfrak{g}$ and its loop algebra $\mathfrak{g}[z, z^{-1}]$ respectively. We shall assume throughout that the deformation parameters $\hbar$ and q are related by $q = e^{\pi\iota\hbar}$, and that q is not a root of unity.

The present paper builds upon the equivalence of categories of finite-dimensional representations $\Gamma : \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \longrightarrow \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ constructed in [13].[1] A natural question stemming from [13] is whether Γ is a tensor functor, that is admits a family of natural isomorphisms $\mathcal{J}_{V_1,V_2} : \Gamma(V_1) \otimes \Gamma(V_2) \to \Gamma(V_1 \otimes V_2)$ of $U_q(L\mathfrak{g})$-modules which

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[1] Strictly speaking, Γ is defined on a subcategory of $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, and becomes an equivalence after restricting the source category suitably. We will gloss over this point here, and refer the reader to [13] or Section 6 below for the precise statement.

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are associative with respect to triples of representations.[2] Partial evidence, pointing towards a positive answer, is obtained in [13] where it is shown that Γ is compatible with taking the q-characters of Frenkel–Reshetikhin and Knight, and therefore induces a homomorphism of Grothendieck rings.

1.2. The goal of this paper is to show that Γ admits a tensor structure. Our main result, which will be explained in more detail below, is that Γ gives rise to an equivalence of *meromorphic* tensor categories. Moreover, when $|q| \neq 1$, this equivalence also preserves the meromorphic braiding arising from the abelianisation of the universal R-matrices of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$.

This may be regarded as a meromorphic version of the Kazhdan–Lusztig equivalence between the category $\mathcal{O}$ of representations of the affine algebra $\widehat{\mathfrak{g}}$ at level κ and the category of finite-dimensional representations of the quantum group $U_q\mathfrak{g}$, where $q = e^{\pi\iota/m(\kappa+h^\vee)}$ [17–19]. Here m is the ratio of the square length of the long roots to the short roots and $h^\vee$ is the dual Coxeter number. More precisely, for $\kappa \notin \mathbf{Q}$, the central ingredient of the KL equivalence is the construction of a tensor functor from the (Drinfeld) category $\mathcal{D}(\mathfrak{g})$ of finite-dimensional $\mathfrak{g}$-modules, with associativity and commutativity constraints given by the monodromy of the KZ equations with deformation parameter $\hbar = 1/(\kappa + h^\vee)$, to the category of finite-dimensional representations of $U_q\mathfrak{g}$ [18].

In the present work, $\mathcal{D}(\mathfrak{g})$ is replaced by $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, $\mathrm{Rep}_{\mathrm{fd}}(U_q\mathfrak{g})$ by $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, and the KZ equations by an appropriate abelianisation of the additive, difference qKZ equations defined by the universal R-matrix $\mathcal{R}(s)$ of $Y_\hbar(\mathfrak{g})$ [11, 27].

1.3. Our equivalence implies in particular that the monodromy of these difference equations, a meromorphic function of the spectral parameter $\zeta = e^{2\pi\iota s}$, is explicitly expressed in terms of the abelianisation of the universal R-matrix $\mathscr{R}(\zeta)$ of $U_q(L\mathfrak{g})$. The latter result is a version of the Kohno–Drinfeld theorem for abelian qKZ equations.

This result was conjectured by Frenkel–Reshetikhin [11] for the non-abelian qKZ equations, and proved in the rational and trigonometric cases by Tarasov–Varchenko when $\mathfrak{g} = \mathfrak{sl}_2$, and attention is restricted to evaluation representations with generic highest weights [29, 30].

One difficulty in addressing the general case is that no functorial way of relating arbitrary representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ was known to exist outside of type A prior to [13].[3] We shall prove the Kohno–Drinfeld theorem for the full (non-abelian) qKZ equations for any $\mathfrak{g}$ in a sequel to this paper [12].[4]

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[2] Although $\Gamma(V) = V$ as vector spaces, $\mathcal{J}_{V_1,V_2} = \mathrm{id}_{V_1\otimes V_2}$ is not the required isomorphism since the actions of $U_q(L\mathfrak{g})$ on $\Gamma(V_1) \otimes \Gamma(V_2)$ and $\Gamma(V_1 \otimes V_2)$ do not coincide.

[3] For $\mathfrak{g} = \mathfrak{sl}_2$, evaluation representations of $Y_\hbar(\mathfrak{g})$ can be explicitly deformed to representations of $U_q(L\mathfrak{g})$. More generally, in type A_n, Moura proved the Kohno–Drinfeld Theorem for the trigonometric qKZ equations with values in the vector representation of $U_q(L\mathfrak{g})$ [5] and, jointly with Etingof, used this to construct a functor from the finite-dimensional representations of $U_q(L\mathfrak{g})$ arising from the RTT construction to those of Felder's elliptic quantum group [9].

[4] A more general result, for the Lie algebras associated by Maulik–Okounkov to quivers [24], was independently announced by Okounkov [25]. It includes in particular the Kohno–Drinfeld theorem for the qKZ equations corresponding to simply-laced Lie algebras.

1.4. A crucial feature of our approach is that the relevant monoidal structures arise from the *deformed Drinfeld coproducts* on $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, rather than from the standard (Kac–Moody) ones.[5] The Drinfeld coproduct was defined for $U_q(L\mathfrak{g})$ by Drinfeld [6], and regularised through deformation by Hernandez [14, 15]. Whereas this coproduct has long been understood to arise from the polarisation of the loop algebra $\mathfrak{g}((z))$ given by

$$\mathfrak{g}((z)) = \left(\mathfrak{n}_-((z)) \oplus z^{-1}\mathfrak{h}\left[z^{-1}\right]\right) \oplus \left(\mathfrak{h}[[z]] \oplus \mathfrak{n}_+((z))\right),$$

a proper understanding of the structure it confers finite-dimensional representations has been lacking so far.

We define a similar deformed coproduct on $Y_\hbar(\mathfrak{g})$, and show that these endow $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ with the structures of *meromorphic tensor categories* that is, roughly speaking, categories endowed with a monoidal structure and associativity constraints depending meromorphically on parameters. This notion was outlined by Frenkel–Reshetikhin who used the term *analytic tensor categories* [11, p. 49], and formalised by Soibelman to describe the structure of finite-dimensional representations of $U_q(L\mathfrak{g})$ corresponding to the standard coproduct and the universal R-matrix $\mathscr{R}(\zeta)$ [28]. Our observation that the deformed Drinfeld coproduct fits within, and provides new examples of such categories seems to be new.

Our first main result may be succinctly stated as saying that Γ is a meromorphic tensor functor.

1.5. Our second main result is that, when $|q| \neq 1$, Γ is a *braided* meromorphic tensor functor. In more detail, $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ are known not to be braided tensor categories. As pointed out, however, the universal R-matrix $\mathscr{R}(\zeta)$ of $U_q(L\mathfrak{g})$ defines a meromorphic commutativity constraint on $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ with respect to the standard tensor product $\otimes$.

We show that the same holds with respect to the deformed Drinfeld tensor product $\otimes_\zeta$, *provided* $\mathscr{R}(\zeta)$ is replaced by the diagonal component $\mathscr{R}_0(\zeta)$ of its Gauss decomposition. Thus, $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ may be endowed with two distinct structures of meromorphic braided tensor category, namely as

$$\left(\mathrm{Rep}_{\mathrm{fd}}\left(U_q(L\mathfrak{g})\right), \otimes, \mathscr{R}(\zeta)\right) \quad \text{and} \quad \left(\mathrm{Rep}_{\mathrm{fd}}\left(U_q(L\mathfrak{g})\right), \otimes_\zeta, \mathscr{R}_0(\zeta)\right)$$

The latter structure does not seem to have been noticed before, though it should be closely related to the large volume limit in quantum cohomology.[6]

We prove a similar result for the Yangian by constructing the commutative part $\mathcal{R}_0(s)$ of its universal R-matrix, and showing that it defines commutativity constraints for the deformed Drinfeld tensor product $\otimes_s$ of $Y_\hbar(\mathfrak{g})$. The construction of $\mathcal{R}_0(s)$ is more

[5] The relation to the standard coproduct is discussed in 2.13.
[6] These two structures are, in fact, meromorphically equivalent, see 2.13.

delicate than that of $\mathscr{R}_0(\zeta)$, and involves a non-trivial analytic regularisation of the formal infinite product formulae for $\mathcal{R}_0(s)$ obtained by Khoroshkin–Tolstoy [21].

We then show that

$$\Gamma : \left(\mathrm{Rep}_{\mathrm{fd}}\left(\mathrm{Y}_\hbar(\mathfrak{g})\right), \otimes_s, \mathcal{R}_0(s)\right) \longrightarrow \left(\mathrm{Rep}_{\mathrm{fd}}\left(\mathrm{U}_q(\mathrm{L}\mathfrak{g})\right), \otimes_\zeta, \mathscr{R}_0(\zeta)\right)$$

is compatible with the meromorphic braiding. As mentioned above, this implies in particular the Kohno–Drinfeld theorem for the additive qKZ equations defined by $\mathcal{R}_0(s)$, namely the fact that their monodromy is expressed in terms of $\mathscr{R}_0(\zeta)$, where $\zeta = e^{2\pi\iota s}$.

2. Statement of main results

This section contains a more detailed description of our main results, their background, and a sketch of some of their proofs.

2.1. *The deformed Drinfeld coproduct of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$. — The Drinfeld coproduct on $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ was defined by Drinfeld in [6], and involves formal infinite sums of elements in $\mathrm{U}_q(\mathrm{L}\mathfrak{g})^{\otimes 2}$. Composing with the $\mathbf{C}^\times$-action on the first factor, Hernandez obtained a deformed coproduct, which is an algebra homomorphism

$$\Delta_\zeta : \mathrm{U}_q(\mathrm{L}\mathfrak{g}) \to \mathrm{U}_q(\mathrm{L}\mathfrak{g})\left((\zeta^{-1})\right) \otimes \mathrm{U}_q(\mathrm{L}\mathfrak{g})$$

where ζ is a formal variable [14, §6]. The map Δ_ζ is coassociative, in the sense that $\Delta_{\zeta_1} \otimes \mathbf{1} \circ \Delta_{\zeta_2} = \mathbf{1} \otimes \Delta_{\zeta_2} \circ \Delta_{\zeta_1\zeta_2}$ [15, Lemma 3.4].

When computed on the tensor product of two finite-dimensional representations $\mathcal{V}_1, \mathcal{V}_2$ of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$, the deformed Drinfeld coproduct Δ_ζ is analytically well-behaved. Specifically, the action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}_1((\zeta^{-1})) \otimes \mathcal{V}_2$ is the Laurent expansion at $\zeta = \infty$ of a family of actions of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}_1 \otimes \mathcal{V}_2$, whose matrix coefficients are rational functions of ζ [15, Lemma 3.10]. We denote $\mathcal{V}_1 \otimes \mathcal{V}_2$ endowed with this action by $\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2$.

2.2. In Section 4, we give simple contour integral formulae for the action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2$. These yield an alternative proof of the rationality of $\otimes_\zeta$, as well as an explicit determination of its poles as a function of ζ.

Specifically, let $\mathcal{V}$ be a finite-dimensional representation of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$, $\mathbf{I}$ the set of vertices of the Dynkin diagram of $\mathfrak{g}$, $\{\Psi_i(z), \mathcal{X}_i^\pm(z)\}_{i\in\mathbf{I}}$ the $\mathrm{End}(\mathcal{V})$-valued rational functions of $z \in \mathbf{P}^1$ whose Taylor expansion at $z = \infty, 0$ give the action of the generators of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}$ (see Section 3.10), and $\sigma(\mathcal{V}) \subset \mathbf{C}^\times$ the set of poles of these functions.

Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$, and let $\zeta \in \mathbf{C}^\times$ be such that $\zeta\sigma(\mathcal{V}_1)$ and $\sigma(\mathcal{V}_2)$ are disjoint. Then, the action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}_1 \otimes_\zeta \mathcal{V}_2$ is given by the following formulae for any $m \in \mathbf{Z}_{\geq 0}$ and $k \in \mathbf{Z}$

$$\Delta_\zeta\left(\Psi^\pm_{i,\pm m}\right) = \sum_{p_1+p_2=m} \zeta^{\pm p_1} \Psi^\pm_{i,\pm p_1} \otimes \Psi^\pm_{i,\pm p_2}$$

$$\Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \oint_{C_2} \Psi_i(\zeta^{-1}w) \otimes \mathcal{X}_i^+(w) w^{k-1} dw$$

$$\Delta_\zeta(\mathcal{X}_{i,k}^-) = \oint_{C_1} \mathcal{X}_i^-(\zeta^{-1}w) \otimes \Psi_i(w) w^{k-1} dw + 1 \otimes \mathcal{X}_{i,k}^-$$

where

- $C_1, C_2 \subset \mathbf{C}^\times$ are Jordan curves which do not enclose 0.
- C_1 encloses $\zeta\sigma(\mathcal{V}_1)$ and none of the points in $\sigma(\mathcal{V}_2)$.
- C_2 encloses $\sigma(\mathcal{V}_2)$ and none of the points in $\zeta\sigma(\mathcal{V}_1)$.

Direct inspection shows that Δ_ζ is a rational function of ζ, with poles contained in $\sigma(\mathcal{V}_2)\sigma(\mathcal{V}_1)^{-1}$.

2.3. A remarkable feature of the deformed Drinfeld coproduct $\otimes_\zeta$ is that it endows $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ with the structure of a meromorphic tensor category in the sense of [28]. This category is strict, in that for any $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, the identification of vector spaces

$$(\mathcal{V}_1 \otimes_{\zeta_1} \mathcal{V}_2) \otimes_{\zeta_2} \mathcal{V}_3 = \mathcal{V}_1 \otimes_{\zeta_1\zeta_2} (\mathcal{V}_2 \otimes_{\zeta_2} \mathcal{V}_3)$$

intertwines the action of $U_q(L\mathfrak{g})$.[7]

Meromorphic braided tensor categories were introduced by Soibelman in [28] to formalise the structure of the category of finite-dimensional representations of $U_q(L\mathfrak{g})$ endowed with the standard (Kac–Moody) tensor product and the universal R-matrix $\mathscr{R}(\zeta)$.

2.4. *The deformed Drinfeld coproduct of* $Y_\hbar(\mathfrak{g})$. — A Drinfeld coproduct was conjecturally defined for the double Yangian $\mathcal{D}Y_\hbar(\mathfrak{g})$ by Khoroshkin–Tolstoy [21]. Like its counterpart for $U_q(L\mathfrak{g})$, it involves formal infinite sums. Moreover, the Yangian $Y_\hbar(\mathfrak{g}) \subset \mathcal{D}Y_\hbar(\mathfrak{g})$ is not closed under it.

By degenerating our contour integral formulae for $\otimes_\zeta$, we obtain in Section 4.5 a family of actions $V_1 \otimes_s V_2$ of $Y_\hbar(\mathfrak{g})$ on the tensor product of two finite-dimensional representations of $Y_\hbar(\mathfrak{g})$, which is a rational function of a parameter $s \in \mathbf{C}$. Its expansion at $s = \infty$ should coincide with the deformation of the Drinfeld coproduct on $\mathcal{D}Y_\hbar(\mathfrak{g})$ via the translation action of $\mathbf{C}$ on $Y_\hbar(\mathfrak{g})$, once the negative modes of $\mathcal{D}Y_\hbar(\mathfrak{g})$ are reexpressed in terms of the positive ones through a Taylor expansion of the corresponding generating functions.

[7] Readers unfamiliar with the associativity identity above may note that it is also satisfied by the (holomorphic) tensor product defined by $\mathcal{V}_1 \odot_\zeta \mathcal{V}_2 = \mathcal{V}_1(\zeta) \otimes \mathcal{V}_2$, where $\otimes$ is the standard tensor product, and $\mathcal{V}_1(\zeta)$ is the pull-back of $\mathcal{V}_1$ by the $\mathbf{C}^\times$-action on $U_q(L\mathfrak{g})$ by dilations.

Let $\{\xi_{i,r}, x^{\pm}_{i,r}\}_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}}$ be the loop generators of $\mathrm{Y}_\hbar(\mathfrak{g})$ (see [7], or Section 3 for definitions). On a finite-dimensional representation V, the generating series

$$\xi_i(u) = 1 + \hbar\sum_{r\geq 0}\xi_{i,r}u^{-r-1} \quad\text{and}\quad x_i^{\pm}(u) = \hbar\sum_{r\geq 0}x^{\pm}_{i,r}u^{-r-1}$$

are expansions at $u=\infty$ of End(V)-valued rational functions [13, Prop. 3.6]. Let $\sigma(\mathrm{V})\subset\mathbf{C}$ be the he set of poles of the functions $\{x_i^{\pm}(u), \xi_i(u)^{\pm 1}\}_{i\in\mathbf{I}}$ on V.

Let $s\in\mathbf{C}$ be such that $\sigma(\mathrm{V}_1)+s$ and $\sigma(\mathrm{V}_2)$ are disjoint. Then, the action of $\mathrm{Y}_\hbar(\mathfrak{g})$ on $\mathrm{V}_1\otimes_s\mathrm{V}_2$ is given by

$$\Delta_s(\xi_{i,r}) = \tau_s(\xi_{i,r})\otimes 1 + \hbar\sum_{p_1+p_2=r-1}\tau_s(\xi_{i,p_1})\otimes\xi_{i,p_2} + 1\otimes\xi_{i,r}$$

$$\Delta_s\left(x^+_{i,r}\right) = \tau_s\left(x^+_{i,r}\right)\otimes 1 + \hbar^{-1}\oint_{\mathrm{C}_2}\xi_i(v-s)\otimes x_i^+(v)v^r dv$$

$$\Delta_s\left(x^-_{i,r}\right) = \hbar^{-1}\oint_{\mathrm{C}_1}x_i^-(v-s)\otimes\xi_i(v)v^r dv + 1\otimes x^-_{i,r}$$

where τ_s is the translation automorphism of $\mathrm{Y}_\hbar(\mathfrak{g})$ given by

$$\tau_s\big(\xi_i(u)\big) = \xi_i(u-s) \quad\text{and}\quad \tau_s\big(x_i^{\pm}(u)\big) = x_i^{\pm}(u-s)$$

and $\mathrm{C}_1, \mathrm{C}_2$ are Jordan curves such that

- C_1 encloses $\sigma(\mathrm{V}_1)+s$ and none of the points in $\sigma(\mathrm{V}_2)$.
- C_2 encloses $\sigma(\mathrm{V}_2)$ and none of the points in $\sigma(\mathrm{V}_1)+s$.

As for $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$, direct inspection shows that the action of $\mathrm{Y}_\hbar(\mathfrak{g})$ on $\mathrm{V}_1\otimes_s\mathrm{V}_2$ is a rational function of s, with poles contained in $\sigma(\mathrm{V}_2)-\sigma(\mathrm{V}_1)$. Moreover, the tensor product $\otimes_s$ gives $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ the structure of a meromorphic tensor category, which is strict in the sense that, for any $\mathrm{V}_1,\mathrm{V}_2,\mathrm{V}_3\in\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$

$$(\mathrm{V}_1\otimes_{s_1}\mathrm{V}_2)\otimes_{s_2}\mathrm{V}_3 = \mathrm{V}_1\otimes_{s_1+s_2}(\mathrm{V}_2\otimes_{s_2}\mathrm{V}_3)$$

2.5. *Meromorphic tensor structure on* Γ. — Recall the notion of *non-congruent* representation of $\mathrm{Y}_\hbar(\mathfrak{g})$ [13, §5.1]. Let $\{\xi_i(u), x_i^{\pm}(u)\}_{i\in\mathbf{I}}$ be the generating functions defined in 2.4. V is called non-congruent if, for any $i\in\mathbf{I}$, the poles of $x_i^+(u)$ (resp. $x_i^-(u)$) do not differ by non-zero integers. If V is non-congruent, the monodromy of the difference equations defined by the commuting fields $\xi_i(u)$ may be used to define an action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on the vector space $\Gamma(\mathrm{V}) = \mathrm{V}$ [13].

2.6. If $V_1, V_2 \in \operatorname{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ are non-congruent, the Drinfeld tensor product $V_1 \otimes_s V_2$ is generically non-congruent in s. Our first main result is the following (see Theorem 7.3).

Theorem.

(i) *There exists a meromorphic* $GL(V_1 \otimes V_2)$*-valued function* $\mathcal{J}_{V_1,V_2}(s)$*, which is natural in* V_1, V_2*, and such that*

$$\mathcal{J}_{V_1,V_2}(s) : \Gamma(V_1) \otimes_\zeta \Gamma(V_2) \longrightarrow \Gamma(V_1 \otimes_s V_2)$$

is an isomorphism of $U_q(L\mathfrak{g})$*-modules, where* $\zeta = e^{2\pi\iota s}$*.*

(ii) $\mathcal{J}$ *is a meromorphic tensor structure on* Γ*. That is, for any non-congruent* $V_1, V_2, V_3 \in \operatorname{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$*, the following is a commutative diagram*

$$\begin{array}{ccc}
(\Gamma(V_1) \otimes_{\zeta_1} \Gamma(V_2)) \otimes_{\zeta_2} \Gamma(V_3) & = & \Gamma(V_1) \otimes_{\zeta_1\zeta_2} (\Gamma(V_2) \otimes_{\zeta_2} \Gamma(V_3)) \\
\Big\downarrow {\scriptstyle \mathcal{J}_{V_1,V_2}(s_1)\otimes \mathbf{1}} & & \Big\downarrow {\scriptstyle \mathbf{1}\otimes \mathcal{J}_{V_2,V_3}(s_2)} \\
\Gamma(V_1 \otimes_{s_1} V_2) \otimes_{\zeta_2} \Gamma(V_3) & & \Gamma(V_1) \otimes_{\zeta_1\zeta_2} \Gamma(V_2 \otimes_{s_2} V_3) \\
\Big\downarrow {\scriptstyle \mathcal{J}_{V_1\otimes_{s_1} V_2, V_3}(s_2)} & & \Big\downarrow {\scriptstyle \mathcal{J}_{V_1, V_2\otimes_{s_2} V_3}(s_1+s_2)} \\
\Gamma((V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3) & = & \Gamma(V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3))
\end{array}$$

where $\zeta_i = \exp(2\pi\iota s_i)$*.*

2.7. Just as the functor Γ is governed by the abelian, additive difference equations defined by the commuting fields $\xi_i(u)$ of the Yangian, the tensor structure $\mathcal{J}(s)$ arises from another such difference equation, namely an appropriate abelianisation of the qKZ equations on $V_1 \otimes V_2$ [11, 27].

Specifically, let

$$\mathcal{R}^0(s) = 1 + \hbar\frac{\Omega_{\mathfrak{h}}}{s} + \cdots$$

be the diagonal part in the Gauss decomposition of the universal R-matrix of $Y_\hbar(\mathfrak{g})$ acting on $V_1 \otimes V_2$, where $\Omega_{\mathfrak{h}} \in \mathfrak{h} \otimes \mathfrak{h}$ is the Cartan part of the Casimir tensor of $\mathfrak{g}$ [21]. Unlike the analogous case of $U_q(L\mathfrak{g})$ [9, 20], the expansion of $\mathcal{R}^0(s)$ does *not* converge near

$s=\infty$. Indeed, when evaluated on the tensor product of highest-weight vectors of two finite-dimensional irreducible representations of $Y_\hbar(\mathfrak{g})$, this series is given by the Stirling expansion of a ratio of Gamma functions [21, Theorem 7.2], which is known not to converge. We show, however, that $\mathcal{R}^0(s)$ possesses two *distinct*, meromorphic regularisations $\mathcal{R}^{0,\pm}(s)$ in Section 5. These are asymptotic to $\mathcal{R}^0(s)$ in the half-planes $\pm\mathrm{Re}(s/\hbar)\gg 0$, and are related by the unitarity constraint $\mathcal{R}^{0,+}(s)\mathcal{R}^{0,-}(-s)^{21}=1$.

Each $\mathcal{R}^{0,\pm}(s)$ gives rise to the abelian qKZ equation

$$\Phi^\pm(s+1)=\mathcal{R}^{0,\pm}(s)\Phi^\pm(s)$$

where $\Phi^\pm$ is an $\mathrm{End}(V_1\otimes V_2)$-valued function of s. This equation admits a canonical right fundamental solution $\Phi^\pm_+(s)$, which is holomorphic and invertible on an obtuse sector contained inside the half-plane $\mathrm{Re}(s)\gg 0$, and possesses an asymptotic expansion of the form $(1+O(s^{-1}))s^{\hbar\Omega_\mathfrak{h}}$ within it (see Proposition 7.1). The tensor structure $\mathcal{J}_{V_1,V_2}(s)$ may be taken to be $\Phi^+_+(s+1)^{-1}$ or $\Phi^-_+(s+1)^{-1}$, and is a regularisation of the infinite product

$$\cdots\mathcal{R}^{0,\pm}(s+3)\mathcal{R}^{0,\pm}(s+2)\mathcal{R}^{0,\pm}(s+1)$$

Specifically,

$$\mathcal{J}^\pm_{V_1,V_2}(s)=e^{\hbar\gamma\Omega_\mathfrak{h}}\overleftarrow{\prod_{m\geq 1}}\mathcal{R}^{0,\pm}(s+m)e^{-\frac{\hbar\Omega_\mathfrak{h}}{m}}$$

where $\gamma=\lim_n(1+1/2+\cdots+1/n-\log(n))$ is the Euler–Mascheroni constant.

2.8. *Regularisation of* $\mathcal{R}^0(s)$. — As mentioned above, the abelian R-matrix $\mathcal{R}^0(s)$ needs to be regularised. A conjectural construction of $\mathcal{R}^0(s)$ as a formal infinite product with values in the double Yangian $\mathcal{D}Y_\hbar(\mathfrak{g})$ was given by Khoroshkin–Tolstoy [21, Thm. 5.2]. To make sense of this product, we notice in Section 5 that $\mathcal{R}^0(s)$ formally satisfies an abelian additive difference equation whose step is a multiple of $\hbar$.[8] We then prove that the coefficient matrix $\mathcal{A}(s)$ of this equation can be interpreted as a rational function of s, and define $\mathcal{R}^{0,\pm}(s)$ as the canonical fundamental solutions of the difference equation. Let us outline this approach in more detail.

2.9. Let $b_{ij}=d_ia_{ij}$ be the entries of the symmetrised Cartan matrix of $\mathfrak{g}$. Let T be an indeterminate, and $\mathsf{B}(T)=([b_{ij}]_T)$ the corresponding matrix of T-numbers. Then, there exists an integer $l=mh^\vee$, which is a multiple of the dual Coxeter number $h^\vee$ of $\mathfrak{g}$, and is such that $\mathsf{B}(T)^{-1}=[l]_T^{-1}\mathsf{C}(T)$, where the entries of $\mathsf{C}(T)$ are Laurent polynomials in T with coefficients in $\mathbf{Z}_{\geq 0}$ [21].

[8] This equation should in turn be a consequence of the (non-linear) difference equation satisfied by the full R-matrix of $Y_\hbar(\mathfrak{g})$ obtained from crossing symmetry.

Consider the following $\mathrm{GL}(\mathrm{V}_1 \otimes \mathrm{V}_2)$-valued function of $s \in \mathbf{C}$

$$\mathcal{A}(s) = \exp\left(-\sum_{\substack{i,j \in \mathbf{I} \\ r \in \mathbf{Z}}} c_{ij}^{(r)} \oint_{\mathcal{C}} t_i'(v) \otimes t_j\left(v + s + \frac{(l+r)\hbar}{2} \right) dv \right)$$

where

- $c_{ij}(\mathrm{T}) = \sum_{r \in \mathbf{Z}} c_{ij}^{(r)} \mathrm{T}^r$ are the entries of $\mathsf{C}(\mathrm{T})$.
- the contour $\mathcal{C}$ encloses the poles of $\xi_i(u)^{\pm 1}$ on V_1.
- $t_i(u) = \log(\xi_i(u))$ is defined by choosing a branch of the logarithm.
- $s \in \mathbf{C}$ is such that $v \to t_j(v + s + (l+r)\hbar/2)$ is analytic on V_2 within $\mathcal{C}$, for every $j \in \mathbf{I}$ and $r \in \mathbf{Z}$ such that $c_{ij}^{(r)} \neq 0$.

We prove in Section 5.5 that $\mathcal{A}$ extends to a rational function of s which has the following expansion near $s = \infty$

$$\mathcal{A}(s) = 1 - l\hbar^2 \frac{\Omega_{\mathfrak{h}}}{s^2} + \mathrm{O}\left(s^{-3}\right)$$

2.10. The infinite product $\mathcal{R}^0(s)$ considered in [21] formally satisfies

$$\mathcal{R}^0(s + l\hbar) = \mathcal{A}(s)\mathcal{R}^0(s)$$

This difference equation is regular (that is, the coefficient of s^{-1} in the expansion of $\mathcal{A}(s)$ at $s = \infty$ is zero), and therefore admits two canonical meromorphic fundamental solutions $\mathcal{R}^{0,\pm}(s)$. The latter are uniquely determined by the requirement that they be holomorphic and invertible for $\pm\mathrm{Re}(s/\hbar) \gg 0$, and asymptotic to $1 + \mathrm{O}(s^{-1})$ as $s \to \infty$ in that domain (see *e.g.*, [2, 3, 22] or [13, §4]). Explicitly,

$$\mathcal{R}^{0,+}(s) = \overrightarrow{\prod_{n \geq 0}} \mathcal{A}(s + nl\hbar)^{-1}$$

$$\mathcal{R}^{0,-}(s) = \overrightarrow{\prod_{n \geq 1}} \mathcal{A}(s - nl\hbar)$$

The functions $\mathcal{R}^{0,\pm}(s)$ are distinct regularisations of $\mathcal{R}^0(s)$, and are related by the unitarity constraint

$$\mathcal{R}^{0,+}_{\mathrm{V}_1,\mathrm{V}_2}(s)\mathcal{R}^{0,-}_{\mathrm{V}_2,\mathrm{V}_1}(-s)^{21} = 1$$

We show in Theorem 5.9 that they define meromorphic commutativity constraints on $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ endowed with the Drinfeld tensor product $\otimes_s$.

2.11. *Kohno–Drinfeld theorem for abelian qKZ equations.* — Our second main result is a Kohno–Drinfeld theorem for the abelian, additive qKZ equations defined by $\mathcal{R}^{0,\pm}(s)$. Together with Theorem 2.6, it establishes an equivalence of meromorphic braided tensor categories between $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ akin to the Kazhdan–Lusztig equivalence between the affine Lie algebra $\widehat{\mathfrak{g}}$ and corresponding quantum group $\mathrm{U}_q\mathfrak{g}$.

Fix $\mathrm{V}_1, \ldots, \mathrm{V}_n \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$. The abelian qKZ equations are the integrable system of additive difference equations for a meromorphic function $\mathrm{F} : \mathbf{C}^n \to \mathrm{End}(\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_n)$ which are given by [11, 27]

$$\mathrm{F}(\underline{s} + e_i) = \mathrm{A}_i(\underline{s})\mathrm{F}(\underline{s}) \tag{2.1}$$

where $\underline{s} = (s_1, \ldots, s_n) \in \mathbf{C}^n$, $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbf{C}^n$, and $\mathrm{A}_i(\underline{s})$ is given by

$$\mathrm{A}_i(\underline{s}) = \mathcal{R}^{0,\pm}_{i-1,i}(s_{i-1} - s_i - 1)^{-1} \cdots \mathcal{R}^{0,\pm}_{1,i}(s_1 - s_i - 1)^{-1} \cdot \mathcal{R}^{0,\pm}_{i,n}(s_i - s_n) \cdots \mathcal{R}^{0,\pm}_{i,i+1}(s_i - s_{i+1})$$

with $\mathcal{R}^{0,\pm}_{i,j} = \mathcal{R}^{0,\pm}_{\mathrm{V}_i,\mathrm{V}_j}$ the regularisations of the commutative R-matrix of $\mathrm{Y}_\hbar(\mathfrak{g})$ described in 2.8.

These equations admit a set of fundamental solutions $\Phi^\pm_\sigma$ which generalise the right/left solutions in the $n = 2$ case. They are parametrised by permutations $\sigma \in \mathfrak{S}_n$, and have prescribed asymptotic behaviour when $s_i - s_j \to \infty$ for any $i < j$, in such a way that $\mathrm{Re}(s_{\sigma^{-1}(i)} - s_{\sigma^{-1}(j)}) \gg 0$. By definition, the monodromy of (2.1) is the 2-cocycle on $\mathfrak{S}_n$ with values in the group of meromorphic $\mathrm{GL}(\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_n)$-valued functions of the variables $\zeta_i = e^{2\pi\iota s_i}$ given by

$$\mathrm{M}^\pm_{\sigma,\tau}(\underline{s}) = \left(\Phi^\pm_\sigma(\underline{s})\right)^{-1} \cdot \Phi^\pm_\tau(\underline{s})$$

2.12. A Kohno–Drinfeld theorem for the qKZ equations determined by the full (non-abelian) R-matrix of $\mathrm{Y}_\hbar(\mathfrak{g})$ was conjectured by Frenkel–Reshetikhin [11, §6]. It states that the monodromy of (2.1), with $\mathcal{R}^0$ replaced by $\mathcal{R}$, is given by the universal R-matrix $\mathscr{R}(\zeta)$ of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ acting on a tensor product of suitable q-deformations of $\mathrm{V}_1, \ldots, \mathrm{V}_n$.

Assuming that $|q| \neq 1$, we prove this theorem for the abelian qKZ equations determined by $\mathcal{R}^{0,\pm}$. To this end, we first construct the commutative part $\mathscr{R}^0(\zeta)$ of the R-matrix of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ in Section 8 by following a procedure similar to that described in 2.8–2.10. Namely, we start from Damiani's formula for $\mathscr{R}^0(\zeta)$ [4], show that it formally satisfies a regular q-difference equation with respect to the parameter ζ, and deduce from this that it is the expansion at $\zeta = 0$ of the corresponding canonical solution (unlike the case of $\mathrm{Y}_\hbar(\mathfrak{g})$, no regularisation of $\mathscr{R}^0(\zeta)$ is necessary here). We also show that $\mathscr{R}^0(\zeta)$ defines meromorphic commutativity constraints on $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ endowed with the deformed Drinfeld coproduct.

We then prove the following (Theorem 9.6)

Theorem. — *Assume that* $|q| \neq 1$, *and set*

$$\varepsilon = \begin{cases} + & \text{if } |q| < 1 \\ - & \text{if } |q| > 1 \end{cases}$$

Let $\mathrm{V}_1, \dots, \mathrm{V}_n \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ *be non-congruent, and let* $\mathcal{V}_\ell = \Gamma(\mathrm{V}_\ell)$ *be the corresponding representations of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$.

Then, the monodromy of the abelian qKZ equations determined by $\mathcal{R}^{0,\varepsilon}(s)$ *on* $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_n$ *is given by* $\mathscr{R}^0(\zeta)$. *Specifically, the following holds for any* $\sigma \in \mathfrak{S}_n$ *and* $i = 1, \dots, n-1$ *such that* $\sigma^{-1}(i) < \sigma^{-1}(i+1)$,

$$\left(\Phi^{\varepsilon}_{\sigma}(\underline{s})\right)^{-1} \cdot \Phi^{\varepsilon}_{(i,\ i+1)\sigma}(\underline{s}) = \mathscr{R}^0_{\mathcal{V}_i, \mathcal{V}_{i+1}}\left(\zeta_i \zeta_{i+1}^{-1}\right)$$

The same result holds for the monodromy of the qKZ equations determined by $\mathcal{R}^{0,-\varepsilon}(s)$, provided $\mathscr{R}^0(\zeta)$ is replaced by $\mathscr{R}^0_{21}(\zeta^{-1})^{-1}$.[9]

2.13. *Relation to the Kac–Moody coproduct.* — We conjecture that the twist $\mathcal{J}(s)$ also yields a non-meromorphic tensor structure on the functor Γ, when the categories $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ are endowed with the standard monoidal structures arising from the Kac–Moody coproducts on $\mathrm{Y}_\hbar(\mathfrak{g}), \mathrm{U}_q(\mathrm{L}\mathfrak{g})$.

More precisely, the Drinfeld and Kac–Moody coproducts on $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ are related by a meromorphic twist, given by the lower triangular part $\mathscr{R}_-^{\mathrm{U}_q(\mathrm{L}\mathfrak{g})}(\zeta)$ of the universal R-matrix [8]. A similar statement holds for $\mathrm{Y}_\hbar(\mathfrak{g})$ [12]. Composing, we obtain a meromorphic tensor structure $\mathrm{J}(s)$ on Γ relative to the standard monoidal structures

$$\begin{array}{ccc}
\Gamma(\mathrm{V}_1)(\zeta) \otimes \Gamma(\mathrm{V}_2) & \xrightarrow{\ \mathcal{R}_-^{\mathrm{U}_q(\mathrm{L}\mathfrak{g})}(\zeta)\ } & \Gamma(\mathrm{V}_1) \otimes_\zeta \Gamma(\mathrm{V}_2) \\
\Big\downarrow {\scriptstyle \mathrm{J}_{\mathrm{V}_1,\mathrm{V}_2}(s)} & & \Big\downarrow {\scriptstyle \mathcal{J}_{\mathrm{V}_1,\mathrm{V}_2}(s)} \\
\Gamma(\mathrm{V}_1(s) \otimes \mathrm{V}_2) & \xrightarrow{\ \mathcal{R}_-^{\mathrm{Y}_\hbar(\mathfrak{g})}(s)\ } & \Gamma(\mathrm{V}_1 \otimes_s \mathrm{V}_2)
\end{array}$$

We conjecture that $\mathrm{J}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ is holomorphic in s, and can therefore be evaluated at $s = 0$, thus yielding a tensor structure on Γ with respect to the standard coproducts. We will return to this in [12].

[9] Theorem 9.6 contains both of these statements in a uniform fashion. Thus $\mathscr{R}^0(\zeta)$ of Theorem 2.12 above is $\mathscr{R}^{0,\varepsilon}(\zeta)$ of Theorem 9.6 with $\varepsilon = \pm$ according to the statement above.

2.14. *Extension to arbitrary Kac–Moody algebras.* — The results of [13] hold for an arbitrary symmetrisable Kac–Moody algebra $\mathfrak{g}$, provided one considers the categories of representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ whose restriction to $\mathfrak{g}$ and $U_q\mathfrak{g}$ respectively are integrable and in category $\mathcal{O}$. Although we restricted ourselves to the case of a finite-dimensional semisimple $\mathfrak{g}$ in this paper, our results on the Drinfeld coproducts of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ are valid for an arbitrary $\mathfrak{g}$, and it seems likely that the same should hold for the construction of the tensor structure $\mathcal{J}(s)$. The main obstacle in working in this generality is the construction and regularisation of $\mathcal{R}^0(s)$ for an arbitrary $\mathfrak{g}$. Once this is achieved, the proof of Theorems 2.6 and 2.12 carries over verbatim.

2.15. *Outline of the paper.* — In Section 3, we review the definitions of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$. Section 4 is devoted to defining the Drinfeld coproducts on $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$. We give a construction of the diagonal part R^0 of the R-matrix of $Y_\hbar(\mathfrak{g})$ in Section 5. Section 6 reviews the definition of the functor Γ given in [13]. The construction of a meromorphic tensor structure on Γ is given in Section 7. In Section 8, we show that, when $|q| \neq 1$, the commutative part $\mathscr{R}^0(\zeta)$ of the R-matrix of $U_q(L\mathfrak{g})$ defines a meromorphic commutativity constraint on $\mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$. Finally, in Section 9, we prove a Kohno–Drinfeld theorem for the abelian qKZ equations defined by $\mathcal{R}^0(s)$. Appendix A gives the inverses of all symmetrised q-Cartan matrices of finite type.

3. Yangians and quantum loop algebras

3.1. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra and $(\cdot,\cdot)$ the invariant bilinear form on $\mathfrak{g}$ normalised so that the squared length of short roots is 2. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ a basis of simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $a_{ij} = 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$ the entries of the corresponding Cartan matrix A. Set $d_i = (\alpha_i,\alpha_i)/2 \in \{1,2,3\}$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i,j \in \mathbf{I}$.

3.2. *The Yangian* $Y_\hbar(\mathfrak{g})$. — Let $\hbar \in \mathbf{C}$. The Yangian $Y_\hbar(\mathfrak{g})$ is the unital, associative $\mathbf{C}$-algebra generated by elements $\{x_{i,r}^{\pm}, \xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}}$, subject to the following relations

(Y1) For any $i,j \in \mathbf{I}$, $r,s \in \mathbf{Z}_{\geq 0}$

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

(Y2) For $i,j \in \mathbf{I}$ and $s \in \mathbf{Z}_{\geq 0}$

$$\left[\xi_{i,0}, x_{j,s}^{\pm}\right] = \pm d_i a_{ij} x_{j,s}^{\pm}$$

(Y3) For $i,j \in \mathbf{I}$ and $r,s \in \mathbf{Z}_{\geq 0}$

$$\left[\xi_{i,r+1}, x_{j,s}^{\pm}\right] - \left[\xi_{i,r}, x_{j,s+1}^{\pm}\right] = \pm\hbar\frac{d_i a_{ij}}{2}\left(\xi_{i,r}x_{j,s}^{\pm} + x_{j,s}^{\pm}\xi_{i,r}\right)$$

(Y4) For $i,j \in \mathbf{I}$ and $r,s \in \mathbf{Z}_{\geq 0}$

$$\left[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}\right] - \left[x^{\pm}_{i,r}, x^{\pm}_{j,s+1}\right] = \pm\hbar\frac{d_i a_{ij}}{2}\left(x^{\pm}_{i,r}x^{\pm}_{j,s} + x^{\pm}_{j,s}x^{\pm}_{i,r}\right)$$

(Y5) For $i,j \in \mathbf{I}$ and $r,s \in \mathbf{Z}_{\geq 0}$

$$\left[x^{+}_{i,r}, x^{-}_{j,s}\right] = \delta_{ij}\xi_{i,r+s}$$

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \cdots, r_m \in \mathbf{Z}_{\geq 0}$ and $s \in \mathbf{Z}_{\geq 0}$

$$\sum_{\pi \in \mathfrak{S}_m}\left[x^{\pm}_{i,r_{\pi(1)}}, \left[x^{\pm}_{i,r_{\pi(2)}}, \left[\cdots, \left[x^{\pm}_{i,r_{\pi(m)}}, x^{\pm}_{j,s}\right]\cdots\right]\right]\right] = 0$$

3.3. *Remark.* — By [23, Lemma 1.9], the relation (Y6) follows from (Y1)–(Y3) and the special case of (Y6) when $r_1 = \cdots = r_m = 0$. In turn, the latter automatically holds on finite-dimensional representations of the algebra defined by relations (Y2) and (Y5) alone (see, *e.g.*, [13, Prop. 2.7]). Thus, a finite-dimensional representation V of $Y_\hbar(\mathfrak{g})$ is given by operators $\{\xi_{i,r}, x^{\pm}_{i,r}\}_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}}$ in End(V) satisfying relations (Y1)–(Y5).

3.4. Assume henceforth that $\hbar \neq 0$, and define $\xi_i(u), x^{\pm}_i(u) \in Y_\hbar(\mathfrak{g})[[u^{-1}]]$ by

$$\xi_i(u) = 1 + \hbar\sum_{r\geq 0}\xi_{i,r}u^{-r-1} \quad \text{and} \quad x^{\pm}_i(u) = \hbar\sum_{r\geq 0}x^{\pm}_{i,r}u^{-r-1}$$

For an associative algebra A, we denote by $A[u,v;u^{-1},v^{-1}]]$ the algebra of formal series $\sum_{r,s} a_{r,s}u^r v^s$ for which there exist $M, N \in \mathbf{Z}$ such that $a_{r,s} \neq 0$ implies $r \leq M$ and $s \leq N$.

Proposition [13, Prop. 2.3]. — The relations (Y1), (Y2)–(Y3), (Y4), (Y5) and (Y6) are respectively equivalent to the following identities in $Y_\hbar(\mathfrak{g})[u,v;u^{-1},v^{-1}]]$

($\mathcal{Y}$1) *For any* $i,j \in \mathbf{I}$,

$$\left[\xi_i(u), \xi_j(v)\right] = 0$$

($\mathcal{Y}$2) *For any* $i,j \in \mathbf{I}$,

$$\left[\xi_{i,0}, x^{\pm}_j(u)\right] = \pm d_i a_{ij} x^{\pm}_j(u)$$

($\mathcal{Y}$3) *For any* $i,j \in \mathbf{I}$, *and* $a = \hbar d_i a_{ij}/2$

$$(u - v \mp a)\xi_i(u)x^{\pm}_j(v) = (u - v \pm a)x^{\pm}_j(v)\xi_i(u) \mp 2a x^{\pm}_j(u \mp a)\xi_i(u)$$

($\mathcal{Y}4$) *For any* $i,j\in\mathbf{I}$, *and* $a=\hbar d_i a_{ij}/2$

$$\begin{aligned}&(u-v\mp a)x_i^\pm(u)x_j^\pm(v)\\&\quad=(u-v\pm a)x_j^\pm(v)x_i^\pm(u)+\hbar\Big(\big[x_{i,0}^\pm, x_j^\pm(v)\big]-\big[x_i^\pm(u), x_{j,0}^\pm\big]\Big)\end{aligned}$$

($\mathcal{Y}5$) *For any* $i,j\in\mathbf{I}$

$$(u-v)\big[x_i^+(u), x_j^-(v)\big]=-\delta_{ij}\hbar\big(\xi_i(u)-\xi_i(v)\big)$$

($\mathcal{Y}6$) *For any* $i\neq j\in\mathbf{I}$, $m=1-a_{ij}$,

$$\sum_{\pi\in\mathfrak{S}_m}\Big[x_i^\pm(u_{\pi(1)}),\Big[x_i^\pm(u_{\pi(2)}),\Big[\cdots,\Big[x_i^\pm(u_{\pi(m)}),x_j^\pm(v)\Big]\cdots\Big]\Big]\Big]=0$$

Remark. — Taking the coefficient of u^0 in relation ($\mathcal{Y}3$) gives

$$\hbar\xi_{i,0}x_j^\pm(v)-vx_j^\pm(v)\mp ax_j^\pm(v)=\hbar x_j^\pm(v)\xi_{i,0}-vx_j^\pm(v)\pm ax_j^\pm(v)$$

Thus we get $[\xi_{i,0}, x_j^\pm(v)]=\pm d_i a_{ij} x_j^\pm(v)$ which is relation ($\mathcal{Y}2$).

3.5. *Shift automorphism.* — The group of translations of the complex plane acts on $\mathrm{Y}_\hbar(\mathfrak{g})$ by

$$\tau_a(y_r)=\sum_{s=0}^{r}\binom{r}{s}a^{r-s}y_s$$

where $a\in\mathbf{C}$, y is one of ξ_i, $x_i^\pm$. In terms of the generating series introduced in 3.4,

$$\tau_a\big(y(u)\big)=y(u-a)$$

Given a representation V of $\mathrm{Y}_\hbar(\mathfrak{g})$ and $a\in\mathbf{C}$, set $\mathrm{V}(a)=\tau_a^*(\mathrm{V})$.

3.6. *Quantum loop algebra* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$. — Let $q\in\mathbf{C}^\times$ be of infinite order. For any $i\in\mathbf{I}$, set $q_i=q^{d_i}$. We use the standard notation for Gaussian integers

$$[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$$

$$[n]_q!=[n]_q[n-1]_q\cdots[1]_q \qquad \begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$$

The quantum loop algebra $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ is the unital, associative $\mathbf{C}$-algebra generated by elements $\{\Psi_{i,\pm r}^\pm\}_{i\in\mathbf{I},r\in\mathbf{Z}_{\geq 0}}$, $\{\mathcal{X}_{i,k}^\pm\}_{i\in\mathbf{I},k\in\mathbf{Z}}$, subject to the following relations

(QL1) For any $i, j \in \mathbf{I}$, $r, s \in \mathbf{Z}_{\geq 0}$,

$$\left[\Psi^{\pm}_{i,\pm r}, \Psi^{\pm}_{j,\pm s}\right] = 0 \qquad \left[\Psi^{\pm}_{i,\pm r}, \Psi^{\mp}_{j,\mp s}\right] = 0 \qquad \Psi^{\pm}_{i,0}\Psi^{\mp}_{i,0} = 1$$

(QL2) For any $i, j \in \mathbf{I}$, $k \in \mathbf{Z}$,

$$\Psi^{+}_{i,0}\mathcal{X}^{\pm}_{j,k}\Psi^{-}_{i,0} = q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,k}$$

(QL3) For any $i, j \in \mathbf{I}$, $\varepsilon \in \{\pm\}$ and $l \in \mathbf{Z}$

$$\Psi^{\varepsilon}_{i,k+1}\mathcal{X}^{\pm}_{j,l} - q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,l}\Psi^{\varepsilon}_{i,k+1} = q_i^{\pm a_{ij}}\Psi^{\varepsilon}_{i,k}\mathcal{X}^{\pm}_{j,l+1} - \mathcal{X}^{\pm}_{j,l+1}\Psi^{\varepsilon}_{i,k}$$

for any $k \in \mathbf{Z}_{\geq 0}$ if $\varepsilon = +$ and $k \in \mathbf{Z}_{<0}$ if $\varepsilon = -$

(QL4) For any $i, j \in \mathbf{I}$ and $k, l \in \mathbf{Z}$

$$\mathcal{X}^{\pm}_{i,k+1}\mathcal{X}^{\pm}_{j,l} - q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k+1} = q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{i,k}\mathcal{X}^{\pm}_{j,l+1} - \mathcal{X}^{\pm}_{j,l+1}\mathcal{X}^{\pm}_{i,k}$$

(QL5) For any $i, j \in \mathbf{I}$ and $k, l \in \mathbf{Z}$

$$\left[\mathcal{X}^{+}_{i,k}, \mathcal{X}^{-}_{j,l}\right] = \delta_{ij}\frac{\Psi^{+}_{i,k+l} - \Psi^{-}_{i,k+l}}{q_i - q_i^{-1}}$$

where $\Psi^{\pm}_{i,\mp k} = 0$ for any $k \geq 1$.

(QL6) For any $i \neq j \in \mathbf{I}$, $m = 1 - a_{ij}$, $k_1, \ldots, k_m \in \mathbf{Z}$ and $l \in \mathbf{Z}$

$$\sum_{\pi \in \mathfrak{S}_m}\sum_{s=0}^{m}(-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \mathcal{X}^{\pm}_{i,k_{\pi(1)}} \cdots \mathcal{X}^{\pm}_{i,k_{\pi(s)}}\mathcal{X}^{\pm}_{j,l}\mathcal{X}^{\pm}_{i,k_{\pi(s+1)}} \cdots \mathcal{X}^{\pm}_{i,k_{\pi(m)}} = 0$$

3.7. *Remark.* — By [13, Lemma 2.12], the relation (QL6) follows from (QL1)–(QL3) and the special case of (QL6) when $k_1 = \cdots = k_m = 0$. In turn, the latter automatically holds on finite-dimensional representations of the algebra defined by relations (QL2) and (QL5) alone (see, *e.g.*, [13, Prop. 2.13]). Thus, a finite-dimensional representation $\mathcal{V}$ of $U_q(L\mathfrak{g})$ is given by operators $\{\Psi^{\pm}_{i,\pm r}, \mathcal{X}^{\pm}_{i,k}\}_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}, k\in\mathbf{Z}}$ in $\mathrm{End}(\mathcal{V})$ satisfying relations (QL1)–(QL5).

3.8. Define $\Psi_i(z)^+, \mathcal{X}^{\pm}_i(z)^+ \in U_q(L\mathfrak{g})[[z^{-1}]]$ and $\Psi_i(z)^-, \mathcal{X}^{\pm}_i(z)^- \in U_q(L\mathfrak{g})[[z]]$ by

$$\Psi_i(z)^+ = \sum_{r\geq 0}\Psi^{+}_{i,r}z^{-r} \qquad \Psi_i(z)^- = \sum_{r\leq 0}\Psi^{-}_{i,r}z^{-r}$$

$$\mathcal{X}^{\pm}_i(z)^+ = \sum_{r\geq 0}\mathcal{X}^{\pm}_{i,r}z^{-r} \qquad \mathcal{X}^{\pm}_i(z)^- = -\sum_{r<0}\mathcal{X}^{\pm}_{i,r}z^{-r}$$

Proposition [13, Prop. 2.7]. — The relations (QL1), (QL2)–(QL3), (QL4), (QL5), (QL6) imply the following relations in $\mathrm{U}_q(\mathrm{L}\mathfrak{g})[z,w;z^{-1},w^{-1}]]$

$(\mathcal{QL}1)$ *For any* $i,j\in\mathbf{I}$,

$$\left[\Psi_i(z)^+,\Psi_j(w)^+\right]=0$$

$(\mathcal{QL}2)$ *For any* $i,j\in\mathbf{I}$,

$$\Psi_{i,0}^+\mathcal{X}_j^\pm(z)^+\Psi_{i,0}^-=q_i^{\pm a_{ij}}\mathcal{X}_j^\pm(z)^+$$

$(\mathcal{QL}3)$ *For any* $i,j\in\mathbf{I}$

$$\begin{aligned}&\left(z-q_i^{\pm a_{ij}}w\right)\Psi_i(z)^+\mathcal{X}_j^\pm(w)^+\\&\quad=\left(q_i^{\pm a_{ij}}z-w\right)\mathcal{X}_j^\pm(w)^+\Psi_i(z)^+\\&\qquad-\left(q_i^{\pm a_{ij}}-q_i^{\mp a_{ij}}\right)q_i^{\pm a_{ij}}w\mathcal{X}_j^\pm\left(q_i^{\mp a_{ij}}z\right)^+\Psi_i(z)^+\end{aligned}$$

$(\mathcal{QL}4)$ *For any* $i,j\in\mathbf{I}$

$$\begin{aligned}&\left(z-q_i^{\pm a_{ij}}w\right)\mathcal{X}_i^\pm(z)^+\mathcal{X}_j^\pm(w)^+-\left(q_i^{\pm a_{ij}}z-w\right)\mathcal{X}_j^\pm(w)^+\mathcal{X}_i^\pm(z)^+\\&\quad=z\left(\mathcal{X}_{i,0}^\pm\mathcal{X}_j^\pm(w)^+-q_i^{\pm a_{ij}}\mathcal{X}_j^\pm(w)^+\mathcal{X}_{i,0}^\pm\right)\\&\qquad+w\left(\mathcal{X}_{j,0}^\pm\mathcal{X}_i^\pm(z)^+-q_i^{\pm a_{ij}}\mathcal{X}_i^\pm(z)^+\mathcal{X}_{j,0}^\pm\right)\end{aligned}$$

$(\mathcal{QL}5)$ *For any* $i,j\in\mathbf{I}$

$$\begin{aligned}&(z-w)\left[\mathcal{X}_i^+(z)^+,\mathcal{X}_j^-(w)^+\right]\\&\quad=\frac{\delta_{ij}}{q_i-q_i^{-1}}\left(z\Psi_i(w)^+-w\Psi_i(z)^+-(z-w)\Psi_{i,0}^-\right)\end{aligned}$$

$(\mathcal{QL}6)$ *For any* $i\neq j\in\mathbf{I}$, *and* $m=1-a_{ij}$

$$\begin{aligned}&\sum_{\pi\in\mathfrak{S}_m}\sum_{s=0}^m(-1)^s\begin{bmatrix}m\\s\end{bmatrix}_{q_i}\mathcal{X}_i^\pm(z_{\pi(1)})^+\cdots\mathcal{X}_i^\pm(z_{\pi(s)})^+\mathcal{X}_j^\pm(w)^+\\&\quad\cdot\mathcal{X}_i^\pm(z_{\pi(s+1)})^+\cdots\mathcal{X}_i^\pm(z_{\pi(m)})^+=0\end{aligned}$$

3.9. *Shift automorphism.* — The group $\mathbf{C}^\times$ of dilations of the complex plane acts on $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ by

$$\tau_\alpha(\mathrm{Y}_k)=\alpha^k\mathrm{Y}_k$$

where $\alpha\in\mathbf{C}^\times$, Y is one of $\Psi_i^\pm,\mathcal{X}_i^\pm$. In terms of the generating series of 3.8, we have

$$\tau_\alpha\left(\mathrm{Y}(z)^\pm\right)=\mathrm{Y}\left(\alpha^{-1}z\right)^\pm$$

Given a representation $\mathcal{V}$ of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ and $\alpha\in\mathbf{C}^\times$, we denote $\tau_\alpha^*(\mathcal{V})$ by $\mathcal{V}(\alpha)$.

3.10. *Rationality.* — The following rationality property is due to Beck–Kac [1] and Hernandez [15] for $U_q(L\mathfrak{g})$ and to the authors for $Y_\hbar(\mathfrak{g})$. In the form below, the result appears in [13, Prop. 3.6].

Proposition.

(i) *Let* V *be a* $Y_\hbar(\mathfrak{g})$*-module on which the operators* $\{\xi_{i,0}\}_{i\in\mathbf{I}}$ *act semisimply with finite-dimensional weight spaces. Then, for every weight* μ *of* V*, the generating series*

$$\xi_i(u) \in \mathrm{End}(V_\mu)[[u^{-1}]] \quad \textit{and} \quad x_i^\pm(u) \in \mathrm{Hom}(V_\mu, V_{\mu\pm\alpha_i})[[u^{-1}]]$$

defined in 3.4 are the expansions at ∞ *of rational functions of* u*. Specifically, let* $t_{i,1} = \xi_{i,1} - \frac{\hbar}{2}\xi_{i,0}^2 \in Y_\hbar(\mathfrak{g})^{\mathfrak{h}}$*. Then,*

$$x_i^\pm(u) = 2d_i\hbar u^{-1}\left(2d_i \mp \frac{\mathrm{ad}(t_{i,1})}{u}\right)^{-1} x_{i,0}^\pm$$

and

$$\xi_i(u) = 1 + \left[x_i^+(u), x_{i,0}^-\right]$$

(ii) *Let* $\mathcal{V}$ *be a* $U_q(L\mathfrak{g})$*-module on which the operators* $\{\Psi_{i,0}^\pm\}_{i\in\mathbf{I}}$ *act semisimply with finite-dimensional weight spaces. Then, for every weight* μ *of* $\mathcal{V}$ *and* $\varepsilon \in \{\pm\}$*, the generating series*

$$\Psi_i(z)^\pm \in \mathrm{End}(\mathcal{V}_\mu)[[z^{\mp 1}]] \quad \textit{and} \quad \mathcal{X}_i^\varepsilon(z)^\pm \in \mathrm{Hom}(\mathcal{V}_\mu, \mathcal{V}_{\mu+\varepsilon\alpha_i})[[z^{\mp 1}]]$$

defined in 3.8 are the expansions of rational functions $\Psi_i(z)$, $\mathcal{X}_i^\varepsilon(z)$ *at* $z=\infty$ *and* $z=0$*. Specifically, let* $H_{i,\pm 1}^\pm = \pm\Psi_{i,0}^\mp\Psi_{i,\pm 1}^\pm/(q_i - q_i^{-1})$*. Then,*

$$\mathcal{X}_i^\varepsilon(z) = \left(1 - \varepsilon\frac{\mathrm{ad}(H_{i,1}^+)}{[2]_{q_i} z}\right)^{-1} \mathcal{X}_{i,0}^\varepsilon$$

$$= -z\left(1 - \varepsilon z\frac{\mathrm{ad}(H_{i,-1}^-)}{[2]_{q_i}}\right)^{-1} \mathcal{X}_{i,-1}^\varepsilon$$

and

$$\Psi_i(z) = \Psi_{i,0}^- + \left(q_i - q_i^{-1}\right)\left[\mathcal{X}_i^+(z), \mathcal{X}_{i,0}^-\right]$$

3.11. *Poles of finite-dimensional representations.* — By Proposition 3.10, we can define, for a given $V \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, a subset $\sigma(V) \subset \mathbf{C}$ consisting of the poles of the rational functions $\xi_i(u)^{\pm 1}$, $x_i^\pm(u)$.

Similarly, for any $\mathcal{V} \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, we define a subset $\sigma(\mathcal{V}) \subset \mathbf{C}^\times$ consisting of the poles of the functions $\Psi_i(z)^{\pm 1}$, $\mathcal{X}_i^\pm(z)$.

3.12. The following is a direct consequence of Proposition 3.10 and contour deformation. We set $\oint_{\mathcal{C}} f = \frac{1}{2\pi\iota}\int_{\mathcal{C}} f$.

Corollary.

(i) *Let* $V \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ *and* $\mathcal{C} \subset \mathbf{C}$ *be a Jordan curve enclosing* $\sigma(V)$.[10] *Then, the following holds on* V *for any* $r \in \mathbf{Z}_{\geq 0}$

$$x_{i,r}^{\pm} = \frac{1}{\hbar}\oint_{\mathcal{C}} x_i^{\pm}(u)u^r\,du \quad \textit{and} \quad \xi_{i,r} = \frac{1}{\hbar}\oint_{\mathcal{C}} \xi_i(u)u^r\,du$$

(ii) *Let* $\mathcal{V} \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ *and* $\mathcal{C} \subset \mathbf{C}^\times$ *be a Jordan curve enclosing* $\sigma(\mathcal{V})$ *and not enclosing* 0. *Then, the following holds on* $\mathcal{V}$ *for any* $k \in \mathbf{Z}$ *and* $r \in \mathbf{Z}_{>0}$

$$\mathcal{X}_{i,k}^{\pm} = \oint_{\mathcal{C}} \mathcal{X}_i^{\pm}(z)z^{k-1}\,dz \qquad \Psi_{i,\pm r}^{\pm} = \pm\oint_{\mathcal{C}} \Psi_i(z)z^{\pm r-1}\,dz$$

and

$$\oint_{\mathcal{C}} \Psi_i(z)\frac{dz}{z} = \Psi_{i,0}^{+} - \Psi_{i,0}^{-}$$

3.13. The following result will be needed later.

Lemma. — *Let* V *be a finite-dimensional representation of* $Y_\hbar(\mathfrak{g})$ *and* $i,j \in \mathbf{I}$. *If* u_0 *is a pole of* $x_j^{\pm}(u)$, *then* $u_0 \pm \frac{\hbar d_i a_{ij}}{2}$ *are poles of* $\xi_i(u)^{\pm 1}$.

Proof. — Consider the relation $(\mathcal{Y}3)$ of Proposition 3.4 (here $a = \hbar d_i a_{ij}/2$).

$$\mathrm{Ad}\big(\xi_i(u)\big)x_j^{+}(v) = \frac{u-v+a}{u-v-a}x_j^{+}(v) - \frac{2a}{u-v-a}x_j^{+}(u-a) \tag{3.1}$$

Set $v = u + a$ to get $\mathrm{Ad}(\xi_i(u))x_j^{+}(u+a) = x_j^{+}(u-a)$. Combining this with Equation (3.1) above we get:

$$\mathrm{Ad}\big(\xi_i(u)\big)^{-1}x_j^{+}(v) = \frac{u-v-a}{u-v+a}x_j^{+}(v) + \frac{2a}{u-v+a}x_j^{+}(u+a) \tag{3.2}$$

Differentiating (3.2) with respect to v and then setting $v = u - a$ yields

$$2a\big(\mathrm{Ad}\big(\xi_i(u)\big)\big)^{-1}\left(\frac{d}{du}x_j^{+}(u-a)\right) = x_j^{+}(u+a) - x_j^{+}(u-a) \tag{3.3}$$

[10] By a Jordan curve, we shall mean a disjoint union of simple, closed curves the inner domains of which are pairwise disjoint.

Differentiating (3.1) with respect to u, and combining equations (3.2), (3.3) with the following fact

$$\frac{d}{du}\mathrm{Ad}\big(\xi_i(u)\big)x_j^+(v) = \mathrm{Ad}\big(\xi_i(u)\big)\big[\xi_i(u)^{-1}\xi_i'(u), x_j^+(v)\big]$$

shows that

$$\big[\xi_i(u)^{-1}\xi_i'(u), x_j^+(v)\big] = \left(\frac{1}{u-v+a} - \frac{1}{u-v-a}\right)x_j^+(v) + \frac{1}{u-v-a}x_j^+(u-a) - \frac{1}{u-v+a}x_j^+(u+a) \tag{3.4}$$

Thus, if $x_j^+(v)$ has a pole at u_0 of order N, then multiplying both sides by $(v-u_0)^{\mathrm{N}}$ and letting $v \to u_0$ we get:

$$\big[\xi_i(u)^{-1}\xi_i'(u), \mathrm{X}\big] = \left(\frac{1}{u-u_0+a} - \frac{1}{u-u_0-a}\right)\mathrm{X}$$

where $\mathrm{X} = (v-u_0)^{\mathrm{N}}x_j^+(v)|_{v=u_0}$. Hence the logarithmic derivative of $\xi_i(u)$ has poles at $u_0 \pm a$, which implies that $u_0 \pm a$ must be poles of $\xi_i(u)^{\pm 1}$. The argument for $x_j^-(v)$ is same as above, upon replacing a by $-a$. □

4. The Drinfeld coproduct

In this section, we review the definition of the deformed Drinfeld coproduct on $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ following [14, 15]. We then express it in terms of contour integrals, and use these to determine the poles of the coproduct as a function of the deformation parameter. By degenerating the integrals, we obtain a deformed Drinfeld coproduct for the Yangian $\mathrm{Y}_\hbar(\mathfrak{g})$. We also point out that these coproducts define a meromorphic tensor product on the category of finite-dimensional representations of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ and $\mathrm{Y}_\hbar(\mathfrak{g})$.

4.1. *Drinfeld coproduct on* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$. — Let $\mathcal{V}, \mathcal{W} \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$. Twisting Drinfeld's coproduct on $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ by the $\mathbf{C}^\times$-action on the first factor yields an action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ on $\mathcal{V}((\zeta^{-1})) \otimes \mathcal{W}$, where ζ is a formal variable [14, 15]. This action is given on the generators of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ by[11]

$$\Psi^\pm_{i,\pm m} \longrightarrow \sum_{p_1+p_2=m} \zeta^{\pm p_1}\Psi^\pm_{i,\pm p_1} \otimes \Psi^\pm_{i,\pm p_2}$$

[11] We use a different convention than [14, 15]. The coproduct $\Delta_\zeta^{(\mathrm{H})}$ given in [14, 15] yields an action on $\mathcal{V}\otimes\mathcal{W}((\zeta))$ obtained by twisting the Drinfeld coproduct by the $\mathbf{C}^\times$-action on the second tensor factor. The above action is equal to $\Delta^{(\mathrm{H})}_{\zeta^{-1}}(\tau_\zeta(\mathrm{X}))$.

$$\mathcal{X}^+_{i,k} \longrightarrow \zeta^k \mathcal{X}^+_{i,k} \otimes 1 + \sum_{l\geq 0} \zeta^{-l} \Psi^-_{i,-l} \otimes \mathcal{X}^+_{i,k+l}$$

$$\mathcal{X}^-_{i,k} \longrightarrow \sum_{l\geq 0} \zeta^{k-l} \mathcal{X}^-_{i,k-l} \otimes \Psi^+_{i,l} + 1 \otimes \mathcal{X}^-_{i,k}$$

Hernandez proved that the above formulae are the Laurent expansions at $\zeta = \infty$ of a family of actions of $U_q(L\mathfrak{g})$ on $\mathcal{V} \otimes \mathcal{W}$ the matrix coefficients of which are rational functions of ζ [15, Lemma 3.10].

4.2. Let $\mathcal{V}, \mathcal{W} \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$ be as above, and $\sigma(\mathcal{V}), \sigma(\mathcal{W}) \subset \mathbf{C}^\times$ be their sets of poles (see 3.11). Let $\zeta \in \mathbf{C}^\times$ be such that $\zeta\sigma(\mathcal{V})$ and $\sigma(\mathcal{W})$ are disjoint, and define an action of the generators of $U_q(L\mathfrak{g})$ on $\mathcal{V} \otimes \mathcal{W}$ as follows

$$\Delta_\zeta\left(\Psi^\pm_{i,\pm m}\right) = \sum_{p_1+p_2=m} \zeta^{\pm p_1} \Psi^\pm_{i,\pm p_1} \otimes \Psi^\pm_{i,\pm p_2}$$

$$\Delta_\zeta\left(\mathcal{X}^+_{i,k}\right) = \zeta^k \mathcal{X}^+_{i,k} \otimes 1 + \oint_{C_2} \Psi_i\left(\zeta^{-1} w\right) \otimes \mathcal{X}^+_i(w) w^{k-1} dw$$

$$\Delta_\zeta\left(\mathcal{X}^-_{i,k}\right) = \oint_{C_1} \mathcal{X}^-_i\left(\zeta^{-1} w\right) \otimes \Psi_i(w) w^{k-1} dw + 1 \otimes \mathcal{X}^-_{i,k}$$

where

- $C_1, C_2 \subset \mathbf{C}^\times$ are Jordan curves which do not enclose 0.
- C_1 encloses $\zeta\sigma(\mathcal{V})$ and none of the points in $\sigma(\mathcal{W})$.
- C_2 encloses $\sigma(\mathcal{W})$ and none of the points in $\zeta\sigma(\mathcal{V})$.

The above operators are holomorphic functions of $\zeta \in \mathbf{C}^\times \setminus \sigma(\mathcal{W})\sigma(\mathcal{V})^{-1}$. The corresponding generating series $\Delta_\zeta(\Psi_i(z)^\pm)$, $\Delta_\zeta(\mathcal{X}^\varepsilon_i(z)^\pm)$ are the expansions at $z = \infty, 0$ of the $\mathrm{End}(\mathcal{V} \otimes \mathcal{W})$-valued holomorphic functions

$$\Delta_\zeta\left(\Psi_i(z)\right) = \Psi_i\left(\zeta^{-1} z\right) \otimes \Psi_i(z)$$

$$\Delta_\zeta\left(\mathcal{X}^+_i(z)\right) = \mathcal{X}^+_i\left(\zeta^{-1} z\right) \otimes 1 + \oint_{C_2} \frac{zw^{-1}}{z-w} \Psi_i\left(\zeta^{-1} w\right) \otimes \mathcal{X}^+_i(w)\, dw$$

$$\Delta_\zeta\left(\mathcal{X}^-_i(z)\right) = \oint_{C_1} \frac{zw^{-1}}{z-w} \mathcal{X}^-_i\left(\zeta^{-1} w\right) \otimes \Psi_i(w)\, dw + 1 \otimes \mathcal{X}^-_i(z)$$

where the integrals are understood to mean the function of z defined for z outside of C_1, C_2. Throughout this paper, inside/outside of a Jordan curve C refers to the bounded/unbounded components of the complement $\mathbf{C} \setminus C$, and thus they exclude C itself. We shall prove below that their dependence in both ζ and z is rational.

4.3.

Theorem.

(i) *The Laurent expansion of* Δ_ζ *at* $\zeta=\infty$ *is given by the deformed Drinfeld coproduct of Section 4.1.*
(ii) Δ_ζ *defines an action of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ *on* $\mathcal{V}\otimes\mathcal{W}$. *The resulting representation is denoted by* $\mathcal{V}\otimes_\zeta\mathcal{W}$.
(iii) *The action of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ *on* $\mathcal{V}\otimes_\zeta\mathcal{W}$ *is a rational function of* ζ, *with poles contained in* $\sigma(\mathcal{W})\sigma(\mathcal{V})^{-1}$.
(iv) *The identification of vector spaces*

$$(\mathcal{V}_1\otimes_{\zeta_1}\mathcal{V}_2)\otimes_{\zeta_2}\mathcal{V}_3=\mathcal{V}_1\otimes_{\zeta_1\zeta_2}(\mathcal{V}_2\otimes_{\zeta_2}\mathcal{V}_3)$$

intertwines the action of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$.
(v) *If* $\mathcal{V}\cong\mathbf{C}$ *is the trivial representation of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$, *then*

$$\mathcal{V}\otimes_\zeta\mathcal{W}=\mathcal{W}\quad\textit{and}\quad\mathcal{W}\otimes_\zeta\mathcal{V}=\mathcal{W}(\zeta)$$

(vi) *The following holds for any* $\zeta,\zeta'\in\mathbf{C}^\times$

$$\mathcal{V}\otimes_{\zeta\zeta'}\mathcal{W}=\mathcal{V}(\zeta)\otimes_{\zeta'}\mathcal{W}\quad\textit{and}\quad\mathrm{V}(\zeta')\otimes_\zeta\mathcal{W}(\zeta')=(\mathcal{V}\otimes_\zeta\mathcal{W})(\zeta')$$

(vii) *The following holds for any* $\zeta\in\mathbf{C}^\times$

$$\sigma(\mathcal{V}\otimes_\zeta\mathcal{W})\subseteq\big(\zeta\sigma(\mathcal{V})\big)\cup\sigma(\mathcal{W})$$

Proof. — (i) Expanding $\Delta_\zeta(\Psi^\pm_{i,m})$ and $\Delta_\zeta(\mathcal{X}^\pm_{i,k})$ as Laurent series in ζ^{-1} yields the following for any $m\in\mathbf{Z}_{\geq 0}$ and $k\in\mathbf{Z}$

$$\Delta_\zeta\big(\Psi^\pm_{i,\pm m}\big)=\sum_{n=0}^{m}\zeta^{\pm n}\Psi^\pm_{i,\pm n}\otimes\Psi^\pm_{\pm(m-n)}$$

$$\begin{aligned}\Delta_\zeta\big(\mathcal{X}^+_{i,k}\big)&=\zeta^k\mathcal{X}^+_{i,k}\otimes 1+\sum_{l\geq 0}\zeta^{-l}\oint_{C_2}\Psi^-_{i,-l}\otimes\mathcal{X}^+_i(w)w^{k+l-1}dw\\&=\zeta^k\mathcal{X}^+_{i,k}\otimes 1+\sum_{l\geq 0}\zeta^{-l}\Psi^-_{i,-l}\otimes\mathcal{X}^+_{i,k+l}\end{aligned}$$

$$\begin{aligned}\Delta_\zeta\big(\mathcal{X}^-_{i,k}\big)&=\oint_{\zeta^{-1}C_1}\mathcal{X}^-_i(w)\otimes\Psi_i(\zeta w)\zeta^k w^{k-1}dw+1\otimes\mathcal{X}^-_{i,k}\\&=\sum_{l\geq 0}\zeta^{k-l}\oint_{\zeta^{-1}C_1}\mathcal{X}^-_i(w)\otimes\Psi^+_{i,l}w^{k-l-1}dw+1\otimes\mathcal{X}^-_{i,k}\\&=\sum_{l\geq 0}\zeta^{k-l}\mathcal{X}^-_{i,k-l}\otimes\Psi^+_{i,l}+1\otimes\mathcal{X}^-_{i,k}\end{aligned}$$

where the third and sixth equalities follow by Corollary 3.12, and the fourth by a change of variables. Note that C_1 is assumed to enclose $\zeta\sigma(\mathcal{V}_1)$, thus $\zeta^{-1}C_1$ in the computation of $\Delta_\zeta(\mathcal{X}_{i,k}^-)$ above encloses $\sigma(\mathcal{V}_1)$.

(ii) By Remark 3.7, it suffices to check the relations (QL1)–(QL5). These follow from (i) and [14, Prop. 6.3], since it is sufficient to prove them when ζ is a formal variable. Alternatively, a direct proof can be given along the lines of Theorem 4.6 below.

(iii) The rationality of $\mathcal{V}\otimes_\zeta\mathcal{W}$ follows from (i) and [15, Lemma 3.10]. Alternatively, let $\{w_j\}_{j\in J}\subset\mathbf{C}^\times$ be the poles of $\mathcal{X}_i^+(w)$ on $\mathcal{W}$, and

$$\mathcal{X}_i^+(w)=\mathcal{X}_{i,0}^+ + \sum_{j\in J, n\geq 1}\mathcal{X}_{i;j,n}^+(w-w_j)^{-n}$$

its corresponding partial fraction decomposition. Since C_2 encloses all w_j, and $\Psi_i(\zeta^{-1}w)w^{k-1}$ is regular inside C_2, we get

$$\Delta_\zeta\big(\mathcal{X}_{i,k}^+\big)=\zeta^k\mathcal{X}_{i,k}^+\otimes 1+\sum_{j,n}\partial_w^{(n-1)}\big(\Psi_i\big(\zeta^{-1}w\big)w^{k-1}\big)\big|_{w=w_j}\otimes\mathcal{X}_{i;j,n}^+$$

where $\partial^{(p)}=\partial^p/p!$. This is clearly a rational function of ζ, whose poles are a subset of the points $\zeta=w_jw_k'^{-1}$, where w_k' is a pole of $\Psi_i(w)$ on $\mathcal{V}$. A similar argument shows that $\Delta_\zeta(\mathcal{X}_{i,k}^-)$ is also a rational function whose poles are contained in $\sigma(\mathcal{W})\sigma(\mathcal{V})^{-1}$.

(iv) Follows from (i) and [15, Lemma 3.4].

(v), (vi) and (vii) are clear. □

4.4. *Degeneration.* — The formulae for the Drinfeld coproduct on $Y_\hbar(\mathfrak{g})$ given in 4.5 below can be formally obtained by degenerating those for the Drinfeld coproduct of $U_q(L\mathfrak{g})$ given in 4.2. This amounts to setting $z=e^{2\pi\iota\epsilon u}$, $w=e^{2\pi\iota\epsilon v}$, and letting $\epsilon\to 0$. Under this limit, the 1-form $\frac{zw^{-1}}{z-w}dw$ goes to $\frac{dv}{u-v}$. In addition, we replace the trigonometric functions $\Psi_i(z)$, $\mathcal{X}_i^\pm(z)$ coming from $U_q(L\mathfrak{g})$ by their rational counterparts $\xi_i(u)$, $x_i^\pm(u)$. This method is solely a heuristic, and a proof that the formulae given in 4.5 satisfy the relations of the Yangian $Y_\hbar(\mathfrak{g})$ is provided in 4.7–4.10.

4.5. *Drinfeld coproduct on* $Y_\hbar(\mathfrak{g})$. — Let now $V,W\in\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, and $\sigma(V)$, $\sigma(W)\subset\mathbf{C}$ be their sets of poles. Let $s\in\mathbf{C}$ be such that $\sigma(V)+s$ and $\sigma(W)$ are disjoint, and define an action of the generators of $Y_\hbar(\mathfrak{g})$ on $V\otimes W$ via

$$\Delta_s\big(\xi_i(u)\big)=\xi_i(u-s)\otimes\xi_i(u)$$

$$\Delta_s\big(x_i^+(u)\big)=x_i^+(u-s)\otimes 1+\oint_{C_2}\frac{1}{u-v}\xi_i(v-s)\otimes x_i^+(v)\,dv$$

$$\Delta_s\big(x_i^-(u)\big)=\oint_{C_1}\frac{1}{u-v}x_i^-(v-s)\otimes\xi_i(v)\,dv+1\otimes x_i^-(u)$$

where

- C_2 encloses $\sigma(W)$ and none of the points in $\sigma(V)+s$.
- C_1 encloses $\sigma(V)+s$ and none of the points in $\sigma(W)$.
- The integrals are understood to mean the holomorphic functions of u they define in the domain where u is outside of C_1, C_2.

In terms of the generators $\{\xi_{i,r}, x_{i,r}^{\pm}\}$, the above formulae read

$$\Delta_s(\xi_{i,r}) = \tau_s(\xi_{i,r}) \otimes 1 + \hbar \sum_{p_1+p_2=r-1} \tau_s(\xi_{i,p_1}) \otimes \xi_{i,p_2} + 1 \otimes \xi_{i,r}$$

$$\Delta_s\left(x_{i,r}^{+}\right) = \tau_s\left(x_{i,r}^{+}\right) \otimes 1 + \hbar^{-1} \oint_{C_2} \xi_i(v-s) \otimes x_i^{+}(v) v^r dv$$

$$\Delta_s\left(x_{i,r}^{-}\right) = \hbar^{-1} \oint_{C_1} x_i^{-}(v-s) \otimes \xi_i(v) v^r dv + 1 \otimes x_{i,r}^{-}$$

4.6.

Theorem.

(i) *The formulae in 4.5 define an action of* $Y_\hbar(\mathfrak{g})$ *on* $V \otimes W$. *The resulting representation is denoted by* $V \otimes_s W$.
(ii) *The action of* $Y_\hbar(\mathfrak{g})$ *on* $V \otimes_s W$ *is a rational function of* s, *with poles contained in* $\sigma(W) - \sigma(V)$.
(iii) *The identification of vector spaces*

$$(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 = V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)$$

intertwines the action of $Y_\hbar(\mathfrak{g})$.
(iv) *If* $V \cong \mathbf{C}$ *is the trivial representation of* $Y_\hbar(\mathfrak{g})$, *then*

$$V \otimes_s W = W \quad \textit{and} \quad W \otimes_s V = W(s)$$

(v) *The following holds for any* $s, s' \in \mathbf{C}$,

$$V \otimes_{s+s'} W = V(s) \otimes_{s'} W \quad \textit{and} \quad V\left(s'\right) \otimes_s W\left(s'\right) = (V \otimes_s W)\left(s'\right)$$

(vi) *The following holds for any* $s \in \mathbf{C}$,

$$\sigma(V \otimes_s W) \subset \left(\sigma(V) + s\right) \cup \sigma(W)$$

Proof. — (ii) is proved as in Theorem 4.3, and (iv)–(vi) are clear.

To prove (i), it suffices by Remark 3.3 to check that relations (Y1)–(Y5) hold on $V \otimes_s W$. By (v), we may assume that $\sigma(V) \cap \sigma(W) = \emptyset$, and that $s = 0$. We choose the contours C_1 and C_2 enclosing $\sigma(V)$ and $\sigma(W)$ respectively, such that they do not intersect. The relation (Y1) holds trivially. The relations (Y2) and (Y3) are checked in 4.7, (Y4) in 4.8 and (Y5) in 4.9.

(iii) is proved in 4.10. □

4.7. *Proof of (Y2) and (Y3).* — We prove these relations for the + case only. By Proposition 3.4 and Remark 3.4, it is equivalent to show that Δ_0 preserves the relation

$$\xi_i(u_1)x_j^+(u_2)\xi_i(u_1)^{-1} = \frac{u_1-u_2+a}{u_1-u_2-a}x_j^+(u_2) - \frac{2a}{u_1-u_2-a}x_j^+(u_1-a)$$

where $a=\hbar d_i a_{ij}/2$. It suffices to prove this for u_1, u_2 large enough, and we shall assume that u_2 lies outside of C_2, and that u_1 lies outside of C_2+a.

Applying Δ_0 to the left-hand side gives

$$\xi_i(u_1)x_j^+(u_2)\xi_i(u_1)^{-1}\otimes 1 + \oint_{C_2}\frac{1}{u_2-v}\xi_i(v)\otimes\xi_i(u_1)x_j^+(v)\xi_i(u_1)^{-1}\,dv$$

$$= \xi_i(u_1)x_j^+(u_2)\xi_i(u_1)^{-1}\otimes 1 + \oint_{C_2}\frac{u_1-v+a}{(u_2-v)(u_1-v-a)}\xi_i(v)\otimes x_j^+(v)\,dv$$

$$-\oint_{C_2}\frac{2a}{(u_2-v)(u_1-v-a)}\xi_i(v)\otimes x_j^+(u_1-a)\,dv$$

where the third summand is equal to zero since the integrand is regular inside C_2.

Applying now Δ_0 to the right-hand side yields

$$\xi_i(u_1)x_j^+(u_2)\xi_i(u_1)^{-1}\otimes 1$$

$$+\frac{1}{u_1-u_2-a}\oint_{C_2}\left(\frac{u_1-u_2+a}{u_2-v}-\frac{2a}{u_1-a-v}\right)\xi_i(v)\otimes x_j^+(v)\,dv$$

The equality of the two expressions now follows from the identity

$$\frac{u_1-u_2+a}{u_2-v}-\frac{2a}{u_1-a-v}=\frac{(u_1-u_2-a)(u_1+a-v)}{(u_2-v)(u_1-a-v)}$$

4.8. *Proof of (Y4).* — We check this relation for the + case only. We need to prove that Δ_0 preserves the relation

$$x^+_{i,r+1}x^+_{j,s} - x^+_{i,r}x^+_{j,s+1} - a x^+_{i,r}x^+_{j,s} = x^+_{j,s}x^+_{i,r+1} - x^+_{j,s+1}x^+_{i,r} + a x^+_{j,s}x^+_{i,r} \tag{4.1}$$

where $a=\hbar d_i a_{ij}/2$. Note that $\Delta_0(x^+_{i,m}x^+_{j,n})$ is equal to

$$x^+_{i,m}x^+_{j,n}\otimes 1 + \frac{1}{\hbar}\oint_{C_2} v^n x^+_{i,m}\xi_j(v)\otimes x_j^+(v)\,dv + \frac{1}{\hbar}\oint_{C_2} v^m \xi_i(v)x^+_{j,n}\otimes x_i^+(v)\,dv$$

$$+\frac{1}{\hbar^2}\oiint_{C_2} v_1^m v_2^n \xi_i(v_1)\xi_j(v_2)\otimes x_i^+(v_1)x_j^+(v_2)\,dv_1 dv_2$$

We now apply Δ_0 to both sides of relation (4.1), and consider the four summands of $\Delta_0(x^+_{i,m}x^+_{j,n})$ separately.

The first summand of Δ_0 of the left and right-hand sides of (4.1) are, respectively

$$\left(x^+_{i,r+1}x^+_{j,s} - x^+_{i,r}x^+_{j,s+1} - ax^+_{i,r}x^+_{j,s}\right) \otimes 1$$
$$\left(x^+_{j,s}x^+_{i,r+1} - x^+_{j,s+1}x^+_{i,r} + ax^+_{j,s}x^+_{i,r}\right) \otimes 1$$

which cancel because of (4.1).

The second summand of the left-hand side and the *third summand* of the right-hand side are, respectively

$$\frac{1}{\hbar}\oint_{C_2} v^s\left(x^+_{i,r+1} - vx^+_{i,r} - ax^+_{i,r}\right)\xi_j(v) \otimes x^+_j(v)\, dv$$
$$\frac{1}{\hbar}\oint_{C_2} v^s\xi_j(v)\left(x^+_{i,r+1} - vx^+_{i,r} + ax^+_{i,r}\right) \otimes x^+_j(v)\, dv$$

which cancel because of the following version of (Y2) and (Y3)

$$\left(x^+_{i,r+1} - vx^+_{i,r} - ax^+_{i,r}\right)\xi_j(v) = \xi_j(v)\left(x^+_{i,r+1} - vx^+_{i,r} + ax^+_{i,r}\right)$$

Similarly the third summand of the left-hand side and the second summand of the right-hand side cancel.

The fourth summands of the left and right-hand sides of (4.1) are, respectively

$$\frac{1}{\hbar^2}\oiint_{C_2} v_1^r v_2^s (v_1 - v_2 - a)\xi_i(v_1)\xi_j(v_2) \otimes x^+_i(v_1)x^+_j(v_2)\, dv_1 dv_2$$
$$\frac{1}{\hbar^2}\oiint_{C_2} v_1^r v_2^s (v_1 - v_2 + a)\xi_j(v_2)\xi_i(v_1) \otimes x^+_j(v_2)x^+_i(v_1)\, dv_1 dv_2$$

By ($\mathcal{Y}$4), their difference is equal to

$$\frac{1}{\hbar}\oiint_{C_2} v_1^r v_2^s\, \xi_i(v_1)\xi_j(v_2) \otimes \left(\left[x^+_{i,0}, x^+_j(v_2)\right] - \left[x^+_i(v_1), x^+_{j,0}\right]\right) dv_1 dv_2$$

which is equal to zero because the first (resp. second) summand is regular when v_1 (resp. v_2) lies inside C_2.

4.9. *Proof of (Y5).* — We need to check that Δ_0 preserves the relation

$$\left[x^+_i(u_1), x^-_j(u_2)\right] = -\hbar\delta_{ij}\frac{\xi_i(u_1) - \xi_i(u_2)}{u_1 - u_2}$$

As in Section 4.7 above, it suffices to prove this for u_1, u_2 large enough, and we shall assume that u_1, u_2 lie outside of C_1, C_2 respectively. Applying Δ_0 to the left-hand side yields

$$\oint_{C_1} \frac{1}{u_2 - v}\left[x_i^+(u_1), x_j^-(v)\right] \otimes \xi_j(v)\, dv + \oint_{C_2} \frac{1}{u_1 - v}\xi_i(v) \otimes \left[x_i^+(v), x_j^-(u_2)\right] dv + \mathcal{B}$$

where

$$\mathcal{B} = \oint_{C_1}\oint_{C_2} \frac{1}{(u_1 - v_2)(u_2 - v_1)}\left[\xi_i(v_2) \otimes x_i^+(v_2), x_j^-(v_1) \otimes \xi_j(v_1)\right] dv_2 dv_1$$

We shall prove below that $\mathcal{B} = 0$. Thus, by relation (Y5) the above is equal to zero if $i \neq j$. If $i = j$, it is equal to

$$\begin{aligned}
&-\oint_{C_1} \frac{\hbar}{(u_2 - v)(u_1 - v)}\big(\xi_i(u_1) - \xi_i(v)\big) \otimes \xi_i(v)\, dv \\
&-\oint_{C_2} \frac{\hbar}{(u_1 - v)(v - u_2)}\xi_i(v) \otimes \big(\xi_i(v) - \xi_i(u_2)\big)\, dv \\
&= \oint_{C_1 \sqcup C_2} \frac{\hbar}{(u_1 - v)(u_2 - v)}\xi_i(v) \otimes \xi_i(v)\, dv \\
&= \frac{\hbar}{u_1 - u_2}\big(\xi_i(u_2) \otimes \xi_i(u_2) - \xi_i(u_1) \otimes \xi_i(u_1)\big)
\end{aligned}$$

where the first equality follows because $\xi_i(u_1) \otimes \xi_i(v)$ (resp. $\xi_i(v) \otimes \xi_i(u_2)$) is regular when v is inside C_1 (resp. C_2), and the second by deformation of contours and the fact that $\xi_i(v) \otimes \xi_i(v)$ is regular outside $C_1 \sqcup C_2$.

Proof that $\mathcal{B} = 0$. — We shall need the following variant of relation ($\mathcal{Y}3$) of Proposition 3.4.

$$(u - v)\left[\xi_i(u), x_j^\pm(v)\right] = \pm a\left\{\xi_i(u), x_j^\pm(v) - x_j^\pm(u)\right\} \tag{4.2}$$

where $a = \hbar d_i a_{ij}/2$ and $\{x, y\} = xy + yx$. The integrand of $\mathcal{B}$ can be simplified in two different ways. First we write

$$\begin{aligned}
&\left[\xi_i(v_2) \otimes x_i^+(v_2), x_j^-(v_1) \otimes \xi_j(v_1)\right] \\
&\quad = \left[\xi_i(v_2), x_j^-(v_1)\right] \otimes x_i^+(v_2)\xi_j(v_1) + x_j^-(v_1)\xi_i(v_2) \otimes \left[x_i^+(v_2), \xi_j(v_1)\right]
\end{aligned}$$

Using (4.2), we get

$$\begin{aligned}\mathcal{B} = &\oint_{C_1}\oint_{C_2} \frac{a}{(u_1-v_2)(u_2-v_1)(v_1-v_2)}\\ &\times \left(\left\{\xi_i(v_2), x_j^-(v_1)-x_j^-(v_2)\right\}\otimes x_i^+(v_2)\xi_j(v_1)\right.\\ &\left.- x_j^-(v_1)\xi_i(v_2)\otimes\left\{\xi_j(v_1), x_i^+(v_2)-x_i^+(v_1)\right\}\right) dv_2dv_1\\ = &\oint_{C_1}\oint_{C_2} \frac{a}{(u_1-v_2)(u_2-v_1)(v_1-v_2)}\left(\left\{\xi_i(v_2), x_j^-(v_1)\right\}\otimes x_i^+(v_2)\xi_j(v_1)\right.\\ &\left.- x_j^-(v_1)\xi_i(v_2)\otimes\left\{\xi_j(v_1), x_i^+(v_2)\right\}\right) dv_2dv_1\\ = &\oint_{C_1}\oint_{C_2} \frac{a}{(u_1-v_2)(u_2-v_1)(v_1-v_2)}\left(\xi_i(v_2)x_j^-(v_1)\otimes x_i^+(v_2)\xi_j(v_1)\right.\\ &\left.- x_j^-(v_1)\xi_i(v_2)\otimes\xi_j(v_1)x_i^+(v_2)\right) dv_2dv_1\end{aligned}$$

where the second equality follows from the fact that $\{\xi_i(v_2), x_j^-(v_2)\}\otimes x_i^+(v_2)\xi_j(v_1)$ (resp. $x_j^-(v_1)\xi_i(v_2)\otimes\{\xi_j(v_1), x_i^+(v_1)\}$) is regular when v_1 is inside C_1 (resp. v_2 is inside C_2).

Now if we write instead

$$\begin{aligned}&\left[\xi_i(v_2)\otimes x_i^+(v_2), x_j^-(v_1)\otimes\xi_j(v_1)\right]\\ &\quad = \xi_i(v_2)x_j^-(v_1)\otimes\left[x_i^+(v_2),\xi_j(v_1)\right] + \left[\xi_i(v_2), x_j^-(v_1)\right]\otimes\xi_j(v_1)x_i^+(v_2)\end{aligned}$$

and use relation (4.2) as before, we obtain

$$\begin{aligned}\mathcal{B} = &\oint_{C_1}\oint_{C_2} \frac{-a}{(v_1-v_2)(u_1-v_2)(u_2-v_1)}\left(\xi_i(v_2)x_j^-(v_1)\otimes x_i^+(v_2)\xi_j(v_1)\right.\\ &\left.- x_j^-(v_1)\xi_i(v_2)\otimes\xi_j(v_1)x_i^+(v_2)\right) dv_2dv_1\end{aligned}$$

Thus $\mathcal{B} = -\mathcal{B}$, whence $\mathcal{B} = 0$. □

4.10. *Coassociativity.* — We need to show that the generators of $Y_\hbar(\mathfrak{g})$ act by the same operators on

$$(V_1 \otimes_{s_1} V_2)\otimes_{s_2} V_3 \quad\text{and}\quad V_1\otimes_{s_1+s_2}(V_2\otimes_{s_2} V_3)$$

The action of $\xi_i(u)$ on both modules is given by $\xi_i(u-s_1-s_2)\otimes\xi_i(u-s_2)\otimes\xi_i(u)$.

To compute the action of $x_i^+(u)$, we shall assume that s_1 and s_2 are such that $\sigma(V_1)+s_1+s_2, \sigma(V_2)+s_2$ and $\sigma(V_3)$ are all disjoint. By (vi), this implies in particular that $\sigma(V_1\otimes_{s_1} V_2)+s_2$ and $\sigma(V_3)$ are disjoint, and that so are $\sigma(V_1)+s_1+s_2$ and $\sigma(V_2\otimes_{s_2} V_3)$, so that the above tensor products are defined.

Under these assumptions, the action of $x_i^+(u)$ on $(\mathrm{V}_1 \otimes_{s_1} \mathrm{V}_2) \otimes_{s_2} \mathrm{V}_3$ is given by

$$\begin{aligned}
&\Delta_{s_1}\big(x_i^+(u-s_2)\big) \otimes 1 + \oint_{\mathrm{C}_3} \frac{1}{u-v_3} \Delta_{s_1}\big(\xi_i(v_3-s_2)\big) \otimes x_i^+(v_3)\, dv_3 \\
&= x_i^+(u-s_2-s_1) \otimes 1 \otimes 1 \\
&\quad + \oint_{\mathrm{C}_2} \frac{1}{u-s_2-v_2} \xi_i(v_2-s_1) \otimes x_i^+(v_2) \otimes 1\, dv_2 \\
&\quad + \oint_{\mathrm{C}_3} \frac{1}{u-v_3} \xi_i(v_3-s_2-s_1) \otimes \xi_i(v_3-s_2) \otimes x_i^+(v_3)\, dv_3
\end{aligned}$$

where C_3 encloses $\sigma(\mathrm{V}_3)$ and none of the points of $\sigma(\mathrm{V}_1)+s_1+s_2$ and $\sigma(\mathrm{V}_2)+s_2$, C_2 encloses $\sigma(\mathrm{V}_2)$ and none of the points of $\sigma(\mathrm{V}_1)+s_1$, and u is assumed to be outside of C_3 and C_2+s_2.

The action of $x_i^+(u)$ on $\mathrm{V}_1 \otimes_{s_1+s_2} (\mathrm{V}_2 \otimes_{s_2} \mathrm{V}_3)$ is given by

$$\begin{aligned}
&x_i^+(u-s_1-s_2) \otimes 1 \otimes 1 \\
&\quad + \oint_{\mathrm{C}_{23}} \frac{1}{u-v_{23}} \xi_i(v_{23}-s_1-s_2) \otimes \Delta_{s_2}\big(x_i^+(v_{23})\big)\, dv_{23} \\
&= x_i^+(u-s_1-s_2) \otimes 1 \otimes 1 \\
&\quad + \oint_{\mathrm{C}_{23}} \frac{1}{u-v_{23}} \xi_i(v_{23}-s_1-s_2) \otimes x_i^+(v_{23}-s_2) \otimes 1\, dv_{23} \\
&\quad + \oint_{\mathrm{C}_{23}} \oint_{\mathrm{C}_3'} \frac{1}{u-v_{23}} \frac{1}{v_{23}-v_3'} \xi_i(v_{23}-s_1-s_2) \otimes \xi\big(v_3'-s_2\big) \\
&\quad \otimes x_i^+\big(v_3'\big)\, dv_3' dv_{23}
\end{aligned}$$

where C_{23} encloses $\sigma(\mathrm{V}_2)+s_2 \cup \sigma(\mathrm{V}_3)$ and none of the points of $\sigma(\mathrm{V}_1)+s_1+s_2$, C_3' is chosen inside C_{23} and encloses $\sigma(\mathrm{V}_3)$ and none of the points of $\sigma(\mathrm{V}_2)+s_2$, and u is assumed to be outside of C_{23}.

Since the singularities of the first integrand which are contained in C_{23} lie in $\sigma(\mathrm{V}_2)+s_2$, the corresponding integral is equal to

$$\oint_{\mathrm{C}_2'} \frac{1}{u-v_2'} \xi_i\big(v_2'-s_1-s_2\big) \otimes x_i^+\big(v_2'-s_2\big) \otimes 1\, dv_2'$$

where C_2' contains $\sigma(\mathrm{V}_2)+s_2$ and none of the points of $\sigma(\mathrm{V}_1)+s_1+s_2$. On the other hand, integrating in v_{23} in the second integral yields

$$\oint_{\mathrm{C}_3'} \frac{1}{u-v_3'} \xi_i\big(v_3'-s_1-s_2\big) \otimes \xi\big(v_3'-s_2\big) \otimes x_i^+\big(v_3'\big)\, dv_3'$$

so that the two actions of $x_i^+(u)$ agree. The proof for $x_i^-(u)$ is similar.

5. The commutative R-matrix of the Yangian

In this section, we construct the commutative part $\mathcal{R}^0(s)$ of the R-matrix of the Yangian, and show that it defines meromorphic commutativity constraints on $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$, when the latter is equipped with the Drinfeld tensor product defined in Section 4.

A conjectural formula expressing $\mathcal{R}^0(s)$ as a formal infinite product with values in the *double Yangian* $\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$ was given by Khoroshkin–Tolstoy [21, Thm. 5.2]. We review their formula in Sections 5.1–5.2, and outline our own construction in 5.3. Our starting point is the observation that $\mathcal{R}^0(s)$ formally satisfies an additive difference equation whose coefficient matrix $\mathcal{A}(s)$ we show to be a rational function on finite-dimensional representations of $\mathrm{Y}_\hbar(\mathfrak{g})$. By taking the left and right canonical fundamental solutions of this equation, we construct two regularisations $\mathcal{R}^{0,\pm}(s)$ of $\mathcal{R}^0(s)$ which are meromorphic functions of the parameter s, and then show that they have the required intertwining properties with respect to the Drinfeld coproduct.

Note that Sections 5.2 and 5.3 are included solely to motivate our construction, and that the definition of $\mathcal{R}^{0,\pm}(s)$ and the proofs of its properties are independent of the results of [21]. In particular, we do not work with the double Yangian.

5.1. *The* T*-Cartan matrix of* $\mathfrak{g}$. — Let $\mathsf{A}=(a_{ij})$ be the Cartan matrix of $\mathfrak{g}$ and $\mathsf{B}=(b_{ij})$ its symmetrisation, where $b_{ij}=d_ia_{ij}$. Let T be an indeterminate, and let $\mathsf{B}(\mathrm{T})=([b_{ij}]_\mathrm{T})\in GL_{\mathbf{I}}(\mathbf{C}[\mathrm{T}^{\pm 1}])$ the corresponding matrix of T-numbers. Then, there exists an integer $l=mh^\vee$, which is a multiple of the dual Coxeter number $h^\vee$ of $\mathfrak{g}$, and is such that

$$\mathsf{B}(\mathrm{T})^{-1}=\frac{1}{[l]_\mathrm{T}}\mathsf{C}(\mathrm{T}) \tag{5.1}$$

where the entries of $\mathsf{C}(\mathrm{T})$ are Laurent polynomials in T with positive integer coefficients.[12] We denote the entries of the matrix $\mathsf{C}(\mathrm{T})$ by $c_{ij}(\mathrm{T})=\sum_{r\in\mathbf{Z}}c_{ij}^{(r)}\mathrm{T}^r$, and note that $c_{ji}(\mathrm{T})=c_{ij}(\mathrm{T})=c_{ij}(\mathrm{T}^{-1})$.

5.2. *The Khoroshkin–Tolstoy construction.* — The starting point of [21] is a conjectural presentation of the Drinfeld double $\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$ of the Yangian $\mathrm{Y}_\hbar(\mathfrak{g})$. $\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$ is generated by $\{\xi_{i,r},x_{i,r}^\pm\}_{i\in\mathbf{I},r\in\mathbf{Z}_{\geq 0}}$ and $\{\xi_{i,r},x_{i,r}^\pm\}_{i\in\mathbf{I},r\in\mathbf{Z}_{<0}}$, where the first are the generators of $\mathrm{Y}_\hbar(\mathfrak{g})$. We will not need the complete presentation of $\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$. For our purposes, it is sufficient to know that $\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$ contains the following two sets of commuting elements: $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbf{Z}_{\geq 0}}$ and $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbf{Z}_{<0}}$. Let $\mathrm{Y}_0^\pm\subset\mathcal{D}\mathrm{Y}_\hbar(\mathfrak{g})$ be the subalgebras they generate. The Hopf pairing

[12] This result is stated without proof in [21, p. 391], and proved for $\mathfrak{g}$ simply-laced in [16, Prop. 2.1]. We give a proof in Appendix A, which also corrects the values of the multiple m tabulated in [21] for the C_n and D_n series. With those corrections, the value of m for any $\mathfrak{g}$ is the ratio of the squared length of long roots and short ones.

$\langle -, - \rangle$ on $\mathcal{D}Y_\hbar(\mathfrak{g})$ restricts to a perfect pairing $Y_0^+ \otimes Y_0^- \to \mathbf{C}$, and the commutative part of the R-matrix of $Y_\hbar(\mathfrak{g})$ is given by

$$\mathcal{R}^0 = \exp\left(\sum_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}} a_{i,r}^+ \otimes a_{i,-r-1}^-\right) \tag{5.2}$$

where $\{a_{i,r}^+\}_{i\in\mathbf{I},r\in\mathbf{Z}_{\geq 0}}$ and $\{a_{i,r}^-\}_{i\in\mathbf{I},r\in\mathbf{Z}_{<0}}$ are generators of Y_0^+, Y_0^- respectively, which are primitive modulo elements which pair trivially with $Y_0^\pm$, and such that $\langle a_{i,r}^+, a_{j,-s-1}^- \rangle = \delta_{ij}\delta_{rs}$.

Constructing these generators amounts to finding formal power series

$$a_i^+(u) = \sum_{r\geq 0} a_{i,r}^+ u^{-r-1} \in Y_0^+[[u^{-1}]] \quad \text{and} \quad a_i^-(v) = \sum_{r<0} a_{i,r}^- v^{-r-1} \in Y_0^-[[v]]$$

such that $\langle a_i^+(u), a_j^-(v)\rangle = \delta_{ij}/(u-v)$. To this end, introduce the generating series

$$\xi_i^+(u) = 1 + \hbar \sum_{r\geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad \xi_i^-(v) = 1 - \hbar \sum_{r<0} \xi_{i,r} v^{-r-1}$$

Then, by definition of $\mathcal{D}Y_\hbar(\mathfrak{g})$, we have

$$\left\langle \xi_i^+(u), \xi_j^-(v) \right\rangle = \frac{u-v+a}{u-v-a} \in \mathbf{C}[[u^{-1}, v]]$$

where $a = \hbar b_{ij}/2$. Define now

$$t_i^+(u) = \log\big(\xi_i^+(u)\big) \in Y_0^+[[u^{-1}]] \quad \text{and} \quad t_i^-(v) = \log\big(\xi_i^-(v)\big) \in Y_0^-[[v]] \tag{5.3}$$

Then, it follows that

$$\left\langle t_i^+(u), t_j^-(v) \right\rangle = \log\left(\frac{u-v+a}{u-v-a}\right)$$

Indeed, $\xi_i^\pm(u)$ are group-like modulo terms which pair trivially with Y_0^+, Y_0^-, and if a, b are primitive elements of a Hopf algebra endowed with a Hopf pairing $\langle -, - \rangle$, then $\langle e^a, e^b \rangle = e^{\langle a, b\rangle}$. Differentiating with respect to u yields

$$\left\langle \frac{d}{du} t_i^+(u), t_j^-(v) \right\rangle = \frac{1}{u-v+a} - \frac{1}{u-v-a}$$

Let T be the shift operator acting on functions of v as $\mathrm{T}f(v) = f(v - \hbar/2)$. Then, the above identity may be rewritten as

$$\left\langle \frac{d}{du} t_i^+(u), t_j^-(v) \right\rangle = \big(\mathrm{T}^{b_{ij}} - \mathrm{T}^{-b_{ij}}\big)\frac{1}{u-v} = \big(\mathrm{T} - \mathrm{T}^{-1}\big)\mathsf{B}(\mathrm{T})_{ij}\frac{1}{u-v}$$

where $\mathsf{B}(\mathrm{T})$ is the matrix introduced in 5.1. It follows that if $\mathsf{D}(\mathrm{T})$ is an $\mathbf{I}\times\mathbf{I}$ matrix with entries in $\mathbf{C}[[\mathrm{T},\mathrm{T}^{-1}]]$, then

$$\sum_k \mathsf{D}(\mathrm{T})_{jk}\left\langle \frac{d}{du}t_i^+(u), t_k^-(v)\right\rangle = \left(\mathrm{T}-\mathrm{T}^{-1}\right)\left(\mathsf{D}(\mathrm{T})\mathsf{B}(\mathrm{T})\right)_{ji}\frac{1}{u-v}$$

By (5.1), choosing $\mathsf{D}(\mathrm{T}) = (\mathrm{T}^l - \mathrm{T}^{-l})^{-1}\mathsf{C}(\mathrm{T})$, and setting

$$a_i^+(u) = \frac{d}{du}t_i^+(u) \quad \text{and} \quad a_j^-(v) = \sum_{k\in\mathbf{I}}\left(\mathrm{T}^l-\mathrm{T}^{-l}\right)^{-1}\mathsf{C}(\mathrm{T})_{jk}t_k^-(v) \tag{5.4}$$

gives the sought for generators, provided one can interpret $(\mathrm{T}^l - \mathrm{T}^{-l})^{-1}$. This can be done by expanding in powers of T^l or of T^{-l}, and leads to two distinct formal expressions for $\mathcal{R}^0$ [21, (5.27)–(5.28)].

5.3. To make sense of the above construction of $\mathcal{R}^0$ on the tensor product $\mathrm{V}_1\otimes\mathrm{V}_2$ of two finite-dimensional representations of $\mathrm{Y}_\hbar(\mathfrak{g})$, we proceed as follows.

(1) By 3.10, $a_i^+(u)$ acting on V_1:

$$a_i^+(u) = \frac{d}{du}t_i^+(u) = \xi_i^+(u)'\xi_i^+(u)^{-1}$$

is a rational $\mathrm{End}(\mathrm{V}_1)$-valued function of u, regular near ∞.

(2) If $a_j^-(v)$ defined by (5.4) can be shown to be a meromorphic function of v, we may interpret the sum over r in (5.2) as the contour integral $\oint_{\mathrm{C}} a_i^+(u)\otimes a_i^-(u)\,du$, where C encloses all poles of $a_i^+(u)$ and none of those of $a_i^-(u)$.

(3) The action of $\mathcal{R}^0$ on $\mathrm{V}_1(s)\otimes\mathrm{V}_2$ would then be given by

$$\begin{aligned}\mathcal{R}^0(s) &= \exp\left(\sum_i \oint_{\mathrm{C}+s} a_i^+(u-s)\otimes a_i^-(u)\,du\right)\\ &= \exp\left(\sum_i \oint_{\mathrm{C}} a_i^+(u)\otimes a_i^-(u+s)\,du\right)\end{aligned}$$

where C encloses all poles of $a_i^+(u)$ on V_1 and none of those of $a_i^-(u)$ on V_2.

(4) We show in 5.4 that, on any finite-dimensional representation of $\mathrm{Y}_\hbar(\mathfrak{g})$, $t_i^+(u)$ is the expansion near $u=\infty$ of a meromorphic function of u defined on the complement of a compact cut-set $0\in\mathsf{X}\subset\mathbf{C}$, and interpret $t_i^-(v)$ as the corresponding analytic continuation of $t_i^+(u)$. This resolves in particular the ambiguity in the definition (5.3) of $t_i^-(v)$ as a formal power series in v, since the constant term of $\xi_i^-(v)$ is not equal to 1, and allows to apply the shift operator T to $t_j^-(v)$, since T does not act on formal power series of v. Moreover, since

we work with $Y_\hbar(\mathfrak{g})$, we do not have the operators $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbf{Z}_{<0}}$ at our disposal. This makes the reinterpretation of $t_i^-(v)$ as a meromorphic function essential for our purposes.

(5) To interpret $a_j^-(v)$, we note that it formally satisfies the difference equation $a_j^-(v+l\hbar)-a_j^-(v)=b_j^-(v)$, where

$$b_j^-(v)=-\sum_{k\in\mathbf{I}}T^{-l}\mathsf{C}(T)_{jk}t_k^-(v)=-\sum_{k\in\mathbf{I},r\in\mathbf{Z}}c_{jk}^{(r)}t_k^-\left(v+(l+r)\frac{\hbar}{2}\right)$$

and we used the fact that $\mathsf{C}(T)=\mathsf{C}(T^{-1})$. This implies that $\mathcal{R}^0(s)$ formally satisfies

$$\mathcal{R}^0(s+l\hbar)\mathcal{R}^0(s)^{-1}=\exp\left(\sum_i\oint_C a_i^+(u)\otimes b_i^-(u+s)\,du\right) \tag{5.5}$$

(6) We show in 5.5–5.7 that the operator $\mathcal{A}(s)$ given by the right-hand side of (5.5) is a rational function of s such that $\mathcal{A}(\infty)=1$. We then define two regularisations $\mathcal{R}^{0,\pm}(s)$ of $\mathcal{R}^0(s)$ as the canonical right and left fundamental solutions of the difference equation (5.5), and show in 5.9 that they define meromorphic commutativity constraints on $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ endowed with the deformed Drinfeld coproduct.

5.4. *Matrix logarithms.* — We shall need the following result

Proposition. — Let V *be a complex, finite-dimensional vector space, and* $\xi:\mathbf{C}\to\mathrm{End}(V)$ *a rational function such that*

- $\xi(\infty)=1$.
- $[\xi(u),\xi(v)]=0$ *for any* $u,v\in\mathbf{C}$.

Let $\sigma(\xi)\subset\mathbf{C}$ *be the set of poles of* $\xi(u)^{\pm1}$*, and define the cut-set* $X(\xi)$ *by*

$$X(\xi)=\bigcup_{a\in\sigma(\xi)}[0,a] \tag{5.6}$$

where $[0,a]$ *is the line segment joining* 0 *and* a*. Then, there is a unique single-valued, holomorphic function* $t(u)=\log(\xi(u)):\mathbf{C}\setminus X(\xi)\to\mathrm{End}(V)$ *such that*

$$\exp\big(t(u)\big)=\xi(u)\quad\text{and}\quad t(\infty)=0 \tag{5.7}$$

Moreover, $[t(u),t(v)]=0$ *for any* $u,v\in\mathbf{C}$*, and* $t(u)'=\xi(u)^{-1}\xi'(u)$.

Proof. — Equation (5.7) uniquely defines $t(u)$ as a holomorphic function near $u=\infty$. To continue $t(u)$ meromorphically, note first that the semisimple and unipotent factors $\xi_{\mathrm{S}}(u), \xi_{\mathrm{U}}(u)$ of the multiplicative Jordan decomposition of $\xi(u)$ are rational functions of u since $[\xi(u),\xi(v)]=0$ for any u,v (see *e.g.*, [13, Lemma 4.12]). Thus,

$$t_{\mathrm{N}}(u)=\log\big(\xi_{\mathrm{U}}(u)\big)=\sum_{k\geq 1}(-1)^{k-1}\frac{(\xi_{\mathrm{U}}(u)-1)^k}{k}$$

is a well-defined rational function of $u\in\mathbf{C}$ whose poles are contained in those of $\xi(u)$.

To define $\log(\xi_{\mathrm{S}}(u))$ consistently, note that the eigenvalues of $\xi(u)$ are rational functions of the form $\prod_j(u-a_j)(u-b_j)^{-1}$. Since, for $a\in\mathbf{C}^\times$, the function $\log(1-au^{-1})$ is single-valued on the complement of the interval $[0,a]$, where log is the standard determination of the logarithm, we may define a single-valued, holomorphic function $\log(\xi_{\mathrm{S}}(u))$ on the complement of the intervals $[0,a]$, where a ranges over the (non-zero) zeros and poles of the eigenvalues of $\xi(u)$.

Finally, we set

$$t(u)=t_{\mathrm{N}}(u)+t_{\mathrm{S}}(u)$$

The fact that $[t(u),t(v)]=0$ is clear from the construction, or from the fact that it clearly holds for u,v near ∞. Finally, the derivative of $t(u)$ can be computed by differentiating the identity $\exp(t(u))=\xi(u)$, and using the formula for the left-logarithmic derivative of the exponential function (see, *e.g.*, [10]). □

Definition. — *If* V *is a finite-dimensional representation of* $Y_\hbar(\mathfrak{g})$, *and* $\xi_i(u)$ *is the rational function* $\xi_i(u)=1+\hbar\sum_{r\geq 0}\xi_{i,r}u^{-r-1}$ *given by Proposition 3.10, the corresponding logarithm will be denoted by* $t_i(u)$.

5.5. *The operator* $\mathcal{A}_{V_1,V_2}(s)$. — Let V_1, V_2 be two finite-dimensional representations of $Y_\hbar(\mathfrak{g})$. Let $\mathcal{C}_1$ be a contour enclosing the set of poles of the operators $\xi_i(u)^{\pm 1}$ on V_1, and consider the following operator on $V_1\otimes V_2$

$$\mathcal{A}_{V_1,V_2}(s)=\exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ r\in\mathbf{Z}}}c_{ij}^{(r)}\oint_{\mathcal{C}_1}t_i'(v)\otimes t_j\left(v+s+\frac{(l+r)\hbar}{2}\right)dv\right)$$

where $s\in\mathbf{C}$ is such that $t_j(v+s+\hbar(l+r)/2)$ is an analytic function of v within $\mathcal{C}_1$ for every $j\in\mathbf{I}$ and $r\in\mathbf{Z}$ such that $c_{ij}^{(r)}\neq 0$ for some $i\in\mathbf{I}$.

Let $\Omega_{\mathfrak{h}}\in\mathfrak{h}\otimes\mathfrak{h}\subset Y_\hbar(\mathfrak{g})\otimes Y_\hbar(\mathfrak{g})$ be the Cartan part of the Casimir tensor. Explicitly,

$$\Omega_{\mathfrak{h}}=\sum_{i\in\mathbf{I}}d_ih_i\otimes\varpi_i^\vee=\sum_{i\in\mathbf{I}}\varpi_i^\vee\otimes d_ih_i \tag{5.8}$$

where $d_i h_i = \xi_{i,0}$, and $\varpi_i^\vee$ are the fundamental coweights, which are defined by $(\varpi_i^\vee, d_j h_j) = \delta_{ij}$. By definition of the bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}\times\mathfrak{h}$, we have $\varpi_i^\vee = \sum_{j\in\mathbf{I}}(\mathsf{B}^{-1})_{ij} d_j h_j$.

Theorem.

(i) $\mathcal{A}_{V_1,V_2}(s)$ *extends to a rational function of s which is regular at* ∞, *and such that*

$$\mathcal{A}_{V_1,V_2}(s) = 1 - l\hbar^2\frac{\Omega_{\mathfrak{h}}}{s^2} + O\big(s^{-3}\big)$$

The poles of $\mathcal{A}_{V_1,V_2}(s)^{\pm 1}$ *are contained in*

$$\sigma(V_2) - \sigma(V_1) - \frac{\hbar}{2}\{l + r\}$$

where r ranges over the integers such that $c_{ij}^{(r)} \neq 0$ *for some* $i,j\in\mathbf{I}$.

(ii) *For any* s, s' *we have* $[\mathcal{A}_{V_1,V_2}(s), \mathcal{A}_{V_1,V_2}(s')] = 0$.

(iii) *For any* $V_1, V_2, V_3 \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, *we have*

$$\mathcal{A}_{V_1\otimes_{s_1}V_2,V_3}(s_2) = \mathcal{A}_{V_1,V_3}(s_1+s_2)\mathcal{A}_{V_2,V_3}(s_2)$$
$$\mathcal{A}_{V_1,V_2\otimes_{s_2}V_3}(s_1+s_2) = \mathcal{A}_{V_1,V_3}(s_1+s_2)\mathcal{A}_{V_1,V_2}(s_1)$$

(iv) *The following shifted unitarity condition holds*

$$\sigma\circ\mathcal{A}_{V_1,V_2}(-s)\circ\sigma^{-1} = \mathcal{A}_{V_2,V_1}(s - l\hbar)$$

where $\sigma : V_1\otimes V_2 \to V_2\otimes V_1$ *is the flip of the tensor factors.*

(v) *For every* $a, b\in\mathbf{C}$ *we have*

$$\mathcal{A}_{V_1(a),V_2(b)}(s) = \mathcal{A}_{V_1,V_2}(s + a - b)$$

Proof. — Properties (ii), (iii) and (v) follow from the definition of $\mathcal{A}$, and the fact that $t_i(u)$ are primitive with respect to the Drinfeld coproduct. To prove (i) and (iv), we work in the following more general situation.

Let V, W be complex, finite-dimensional vector spaces, $A, B : \mathbf{C}\to \mathrm{End}(V)$ rational functions satisfying the assumptions of Proposition 5.4, and let $\log A(v), \log B(v)$ be the corresponding logarithms. Let $\sigma(A), \sigma(B)$ denote the set of poles of $A(v)^{\pm 1}$ and $B(v)^{\pm 1}$ respectively. Set

$$X(s) = \exp\left(\oint_{\mathcal{C}_1} A(v)^{-1}A'(v)\otimes\log\big(B(v+s)\big)\,dv\right)$$

where $\mathcal{C}_1$ encloses $\sigma(A)$, and s is such that $\log(B(v+s))$ is analytic within $\mathcal{C}_1$.

Claim **1.** — *The operator* $\mathrm{X}(s) \in \mathrm{End}(\mathrm{V} \otimes \mathrm{W})$ *is a rational function of* s*, regular at* ∞*, and has the following Taylor series expansion near* ∞

$$\mathrm{X}(s) = 1 + (\mathrm{A}_0 \otimes \mathrm{B}_0)s^{-2} + \mathrm{O}\big(s^{-3}\big)$$

where $\mathrm{A}(s) = 1 + \mathrm{A}_0 s^{-1} + \mathrm{O}(s^{-2})$ *and* $\mathrm{B}(s) = 1 + \mathrm{B}_0 s^{-1} + \mathrm{O}(s^{-2})$*. Moreover, the poles of* $\mathrm{X}(s)^{\pm 1}$ *are contained in* $\sigma(\mathrm{B}) - \sigma(\mathrm{A})$.

Note that this claim implies the first part of Theorem 5.5(i), since

$$\begin{aligned}
\mathcal{A}_{\mathrm{V}_1,\mathrm{V}_2}(s) &= \prod_{\substack{i,j\in\mathbf{I}\\ r\in\mathbf{Z}}} \exp\left(\oint_{\mathcal{C}} t_i'(v) \otimes t_j\left(v + s + \frac{(l+r)\hbar}{2}\right) dv\right)^{-c_{ij}^{(r)}} \\
&= 1 - \hbar^2 s^{-2} \sum_{\substack{i,j\in\mathbf{I}\\ r\in\mathbf{Z}}} c_{ij}^{(r)} \xi_{i,0} \otimes \xi_{j,0} + \mathrm{O}\big(s^{-3}\big) \\
&= 1 - l\hbar^2 \Omega_{\mathfrak{h}} s^{-2} + \mathrm{O}\big(s^{-3}\big)
\end{aligned}$$

since $\sum_{r\in\mathbf{Z}} c_{ij}^{(r)} = c_{ij}(\mathrm{T})|_{\mathrm{T}=1}$ is the (i,j) entry of $l \cdot \mathsf{B}^{-1}$.

Part (iv) of Theorem 5.5 is a consequence of the following claim, together with the fact that $c_{ji}^{(r)} = c_{ij}^{(r)} = c_{ij}^{(-r)}$.

Claim **2.** — $\mathrm{X}(s) = \exp(\oint_{\mathcal{C}_2} \log(\mathrm{A}(v-s)) \otimes \mathrm{B}(v)^{-1}\mathrm{B}'(v)\, dv)$*, where* $\mathcal{C}_2$ *encloses* $\sigma(\mathrm{B})$ *and* $s \in \mathbf{C}$ *is such that* $\log(\mathrm{A}(v-s))$ *is analytic within* $\mathcal{C}_2$.

We prove these claims in Sections 5.6 and 5.7 respectively. □

5.6. *Proof of Claim 1.* — Since $\mathrm{A}(v)$ commutes with itself for different values of v, the semisimple and unipotent parts $\mathrm{A}(v) = \mathrm{A}_{\mathrm{S}}(v)\mathrm{A}_{\mathrm{U}}(v)$ of the Jordan decomposition of $\mathrm{A}(v)$ are rational functions of v [13, Lemma 4.12]. Since the logarithmic derivative of $\mathrm{A}(v)$ separates the two additively, we can treat the semisimple and unipotent cases separately.

The semisimple case reduces to the scalar case, *i.e.*, when V is one-dimensional and

$$\mathrm{A}(v) = \prod_j \frac{v - a_j}{v - b_j} = 1 + \left(\sum_j b_j - a_j\right) v^{-1} + \mathrm{O}\big(v^{-2}\big)$$

for some $a_j, b_j \in \mathbf{C}$. In this case,

$$\begin{aligned}
X(s) &= \exp\left(\sum_j \oint_{C_1} \left(\frac{1}{v-a_j} - \frac{1}{v-b_j}\right) \otimes \log\big(B(v+s)\big) dv\right) \\
&= \exp\left(\sum_j 1 \otimes \big(\log\big(B(s+a_j)\big) - \log\big(B(s+b_j)\big)\big)\right) \\
&= \prod_j 1 \otimes B(s+a_j)B(s+b_j)^{-1}
\end{aligned}$$

which is a rational function of s such that the poles of $X(s)^{\pm 1}$ are contained in $\sigma(B) - \sigma(A)$. Moreover,

$$X(s) = 1 + s^{-2}\left(\sum_j b_j - a_j\right) \otimes B_0 + O\big(s^{-3}\big)$$

Assume now that $A(v)$ is unipotent. In this case,

$$\log\big(A(v)\big) = \sum_{k\geq 1} (-1)^{k-1} \frac{(A(v)-1)^k}{k} = A_0 v^{-1} + O\big(v^{-2}\big)$$

is given by a finite sum, and is therefore a rational function of v. Decomposing it into partial fractions yields

$$\log\big(A(v)\big) = \sum_{\substack{j\in J \\ n\in \mathbf{Z}_{\geq 0}}} \frac{N_{j,n}}{(v-a_j)^{n+1}}$$

where J is a finite indexing set, $a_j \in \mathbf{C}$ and $\sum_j N_{j,0} = A_0$. In this case we obtain

$$X(s) = \exp\left(\sum_{\substack{j\in J \\ n\in \mathbf{Z}_{\geq 0}}} -(n+1)N_{j,n} \otimes \frac{\partial_v^{n+1}}{(n+1)!} \log\big(B(v)\big)\Big|_{v=s+a_j}\right)$$

This is again a rational function of s since the $N_{j,n}$ are nilpotent and pairwise commute, such that the poles of $X(s)^{\pm 1}$ are contained in $\sigma(B) - \sigma(A)$. Moreover,

$$X(s) = 1 + s^{-2} \sum_j N_{j,0} \otimes B_0 + O\big(s^{-3}\big)$$

5.7. *Proof of Claim 2.* — Let $\mathsf{X}(A), \mathsf{X}(B) \subset \mathbf{C}$ be defined by (5.6), and $\mathcal{C}_1, \mathcal{C}_2$ be two contours enclosing $\mathsf{X}(A)$ and $\mathsf{X}(B)$ respectively. For each $s \in \mathbf{C}$ such that $\mathcal{C}_1 + s$ is outside of $\mathcal{C}_2$, we have

$$\begin{aligned}
&\oint_{\mathcal{C}_1} A(v)^{-1}A'(v) \otimes \log\big(B(v+s)\big)\, dv \\
&\quad = -\oint_{\mathcal{C}_1} \log\big(A(v)\big) \otimes B(v+s)^{-1}B'(v+s)\, dv \\
&\quad = \oint_{\mathcal{C}_2 - s} \log\big(A(v)\big) \otimes B(v+s)^{-1}B'(v+s)\, dv \\
&\quad = \oint_{\mathcal{C}_2} \log\big(A(w-s)\big) \otimes B(w)^{-1}B'(w)\, dw
\end{aligned}$$

where the first equality follows by integration by parts, the second by a deformation of contour since the integrand is regular at $v = \infty$ and has zero residue there, and the third by the change of variables $w = v + s$.

5.8. *The abelian* R*-matrix of* $Y_\hbar(\mathfrak{g})$. — Let $V_1, V_2 \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, and let $\mathcal{A}_{V_1,V_2}(s) \in GL(V_1 \otimes V_2)$ be the operator defined in Section 5.5. Consider the additive difference equation

$$\mathcal{R}_{V_1,V_2}(s + l\hbar) = \mathcal{A}_{V_1,V_2}(s)\mathcal{R}_{V_1,V_2}(s) \tag{5.9}$$

where $l = mh^\vee$ was defined in Section 5.1.

This equation is regular, in that $\mathcal{A}_{V_1,V_2}(s) = 1 + O(s^{-2})$ by Theorem 5.5. In particular, it admits two canonical meromorphic fundamental solutions

$$\mathcal{R}^{0,\pm}_{V_1,V_2} : \mathbf{C} \to GL(V_1 \otimes V_2)$$

which are uniquely determined by the following requirements (see *e.g.*, [2, 3, 22] or [13, §4])

- $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ is holomorphic and invertible for $\pm\mathrm{Re}(s/\hbar) \gg 0$.
- $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ possesses an asymptotic expansion of the form

$$\mathcal{R}^{0,\pm}_{V_1,V_2}(s) \sim 1 + \mathcal{R}^{\pm}_0 s^{-1} + \mathcal{R}^{\pm}_1 s^{-2} + \cdots$$

  in any half-plane $\pm\mathrm{Re}(s/\hbar) > m$, $m \in \mathbf{R}$. In other words, we can find $R > 0$ so that for any $N \geq 0$, there is a constant C_N such that

$$\left\| \mathcal{R}^{0,\pm}_{V_1,V_2}(s) - \left(1 + \sum_{k=0}^{N-1} \mathcal{R}^{\pm}_k s^{-k-1}\right) \right\| < \frac{C_N}{|s|^{N+1}}$$

  for $|s| > R$ in the corresponding domain, where $\|\cdot\|$ is a fixed norm on $\mathrm{End}(V_1 \otimes V_2)$.

Explicitly,

$$\mathcal{R}^{0,+}_{V_1,V_2}(s) = \prod^{\to}_{n\geq 0} \mathcal{A}_{V_1,V_2}(s+nl\hbar)^{-1}$$
$$\mathcal{R}^{0,-}_{V_1,V_2}(s) \ = \prod^{\to}_{n\geq 1} \mathcal{A}_{V_1,V_2}(s-nl\hbar)$$

where the products converge uniformly on compact sets of $\pm\mathrm{Re}(s/\hbar) \gg 0$ since $\mathcal{A}_{V_1,V_2}(s) = 1 + O(s^{-2})$. Note that the order of products indicated above is immaterial, since $\mathcal{A}_{V_1,V_2}(s)$ takes values in a commutative subalgebra of $\mathrm{End}(V_1 \otimes V_2)$.

5.9. The following is the main result of this section.

Theorem. — $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ *have the following properties*

(i) *The map*

$$\sigma \circ \mathcal{R}^{0,\pm}_{V_1,V_2}(s) : V_1(s) \otimes_0 V_2 \to V_2 \otimes_0 V_1(s)$$

where σ *is the flip of tensor factors, is a morphism of* $Y_\hbar(\mathfrak{g})$*-modules, which is natural in* V_1 *and* V_2.

(ii) *For any* $V_1, V_2, V_3 \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ *we have*

$$\mathcal{R}^{0,\pm}_{V_1\otimes_{s_1} V_2, V_3}(s_2) = \mathcal{R}^{0,\pm}_{V_1,V_3}(s_1+s_2)\mathcal{R}^{0,\pm}_{V_2,V_3}(s_2)$$
$$\mathcal{R}^{0,\pm}_{V_1, V_2\otimes_{s_2} V_3}(s_1+s_2) = \mathcal{R}^{0,\pm}_{V_1,V_3}(s_1+s_2)\mathcal{R}^{0,\pm}_{V_1,V_2}(s_1)$$

(iii) *The following unitary condition holds*

$$\sigma \circ \mathcal{R}^{0,\pm}_{V_1,V_2}(-s) \circ \sigma^{-1} = \mathcal{R}^{0,\mp}_{V_2,V_1}(s)^{-1}$$

(iv) *For* $a, b \in \mathbf{C}$ *we have*

$$\mathcal{R}^{0,\pm}_{V_1(a),V_2(b)}(s) = \mathcal{R}^{0,\pm}_{V_1,V_2}(s+a-b)$$

(v) *For any* s, s',

$$\left[\mathcal{R}^{0,\pm}_{V_1,V_2}(s), \mathcal{R}^{0,\pm}_{V_1,V_2}(s')\right] = 0 = \left[\mathcal{R}^{0,\pm}_{V_1,V_2}(s), \mathcal{R}^{0,\mp}_{V_1,V_2}(s')\right]$$

(vi) $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ *have the same asymptotic expansion, which is of the form*

$$\mathcal{R}^{0,\pm}_{V_1,V_2}(s) \sim 1 + \hbar\Omega_{\mathfrak{h}} s^{-1} + O\left(s^{-2}\right) \tag{5.10}$$

(vii) *There is a* $\rho > 0$ *such that the asymptotic expansion of* $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ *is valid on any domain*

$$\left\{\pm\mathrm{Re}(s/\hbar) > m\right\} \cup \left\{|\,\mathrm{Im}(s/\hbar)| > \rho,\ \arg(\pm s/\hbar) \in (-\pi+\delta, \pi-\delta)\right\}$$

where $m \in \mathbf{R}$ *and* $\delta \in (0,\pi)$ *are arbitrary.*

(viii) *The poles of* $\mathcal{R}^{0,+}_{V_1,V_2}(s)^{\pm 1}$ *and* $\mathcal{R}^{0,-}_{V_1,V_2}(s)^{\pm 1}$ *are contained in*

$$\sigma(V_2) - \sigma(V_1) - \mathbf{Z}_{\geq 0} l\hbar - \frac{\hbar}{2}\{l + r\} \quad \textit{and}$$

$$\sigma(V_2) - \sigma(V_1) + \mathbf{Z}_{>0} l\hbar - \frac{\hbar}{2}\{l + r\}$$

where r ranges over the integers such that $c_{ij}^{(r)} \neq 0$ for some $i, j \in \mathbf{I}$.

Proof. — Part (i) is proved in 5.12 after some preparatory results. Properties (ii)–(vi) and (viii) follow from Theorem 5.5 and Section 5.8. (vii) is proved in [31, Lemma 8.1]. □

5.10. *Commutation relations with* $\mathcal{A}_{V_1,V_2}(s)$. — Let $\mathcal{C} \subset \mathbf{C}$ be a contour, and $a_\ell : \mathbf{C} \to \operatorname{End}(V_\ell)$, $\ell = 1, 2$ two meromorphic functions which are analytic within $\mathcal{C}$ and commute with the operators $\{\xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbf{Z}_{\geq 0}}$. For any $k \in \mathbf{I}$, define operators $X_k^{\pm,\ell} \in \operatorname{End}(V_1 \otimes V_2)$ by

$$X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(v) x_k^{\pm}(v) \otimes a_2(v)\, dv \quad \text{and} \quad X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(v) \otimes a_2(v) x_k^{\pm}(v)\, dv$$

Proposition. — *The following commutation relations hold*

$$\operatorname{Ad}\big(\mathcal{A}_{V_1,V_2}(s)\big) X_k^{\pm,1} = \oint_{\mathcal{C}} a_1(v) x_k^{\pm}(v) \otimes a_2(v) \xi_k(v + s + l\hbar)^{\pm 1} \xi_k(v + s)^{\mp 1}\, dv$$

$$\operatorname{Ad}\big(\mathcal{A}_{V_1,V_2}(s)\big) X_k^{\pm,2} = \oint_{\mathcal{C}} a_1(v) \xi_k(v - s)^{\pm 1} \xi_k(v - s - l\hbar)^{\mp 1} \otimes a_2(v) x_k^{\pm}(v)\, dv$$

Proof. — We only prove the first relation. The second one follows from the first and the unitarity property of Theorem 5.5. We begin by computing the commutation between $X_k^{\pm,1}$ and a typical summand in $\log \mathcal{A}_{V_1,V_2}(s)$. Set $b = \pm\hbar d_i a_{ik}/2$. Note that the definition of $X_k^{\pm,1}$ does not change if we replace the contour $\mathcal{C}$ by a smaller one $\mathcal{C}'$, as long as both $\mathcal{C}$ and $\mathcal{C}'$ enclose the same set of poles of $x_k^{\pm}(v)$. Let $\mathcal{C}_1$ be the contour chosen for the definition of $\mathcal{A}_{V_1,V_2}(s)$ given in Section 5.5. According to Lemma 3.13, if v_0 is a pole of $x_k^{\pm}(v)$ then $\mathcal{C}_1$ must enclose $v_0 \pm b$. Combining these observations, we will assume, in the calculation below, that $\mathcal{C}_1$ encloses $\mathcal{C}$ and its translates by $\pm b$. By (3.5),

$$\begin{aligned}
&\left[\oint_{\mathcal{C}_1} t_i'(u) \otimes t_j(u + s)\, du, X_k^{\pm,1}\right] \\
&\quad = \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} a_1(v) \big[t_i'(u), x_k^{\pm}(v)\big] \otimes t_j(u + s) a_2(v)\, dv du \\
&\quad = \oint_{\mathcal{C}_1} \oint_{\mathcal{C}} \frac{1}{u - v + b} a_1(v) x_k^{\pm}(v) \otimes t_j(u + s) a_2(v)\, dv du
\end{aligned}$$

$$
\begin{aligned}
&-\oint_{\mathcal{C}_1}\oint_{\mathcal{C}} \frac{1}{u-v-b} a_1(v) x_k^{\pm}(v) \otimes t_j(u+s) a_2(v)\, dv du \\
&+\oint_{\mathcal{C}_1}\oint_{\mathcal{C}} \frac{1}{u-v-b} a_1(v) x_k^{\pm}(u-b) \otimes t_j(u+s) a_2(v)\, dv du \\
&-\oint_{\mathcal{C}_1}\oint_{\mathcal{C}} \frac{1}{u-v+b} a_1(v) x_k^{\pm}(u+b) \otimes t_j(u+s) a_2(v)\, dv du \\
=&\oint_{\mathcal{C}} a_1(v) x_k^{\pm}(v) \otimes \big(t_j(v-b+s) - t_j(v+b+s)\big) a_2(v)\, dv
\end{aligned}
$$

where the third equality follows from the fact that s is such that $t_j(u+s)$ is holomorphic inside $\mathcal{C}_1$. Note that the third and the fourth terms on the right-hand side of the second equality vanish since their integrands are holomorphic in the variable v.

Let the indeterminate T of Section 5.1 act as the difference operator $\mathrm{T}t_j(v) = t_j(v-\hbar/2)$. Then,

$$
\begin{aligned}
&\sum_{i,j\in\mathbf{I}}\left[\oint_{\mathcal{C}_1} t_i'(u)\otimes c_{ij}(\mathrm{T})t_j(u+s)\,du, \mathrm{X}_k^{\pm,1}\right] \\
&\quad = \sum_{i,j\in\mathbf{I}}\oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v)\otimes a_2(v)c_{ij}(\mathrm{T})\big(\mathrm{T}^{\pm b_{ik}} - \mathrm{T}^{\mp b_{ik}}\big)t_j(v+s)\,dv \\
&\quad = \pm\oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v)\otimes a_2(v)\big(\mathrm{T}^{l} - \mathrm{T}^{-l}\big)t_k(v+s)\,dv
\end{aligned}
$$

where the second equality follows from (5.1). The claimed identity easily follows from this. □

5.11. Let $\mathrm{X}_k^{\pm,1}, \mathrm{X}_k^{\pm,2}$ be the operators defined in 5.10. The following is a corollary of Proposition 5.10 and the definition of $\mathcal{R}^{0,\pm}(s)$.

Proposition. — The following commutation relations hold for any $\varepsilon \in \{\pm\}$

$$
\begin{aligned}
\mathrm{Ad}\big(\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\big)\mathrm{X}_k^{\pm,1} &= \oint_{\mathcal{C}} a_1(v)x_k^{\pm}(v)\otimes a_2(v)\xi_k(v+s)^{\pm 1}\,dv \\
\mathrm{Ad}\big(\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\big)\mathrm{X}_k^{\pm,2} &= \oint_{\mathcal{C}} a_1(v)\xi_k(v-s)^{\mp 1}\otimes a_2(v)x_k^{\pm}(v)\,dv
\end{aligned}
$$

5.12. *Proof of (i) of Theorem 5.9.* — We first rewrite the Drinfeld coproduct in a more symmetric way. Let V be a finite-dimensional representation of $\mathrm{Y}_\hbar(\mathfrak{g})$ and $\mathcal{C}^{\pm}\subset\mathbf{C}$ a contour containing the poles of $x_i^{\pm}(u)$ on V. Then, a simple contour deformation shows that, for any u not contained inside $\mathcal{C}^{\pm}$,

$$\oint_{\mathcal{C}^{\pm}} x_i^{\pm}(v)\frac{dv}{u-v} = x_i^{\pm}(u)$$

It follows that

$$\Delta_s\big(x_i^+(u)\big) = \oint_{\mathcal{C}_1} x_i^+(v-s)\otimes 1\,\frac{dv}{u-v} + \oint_{\mathcal{C}_2} \xi_i(v-s)\otimes x_i^+(v)\,\frac{dv}{u-v}$$

$$\Delta_s\big(x_i^-(u)\big) = \oint_{\mathcal{C}_1} x_i^-(v-s)\otimes \xi_i(v)\,\frac{dv}{u-v} + \oint_{\mathcal{C}_2} 1\otimes x_i^-(v)\,\frac{dv}{u-v}$$

where $\mathcal{C}_1, \mathcal{C}_2$ are as in 4.5.

We need to show that $\sigma\circ\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s) : \mathrm{V}_1(s)\otimes_0 \mathrm{V}_2 \to \mathrm{V}_2\otimes_0 \mathrm{V}_1(s)$ intertwines the action of $\mathrm{Y}_\hbar(\mathfrak{g})$. This is obvious for $\xi_i(u)$, since $\xi_i(u)$ is group-like and commutes with $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$. Denote now by $x_i^+(u)'$ and $x_i^+(u)''$, the action of $x_i^+(u)$ on $\mathrm{V}_1(s)\otimes_0\mathrm{V}_2$ and $\mathrm{V}_2\otimes_0\mathrm{V}_1(s)$ respectively. By above formulas, we have

$$x_i^+(u)' = \oint_{\mathrm{C}_1} x_i^+(v-s)\otimes 1\,\frac{dv}{u-v} + \oint_{\mathrm{C}_2} \xi_i(v-s)\otimes x_i^+(v)\,\frac{dv}{u-v}$$

$$x_i^+(u)'' = \oint_{\mathrm{C}_2} x_i^+(v)\otimes 1\,\frac{dv}{u-v} + \oint_{\mathrm{C}_1} \xi_i(v)\otimes x_i^+(v-s)\,\frac{dv}{u-v}$$

Using Proposition 5.11, we can compute $\mathrm{Ad}(\sigma\circ\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s))x_i^+(u)'$ as follows

$$\begin{aligned}
&\sigma\Bigg(\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\Bigg(\oint_{\mathrm{C}_1} x_i^+(v-s)\otimes 1\,\frac{dv}{u-v}\\
&\qquad + \oint_{\mathrm{C}_2} \xi_i(v-s)\otimes x_i^+(v)\,\frac{dv}{u-v}\Bigg)\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)^{-1}\Bigg)\sigma\\
&\quad = \sigma\Bigg(\oint_{\mathrm{C}_1} x_i^+(v-s)\otimes\xi_i(v)\,\frac{dv}{u-v} + \oint_{\mathrm{C}_2} 1\otimes x_i^+(v)\,\frac{dv}{u-v}\Bigg)\sigma\\
&\quad = \oint_{\mathrm{C}_1} \xi_i(v)\otimes x_i^+(v-s)\,\frac{dv}{u-v} + \oint_{\mathrm{C}_2} x_i^+(v)\otimes 1\,\frac{dv}{u-v}
\end{aligned}$$

This implies that $\mathrm{Ad}(\sigma\circ\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s))x_i^+(u)' = x_i^+(u)''$ and the result follows. The proof for $x_i^-(u)$ is identical.

6. The functor Γ

We review below the main construction of [13]. Assume henceforth that $\hbar\in\mathbf{C}\setminus\mathbf{Q}$, and that $q = e^{\pi\iota\hbar}$.

6.1. *Difference equations.* — Consider the abelian, additive difference equations, for unknown functions $\phi_i : \mathbf{C} \to GL(V)$

$$\phi_i(u+1) = \xi_i(u)\phi_i(u) \tag{6.1}$$

defined by the commuting fields $\xi_i(u) = 1 + \hbar\xi_{i,0}u^{-1} + \cdots$ on a finite-dimensional representation V of $Y_\hbar(\mathfrak{g})$.

Let $\phi_i^\pm(u) : \mathbf{C} \to GL(V)$ be the canonical fundamental solutions of (6.1). $\phi_i^\pm(u)$ are uniquely determined by the requirement that they be holomorphic and invertible for $\pm \operatorname{Re}(u) \gg 0$, and admit an asymptotic expansion of the form

$$\phi_i^\pm(u) \sim \left(1 + \varphi_0^\pm u^{-1} + \varphi_1^\pm u^{-2} \cdots\right)(\pm u)^{\hbar\xi_{i,0}}$$

in any right (resp. left) half-plane $\pm \operatorname{Re}(s) > m$, $m \in \mathbf{R}$ (see *e.g.*, [2, 3, 22] or [13, §4]). $\phi_i^+(u), \phi_i^-(u)$ are regularisations of the formal infinite products

$$\xi_i(u)^{-1}\xi_i(u+1)^{-1}\xi_i(u+2)^{-1}\cdots \quad \text{and} \quad \xi_i(u-1)\xi_i(u-2)\xi_i(u-3)\cdots$$

respectively.

Let $S_i(u) = (\phi_i^+(u))^{-1}\phi_i^-(u)$ be the connection matrix of (6.1). Thus, $S_i(u)$ is 1-periodic in u, and therefore a function of $z = \exp(2\pi\iota u)$. It is moreover regular at $z = 0, \infty$ [13, Prop. 4.8], and therefore a rational function of z such that

$$S_i(0) = e^{-\pi\iota\hbar\xi_{i,0}} = S_i(\infty)^{-1}$$

Explicitly,

$$S_i(u) = \lim_{n\to\infty} \xi_i(u+n)\cdots\xi_i(u+1)\xi_i(u)\xi_i(u-1)\cdots\xi_i(u-n)$$

6.2. *Non-congruent representations.* — We shall say that $V \in \operatorname{Rep}_{fd}(Y_\hbar(\mathfrak{g}))$ is *non-congruent* if, for any $i \in \mathbf{I}$, the poles of $x_i^+(u)$ (resp. $x_i^-(u)$) are not congruent modulo $\mathbf{Z}_{\neq 0}$. Let $\operatorname{Rep}_{fd}^{NC}(Y_\hbar(\mathfrak{g}))$ be the full subcategory of $\operatorname{Rep}_{fd}(Y_\hbar(\mathfrak{g}))$ consisting of non-congruent representations.

6.3. *The functor* Γ. — Given $V \in \operatorname{Rep}_{fd}^{NC}(Y_\hbar(\mathfrak{g}))$, define the action of the generators of $U_q(L\mathfrak{g})$ on $\Gamma(V) = V$ as follows.

(i) For any $i \in \mathbf{I}$, the generating series $\Psi_i(z)^+$ (resp. $\Psi_i(z)^-$) of the commuting generators of $U_q(L\mathfrak{g})$ acts as the Taylor expansions at $z = \infty$ (resp. $z = 0$) of the rational function

$$\Psi_i(z) = S_i(u)|_{e^{2\pi\iota u} = z}$$

To define the action of the remaining generators of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$, let $g_i^{\pm}(u) : \mathbf{C} \to \mathrm{GL}(\mathrm{V})$ be given by $g_i^+(u) = \phi_i^+(u+1)^{-1}$ and $g_i^-(u) = \phi_i^-(u)$. Explicitly,

$$
\begin{aligned}
g_i^+(u) &= \left(\overleftarrow{\prod_{n\geq 1}} \xi_i(u+n)\, e^{-\hbar\xi_{i,0}/n} \right) e^{\gamma\hbar\xi_{i,0}} \\
g_i^-(u) &= e^{-\gamma\hbar\xi_{i,0}} \left(\overrightarrow{\prod_{n\geq 1}} \xi_i(u-n)\, e^{\hbar\xi_{i,0}/n} \right)
\end{aligned}
\tag{6.2}
$$

where $\gamma = \lim_{n\to\infty}(1 + \cdots + 1/n - \log n)$ is the Euler–Mascheroni constant, are regularisations of the infinite products

$$\cdots \xi_i(u+2)\xi_i(u+1) \quad \text{and} \quad \xi_i(u-1)\xi_i(u-2)\cdots$$

Note also that, by definition of $g_i^{\pm}(u)$

$$\mathrm{S}_i(u) = g_i^+(u) \cdot \xi_i(u) \cdot g_i^-(u) \tag{6.3}$$

Let $c_i^{\pm} \in \mathbf{C}^\times$ be scalars such that $c_i^- c_i^+ = d_i \Gamma(\hbar d_i)^2$.

(ii) For any $i \in \mathbf{I}$ and $k \in \mathbf{Z}$, $\mathcal{X}_{i,k}^{\pm}$ acts as the operator

$$\mathcal{X}_{i,k}^{\pm} = c_i^{\pm} \oint_{\mathcal{C}_i^{\pm}} e^{2\pi\iota k u} g_i^{\pm}(u) x_i^{\pm}(u)\, du$$

where the Jordan curve $\mathcal{C}_i^{\pm}$ encloses the poles of $x_i^{\pm}(u)$ and none of their $\mathbf{Z}_{\neq 0}$-translates.[13] The corresponding generating series are the expansions at $z = \infty, 0$ of the $\mathrm{End}(\mathrm{V})$-valued rational function given by

$$\mathcal{X}_i^{\pm}(z) = c_i^{\pm} \oint_{\mathcal{C}_i^{\pm}} \frac{z}{z - e^{2\pi\iota u}} g_i^{\pm}(u) x_i^{\pm}(u)\, du$$

where z lies outside of $\exp(2\pi\iota\mathcal{C}_i^{\pm})$.

6.4. Let $\Pi \subset \mathbf{C}$ be a subset such that $\Pi \pm \frac{\hbar}{2} \subset \Pi$. Let

$$\mathrm{Rep}_{\mathrm{fd}}^{\Pi}\big(\mathrm{Y}_\hbar(\mathfrak{g})\big) \subset \mathrm{Rep}_{\mathrm{fd}}\big(\mathrm{Y}_\hbar(\mathfrak{g})\big)$$

be the full subcategory of consisting of the representations V such that $\sigma(\mathrm{V}) \subset \Pi$.

Similarly, let $\Omega \subset \mathbf{C}^\times$ be a subset stable under multiplication by $q^{\pm 1}$. We define $\mathrm{Rep}_{\mathrm{fd}}^{\Omega}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ to be the full subcategory of $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ consisting of those $\mathcal{V}$ such that $\sigma(\mathcal{V}) \subset \Omega$.

[13] Note that such a curve exists for any $i \in \mathbf{I}$ since V is non-congruent.

6.5.

Theorem [13, Thm. 5.4, Thm. 6.3, Prop. 7.7].

(i) *The above operators give rise to an action of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ *on* V. *They therefore define an exact, faithful functor*

$$\Gamma : \mathrm{Rep}_{\mathrm{fd}}^{\mathrm{NC}}\big(\mathrm{Y}_\hbar(\mathfrak{g})\big) \longrightarrow \mathrm{Rep}_{\mathrm{fd}}\big(\mathrm{U}_q(\mathrm{L}\mathfrak{g})\big)$$

(ii) *The functor* Γ *is compatible with shift automorphisms. That is, for any* $\mathrm{V} \in \mathrm{Rep}_{\mathrm{fd}}^{\mathrm{NC}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ *and* $a \in \mathbf{C}$,

$$\Gamma\big(\mathrm{V}(a)\big) = \Gamma(\mathrm{V})\big(e^{2\pi\iota a}\big)$$

(iii) *Let* $\Pi \subset \mathbf{C}$ *be a non-congruent subset such that* $\Pi \pm \frac{1}{2}\hbar \subset \Pi$. *Then,* $\mathrm{Rep}_{\mathrm{fd}}^{\Pi}(\mathrm{Y}_\hbar(\mathfrak{g}))$ *is a subcategory of* $\mathrm{Rep}_{\mathrm{fd}}^{\mathrm{NC}}(\mathrm{Y}_\hbar(\mathfrak{g}))$, *and* Γ *restricts to an isomorphism of abelian categories.*

$$\Gamma_\Pi : \mathrm{Rep}_{\mathrm{fd}}^{\Pi}\big(\mathrm{Y}_\hbar(\mathfrak{g})\big) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{fd}}^{\Omega}\big(\mathrm{U}_q(\mathrm{L}\mathfrak{g})\big)$$

where $\Omega = \exp(2\pi\iota\Pi)$.

(iv) Γ_Π *is compatible with the q-characters of Knight and Frenkel–Reshetikhin.*

7. Meromorphic tensor structure on Γ

7.1. *The abelian qKZ equations.* — Let $\mathrm{V}_1, \mathrm{V}_2$ be finite-dimensional representations of $\mathrm{Y}_\hbar(\mathfrak{g})$, choose $\varepsilon \in \{\pm\}$, and let $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ be the corresponding R-matrix defined in Section 5.8. Consider the abelian, additive qKZ equation for an unknown function $f : \mathbf{C} \to \mathrm{End}(\mathrm{V}_1 \otimes \mathrm{V}_2)$

$$f(s+1) = \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s) f(s) \tag{7.1}$$

Note that this equation does not fit the usual assumptions in the study of difference equations since $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ is not rational. Moreover, $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ may not have a Laurent expansion at ∞ but, by Theorem 5.9, only an asymptotic expansion of the form $1 + \hbar\Omega_{\mathfrak{h}}/s + \mathrm{O}(s^{-2})$ valid in any domain of the form

$$\big\{\mathrm{Re}(s/\varepsilon\hbar) > m\big\} \cup \big\{|\,\mathrm{Im}(s/\hbar)| > \rho, \arg(s/\varepsilon\hbar) \in (-\pi+\delta, \pi-\delta)\big\}$$

where $\rho > 0$ is fixed, and $m \in \mathbf{R}$, $\delta \in (0,\pi)$ are arbitrary.[14] Nevertheless, these asymptotics and the fact that the poles of $\mathcal{R}^{0,\varepsilon}(s)^{\pm 1}$ are contained in the complement of a domain of the above form, are sufficient to carry over the standard proofs (see, *e.g.*, [13, §4]) and yield the following.

[14] For the qKZ equations determined by the full R-matrix, these issues are usually addressed by proving the existence of factorisation $\mathrm{R}_{\mathrm{V}_1,\mathrm{V}_2}(s) = \mathrm{R}^{\mathrm{rat}}_{\mathrm{V}_1,\mathrm{V}_2}(s) \cdot \mathrm{R}^{\mathrm{mer}}_{\mathrm{V}_1,\mathrm{V}_2}(s)$, where $\mathrm{R}^{\mathrm{rat}}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ is a rational function of s which intertwines the Kac–Moody coproduct Δ and its opposite, and the meromorphic factor $\mathrm{R}^{\mathrm{mer}}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ intertwines Δ (see [20] for the case of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$), and then working with $\mathrm{R}^{\mathrm{rat}}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ instead of $\mathrm{R}_{\mathrm{V}_1,\mathrm{V}_2}(s)$. A similar factorisation can be obtained for the abelian R-matrices $\mathcal{R}^{0,\pm}(s)$. We shall, however, prove in [12] that neither of these factorisations are natural with respect to $\mathrm{V}_1, \mathrm{V}_2$, which is why we work with the meromorphic R-matrices $\mathcal{R}^{0,\pm}(s)$.

Proposition. — *Let* $n \in \mathbf{C}^\times$ *be perpendicular to* $\hbar$ *and such that* $\mathrm{Re}(n) \geq 0$.

(i) *If* $\varepsilon\hbar \notin \mathbf{R}_{<0}$, *Equation* (7.1) *admits a canonical right meromorphic solution* $\Phi_+^\varepsilon : \mathbf{C} \to \mathrm{GL}(\mathrm{V}_1 \otimes \mathrm{V}_2)$, *which is uniquely determined by the following requirements*

- Φ_+^ε *is holomorphic and invertible for* $\mathrm{Re}(s) \gg 0$ *if* $\mathrm{Re}(\varepsilon\hbar) \geq 0$, *and otherwise on a sector of the form*

$$\mathrm{Re}(s) \gg 0 \quad \textit{and} \quad \mathrm{Re}(s/n) \gg 0 \tag{7.2}$$

- Φ_+^ε *has an asymptotic expansion of the form* $(1 + \mathrm{O}(s^{-1}))s^{\hbar\Omega_\mathfrak{h}}$ *in any right half-plane if* $\mathrm{Re}(\varepsilon\hbar) > 0$, *and otherwise in a sector of the form* (7.2).

(ii) *If* $\varepsilon\hbar \notin \mathbf{R}_{>0}$, *Equation* (7.1) *admits a canonical left meromorphic solution* $\Phi_-^\varepsilon : \mathbf{C} \to \mathrm{GL}(\mathrm{V}_1 \otimes \mathrm{V}_2)$, *which is uniquely determined by the following requirements*

- Φ_-^ε *is holomorphic and invertible for* $\mathrm{Re}(s) \ll 0$ *if* $\mathrm{Re}(\varepsilon\hbar) \leq 0$, *and otherwise on a sector of the form*

$$\mathrm{Re}(s) \ll 0 \quad \textit{and} \quad \mathrm{Re}(s/n) \ll 0 \tag{7.3}$$

- Φ_-^ε *has an asymptotic expansion of the form* $(1 + \mathrm{O}(s^{-1}))(-s)^{\hbar\Omega_\mathfrak{h}}$ *in any left half-plane if* $\mathrm{Re}(\varepsilon\hbar) < 0$, *and otherwise in a sector of the form* (7.3).

The right and left solutions, when defined, are given by the products

$$\Phi_+^\varepsilon(s) = e^{-\hbar\gamma\Omega_\hbar}\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)^{-1} \overrightarrow{\prod_{m\geq 1}} \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s+m)^{-1}\, e^{\hbar\Omega_\mathfrak{h}/m} \tag{7.4}$$

$$\Phi_-^\varepsilon(s) = e^{-\hbar\gamma\Omega_\hbar} \overrightarrow{\prod_{m\geq 1}} \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s-m)\, e^{\hbar\Omega_\mathfrak{h}/m} \tag{7.5}$$

7.2. *Proof of Proposition 7.1.* — As mentioned before, the proof follows the same strategy as in [13, §4]. More precisely, we use the fact that $\hbar\Omega_\mathfrak{h}$ commutes with $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ to regularise (7.1), that is set (as in [13, §4.6])

$$\overline{\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}}(s) := \big(1 - \hbar\Omega_\mathfrak{h} s^{-1}\big)\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$$

The auxiliary equation $f(s+1) = (1 - \hbar\Omega_\mathfrak{h} s^{-1})f(s)$ can be solved using the Γ-function (see [13, §4.5, 4.6]), while the regularised equation (Equation (7.1) with $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}$ replaced by $\overline{\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}}$) is solved by taking the infinite products [13, §4.4]:

$$\overline{\Phi_+^\varepsilon}(s) = \overrightarrow{\prod_{n\geq 0}} \overline{\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}}(s+n)^{-1}$$

$$\overline{\Phi^{\varepsilon}_{-}}(s) = \prod_{n\geq 1}^{\rightarrow} \overline{\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}}(s-n)$$

This is the only point of departure from the rational case. In order to prove the convergence of these infinite products, we only need the asymptotics of $\overline{\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}}$ up to the second order in the desired zones, as stated in the following lemma. Its proof is standard and hence omitted.

Lemma. — *Let* $\Omega \subset \mathbf{C}$ *be an open set,* W *a finite-dimensional complex vector space, and* $f : \Omega \to \mathrm{End}(\mathrm{W})$ *a holomorphic and invertible function such that the following assumptions hold.*

(a) *For each* $n \in \mathbf{Z}_{\geq 0}$, $\Omega + n \subset \Omega$.
(b) *There exists a constant* $\mathrm{C} \in \mathbf{R}_{>0}$ *such that*

$$\|f(s) - 1\| < \frac{\mathrm{C}}{|s|^2} \quad \textit{as} \quad s \to \infty, s \in \Omega$$

for some norm $\|\cdot\|$ *on* End(W).

Then the sequence of functions $\{f(s)f(s+1)\cdots f(s+n)\}_{n\geq 1}$ *converges uniformly on compact sets in* Ω *and hence defines a holomorphic function* F(s) *on* Ω.

If, in addition,

(c) $f(s)$ *extends to a meromorphic function on* $\mathbf{C}$.
(d) Ω *contains a fundamental domain for* $s \mapsto s+1$.

then F(s) *can be extended to a meromorphic function on* $\mathbf{C}$ *by using the equation* $\mathrm{F}(s) = f(s)\mathrm{F}(s+1)$.

The same assertions hold for the infinite product $f(s-1)f(s-2)\cdots$ *after changing* $\mathbf{Z}_{\geq 0}$ *to* $\mathbf{Z}_{\leq 0}$ *in condition (a) above.*

This, in particular, explains that we have to consider sectors given in Figure 1 in order to avoid the poles of $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)^{\pm 1}$. Thus we obtain the solutions $\Phi^{\varepsilon}_{\pm}$ of the difference equation (7.1), which are explicitly given in (7.4) and (7.5), and whose asymptotics can be computed using the calculation in [13, §4.7].

7.3. *The tensor structure* $\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$. — Let $\varepsilon \in \{\pm\}$ be such that $\varepsilon\hbar \notin \mathbf{R}_{<0}$, and $\Phi^{\varepsilon}_{+}(s)$ the right fundamental solution of the abelian qKZ equation (7.1). Define a meromorphic function

$$\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2} : \mathbf{C} \to \mathrm{GL}(\mathrm{V}_1 \otimes \mathrm{V}_2)$$

by $\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s) = \Phi^{\varepsilon}_{+}(s+1)^{-1}$. Thus,

$$\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s) = e^{\hbar\gamma\Omega_{\mathfrak{h}}} \overleftarrow{\prod_{m\geq 1}} \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s+m) e^{-\frac{\hbar\Omega_{\mathfrak{h}}}{m}} \tag{7.6}$$

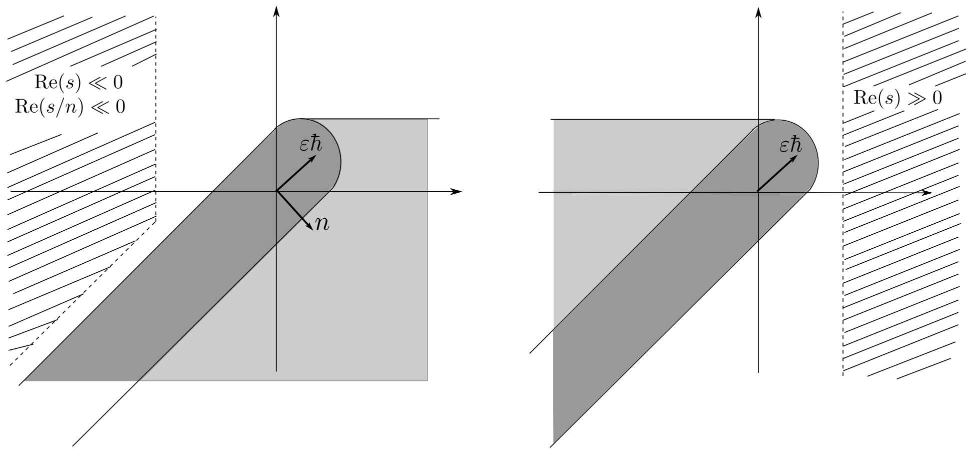

FIG. 1. — Domains of holomorphy and invertibility of Φ^{ε}_{+} (resp. Φ^{ε}_{-}) given by the *ruled region* in the *right* (resp. *left*) *picture*, when $\mathrm{Re}(\varepsilon\hbar) > 0$. The *darker grey region* contains poles of $\mathcal{R}^{0,\varepsilon}(s)^{\pm 1}$

Theorem.

(i) $\mathcal{J}^{\varepsilon}_{V_1,V_2}(s)$ *is natural in* V_1, V_2.

(ii) *If* V_1 *and* V_2 *are non-congruent, and* $\zeta = e^{2\pi\iota s}$,

$$\mathcal{J}^{\varepsilon}_{V_1,V_2}(s) : \Gamma(V_1) \otimes_{\zeta} \Gamma(V_2) \longrightarrow \Gamma(V_1 \otimes_s V_2)$$

is an isomorphism of $U_q(L\mathfrak{g})$*-modules for any* $s \notin \sigma(V_2) - \sigma(V_1) + \mathbf{Z}$.

(iii) *For any non-congruent* $V_1, V_2, V_3 \in \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$, *the following is a commutative diagram*

$$\begin{array}{ccc}
(\Gamma(V_1) \otimes_{\zeta_1} \Gamma(V_2)) \otimes_{\zeta_2} \Gamma(V_3) & = & \Gamma(V_1) \otimes_{\zeta_1\zeta_2} (\Gamma(V_2) \otimes_{\zeta_2} \Gamma(V_3)) \\
\Big\downarrow {\scriptstyle \mathcal{J}^{\varepsilon}_{V_1,V_2}(s_1)\otimes \mathbf{1}} & & \Big\downarrow {\scriptstyle \mathbf{1}\otimes \mathcal{J}^{\varepsilon}_{V_2,V_3}(s_2)} \\
\Gamma(V_1 \otimes_{s_1} V_2) \otimes_{\zeta_2} \Gamma(V_3) & & \Gamma(V_1) \otimes_{\zeta_1\zeta_2} \Gamma(V_2 \otimes_{s_2} V_3) \\
\Big\downarrow {\scriptstyle \mathcal{J}^{\varepsilon}_{V_1\otimes_{s_1}V_2,V_3}(s_2)} & & \Big\downarrow {\scriptstyle \mathcal{J}^{\varepsilon}_{V_1,V_2\otimes_{s_2}V_3}(s_1+s_2)} \\
\Gamma((V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3) & = & \Gamma(V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3))
\end{array}$$

where $\zeta_i = \exp(2\pi\iota s_i)$.

(iv) *The poles of* $\mathcal{J}^+_{V_1,V_2}(s)^{\pm 1}$ *and* $\mathcal{J}^-_{V_1,V_2}(s)^{\pm 1}$ *are contained in*

$$\sigma(V_2) - \sigma(V_1) - \mathbf{Z}_{\geq 0} l\hbar - \frac{\hbar}{2}\{l + r\} - \mathbf{Z}_{>0} \quad \text{and}$$

$$\sigma(V_2) - \sigma(V_1) + \mathbf{Z}_{>0} l\hbar - \frac{\hbar}{2}\{l + r\} - \mathbf{Z}_{>0}$$

where r *ranges over the integers such that* $c_{ij}^{(r)} \neq 0$ *for some* $i, j \in \mathbf{I}$.

Remark. — Note that the condition $s \notin \sigma(V_2) - \sigma(V_1) + \mathbf{Z}$ implies that $V_1 \otimes_s V_2$ exists and is non-congruent, which is required in order to define $\Gamma(V_1 \otimes_s V_2)$.

Proof. — (i) and (iii)–(iv) follow from (7.6) and Theorem 5.9. (ii) is proved in 7.4. □

7.4. Given an element $X \in U_q(L\mathfrak{g})$, we denote its action on $\Gamma(V_1) \otimes_\zeta \Gamma(V_2)$ and $\Gamma(V_1 \otimes_s V_2)$ by X' and X'' respectively. We need to prove that

$$\mathcal{J}^\varepsilon_{V_1,V_2}(s) X' \mathcal{J}^\varepsilon_{V_1,V_2}(s)^{-1} = X''$$

Since $\xi_i(u)$ are group-like with respect to the Drinfeld coproduct, so are the fundamental solutions and the connection matrix of the difference equation $\phi_i(u+1) = \xi_i(u)\phi_i(u)$, which implies that $\Psi_i(z)' = \Psi_i(z)''$. Since $\mathcal{R}^{0,\pm}_{V_1,V_2}(s)$ and hence $\mathcal{J}^\varepsilon_{V_1,V_2}(s)$ commute with these elements, this proves the required relation for $\{\Psi_i(z)\}_{i \in \mathbf{I}}$.

We now prove the relation for $\mathcal{X}^+_{i,k}$. The proof for $\mathcal{X}^-_{i,k}$ is similar. By 4.2 and 6.3, the action of $(c_i^+)^{-1}\mathcal{X}^+_{i,k}$ on $\Gamma(V_1) \otimes_\zeta \Gamma(V_2)$ is given by

$$\begin{aligned}
&\zeta^k \oint_{C_1} e^{2\pi\iota k u} g_i^+(u) x_i^+(u) \otimes 1 \, du \\
&\qquad + \oint_{\mathcal{C}_2} \Psi_i\big(\zeta^{-1} w\big) \otimes \oint_{C_2} g_i^+(u) x_i^+(u) \frac{w}{w - e^{2\pi\iota u}} w^{k-1} \, dw du \\
&\quad = \zeta^k \oint_{C_1} e^{2\pi\iota k u} g_i^+(u) x_i^+(u) \otimes 1 \, du \\
&\qquad + \oint_{C_2} e^{2\pi\iota k u} g_i^+(u-s)\xi_i(u-s) g_i^-(u-s) \otimes g_i^+(u) x_i^+(u) \, du
\end{aligned}$$

where

- C_ℓ encloses $\sigma(V_\ell)$ and none of its $\mathbf{Z}_{\neq 0}$-translates.
- $\mathcal{C}_2$ encloses C_2, $\exp(2\pi\iota\sigma(V_2))$ and none of the points in $\exp(2\pi\iota(s + \sigma(V_1)))$. Note that these sets contain $\sigma(\Gamma(V_2))$ and $\zeta\sigma(\Gamma(V_1))$ by definition. We also remark that we are assuming $s \notin \sigma(V_2) - \sigma(V_1) + \mathbf{Z}$ in (ii) of Theorem 7.3 which makes such a choice of contours possible.

and we used (6.3).

On the other hand, the action of $(c_i^+)^{-1}\mathcal{X}_{i,k}^+$ on $\Gamma(\mathrm{V}_1 \otimes_s \mathrm{V}_2)$ is given by

$$\oint_{\mathrm{C}_{12}} e^{2\pi\iota k u} g_i^+(u-s)\otimes g_i^+(u)\Bigg(x_i^+(u-s)\otimes 1$$

$$+\oint_{\mathrm{C}_2'} \xi_i(v-s)\otimes x_i^+(v)\frac{dv}{u-v}\Bigg)\,du$$

$$=\zeta^k \oint_{\mathrm{C}_1} e^{2\pi\iota k u} g_i^+(u)x_i^+(u)\otimes g_i^+(u+s)\,du$$

$$+\oint_{\mathrm{C}_2} e^{2\pi\iota k v} g_i^+(v-s)\xi_i(v-s)\otimes g_i^+(v)x_i^+(v)\,dv$$

where

- C_{12} encloses $(\sigma(\mathrm{V}_1)+s)\cup\sigma(\mathrm{V}_2)$ (which contains $\sigma(\mathrm{V}_1\otimes_s \mathrm{V}_2)$) and none of its $\mathbf{Z}_{\neq 0}$-translates. Again it is possible thanks to our assumption on s imposed in (ii) of Theorem 7.3 above.
- C_2' encloses $\sigma(\mathrm{V}_2)$ and none of the points of $\sigma(\mathrm{V}_1)+s$.

C_1 is as above, and we assumed that C_{12} encloses C_2', and that $\mathrm{C}_2'=\mathrm{C}_2$.

Let us compute the action of $\mathrm{Ad}(\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s))$ on the first summand of $(c_i^+)^{-1}(\mathcal{X}_{i,k}^+)'$. Note that $\mathrm{ad}(\Omega_{\mathfrak{h}})\,x_i^+(v)\otimes 1=\sum_{j\in\mathbf{I}}[\varpi_j^\vee, x_i^+(v)]\otimes\xi_{j,0}=x_i^+(v)\otimes\xi_{i,0}$, by Equation (5.8). Therefore, for any $a\in\mathbf{C}$

$$\mathrm{Ad}\left(e^{a\Omega_{\mathfrak{h}}}\right)x_i^+(v)\otimes 1=e^{\mathrm{ad}(a\Omega_{\mathfrak{h}})}\,x_i^+(v)\otimes 1=x_i^+(v)\otimes e^{a\xi_{i,0}}$$

Using this and Proposition 5.11 we get

$$\mathrm{Ad}\left(\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\right)\left(\zeta^k\oint_{\mathrm{C}_1} e^{2\pi\iota k u} g_i^+(u)x_i^+(u)\otimes 1\,du\right)$$

$$=\zeta^k\oint_{\mathrm{C}_1} e^{2\pi\iota k u} g_i^+(u)x_i^+(u)\otimes e^{\gamma\hbar\xi_{i,0}}\prod_{n\geq 1}\xi_i(u+s+n)e^{-\hbar\xi_{i,0}/n}\,du$$

$$=\zeta^k\oint_{\mathrm{C}_1} e^{2\pi\iota k u} g_i^+(u)x_i^+(u)\otimes g_i^+(u+s)\,du$$

by the definition of $g_i^+(u)$ given in (6.2). This yields the first term on the right-hand side of $(c_i^+)^{-1}(\mathcal{X}_{i,k}^+)''$. A similar computation can be carried out for the second summand of $(c_i^+)^{-1}(\mathcal{X}_{i,k}^+)'$ which proves that

$$\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\left(\mathcal{X}_{i,k}^+\right)'\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)^{-1}=\left(\mathcal{X}_{i,k}^+\right)''$$

8. The commutative R-matrix of the quantum loop algebra

In this section, we review the construction of the commutative part $\mathscr{R}^0(\zeta)$ of the R-matrix of the quantum loop algebra. We prove that if $|q| \neq 1$, $\mathscr{R}^0(\zeta)$ defines a meromorphic commutativity constraint on $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$, when the latter is equipped with the Drinfeld tensor product studied in Section 4.

8.1. *Drinfeld pairing.* — The Drinfeld pairing for the quantum loop algebra was computed in terms of the loop generators by Damiani [4]. Its restriction to U^0 is given in [4, Corollary 9]. Define $\{\mathrm{H}_{i,r}\}_{i\in\mathbf{I},r\in\mathbf{Z}_{\neq 0}}$ by

$$\Psi_i^{\pm}(z) = \Psi_{i,0}^{\pm} \exp\left(\pm\left(q_i - q_i^{-1}\right)\sum_{r\geq 1}\mathrm{H}_{i,\pm r} z^{\mp r}\right) \tag{8.1}$$

Then, for each $m, n \geq 1$

$$\langle \mathrm{H}_{i,m}, \mathrm{H}_{j,-n}\rangle = -\delta_{m,n}\frac{q^{mb_{ij}} - q^{-mb_{ij}}}{m(q_i - q_i^{-1})(q_j - q_j^{-1})} \tag{8.2}$$

where $b_{ij} = d_i a_{ij} = d_j a_{ji}$. Define $\mathrm{H}_i^{\pm}(z) \in z^{\mp 1}\mathrm{U}^0[[z^{\mp 1}]]$ by

$$\mathrm{H}_i^{\pm}(z) = \pm\left(q_i - q_i^{-1}\right)\sum_{r\geq 1}\mathrm{H}_{i,\pm r} z^{\mp r}$$

Then, by (8.2),

$$\left\langle \mathrm{H}_i^{+}(z), \mathrm{H}_j^{-}(w)\right\rangle = \sum_{m\geq 1}\frac{q^{mb_{ij}} - q^{-mb_{ij}}}{m}\left(\frac{w}{z}\right)^m = \log\left(\frac{z - q^{-b_{ij}}w}{z - q^{b_{ij}}w}\right) \tag{8.3}$$

8.2. *Construction of* $\mathscr{R}^0$. — We now follow the argument outlined in Section 5.2 to construct the canonical element $\mathscr{R}^0$ of this pairing. Namely, differentiating (8.3) with respect to z yields

$$\left\langle \frac{d}{dz}\mathrm{H}_i^{+}(z), \mathrm{H}_j^{-}(w)\right\rangle = \frac{1}{z - q^{-b_{ij}}w} - \frac{1}{z - q^{b_{ij}}w} = \left(\mathrm{T}^{b_{ij}} - \mathrm{T}^{-b_{ij}}\right)\frac{1}{z - w}$$

where $\mathrm{T}f(z,w) = f(z, q^{-1}w)$ is the multiplicative shift operator with respect to w. Hence, if we define

$$\mathrm{H}^{j,-}(w) = \left(\mathrm{T}^{l} - \mathrm{T}^{-l}\right)^{-1}\sum_{k\in\mathbf{I}} c_{jk}(\mathrm{T})\mathrm{H}_k^{-}(w) \in w\mathrm{U}^0[[w]] \tag{8.4}$$

where $(\mathrm{T}^l - \mathrm{T}^{-l})^{-1}$ acts on w^k, $k \neq 0$, as multiplication by $(q^{-lk} - q^{lk})^{-1}$, then

$$\left\langle \frac{d}{dz}\mathrm{H}_i^{+}(z), \mathrm{H}^{j,-}(w)\right\rangle = \delta_{ij}\frac{1}{z - w}$$

Note that $\mathrm{H}^{j,-}(w)$ is explicitly given by

$$\mathrm{H}^{j,-}(w) = \sum_{k\in\mathbf{I}}\left(q_k - q_k^{-1}\right)\sum_{n\geq 1}\left(\frac{c_{jk}(q^n)}{q^{nl}-q^{-nl}}\mathrm{H}_{k,-n}\right)w^n$$

so that $\mathscr{R}^0$ is equal to

$$\mathscr{R}^0 = q^{-\Omega_{\mathfrak{h}}}\exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ m\geq 1}}\frac{m(q_i-q_i^{-1})(q_j-q_j^{-1})c_{ij}(q^m)}{q^{ml}-q^{-ml}}\mathrm{H}_{i,m}\otimes\mathrm{H}_{j,-m}\right)$$

8.3. *q-Difference equation for* $\mathscr{R}^0$. — Set $\mathscr{R}^0(\zeta) = (\tau_\zeta\otimes 1)\mathscr{R}^0$, so that

$$\mathscr{R}^0(\zeta) = q^{-\Omega_{\mathfrak{h}}}\exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ m\geq 1}}\frac{m(q_i-q_i^{-1})(q_j-q_j^{-1})c_{ij}(q^m)}{q^{ml}-q^{-ml}}\zeta^m\mathrm{H}_{i,m}\otimes\mathrm{H}_{j,-m}\right) \tag{8.5}$$

It is easy to see that $\mathscr{R}^0(\zeta)$ satisfies the following q-difference equation

$$\begin{aligned}\mathscr{R}^0\left(q^{2l}\zeta\right)&\mathscr{R}^0(\zeta)^{-1}\\ &= \exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ m\geq 1}}m\left(q_i-q_i^{-1}\right)\left(q_j-q_j^{-1}\right)c_{ij}\left(q^m\right)q^{ml}\zeta^m\mathrm{H}_{i,m}\otimes\mathrm{H}_{j,-m}\right)\end{aligned} \tag{8.6}$$

8.4. Using the method employed in 5.3–5.5, we will show that the right-hand side of (8.6) is the expansion of a rational function at $\zeta = 0$, once it is evaluated on a tensor product of finite-dimensional representations.

We note first that a typical summand may be interpreted as a contour integral as follows

$$\begin{aligned}&\sum_{m\geq 1}m\left(q_i-q_i^{-1}\right)\left(q_j-q_j^{-1}\right)c_{ij}\left(q^m\right)q^{ml}\zeta^m\mathrm{H}_{i,m}\otimes\mathrm{H}_{j,-m}\\ &\quad= \sum_{r\in\mathbf{Z}}c_{ij}^{(r)}\oint_C\frac{d\mathrm{H}_i^+(w)}{dw}\otimes\mathrm{H}_j^-\left(q^{l+r}\zeta w\right)dw\end{aligned}$$

On a tensor product of two finite-dimensional representations $\mathcal{V}_1, \mathcal{V}_2$, the first tensor factor is a rational function of w since

$$\frac{d\mathrm{H}_i^+(w)}{dw} = \Psi_i(w)^{-1}\frac{d\Psi_i(w)}{dw}$$

The second tensor factor $\mathrm{H}_j^-(q^{l+r}\zeta w)$ can be viewed as a single-valued function defined outside of a set of cuts radiating from $\zeta = \infty$. To see this, note that $\mathrm{H}_j^-(w)$ is a logarithm of the rational $\mathrm{End}(\mathcal{V}_2)$-valued function $\Psi_{j,0}^+\Psi_j(w)$, and that the latter is regular at $w = 0, \infty$ and takes the value 1 at $w = 0$. The result then follows from the variant of Proposition 5.4 below.

Proposition. — Let $\mathcal{V}$ *be a complex, finite-dimensional vector space, and* $\psi : \mathbf{C} \to \mathrm{End}(\mathcal{V})$ *a rational function, regular at* 0 *and* ∞ *such that*

- $\psi(0) = 1$.
- $[\psi(w), \psi(w')] = 0$ *for any* $w, w' \in \mathbf{C}$.

Let $\sigma(\psi) \subset \mathbf{C}^\times$ *be the set of poles of* $\psi(w)^{\pm 1}$*, and define the cut-set* $\mathsf{Y}(\psi)$ *by*

$$\mathsf{Y}(\psi) = \bigcup_{\alpha \in \sigma(\psi)} [\alpha, \infty)$$

where $[\alpha, \infty) = \{t\alpha : t \in \mathbf{R}_{\geq 1}\}$*. Then, there is a unique single-valued, holomorphic function* $\mathrm{H}(w) = \log_0(\psi(w)) : \mathbf{C} \setminus \mathsf{Y}(\psi) \to \mathrm{End}(\mathcal{V})$ *such that*

$$\exp\big(\mathrm{H}(w)\big) = \psi(w) \quad \textit{and} \quad \mathrm{H}(0) = 0$$

Moreover, $[\mathrm{H}(w), \mathrm{H}(w')] = 0$ *for any* $w, w' \in \mathbf{C}$ *and* $\frac{d\mathrm{H}}{dw} = \psi^{-1}\frac{d\psi}{dw}$.

The proof of this proposition is analogous to that of Proposition 5.4.

8.5. *The operator* $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$. — Let $\mathcal{V}_1, \mathcal{V}_2$ be two finite-dimensional representations of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$. Let $\mathcal{C}_1$ be a contour enclosing the set of poles $\sigma(\mathcal{V}_1)$ of $\mathcal{V}_1$, and consider the following operator on $\mathcal{V}_1 \otimes \mathcal{V}_2$

$$\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ r\in\mathbf{Z}}} c_{ij}^{(r)} \oint_{\mathcal{C}_1} \frac{d\mathrm{H}_i^+(w)}{dw} \otimes \mathrm{H}_j^-\left(q^{l+r}\zeta w\right) dw\right)$$

where

- $\frac{d\mathrm{H}_i^+}{dw} : \mathbf{C} \to \mathrm{End}(\mathcal{V}_1)$ is the rational function $\Psi_i^{-1}\frac{d\Psi_i}{dw}$,
- $\mathrm{H}_j^- : \mathbf{C} \setminus \mathsf{Y}(\Psi_{j,0}^+\Psi_j(w)) \to \mathrm{End}(\mathcal{V}_2)$ is given by Proposition 8.4,
- $\zeta \in \mathbf{C}$ is small enough so that $\mathrm{H}_j^-(q^{l+r}\zeta w)$ is an analytic function of w within $\mathcal{C}_1$, for every $j \in \mathbf{I}$ such that $c_{ij}^{(r)} \neq 0$ for some $i \in \mathbf{I}$.

Note that, for ζ small, the cut-set $\zeta^{-1}q^{-l-r}\mathsf{Y}(\Psi_{j,0}^+\Psi_j(w))$ of $\mathrm{H}_j^-(q^{l+r}\zeta w)$ can be made to avoid the contour $\mathcal{C}_1$. In particular, the right-hand side of the equation above defines a holomorphic function of ζ in a neighbourhood of $\zeta = 0$.

The following is the counterpart for $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ of Theorem 5.5.

Theorem.

(i) $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ *is a rational function of* ζ*, which is regular at* 0 *and* ∞*, and such that* $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(0) = 1 = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\infty)$.

(ii) *The poles of* $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)^{\pm 1}$ *are contained in* $\bigcup_r \sigma(\mathcal{V}_2)\sigma(\mathcal{V}_1)^{-1}q^{-l-r}$*, where* r *ranges over the integers such that* $c_{ij}^{(r)} \neq 0$ *for some* $i, j \in \mathbf{I}$.

(iii) *For any* ζ, ζ' *we have* $[\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta), \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta')] = 0$.

(iv) *For any* $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$*, we have*

$$\mathscr{A}_{\mathcal{V}_1\otimes_{\zeta_1}\mathcal{V}_2,\mathcal{V}_3}(\zeta_2) = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_3}(\zeta_1\zeta_2)\mathscr{A}_{\mathcal{V}_2,\mathcal{V}_3}(\zeta_2)$$
$$\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2\otimes_{\zeta_2}\mathcal{V}_3}(\zeta_1\zeta_2) = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_3}(\zeta_1\zeta_2)\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta_1)$$

(v) *The following shifted unitarity condition holds*

$$\sigma \circ \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}\left(\zeta^{-1}\right) \circ \sigma^{-1} = \mathscr{A}_{\mathcal{V}_2,\mathcal{V}_1}\left(q^{-2l}\zeta\right)$$

where $\sigma : \mathcal{V}_1 \otimes \mathcal{V}_2 \to \mathcal{V}_2 \otimes \mathcal{V}_1$ *is the flip of the tensor factors.*

(vi) *For every* $\alpha, \beta \in \mathbf{C}^\times$ *we have*

$$\mathscr{A}_{\mathcal{V}_1(\alpha),\mathcal{V}_2(\beta)}(\zeta) = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}\left(\zeta\alpha\beta^{-1}\right)$$

8.6. The following result is needed to prove (i) of the theorem above. The rest of the theorem follows from the same reasoning as Theorem 5.5.

Lemma. — $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ *extends to a rational function of* ζ*. Let* $\mathcal{C}_1$ *be a contour enclosing the set of poles* $\sigma(\mathcal{V}_1) \subset \mathbf{C}^\times$*, and such that* 0 *is outside of* $\mathcal{C}_1$*. Then we have the following*

$$\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \exp\left(-\sum_{\substack{i,j\in\mathbf{I}\\ r\in\mathbf{Z}}} c_{ij}^{(r)} \oint_{\mathcal{C}_1} \frac{d\mathrm{H}_i^+(w)}{dw} \otimes \mathrm{H}_j^+\left(q^{l+r}\zeta w\right) dw \right)$$

where $\mathrm{H}_j^+(w) = \log(\Psi_{j,0}^-\Psi_j(w))$ *is defined using Proposition 5.4, and* $\zeta \in \mathbf{C}$ *is large enough such that* $\mathrm{H}_j^+(q^{l+r}\zeta w)$ *is an analytic function of* w *within* $\mathcal{C}_1$ *for every* $j \in \mathbf{I}$ *such that* $c_{ij}^{(r)} \neq 0$ *for some* $i \in \mathbf{I}$.

Proof. — The proof of this lemma is analogous to that of Theorem 5.5(i) (see Claim 1 in the proof of that theorem). Again we revert to a more general set up as follows. Let V, W be complex, finite-dimensional vector spaces, $\mathrm{A} : \mathbf{C} \to \mathrm{End}(\mathrm{V})$, $\mathrm{B} : \mathbf{C} \to \mathrm{End}(\mathrm{W})$ rational functions such that

- $\mathrm{A}(z), \mathrm{B}(z)$ are regular and invertible at $z = 0$ and $z = \infty$.
- $\mathrm{A}(\infty) \in \mathrm{GL}(\mathrm{V})$ is a semisimple operator.
- $[\mathrm{A}(z), \mathrm{A}(w)] = 0 = [\mathrm{B}(z), \mathrm{B}(w)]$.

Let $b_0(z) = \log_0(\mathrm{B}(0)^{-1}\mathrm{B}(z))$ be defined according to Proposition 8.4 and $b_\infty(z) = \log(\mathrm{B}(\infty)^{-1}\mathrm{B}(z))$ using Proposition 5.4. Let $\mathcal{C}_1$ denote a contour in $\mathbf{C}^\times$ enclosing all the poles of $\mathrm{A}(z)^{\pm 1}$ and not enclosing 0. Define

$$\mathrm{X}_0(\zeta) = \exp\left(\oint_{\mathcal{C}_1} \mathrm{A}(w)^{-1}\mathrm{A}'(w) \otimes b_0(\zeta w)\, dw\right)$$

$$\mathrm{X}_\infty(\zeta) = \exp\left(\oint_{\mathcal{C}_1} \mathrm{A}(w)^{-1}\mathrm{A}'(w) \otimes b_\infty(\zeta w)\, dw\right)$$

where, for X_0 we need to take ζ small enough so that $\zeta^{-1}\mathsf{Y}(\mathrm{B}(0)^{-1}\mathrm{B}(w))$ is outside of $\mathcal{C}_1$ and hence $b_0(\zeta w)$ is analytic within $\mathcal{C}_1$, and for X_∞ we need to take ζ large enough so that $\zeta^{-1}\mathsf{X}(\mathrm{B}(\infty)^{-1}\mathrm{B}(w))$ is outside of $\mathcal{C}_1$ and hence $b_\infty(\zeta w)$ is analytic within $\mathcal{C}_1$.

We need to prove that both $\mathrm{X}_0(\zeta)$ and $\mathrm{X}_\infty(\zeta)$ extend to the same rational function of ζ, taking values in $\mathrm{End}(\mathrm{V}\otimes\mathrm{W})$. For this we consider the Jordan decomposition $\mathrm{A}(z) = \mathrm{A}_\mathrm{S}(z)\mathrm{A}_\mathrm{U}(z)$. By [13, Lemma 4.12], $\mathrm{A}_\mathrm{S}(z)$ and $\mathrm{A}_\mathrm{U}(z)$ are again rational functions of z. Furthermore, $\mathrm{A}_\mathrm{U}(\infty) = 1$ by our assumption that $\mathrm{A}(\infty)$ is semisimple. Since logarithmic derivative of $\mathrm{A}(z)$ splits the two additively, we can treat the semisimple and unipotent cases separately, analogous to the proof of Claim 1 in Theorem 5.5 given in Section 5.6.

The semisimple case reduces to the scalar case, *i.e.*, when V is one-dimensional and

$$\mathrm{A}(z) = \mathrm{A}(\infty)\prod_j \frac{z-\alpha_j}{z-\beta_j}$$

for some $\alpha_j, \beta_j \in \mathbf{C}^\times$. Following the computation given in Section 5.6, we get

$$\mathrm{X}_0(\zeta) = \prod_j 1 \otimes \mathrm{B}(\zeta\alpha_j)\mathrm{B}(\zeta\beta_j)^{-1} = \mathrm{X}_\infty(\zeta)$$

Now assuming $\mathrm{A}(z)$ is unipotent and $\mathrm{A}(\infty) = 1$, we get that $\log(\mathrm{A}(z))$ is again a rational function of z, vanishing at $z = \infty$. Decomposing it into partial fractions yields

$$\log\bigl(\mathrm{A}(z)\bigr) = \sum_{\substack{j\in\mathrm{J}\\ n\in\mathbf{Z}_{\geq 0}}} \frac{\mathrm{N}_{j,n}}{(z-\alpha_j)^{n+1}}$$

where J is a finite indexing set and $\alpha_j \in \mathbf{C}^\times$. We obtain

$$\mathrm{X}_0(\zeta) = \exp\left(\sum_{\substack{j\in\mathrm{J}\\ n\in\mathbf{Z}_{\geq 0}}} -(n+1)\mathrm{N}_{j,n} \otimes \frac{\partial_w^{n+1}}{(n+1)!} b_0(\zeta w)\Big|_{w=\alpha_j}\right)$$

$$\mathrm{X}_\infty(\zeta) = \exp\left(\sum_{\substack{j\in \mathrm{J}\\ n\in \mathbf{Z}_{\geq 0}}} -(n+1)\mathrm{N}_{j,n} \otimes \frac{\partial_w^{n+1}}{(n+1)!} b_\infty(\zeta w)\bigg|_{w=\alpha_j} \right)$$

which are both rational functions, since the $\mathrm{N}_{j,n}$ are nilpotent and pairwise commute. As rational functions, the two are the same since $b_0'(w) = b_\infty'(w) = \mathrm{B}(w)^{-1}\mathrm{B}'(w)$. □

8.7. *Commutation relation with $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$.* — Let $\mathcal{C} \subset \mathbf{C}$ be a contour, and $a_\ell : \mathbf{C} \to \mathrm{End}(\mathcal{V}_\ell)$, $\ell = 1, 2$ two meromorphic functions which are analytic within $\mathcal{C}$ and commute with the operators $\{\Psi_{i,\pm r}^{\pm}\}_{i\in \mathbf{I}, r\in \mathbf{Z}_{\geq 0}}$. For any $k \in \mathbf{I}$, define operators $\mathrm{X}_k^{\pm,\ell} \in \mathrm{End}(\mathcal{V}_1 \otimes \mathcal{V}_2)$ by

$$\mathrm{X}_k^{\pm,1} = \oint_{\mathcal{C}} a_1(w)\mathcal{X}_k^{\pm}(w) \otimes a_2(w)\, dw \quad \text{and}$$

$$\mathrm{X}_k^{\pm,2} = \oint_{\mathcal{C}} a_1(w) \otimes a_2(w)\mathcal{X}_k^{\pm}(w)\, dw$$

Proposition. — The following commutation relations hold

$$\begin{aligned}
&\mathrm{Ad}\big(\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)\big)\mathrm{X}_k^{\pm,1} \\
&\quad = \oint_{\mathcal{C}} a_1(w)\mathcal{X}_k^{\pm}(w) \otimes a_2(w)\Psi_k\big(q^{2l}\zeta w\big)^{\pm 1}\Psi_k(\zeta w)^{\mp 1}\, dw \\
&\mathrm{Ad}\big(\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)\big)\mathrm{X}_k^{\pm,2} \\
&\quad = \oint_{\mathcal{C}} a_1(w)\Psi_k\big(\zeta^{-1}w\big)^{\pm 1}\Psi_k\big(q^{-2l}\zeta^{-1}w\big)^{\mp 1} \otimes a_2(w)\mathcal{X}_k^{\pm}(w)\, dw
\end{aligned}$$

The proof of this proposition is identical to that of Proposition 5.10, except that the following version of relation (3.5) is needed. For each $i, k \in \mathbf{I}$,

$$\begin{aligned}
&\big[\Psi_i(z)^{-1}\Psi_i'(z), \mathcal{X}_k^{\pm}(w)\big] \\
&\quad = \left(\frac{1}{z - q^{\mp b_{ik}}w} - \frac{1}{z - q^{\pm b_{ik}}w}\right)\mathcal{X}_k^{\pm}(w) \\
&\qquad + \frac{wq^{\pm b_{ik}}}{z(z - q^{\pm b_{ik}}w)}\mathcal{X}_k^{\pm}\big(q^{\mp b_{ik}}z\big) - \frac{w}{z(q^{\pm b_{ik}}z - w)}\mathcal{X}_k^{\pm}\big(q^{\pm b_{ik}}z\big)
\end{aligned}$$

One can derive this relation easily from ($\mathcal{QL}$3) of Proposition 3.8 following the computation given in the proof of Lemma 3.13.

8.8. *Regular q-difference equations.* — We review below the existence and uniqueness of solutions of regular q-difference equations. Let $p \in \mathbf{C}^\times$ be such that $|p| \neq 1$, W a complex, finite-dimensional vector space, and consider the difference equation

$$F(pz) = B(z)F(z) \tag{8.7}$$

with values in End(W). Here, $B(z)$ is a meromorphic, End(W)-valued function. We shall assume that the equation is *regular*, that is that B is holomorphic near $z = 0$, and such that $B(0) = 1$. The following result is well-known (see, *e.g.,* [26, §1.2.2]).

Lemma. — *There is a unique formal series* $F(z) \in \mathrm{End}(W)[[z]]$ *which satisfies* (8.7) *and* $F(0) = 1$. *Moreover,* F *converges near* $z = 0$ *to a meromorphic function defined on* $\mathbf{C}$. *Any meromorphic solution* $G(z)$ *of* (8.7) *which is holomorphic in a neighbourhood of* $z = 0$ *is of the form* $F(z)C$ *where* $C = G(0) \in \mathrm{End}(W)$ *is a constant matrix.*

Let us remark that the existence of the formal series is automatic, since Equation (8.7) is equivalent to the recurrence relation $(p^n - 1)F_n = \sum_{m=0}^{n-1} B_{n-m}F_m$, where $F = \sum_{n\geq 0} F_n z^n$ and $B = \sum_{n\geq 0} B_n z^n$, with $F_0 = 1 = B_0$. The convergence of F is proved in [26, §1.2.2 Lemme 1]. The uniqueness is also clear, since the ratio $F(z)^{-1}G(z)$ is a holomorphic function on the elliptic curve $\mathbf{C}^\times/p^{\mathbf{Z}}$, and hence a constant.

One gets a similar assertion if $z = 0$ is changed to $z = \infty$ and one considers formal series in z^{-1}.

8.9. *The abelian* R*-matrix of* $U_q(L\mathfrak{g})$. — Assume now that $|q| \neq 1$. Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathrm{Rep}_{\mathrm{fd}}(U_q(L\mathfrak{g}))$, and let $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ be the operator defined in 8.5. Consider the q-difference equation

$$\overline{\mathscr{R}}_{\mathcal{V}_1,\mathcal{V}_2}(q^{2l}\zeta) = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)\overline{\mathscr{R}}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$$

This equation is regular at 0 and ∞ since $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(0) = 1 = \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\infty)$. By Lemma 8.8, it admits two unique formal solutions $\overline{\mathscr{R}}^{\pm}(\zeta)$ near $q^{\pm\infty}$, which are normalised by

$$\overline{\mathscr{R}}^{+}(q^{\infty}) = 1 = \overline{\mathscr{R}}^{-}(q^{-\infty})$$

These solutions converge in a neighbourhood of $q^{\pm\infty}$, and extend to meromorphic functions on the entire complex plane which are given by the products

$$\overline{\mathscr{R}}^{+}(\zeta) = \overrightarrow{\prod_{n\geq 0}} \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(q^{2ln}\zeta)^{-1} \quad \text{and} \quad \overline{\mathscr{R}}^{-}(\zeta) = \overrightarrow{\prod_{n\geq 1}} \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(q^{-2ln}\zeta)$$

Set

$$\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \begin{cases} q^{\mp\Omega_{\mathfrak{h}}}\overline{\mathscr{R}}^{\pm}(\zeta) & \text{if } |q| < 1 \\ q^{\pm\Omega_{\mathfrak{h}}}\overline{\mathscr{R}}^{\pm}(\zeta) & \text{if } |q| > 1 \end{cases}$$

By uniqueness, the evaluation on $\mathcal{V}_1 \otimes \mathcal{V}_2$ of the operator $\mathscr{R}^0(\zeta)$ given by (8.5), is the expansion at $\zeta = 0$ of $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$, where $\varepsilon \in \{\pm\}$ is such that $q^{\varepsilon\infty} = 0$.

The following is the analog of Theorem 5.9 for $\mathrm{U}_q(L\mathfrak{g})$.

Theorem. — The operators $\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ have the following properties

(i) *The map*

$$\sigma \circ \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) : \mathcal{V}_1(\zeta) \otimes_1 \mathcal{V}_2 \to \mathcal{V}_2 \otimes_1 \mathcal{V}_1(\zeta)$$

where σ is the flip of tensor factors, is a morphism of $\mathrm{U}_q(L\mathfrak{g})$-modules, which is natural in $\mathcal{V}_1$ and $\mathcal{V}_2$.

(ii) *For any $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(L\mathfrak{g}))$ we have*

$$\mathscr{R}^{0,\pm}_{\mathcal{V}_1\otimes_{\zeta_1}\mathcal{V}_2,\mathcal{V}_3}(\zeta_2) = \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_3}(\zeta_1\zeta_2)\mathscr{R}^{0,\pm}_{\mathcal{V}_2,\mathcal{V}_3}(\zeta_2)$$

$$\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2\otimes_{\zeta_2}\mathcal{V}_3}(\zeta_1\zeta_2) = \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_3}(\zeta_1\zeta_2)\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta_1)$$

(iii) *The following unitary condition holds*

$$\sigma \circ \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}\left(\zeta^{-1}\right) \circ \sigma^{-1} = \mathscr{R}^{0,\mp}_{\mathcal{V}_2,\mathcal{V}_1}(\zeta)^{-1}$$

(iv) *For $\alpha, \beta \in \mathbf{C}^\times$, we have*

$$\mathscr{R}^{0,\pm}_{\mathcal{V}_1(\alpha),\mathcal{V}_2(\beta)}(\zeta) = \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}\left(\zeta\alpha\beta^{-1}\right)$$

(v) *For any ζ, ζ',*

$$\left[\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta), \mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta')\right] = 0 = \left[\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta), \mathscr{R}^{0,\mp}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta')\right]$$

(vi) *$\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ is holomorphic near $q^{\pm\infty}$, and*

$$\mathscr{R}^{0,\pm}_{\mathcal{V}_1,\mathcal{V}_2}\left(q^{\pm\infty}\right) = \begin{cases} q^{-\Omega_{\mathfrak{h}}} & \textit{if } q^{\pm\infty} = 0 \\ q^{\Omega_{\mathfrak{h}}} & \textit{if } q^{\pm\infty} = \infty \end{cases}$$

(vii) *The poles of $\mathscr{R}^{0,+}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)^{\pm 1}$ and $\mathscr{R}^{0,-}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)^{\pm 1}$ are contained in*

$$\sigma(\mathcal{V}_2)\sigma(\mathcal{V}_1)^{-1}q^{-l-r}q^{-2l\mathbf{Z}_{\geq 0}} \quad \textit{and} \quad \sigma(\mathcal{V}_2)\sigma(\mathcal{V}_1)^{-1}q^{-l-r}q^{2l\mathbf{Z}_{>0}}$$

respectively, where r ranges over the integers such that $c^{(r)}_{ij} \neq 0$ for some $i, j \in \mathbf{I}$.

9. Kohno–Drinfeld theorem for abelian, additive *q*KZ equations

In this section, we prove that, when $\mathrm{Im}\,\hbar \neq 0$, the monodromy of the additive qKZ equations on n points defined by the commutative R-matrix of the Yangian is given by the commutative R-matrix of the quantum loop algebra. The general case is treated in 9.6, and follows from the $n = 2$ case which is treated in 9.2–9.3. In turn, the latter rests on relating the coefficient matrices of the difference equations whose solutions are the commutative R-matrix of $\mathrm{Y}_\hbar(\mathfrak{g})$ and $\mathrm{U}_q(L\mathfrak{g})$ respectively, which is done in 9.1 below.

9.1. Let V_1, V_2 be two finite-dimensional representations of $Y_\hbar(\mathfrak{g})$, $\mathcal{A}_{V_1,V_2}(s)$ the meromorphic $GL(V_1 \otimes V_2)$-valued function constructed in 5.5, and consider the difference equation[15]

$$f(s+1) = \mathcal{A}_{V_1,V_2}(s) f(s) \tag{9.1}$$

Assume further that V_1, V_2 are non-congruent, let $\mathcal{V}_\ell = \Gamma(V_\ell)$ be the representations of $U_q(L\mathfrak{g})$ obtained by using the functor Γ of Section 6, and $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) \in GL(\mathcal{V}_1 \otimes \mathcal{V}_2)$ the operator constructed in 8.5.

Proposition. — The operator $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ is the monodromy of the difference equation (9.1). That is,

$$\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \prod_{m\in\mathbf{Z}} \mathcal{A}_{V_1,V_2}(s+m)|_{\zeta=e^{2\pi\iota s}}$$

Proof. — For the purposes of the proof, we restrict ourselves to a typical factor in the definition of $\mathcal{A}_{V_1,V_2}(s)$. That is, fix $i,j \in \mathbf{I}$ and define

$$\mathcal{A}^{ij}_{V_1,V_2}(s) = \exp\left(\oint_{\mathcal{C}_1} t_i'(v) \otimes t_j(v+s)\, dv\right)$$

where $\mathcal{C}_1$ encloses the poles of $\xi_i(v)^{\pm 1}$ on V_1, and s is such that $t_j(v+s)$ is analytic within $\mathcal{C}_1$. Since V_1 is non-congruent, we may further assume that no two distinct points in the interior of $\mathcal{C}_1$ or on $\mathcal{C}_1$ are congruent modulo $\mathbf{Z}$.

By Theorem 5.5, $\mathcal{A}^{ij}_{V_1,V_2}(s)$ is a rational function of the form $1 + O(s^{-2})$. The corresponding monodromy matrix $M(\zeta)$ is a rational function of $\zeta = \exp(2\pi\iota s)$ which is given by

$$M(\zeta) = \prod_{m\in\mathbf{Z}} \mathcal{A}^{ij}_{V_1,V_2}(s+m)|_{\zeta=e^{2\pi\iota s}} = \lim_{N\to\infty}\left(\prod_{m=-N}^{N} \mathcal{A}^{ij}_{V_1,V_2}(s+m)|_{\zeta=e^{2\pi\iota s}}\right)$$

The corresponding factor of $\mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ is given by

$$\mathscr{A}^{ij}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \exp\left(\oint_{\widetilde{\mathcal{C}}_1} \Psi_i(w)^{-1}\Psi_i(w)' \otimes H_j^-(w\zeta)\, dw\right)$$

where $\widetilde{\mathcal{C}}_1 = \exp(2\pi\iota\mathcal{C}_1)$, and $H_j^-(w) = \log_0(\Psi_{j,0}^+\Psi_j(w))$ is given by Proposition 8.4. Note that $\widetilde{\mathcal{C}}_1$ is again a Jordan curve because of the assumptions imposed on $\mathcal{C}_1$.

[15] Note that (9.1) differs from the difference equation considered in 5.8 since its step is 1, not $l\hbar$.

We wish to show that $M(\zeta) = \mathscr{A}^{ij}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$. Since both sides are rational functions of ζ, it suffices to prove this for ζ near 0, that is $\mathrm{Im}(s) \gg 0$. Now

$$M(\zeta) = \lim_{N\to\infty} \exp\left(\sum_{m=-N}^{N} \oint_{\mathcal{C}_1} t_i'(v) \otimes t_j(v+s+m)\, dv\right)$$

Since $t_j(v) = \hbar\xi_{j,0}v^{-1} + O(v^{-2})$, the sum $\sum_{m=-N}^{N} t_j(u+m)$ converges uniformly on compact subsets of $\{|\,\mathrm{Im}\, u| > R\}$ for R large enough. To see this, we note that $t_j(v)$, as defined using Proposition 5.4, is a holomorphic function in a neighbourhood of $v = \infty$. Its Taylor series $t_j(v) = \hbar\sum_{r\geq 0} t_{j,r}u^{-r-1}$ therefore converges uniformly for $|u| > R$, for some $R > 0$. Each partial sum $f_N(u) = \sum_{m=-N}^{N} t_j(u+m)$ is a holomorphic function on $\Omega = \{|\,\mathrm{Im}(u)| > R\}$ since the set of poles of f_N is contained in the shifts of the closed disc $D_0(R) := \{|u| \leq R\}$ by integers $m \in \{-N, \dots, N\}$, and hence does not intersect Ω.

Given a compact subset $K \subset \Omega$, let $R_2 > R_1 > R$ be such that $R_1 < |u| < R_2$ for each $u \in K$. Take $M > N > R_1 + R_2$ and let us find an upper bound on $\|f_{N+1}(u) - f_M(u)\|$, which goes to 0 as N goes to ∞, uniformly for each $u \in K$. By definition of the radius of convergence there exists $P > 0$ such that $\|\hbar t_{j,r}\| \leq PR_1^{r+1}$ for each $r \geq 0$

First order term. For each $u \in K$ and $m \geq N$ we have $|u^2 - m^2| > m^2 - R_2^2$. Therefore

$$\left|\sum_{m=N+1}^{M} \frac{1}{u+m} + \frac{1}{u-m}\right| \leq \sum_{m=N+1}^{\infty} \frac{2|u|}{|u^2-m^2|} < \sum_{m=N+1}^{\infty} \frac{2R_2}{m^2 - R_2^2}$$
$$\leq \int_N^{\infty} \frac{2R_2}{x^2 - R_2^2}\, dx = \ln\left(\frac{N+R_2}{N-R_2}\right)$$

Higher order terms. Again we have $|u \pm m| > m - R_2$ for each $u \in K$. Thus, for each $r \geq 1$ we get

$$\left|\sum_{m=N+1}^{M} \frac{1}{(u\pm m)^{r+1}}\right| \leq \sum_{m=N+1}^{\infty} \frac{1}{(m-R_2)^{r+1}} \leq \int_N^{\infty} \frac{1}{(x-R_2)^{r+1}}\, dx$$
$$= \frac{1}{r(N-R_2)^r}$$

Hence, for any $M > N$ we obtain the following bound:

$$\|f_{N+1}(u) - f_M(u)\| < 2PR_1\left(\ln\left(\frac{N+R_2}{N-R_2}\right) + \sum_{r\geq 1} \frac{1}{r}\left(\frac{R_1}{N-R_2}\right)^r\right)$$
$$= 2PR_1 \ln\left(\frac{N+R_2}{N-R_1-R_2}\right)$$

Thus given $\epsilon > 0$ we can choose N large enough so that the above bound is less than ϵ uniformly for each $u \in K$, as claimed.

Now the exponential of $\lim_{N\to\infty}\sum_{m=-N}^{N} t_j(u+m)$ is $\prod_m \xi_j(u+m) = \Psi_j(e^{2\pi\iota u})$ (see Section 6.3). By the uniqueness of $\log_0$ this implies that, for $\operatorname{Im} s \gg 0$,

$$\lim_{N\to\infty}\sum_{m=-N}^{N} t_j(v+s+m) = H_j^-\left(\zeta e^{2\pi\iota v}\right) + \log\left(e^{-\pi\iota\hbar\xi_{j,0}}\right)$$

To see this we observe that, for a fixed v, both sides of the equation above have the same exponential and the same value at $\zeta = 0$, or equivalently $\operatorname{Im} s \to \infty$, in the domain $\{\operatorname{Im}(s) > R - \operatorname{Im}(v)\}$. For v ranging over a compact set (*e.g.*, interior of $\mathcal{C}_1$ and $\mathcal{C}_1$ included, which is needed below) we can take $\operatorname{Im} s \gg 0$ so that for each v in this compact set, the equation holds. Thus we get

$$M(\zeta) = \exp\left(\oint_{\mathcal{C}_1} t_i'(v) \otimes \left(H_j^-\left(\zeta e^{2\pi\iota v}\right) + \log\left(e^{-\pi\iota\hbar\xi_{j,0}}\right)\right) dv\right)$$

Since $t_i'(v) = O(v^{-2})$, we get $\oint t_i'(v)\otimes \log(e^{-\pi\iota\hbar\xi_{j,0}})\, dv = 0$, which implies that

$$M(\zeta) = \exp\left(\oint_{\mathcal{C}_1} \xi_i(v)^{-1}\xi_i(v)' \otimes H_j^-\left(\zeta e^{2\pi\iota v}\right) dv\right)$$

Noting that, by (6.3)

$$\Psi_i\left(e^{2\pi\iota v}\right)^{-1}\frac{d\Psi_i(e^{2\pi\iota v})}{dv} = g_i^+(v)^{-1}g_i^+(v)' + \xi_i(v)^{-1}\xi_i(v)' + g_i^-(v)^{-1}g_i^-(v)'$$

and that $g_i^{\pm}(v)$ are analytic and invertible within $\mathcal{C}_1$ by the non-congruence assumption, so that

$$\oint_{\mathcal{C}_1} g_i^{\pm}(v)^{-1}g_i^{\pm}(v)' \otimes H_j^-\left(\zeta e^{2\pi\iota v}\right) dv = 0$$

we get

$$\begin{aligned} M(\zeta) &= \exp\left(\oint_{\mathcal{C}_1} \Psi_i\left(e^{2\pi\iota v}\right)^{-1}\frac{d\Psi_i(e^{2\pi\iota v})}{dv} \otimes H_j^-\left(\zeta e^{2\pi\iota v}\right) dv\right) \\ &= \exp\left(\oint_{\widetilde{\mathcal{C}}_1} \Psi_i(w)^{-1}\frac{d\Psi_i(w)}{dw} \otimes H_j^-(\zeta w)\, dw\right) \end{aligned}$$

as claimed. □

9.2. *The (reduced) qKZ equations on $n = 2$ points.* — Assume henceforth that $\mathrm{Im}\,\hbar \neq 0$. Fix $\varepsilon \in \{\pm\}$, let $\mathrm{V}_1, \mathrm{V}_2 \in \mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$, and consider the abelian qKZ equation

$$f(s+1) = \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s) f(s)$$

with values in $\mathrm{End}(\mathrm{V}_1 \otimes \mathrm{V}_2)$.

By Proposition 7.1, this equation admits both right and left canonical solutions $\Phi^\varepsilon_\pm(s)$. The corresponding connection matrix is given by

$$\begin{aligned}\mathcal{S}^\varepsilon_{\mathrm{V}_1,\mathrm{V}_2}(s) &= \Phi^\varepsilon_+(s)^{-1}\Phi^\varepsilon_-(s)\\ &= \lim_{\mathrm{N}\to\infty} \mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s+\mathrm{N})\cdots\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)\cdots\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s-\mathrm{N})\end{aligned}$$

and is a meromorphic function of $\zeta = e^{2\pi\iota s}$ which admits a limit as $\mathrm{Im}\, s \to \pm\infty$, depending on whether $\mathrm{Im}(\varepsilon\hbar) \gtrless 0$. In particular, $\mathcal{S}^\varepsilon_{\mathrm{V}_1,\mathrm{V}_2}(\zeta)$ is regular at $\zeta = q^{\varepsilon\infty}$.

Lemma.

$$\mathcal{S}^\varepsilon_{\mathrm{V}_1,\mathrm{V}_2}\left(q^{\varepsilon\infty}\right) = \begin{cases} q^{-\Omega_\mathfrak{h}} & \textit{if } q^{\varepsilon\infty} = 0\\ q^{\Omega_\mathfrak{h}} & \textit{if } q^{\varepsilon\infty} = \infty\end{cases}$$

Proof. — Let us assume $\mathrm{Im}(\hbar) > 0$ and $\varepsilon = +$, for definiteness. Then, by Proposition 7.1, $\Phi^+_+(s)$ has the asymptotic expansion of the form $(1 + \mathrm{O}(s^{-1}))s^{\hbar\Omega_\mathfrak{h}}$ in any right half-plane, while $\Phi^+_-(s) \sim (1 + \mathrm{O}(s^{-1}))(-s)^{\hbar\Omega_\mathfrak{h}}$ only in an obtuse sector shown in Figure 1. Thus we can find a common domain for both, where the limit $\mathrm{Im}(s) \to \infty$ can be taken. Now we have

$$\begin{aligned}\mathcal{S}^+_{\mathrm{V}_1,\mathrm{V}_2}(0) &= \lim_{\mathrm{Im}(s)\to\infty}\left(\Phi^+_+(s)\right)^{-1}\Phi^+_-(s) = \lim_{\mathrm{Im}(s)\to\infty} s^{-\hbar\Omega_\mathfrak{h}}(-s)^{\hbar\Omega_\mathfrak{h}}\\ &= \lim_{\mathrm{Im}(s)\to\infty} e^{\hbar\Omega_\mathfrak{h}(\ln(-s)-\ln(s))} = e^{-\pi\iota\hbar\Omega_\mathfrak{h}}\end{aligned}$$

□

Remark. — Note that in the proof above, there is no common domain where both $\Phi^+_\pm(s)$ admit the claimed asymptotic expansions and $\mathrm{Im}(s)$ can go to $-\infty$. Consequently, the computation above cannot be carried out for $\mathcal{S}^+_{\mathrm{V}_1,\mathrm{V}_2}(\infty)$. This is in contrast with the computation of the monodromy of an additive difference equation when the coefficient matrix is rational, given, for example, in [13, Prop. 4.8].

9.3. *Kohno–Drinfeld theorem for abelian qKZ equations on 2 points.* — The following equates the monodromy of the abelian qKZ equations with the commutative R-matrix of $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$ constructed in 8.9.

Theorem. — *If* $\mathrm{V}_1, \mathrm{V}_2$ *are non-congruent,* $\mathcal{V}_\ell = \Gamma(\mathrm{V}_\ell)$ *are the corresponding representations of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$*, and* $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ *is the commutative* R*-matrix of* $\mathrm{U}_q(\mathrm{L}\mathfrak{g})$*, then*

$$\mathcal{S}^\varepsilon_{\mathrm{V}_1,\mathrm{V}_2}(\zeta) = \mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$$

Proof. — Let $\mathcal{A}_\pm(s)$ be the right and left fundamental solutions of the difference equation $f(s+1) = \mathcal{A}_{V_1,V_2}(s)f(s)$ considered in 9.1. We claim that

$$\Phi^\varepsilon_\pm(s+l\hbar)\Phi^\varepsilon_\pm(s)^{-1} = \mathcal{A}_\pm(s) \tag{9.2}$$

Assuming this for now, we see that $\mathcal{S}^\varepsilon_{V_1,V_2}(\zeta)$ and $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$ satisfy the same q-difference equation. Indeed,

$$\begin{aligned}\mathcal{S}^\varepsilon_{V_1,V_2}(q^{2l}\zeta)\mathcal{S}^\varepsilon_{V_1,V_2}(\zeta)^{-1} &= \Phi^\varepsilon_+(s+l\hbar)^{-1}\Phi^\varepsilon_-(s+l\hbar)\Phi^\varepsilon_-(s)^{-1}\Phi^\varepsilon_+(s)\\ &= \mathcal{A}_+(s)^{-1}\mathcal{A}_-(s)\\ &= \mathscr{A}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)\\ &= \mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(q^{2l}\zeta)\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)^{-1}\end{aligned}$$

where the third equality follows by Proposition 9.1, and the last one by definition of $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}$. Note that the reordering of factors in the calculation above is permissible since all the meromorphic functions involved take values in a commutative subalgebra of $\mathrm{End}(V_1 \otimes V_2)$. Since both $\mathcal{S}^\varepsilon_{V_1,V_2}$ and $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}$ are holomorphic near, and have the same value at $\zeta = q^{\varepsilon\infty}$, they are equal.

Returning to the claim, let $L^\varepsilon_\pm(s)$ denote the left-hand side of (9.2). Then,

$$\begin{aligned}L^\varepsilon_\pm(s+1)L^\varepsilon_\pm(s)^{-1} &= \Phi^\varepsilon_\pm(s+l\hbar+1)\Phi^\varepsilon_\pm(s+1)^{-1}\Phi^\varepsilon_\pm(s)\Phi^\varepsilon_\pm(s+l\hbar)^{-1}\\ &= \mathcal{R}^{0,\varepsilon}_{V_1,V_2}(s+l\hbar)\mathcal{R}^{0,\varepsilon}_{V_1,V_2}(s)^{-1}\\ &= \mathcal{A}_{V_1,V_2}(s)\end{aligned}$$

Thus, $L^\varepsilon_\pm(s)$ and $\mathcal{A}_\pm(s)$ satisfy the same difference equation. Since they also have the same asymptotics as $s \to \infty$ by Proposition 7.1, it follows that they are equal. □

Remark. — The monodromy $\mathcal{S}^\varepsilon_{V_1,V_2}(\zeta)$ may be written in terms of the tensor structures $\mathcal{J}^\pm_{V_1,V_2}(s)$ constructed in 7.3 as

$$\mathcal{S}^\varepsilon_{V_1,V_2}(\zeta) = \mathcal{J}^\varepsilon_{V_1,V_2}(s)\mathcal{R}^{0,\varepsilon}_{V_1,V_2}(s)\left(\mathcal{J}^{-\varepsilon}_{V_2,V_1}(-s)\right)^{-1}_{21}$$

where we used the unitarity constraint (iii) of Theorem 5.9. We can rearrange the factors of the triple product in the right-hand side above, again using the fact that all relevant meromorphic functions take values in a commutative subalgebra of $\mathrm{End}(V_1 \otimes V_2)$. This, and Theorem 9.3 imply the following equation

$$\sigma \circ \mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta) = \mathcal{J}^{-\varepsilon}_{V_2,V_1}(-s)^{-1} \circ \left(\sigma \circ \mathcal{R}^{0,\varepsilon}_{V_1,V_2}(s)\right) \circ \mathcal{J}^\varepsilon_{V_1,V_2}(s)$$

Here $\mathcal{V}_\ell = \Gamma(V_\ell)$ for $\ell = 1, 2$. This equation implies the commutativity of the following diagram, which means that the tensor structures $\mathcal{J}^\pm_{V_1,V_2}(s)$ are compatible with the meromorphic braidings on $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(\mathrm{U}_q(\mathrm{L}\mathfrak{g}))$ given by $\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)$ and $\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)$.

$$\begin{array}{ccc}
\Gamma(\mathrm{V}_1)\otimes_\zeta\Gamma(\mathrm{V}_2) & \xrightarrow{\mathcal{J}^{\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s)} & \Gamma(\mathrm{V}_1\otimes_s\mathrm{V}_2)\\
\| & & \|\\
\Gamma(\mathrm{V}_1)(\zeta)\otimes_1\Gamma(\mathrm{V}_2) & & \Gamma(\mathrm{V}_1(s)\otimes_0\mathrm{V}_2)\\
\downarrow{\scriptstyle \sigma\circ\mathscr{R}^{0,\varepsilon}_{\mathcal{V}_1,\mathcal{V}_2}(\zeta)} & & \downarrow{\scriptstyle \Gamma(\sigma\circ\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_1,\mathrm{V}_2}(s))}\\
\Gamma(\mathrm{V}_2)\otimes_1\Gamma(\mathrm{V}_1)(\zeta) & & \Gamma(\mathrm{V}_2\otimes_0\mathrm{V}_1(s))\\
\| & & \|\\
\big(\Gamma(\mathrm{V}_2)\otimes_{\zeta^{-1}}\Gamma(\mathrm{V}_1)\big)(\zeta) & \xrightarrow[\mathcal{J}^{-\varepsilon}_{\mathrm{V}_2,\mathrm{V}_1}(-s)]{} & (\Gamma(\mathrm{V}_2\otimes_{-s}\mathrm{V}_1))(\zeta)
\end{array}$$

9.4. *The abelian qKZ equations.* — Fix $\varepsilon\in\{\pm\}$ and $n\geq 2$, and let $\mathrm{V}_1,\ldots,\mathrm{V}_n\in\mathrm{Rep}_{\mathrm{fd}}(\mathrm{Y}_\hbar(\mathfrak{g}))$.

The following system of difference equations for a meromorphic function of n variables $\Phi:\mathbf{C}^n\to\mathrm{End}(\mathrm{V}_1\otimes\cdots\otimes\mathrm{V}_n)$ is an abelian version of the qKZ equations [11, 27]

$$\Phi(\underline{s}+e_i)=\mathrm{A}_i(\underline{s})\Phi(\underline{s}) \tag{9.3}$$

where $\underline{s}=(s_1,\ldots,s_n)$, $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbf{C}^n$, and

$$\begin{aligned}\mathrm{A}_i(\underline{s})=\mathcal{R}^{0,\varepsilon}_{i-1,i}(s_{i-1}-s_i-1)^{-1}\cdots\mathcal{R}^{0,\varepsilon}_{1,i}(s_1-s_i-1)^{-1}\\ \cdot\,\mathcal{R}^{0,\varepsilon}_{i,n}(s_i-s_n)\cdots\mathcal{R}^{0,\varepsilon}_{i,i+1}(s_i-s_{i+1})\end{aligned}$$

with $\mathcal{R}^{0,\varepsilon}_{i,j}=\mathcal{R}^{0,\varepsilon}_{\mathrm{V}_i,\mathrm{V}_j}$.

The above system is integrable, that is it satisfies

$$\mathrm{A}_i(\underline{s}+e_j)\mathrm{A}_j(\underline{s})=\mathrm{A}_j(\underline{s}+e_i)\mathrm{A}_i(\underline{s})$$

9.5. *Canonical fundamental solutions.* — The above system admits a set of canonical fundamental solutions which are parametrised by permutations $\sigma\in\mathfrak{S}_n$, and correspond to the right/left solutions in the case $n=2$.

To describe them, let $\Sigma^{\varepsilon}_{\pm,ij}\subset\mathbf{C}^n$ denote the asymptotic zones given in Proposition 7.1 with $s=s_i-s_j$, where $1\leq i\neq j\leq n$. Thus,

$$\Sigma^{\varepsilon}_{\pm,ij}=\big\{\underline{s}\in\mathbf{C}^n\,|\pm\mathrm{Re}(s_i-s_j)\gg 0\text{ and }\pm\mathrm{Re}\big((s_i-s_j)/n\big)\gg 0\big\}$$

where $n \in \mathbf{C}^\times$ is perpendicular to $\hbar$ and such that $\mathrm{Re}(n) \geq 0$, and the second condition in the definition of $\Sigma^\varepsilon_{\pm,ij}$ is required only if $\pm \mathrm{Re}(\varepsilon\hbar) < 0$.

For a permutation $\sigma \in \mathfrak{S}_n$, set

$$\mathrm{C}^\pm(\sigma) = \left\{i < j \mid \sigma^{-1}(i) \lessgtr \sigma^{-1}(j)\right\}$$

and define $\Sigma^\varepsilon(\sigma) \in \mathbf{C}^n$ by

$$\Sigma^\varepsilon(\sigma) = \bigcap_{(i,j)\in \mathrm{C}^+(\sigma)} \Sigma^\varepsilon_{+,ij} \cap \bigcap_{(i,j)\in \mathrm{C}^-(\sigma)} \Sigma^\varepsilon_{-,ij}$$

Proposition. — *For any $\sigma \in \mathfrak{S}_n$, Equation (9.3) admits a fundamental solution Φ^ε_σ which is uniquely determined by the following requirements*

(i) Φ^ε_σ *is holomorphic and invertible in* $\Sigma^\varepsilon(\sigma)$.
(ii) Φ^ε_σ *has an asymptotic expansion of the form*

$$\Phi^\varepsilon_\sigma(\underline{s}) \sim \big(1 + o(1)\big) \prod_{(i,j)\in \mathrm{C}^+(\sigma)} (s_i - s_j)^{\hbar\Omega_\mathfrak{h}} \prod_{(i,j)\in \mathrm{C}^-(\sigma)} (s_j - s_i)^{\hbar\Omega_\mathfrak{h}}$$

for $\underline{s} \in \Sigma^\varepsilon(\sigma)$, with $s_i - s_j \to \infty$ for any $i \neq j$.

Proof. — The solution Φ^ε_σ is constructed as follows. For each $i < j$, let $\Phi^\varepsilon_{\pm,ij}$ be the right and left canonical solutions of the abelian qKZ equation $\Phi_{ij}(s+1) = \mathcal{R}^{0,\varepsilon}_{i,j}(s)\Phi_{ij}(s)$ given in Proposition 7.1. Then,

$$\Phi^\varepsilon_\sigma(\underline{s}) = \prod_{(i,j)\in \mathrm{C}^+(\sigma)} \Phi^\varepsilon_+(s_i - s_j) \prod_{(i,j)\in \mathrm{C}^-(\sigma)} \Phi^\varepsilon_-(s_i - s_j)$$

We now prove the uniqueness (see, *e.g.*, [13, §4.3] for the one variable case). The ratio $\Xi^\varepsilon_\sigma = (\Phi^\varepsilon_\sigma)^{-1}\Psi^\varepsilon_\sigma$ of two solutions is holomorphic for $\underline{s} \in \Sigma^\varepsilon(\sigma)$, and periodic under the lattice $\mathbf{Z}^n \subset \mathbf{C}^n$. It therefore descends to a holomorphic function on the torus $\mathrm{T} = \mathbf{C}^n/\mathbf{Z}^n = (\mathbf{C}^\times)^n$. We claim that $\Xi^\varepsilon_\sigma(\underline{\zeta}) = 1$ for any $\underline{\zeta} \in (\mathbf{C}^\times)^n$. Note that $\Xi^\varepsilon_\sigma(\underline{\zeta}) = \Phi^\varepsilon_\sigma(\underline{s})^{-1}\Psi^\varepsilon_\sigma(\underline{s})$ for any $\underline{s} \in \Sigma^\varepsilon_\sigma$ such that $\zeta_j = e^{2\pi\iota s_j}$ for every j. By definition of the asymptotic zone $\Sigma^\varepsilon_\sigma$, we can find a sequence of points $\{\underline{s}^{(1)}, \underline{s}^{(2)}, \cdots\}$ in $\Sigma^\varepsilon_\sigma$ such that

(a) For every $j = 1, \cdots, n$, and $\mathrm{N} \geq 1$, $e^{2\pi\iota s_j^{(\mathrm{N})}} = \zeta_j$.
(b) For $i \neq j$, $s_i^{(\mathrm{N})} - s_j^{(\mathrm{N})} \to \infty$ as $\mathrm{N} \to \infty$.

Property (a) ensures that we have the following for each $\mathrm{N} \geq 1$

$$\Xi^\varepsilon_\sigma(\underline{\zeta}) = \Phi^\varepsilon_\sigma\left(\underline{s}^{(\mathrm{N})}\right)^{-1}\Psi^\varepsilon_\sigma\left(\underline{s}^{(\mathrm{N})}\right)$$

The asymptotics of Φ^ε_σ and Ψ^ε_σ, and property (b) above then imply that, as we let $\mathrm{N} \to \infty$, the ratio goes to 1. Note that, because of the abelian nature of the difference equations, the multivalued factors in the asymptotics from (ii) of the statement of the proposition cancel out. Thus $\Xi^\varepsilon_\sigma(\underline{\zeta}) = 1$ for every $\underline{\zeta} \in (\mathbf{C}^\times)^n$ and we are done. □

9.6. *Kohno–Drinfeld theorem for abelian qKZ equations.* — Assume now that $V_1, \dots, V_n$ are non-resonant, and let $\mathcal{V}_i = \Gamma(V_i)$ be the corresponding representations of $U_q(L\mathfrak{g})$. The following computes the monodromy of the abelian qKZ equations on $V_1 \otimes \cdots \otimes V_n$ in terms of the commutative R-matrix of $U_q(L\mathfrak{g})$ acting on $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$.

Theorem. — *Let $\sigma \in \mathfrak{S}_n$, and set $\sigma_i = (i\ i+1)$. Then,*

$$\left(\Phi^{\varepsilon}_{\sigma}(\underline{s})\right)^{-1} \Phi^{\varepsilon}_{\sigma_i\sigma}(\underline{s}) = \mathscr{R}^{0,\varepsilon}_{\mathcal{V}_i,\mathcal{V}_{i+1}}\left(\zeta_i\zeta_{i+1}^{-1}\right)^{\pm 1}$$

if $(i, i+1) \in C^{\pm}(\sigma)$, where $\zeta_j = e^{2\pi\iota s_j}$.

Proof. — This follows from the explicit form of the canonical fundamental solutions given by Proposition 9.5 and Theorem 9.3. □

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Appendix A: The inverse of the q-Cartan matrix of $\mathfrak{g}$

A.1 Let $\mathsf{A} = (a_{ij})_{i\in\mathbf{I}}$ be a Cartan matrix of finite type, and $d_i \in \mathbf{Z}_{>0}$ ($i \in \mathbf{I}$) be relatively prime symmetrising integers, *i.e.*, $d_i a_{ij} = d_j a_{ji}$ for every $i, j \in \mathbf{I}$. Consider the symmetrised Cartan matrix $\mathsf{B} = (d_i a_{ij})$, and its q-analog $\mathsf{B}(q) = ([d_i a_{ij}]_q)$. The latter defines a $\mathbf{C}(q)$-valued, symmetric bilinear form on $\bigoplus_{j\in\mathbf{I}} \mathbf{Q}(q)\alpha_j$ by

$$(\alpha_i, \alpha_j)_q = [d_i a_{ij}]_q$$

We give below explicit expressions for the fundamental coweights $\{\lambda_i^{\vee}(q)\}_{i\in\mathbf{I}}$ in terms of $\{\alpha_i\}$. That is, we compute certain elements $\lambda_i^{\vee}(q) \in \bigoplus_{j\in\mathbf{I}} \mathbf{Q}(q)\alpha_j$ such that $(\lambda_i^{\vee}(q), \alpha_j)_q = \delta_{ij}$ for every $i, j \in \mathbf{I}$. The main result of these calculations is the following.

Theorem. — *Let $l = mh^{\vee}$ where $m = 1, 2, 3$ for types* **ADE**, **BCF** *and* **G** *respectively, and $h^{\vee}$ is the dual Coxeter number. Then, for each $i \in \mathbf{I}$*

$$[l]_q \lambda_i^{\vee}(q) \in \oplus_{j\in\mathbf{I}} \mathbf{Z}_{\geq 0}\left[q, q^{-1}\right]\alpha_j$$

A.2 Below we follow Bourbaki's conventions, especially for the labels of the Dynkin diagrams. Recall the standard notations for q-numbers introduced in Section 3.6: $[m]_q = \frac{q^m - q^{-m}}{q-q^{-1}}$. For $m \geq 0$, $[m]_q = \sum_{i=0}^{m-1} q^{m-1-2i} \in \mathbf{Z}_{\geq 0}[q, q^{-1}]$. Moreover, define $\{m\}_q := q^m + q^{-m}$. The following identity is immediate and will be needed later:

$$[a]_q\{b\}_q = [a+b]_q + [a-b]_q \tag{A.1}$$

which belongs to $\mathbf{Z}_{\geq 0}[q, q^{-1}]$ if $a \geq b \geq 0$.

Also we note that for $a, b \in \mathbf{Z}_{\geq 0}$, with $a \neq 0$, we have

$$\frac{[ab]_q}{[a]_q} = [b]_{q^a} \in \mathbf{Z}_{\geq 0}\left[q, q^{-1}\right]$$

A.3 A_n. — In this case $l = n+1$. We have

$$\lambda_i^\vee(q) = \frac{1}{[n+1]_q}\left([n-i+1]_q\left(\sum_{j=1}^{i-1}[j]_q\alpha_j\right) + [i]_q\left(\sum_{j=i}^{n}[n-j+1]_q\alpha_j\right)\right)$$

Thus the assertion of Theorem A.1 holds in this case.

A.4 B_n. — In this case $l = 2(n+1)$. For $1 \leq i \leq n-1$ we have

$$\lambda_i^\vee(q) = \frac{1}{\{n+1\}_q}\left(\{n-i+1\}_q\left(\sum_{j=1}^{i-1}[j]_q\alpha_j\right)\right.$$
$$\left. + [i]_q\left(\left(\sum_{j=i}^{n-1}\{n-j+1\}_q\alpha_j\right) + \alpha_n\right)\right)$$

and

$$\lambda_n^\vee(q) = \frac{1}{\{n+1\}_q}\left(\left(\sum_{j=1}^{n-1}[j]_q\alpha_j\right) + \frac{[n]_q}{[2]_q}\alpha_n\right)$$

The statement of Theorem A.1 in this case follows for $1 \leq i \leq n-1$ from the identity $[m]_q\{m\}_q = [2m]_q$. For $\lambda_n^\vee(q)$, we can write (using the same identity)

$$\lambda_n^\vee(q) = \frac{1}{[2(n+1)]_q}\left([n+1]_q\left(\sum_{j=1}^{n-1}[j]_q\alpha_j\right) + \frac{[n+1]_q[n]_q}{[2]_q}\alpha_n\right)$$

Now it is clear that the coefficient of α_n is a Laurent polynomial in q with positive integer coefficients.

A.5 C_n. — In this case $l = 2(2n-1)$. We have the following for each $1 \leq i \leq n-1$

$$\lambda_i^\vee(q) = \frac{1}{[2]_q\{2n-1\}_q}\left(\{2n-2i-1\}_q\left(\sum_{j=1}^{i-1}[j]_{q^2}\alpha_j\right)\right.$$
$$\left.+[i]_{q^2}\left(\sum_{j=i}^{n-1}\{2n-2j-1\}_q\alpha_j\right)+[2i]_q\alpha_n\right)$$

and

$$\lambda_n^\vee(q) = \frac{1}{[2]_q\{2n-1\}_q}\sum_{j=1}^{n}[2j]_q\alpha_j$$

The statement of Theorem A.1 follows for $\lambda_n^\vee(q)$. For $1 \leq i \leq n-1$ we will have to use the following variant of (A.1):

$$\frac{[2n-1]_q\{2n-2j-1\}_q}{[2]_q} = \frac{[4n-2j-2]_q+[2j]_q}{[2]_q} \in \mathbf{Z}_{\geq 0}\left[q,q^{-1}\right]$$

A.6 D_n. — In this case $l = 2n-2$. We have the following for $1 \leq i \leq n-2$:

$$\lambda_i^\vee(q) = \frac{1}{\{n-1\}_q}\left(\{n-i-1\}_q\left(\sum_{j=1}^{i-1}[j]_q\alpha_j\right)\right.$$
$$\left.+[i]_q\left(\left(\sum_{j=i}^{n-2}\{n-j-1\}_q\alpha_j\right)+\alpha_{n-1}+\alpha_n\right)\right)$$

and

$$\lambda_{n-1}^\vee(q) = \frac{1}{\{n-1\}_q}\left(\left(\sum_{j=1}^{n-2}[j]_q\alpha_j\right)+\frac{[n]_q}{[2]_q}\alpha_{n-1}+\frac{[n-2]_q}{[2]_q}\alpha_n\right)$$
$$\lambda_n^\vee(q) = \frac{1}{\{n-1\}_q}\left(\left(\sum_{j=1}^{n-2}[j]_q\alpha_j\right)+\frac{[n-2]_q}{[2]_q}\alpha_{n-1}+\frac{[n]_q}{[2]_q}\alpha_n\right)$$

Again we obtain Theorem A.1 by the same argument as for B_n.

A.7 F_4. — In this case $l = 18$. We get the following

$$\lambda_1^\vee(q) = \frac{\{3\}_q}{\{9\}_q}\big(\{5\}_q\alpha_1 + [3]_{q^2}\alpha_2 + \{2\}_q\alpha_3 + \alpha_4\big)$$

$$\lambda_2^\vee(q) = \frac{\{3\}_q}{\{9\}_q}\big([3]_{q^2}\alpha_1 + [6]_q\alpha_2 + [4]_q\alpha_3 + [2]_q\alpha_4\big)$$

$$\lambda_3^\vee(q) = \frac{1}{\{9\}_q}\big(\{2\}_q\{3\}_q\alpha_1 + [4]_q\{3\}_q\alpha_2 + [3]_{q^2}\big(\{2\}_q\alpha_3 + \alpha_4\big)\big)$$

$$\lambda_4^\vee(q) = \frac{1}{\{9\}_q}\left(\{3\}_q\alpha_1 + [2]_q\{3\}_q\alpha_2 + [3]_{q^2}\alpha_3 + \frac{\{3\}_q\{4\}_q}{[2]_q}\alpha_4\right)$$

Again the statement of Theorem A.1 is clearly true, except for the coefficient of α_4 in $\lambda_4^\vee(q)$. For that entry we have

$$\frac{[9]_q\{3\}_q}{[2]_q} = \frac{[12]_q + [6]_q}{[2]_q} \in \mathbf{Z}_{\geq 0}\big[q, q^{-1}\big]$$

A.8 G_2. — In this case $l = 12$. We have the following answer

$$\lambda_1^\vee(q) = \frac{\{2\}_q}{\{6\}_q}\left(\frac{[2]_q}{[3]_q}\alpha_1 + \alpha_2\right) \qquad \lambda_2^\vee(q) = \frac{\{2\}_q}{\{6\}_q}\big(\alpha_1 + \{3\}_q\alpha_2\big)$$

As before we multiply and divide these expressions by $[6]_q$ to get the denominator $[12]_q$. Then it is easy to see the coefficients of α_1, α_2 are in $\mathbf{Z}_{\geq 0}[q, q^{-1}]$ as claimed.

A.9 *E series.* — The computations below were carried out using `sage`.

A.10 E_6. — In this case $l = 12$. We have the following expressions:

$$\begin{aligned}
[12]_q\lambda_1^\vee(q) &= \{3\}_q[8]_q\alpha_1 + \{2\}_q[6]_q\alpha_2 + \{2\}_q\{3\}_q[5]_q\alpha_3 + [4]_q[6]_q\alpha_4\\
&\quad + [2]_q\{3\}_q[4]_q\alpha_5 + \{3\}_q[4]_q\alpha_6\\
[12]_q\lambda_2^\vee(q) &= \{2\}_q[6]_q\alpha_1 + \{2\}_q\{3\}_q[6]_q\alpha_2 + [4]_q[6]_q\alpha_3 + \{2\}_q[3]_q[6]_q\alpha_4\\
&\quad + [4]_q[6]_q\alpha_5 + \{2\}_q[6]_q\alpha_6\\
[12]_q\lambda_3^\vee(q) &= \{2\}_q\{3\}_q[5]_q\alpha_1 + [4]_q[6]_q\alpha_2 + \{3\}_q[4]_q[5]_q\alpha_3\\
&\quad + \{1\}_q[4]_q[6]_q\alpha_4 + [2]_q^2\{3\}_q[4]_q\alpha_5 + [2]_q\{3\}_q[4]_q\alpha_6\\
[12]_q\lambda_4^\vee(q) &= [4]_q[6]_q\alpha_1 + \{2\}_q[3]_q[6]_q\alpha_2 + [2]_q[4]_q[6]_q\alpha_3\\
&\quad + [3]_q[4]_q[6]_q\alpha_4 + [2]_q[4]_q[6]_q\alpha_5 + [4]_q[6]_q\alpha_6\\
[12]_q\lambda_5^\vee(q) &= [2]_q\{3\}_q[4]_q\alpha_1 + [4]_q[6]_q\alpha_2 + [2]_q^2\{3\}_q[4]_q\alpha_3\\
&\quad + [2]_q[4]_q[6]_q\alpha_4 + \{3\}_q[4]_q[5]_q\alpha_5 + \{2\}_q\{3\}_q[5]_q\alpha_6\\
[12]_q\lambda_6^\vee(q) &= \{3\}_q[4]_q\alpha_1 + \{2\}_q[6]_q\alpha_2 + [2]_q\{3\}_q[4]_q\alpha_3 + [4]_q[6]_q\alpha_4\\
&\quad + \{2\}_q\{3\}_q[5]_q\alpha_5 + \{3\}_q[8]_q\alpha_6
\end{aligned}$$

A.11 E_7. — In this case $l = 18$ and we have the following expressions:

$$\begin{aligned}
\{9\}_q\lambda_1^\vee(q) &= \{3\}_q\{5\}_q\alpha_1 + \{2\}_q\{3\}_q\alpha_2 + \{3\}_q[3]_{q^2}\alpha_3 + \{3\}_q[4]_q\alpha_4 \\
&\quad + [6]_q\alpha_5 + [2]_q\{3\}_q\alpha_6 + \{3\}_q\alpha_7 \\
\{9\}_q\lambda_2^\vee(q) &= \{2\}_q\{3\}_q\alpha_1 + \frac{\{3\}_q[7]_q}{[2]_q}\alpha_2 + \{3\}_q[4]_q\alpha_3 + \{2\}_q[6]_q\alpha_4 \\
&\quad + [3]_q[3]_{q^2}\alpha_5 + [6]_q\alpha_6 + [3]_{q^2}\alpha_7 \\
\{9\}_q\lambda_3^\vee(q) &= \{3\}_q[3]_{q^2}\alpha_1 + \{3\}_q[4]_q\alpha_2 + \{3\}_q[6]_q\alpha_3 + [2]_q\{3\}_q[4]_q\alpha_4 \\
&\quad + [2]_q[6]_q\alpha_5 + [2]_q^2\{3\}_q\alpha_6 + [2]_q\{3\}_q\alpha_7 \\
\{9\}_q\lambda_4^\vee(q) &= \{3\}_q[4]_q\alpha_1 + \{2\}_q[6]_q\alpha_2 + [2]_q\{3\}_q[4]_q\alpha_3 + [4]_q[6]_q\alpha_4 \\
&\quad + [3]_q[6]_q\alpha_5 + [2]_q[6]_q\alpha_6 + [6]_q\alpha_7 \\
\{9\}_q\lambda_5^\vee(q) &= [6]_q\alpha_1 + [3]_q[3]_{q^2}\alpha_2 + [2]_q[6]_q\alpha_3 + [3]_q[6]_q\alpha_4 \\
&\quad + [3]_{q^2}[5]_q\alpha_5 + \{3\}_q[5]_q\alpha_6 + \frac{\{3\}_q[5]_q}{[2]_q}\alpha_7 \\
\{9\}_q\lambda_6^\vee(q) &= [2]_q\{3\}_q\alpha_1 + [6]_q\alpha_2 + [2]_q^2\{3\}_q\alpha_3 + [2]_q[6]_q\alpha_4 \\
&\quad + \{3\}_q[5]_q\alpha_5 + [2]_q\{3\}_q\{4\}_q\alpha_6 + \{3\}_q\{4\}_q\alpha_7 \\
\{9\}_q\lambda_7^\vee(q) &= \{3\}_q\alpha_1 + [3]_{q^2}\alpha_2 + [2]_q\{3\}_q\alpha_3 + [6]_q\alpha_4 \\
&\quad + \frac{\{3\}_q[5]_q}{[2]_q}\alpha_5 + \{3\}_q\{4\}_q\alpha_6 + [3]_{q^4}\alpha_7
\end{aligned}$$

It only remains to observe that

$$\frac{[9]_q\{3\}_q}{[2]_q} = \frac{[12]_q + [6]_q}{[2]_q} = [6]_{q^2} + [3]_{q^2} \in \mathbf{Z}_{\geq 0}\big[q, q^{-1}\big]$$

A.12 E_8. — In this case $l = 30$ and we have the following expression:

$$\begin{aligned}
\{15\}_q\lambda_1^\vee(q) &= \{5\}_q[4]_{q^3}\alpha_1 + \{3\}_q[5]_{q^2}\alpha_2 + [2]_{q^3}\frac{\{5\}_q[7]_q}{[2]_q}\alpha_3 + \{3\}_q[10]_q\alpha_4 \\
&\quad + \{3\}_q[4]_q\{5\}_q\alpha_5 + \{5\}_q[6]_q\alpha_6 + [2]_q\{3\}_q\{5\}_q\alpha_7 \\
&\quad + \{3\}_q\{5\}_q\alpha_8 \\
\{15\}_q\lambda_2^\vee(q) &= \{3\}_q[5]_{q^2}\alpha_1 + \{3\}_q\{5\}_q[4]_{q^2}\alpha_2 + \{3\}_q[10]_q\alpha_3 + [3]_{q^2}[10]_q\alpha_4 \\
&\quad + \{2\}_q\{5\}_q[6]_q\alpha_5 + [3]_q[3]_{q^2}\{5\}_q\alpha_6 + \{5\}_q[6]_q\alpha_7 \\
&\quad + \{5\}_q[3]_{q^2}\alpha_8 \\
\{15\}_q\lambda_3^\vee(q) &= [2]_{q^3}\frac{\{5\}_q[7]_q}{[2]_q}\alpha_1 + \{3\}_q[10]_q\alpha_2 + [2]_{q^3}\{5\}_q[7]_q\alpha_3 \\
&\quad + [2]_q\{3\}_q[10]_q\alpha_4 + [2]_q\{3\}_q[4]_q\{5\}_q\alpha_5 + [2]_q\{5\}_q[6]_q\alpha_6 \\
&\quad + [2]_q^2\{3\}_q\{5\}_q\alpha_7 + [2]_q\{3\}_q\{5\}_q\alpha_8
\end{aligned}$$

$$\begin{aligned}
\{15\}_q\lambda_4^\vee(q) &= \{3\}_q[10]_q\alpha_1 + [3]_{q^2}[10]_q\alpha_2 + [2]_q\{3\}_q[10]_q\alpha_3 + [6]_q[10]_q\alpha_4 \\
&\quad + [4]_q\{5\}_q[6]_q\alpha_5 + [3]_q\{5\}_q[6]_q\alpha_6 + [2]_q\{5\}_q[6]_q\alpha_7 \\
&\quad + \{5\}_q[6]_q\alpha_8 \\
\{15\}_q\lambda_5^\vee(q) &= \{3\}_q[4]_q\{5\}_q\alpha_1 + \{2\}_q\{5\}_q[6]_q\alpha_2 + [2]_q\{3\}_q[4]_q\{5\}_q\alpha_3 \\
&\quad + [4]_q\{5\}_q[6]_q\alpha_4 + \{2\}_q\{3\}_q[10]_q\alpha_5 + [3]_{q^2}[10]_q\alpha_6 \\
&\quad + \{3\}_q[10]_q\alpha_7 + \{3\}_q[5]_{q^2}\alpha_8 \\
\{15\}_q\lambda_6^\vee(q) &= \{5\}_q[6]_q\alpha_1 + [3]_q[3]_{q^2}\{5\}_q\alpha_2 + [2]_q\{5\}_q[6]_q\alpha_3 \\
&\quad + [3]_q\{5\}_q[6]_q\alpha_4 + [3]_{q^2}[10]_q\alpha_5 + \{4\}_q\{5\}_q[6]_q\alpha_6 \\
&\quad + [2]_q[2]_{q^3}\{4\}_q\{5\}_q\alpha_7 + [2]_{q^3}\{4\}_q\{5\}_q\alpha_8 \\
\{15\}_q\lambda_7^\vee(q) &= [2]_q\{3\}_q\{5\}_q\alpha_1 + \{5\}_q[6]_q\alpha_2 + [2]_q^2\{3\}_q\{5\}_q\alpha_3 \\
&\quad + [2]_q\{5\}_q[6]_q\alpha_4 + \{3\}_q[10]_q\alpha_5 + [2]_q[2]_{q^3}\{4\}_q\{5\}_q\alpha_6 \\
&\quad + [2]_q[3]_{q^4}\{5\}_q\alpha_7 + [3]_{q^4}\{5\}_q\alpha_8 \\
\{15\}_q\lambda_8^\vee(q) &= \{3\}_q\{5\}_q\alpha_1 + \{5\}_q[3]_{q^2}\alpha_2 + [2]_q\{3\}_q\{5\}_q\alpha_3 + \{5\}_q[6]_q\alpha_4 \\
&\quad + \{3\}_q[5]_{q^2}\alpha_5 + [2]_{q^3}\{4\}_q\{5\}_q\alpha_6 + [3]_{q^4}\{5\}_q\alpha_7 + \{5\}_q\{9\}_q\alpha_8
\end{aligned}$$

It only remains to observe that

$$\frac{[15]_q\{5\}_q}{[2]_q} = \frac{[20]_q + [10]_q}{[2]_q} = [10]_{q^2} + [5]_{q^2} \in \mathbf{Z}_{\geq 0}\left[q, q^{-1}\right]$$

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