A VISCOSITY METHOD IN THE MIN-MAX THEORY OF MINIMAL SURFACES

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ABSTRACT

We present the min-max construction of critical points of the area using penalization arguments. Precisely, for any immersion of a closed surface Σ into a given closed manifold, we add to the area Lagrangian a term equal to the L^q norm of the second fundamental form of the immersion times a "viscosity" parameter. This relaxation of the area functional satisfies the Palais–Smale condition for $q > 2$. This permits to construct critical points of the relaxed Lagrangian using classical min-max arguments such as the mountain pass lemma. The goal of this work is to describe the passage to the limit when the "viscosity" parameter tends to zero. Under some natural entropy condition, we establish a varifold convergence of these critical points towards a parametrized integer stationary varifold realizing the min-max value. It is proved in Pigati and Rivière (arXiv:1708.02211, 2017) that parametrized integer stationary varifold are given by smooth maps exclusively. As a consequence we conclude that every surface area minmax is realized by a smooth possibly branched minimal immersion.

I. Introduction

The study of minimal surfaces, critical points of the area, has stimulated the development of entire fields in analysis and in geometry. The calculus of variations is one of them. The origin of the field is very much linked to the question of proving the existence of minimal 2-dimensional discs bounding a given curve in the Euclidean 3-dimensional space and minimizing the area. This question, known as *Plateau Problem*, has been posed since the XVIIIth century by Joseph-Louis Lagrange, the founder of the Calculus of Variation after Leonhard Euler. This question has been ultimately solved independently by Jesse Douglas and Tibor Radó around 1930. In brief the main strategy of the proofs was to minimize the Dirichlet energy instead of the area, which is lacking coercivity properties, the two lagrangians being identical on conformal maps. After these proofs, successful attempts have been made to solve the Plateau Problem in much more general frameworks. This has been in particular at the origin of the field of *Geometric Measure Theory* during the 50's, where the notions of rectifiable current which were proved to be the *ad-hoc* objects for the minimization process of the area (or the mass in general) in the most general setting.

The search of absolute or even local minimizers is of course the first step in the study of the variations of a given lagrangians but is far from being exhaustive while studying the whole set of critical points. In many problems there is even no minimizer at all, this is for instance the case of closed surfaces in simply connected manifolds with also trivial two dimensional homotopy groups. This problem is already present in the 1-dimensional counter-part of minimal surfaces, the study of closed geodesics. For instance in a submanifold of $\mathbf{R}^3$ diffeomorphic to S^2 there is obviously no closed geodesic minimizing the length. In order to construct closed geodesics in such manifold, Birkhoff around 1915 introduced a technic called "min-max" which permits to generate critical points of the

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length with non trivial index. In two words this technic consists in considering the space of paths of closed curves within a non-trivial homotopy classes of paths in the sub-manifold (called "sweep-out") and to minimize, out of all such paths or "sweep-outs", the maximal length of the curves realizing each "sweep-out". In order to do so, one is facing the difficulty posed by a lack of coercivity of the length with respect to this minimization process within this "huge space" of sweep-outs. In order to "project" the problem to a much smaller space of "sweep-outs" in which the length would become more coercive, George Birkhoff replaced each path by a more regular one made of very particular closed curves joining finitely many points with portions of geodesics minimizing the length between these points. This replacement method also called nowadays "curve shortening process" has been generalized in many situations in order to perform min-max arguments.

Back to minimal surfaces, in a series of two works (see [6] and [7]), Tobias Colding and Bill Minicozzi, construct by min-max methods minimal 2 dimensional spheres in homotopy 3-spheres (the analysis carries over to general target Riemannian manifolds). The main strategy of the proof combines the original approach of Douglas and Radó, consisting in replacing the area functional by the Dirichlet energy, with a "Birkhoff type" argument of optimal replacements. Locally to any map from a given "sweep-out" one performs a surgery, replacing the map itself by an harmonic extension minimizing the Dirichlet energy. The convergence of such a "harmonic replacement" procedure, corresponding in some sense to Birkhoff "curve shortening procedure" in one dimension, is ensured by a fundamental result regarding the local convexity of the Dirichlet energy into a manifold under small energy assumption and a unique continuation type property. What makes possible the use of the Dirichlet energy instead of the area functional, as in [38], is the fact that the domain S^2 posses only one conformal structure and modulo a re-parametrization any $W^{1,2}$ map can be made ε-conformal (due to a fundamental result of Charles Morrey see Theorem I.2 [28]). This is not anymore the case if one wants to extend Colding–Minicozzi's approach to general surfaces. This has been done however successfully by Zhou Xin in [48] and [49] following the original Colding–Minicozzi approach. These papers are based on an involved argument in which to any "sweep-out" of $W^{1,2}$-maps a path of smooth conformal structures together with a path of re-parametrization are assigned in order to be as close as possible to paths of conformal maps.

Because of the finite dimensional nature of the moduli space of conformal structures in 2-D, and the "optimal properties" of the Dirichlet energy, Colding–Minicozzi's min-max method is intrinsically linked to two dimensions as Douglas–Radó's resolution of the Plateau problem was too. The field of Geometric Measure Theory, which was originally designed to remedy to this limitation and to solve the Plateau Problem for arbitrary dimensions in various homology classes, has been initially developed with a minimization perspective and the framework of rectifiable currents as well as the lower semicontinuity of the mass for weakly converging sequences was matching perfectly this goal. In order to solve min-max problems in the general framework of Geometric Measure Theory, the notion of *varifold* has been successfully introduced by William Allard and by Fred Almgren. A complete GMT min-max procedure has been finally set up by Jon Pitts in [31] who introduced the notion of *almost minimizing varifolds* and developed their regularity theory in co-dimension 1. Constructive comparison arguments as well as combinatorial type arguments are also needed in this rather involved and general procedure (The reader is invited also to consult [5] and [22] for thorough presentations of the GMT approaches to min-max procedures).

The aim of the present work is to present a direct min-max approach for constructing minimal surfaces in a given closed sub-manifold N^n of $\mathbf{R}^m$. The general scheme is simple : one works with a special subspace of C^1 immersions of a given surface Σ, one adds to the area of each of such an immersion $\vec{\Phi}$ a relaxing "curvature type" functional multiplied by a small viscous parameter σ^2

$$(\mathbf{I.1}) \qquad A^\sigma(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \text{curvature terms } dvol_{g_{\vec{\Phi}}}$$

where $dvol_{g_{\vec{\Phi}}}$ is the volume form on Σ induced by the immersion $\vec{\Phi}$. The "curvature terms" is chosen in order to ensure that A^σ satisfies the *Palais–Smale property* on the *ad-hoc* corresponding Finsler manifold of C^1-immersions. This offers the suitable framework in which Palais deformation theory can be applied to produce critical points realizing an arbitrary minmax value. Once a min-max critical point of A^σ is produced one passes to the limit $\sigma \to 0 \cdots$

More precisely, we introduce the space $\mathcal{E}_{\Sigma,p}$ of $W^{2,2p}$-immersions $\vec{\Phi}$ of a given closed surface Σ for $p > 1$[1] into $N^n \subset \mathbf{R}^m$. It is proved below that this space has a nice structure of *Banach Manifold*. For such immersions we consider the relaxed energy

$$A^\sigma(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma^2 \int_\Sigma \left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}}|^2\right]^p dvol_{g_{\vec{\Phi}}}$$

where $g_{\vec{\Phi}}$ and $\vec{\mathbf{I}}_{\vec{\Phi}}$ are respectively the first and second fundamental forms of $\vec{\Phi}(\Sigma)$ in N^n. Unlike previous existing viscous relaxations for min-max problems in the literature, the energy A^σ is intrinsic in the sense that it is invariant under re-parametrization of $\vec{\Phi}$: $A^\sigma(\vec{\Phi}) = A^\sigma(\vec{\Phi} \circ \Psi)$ for any smooth diffeomorphism Ψ of Σ. Modulo a choice of parametrization it is proved in [20] and [18] that for a fixed $\sigma \neq 0$ the Lagrangian A^σ satisfies the *Palais–Smale condition*. Hence we can consider applying the mountain path lemma to this Lagrangian.

We introduce now the following definition

[1] The condition $p > 1$ ensures that $\vec{\Phi}$ is C^1. This last fact permits to use the classical definition of an immersion. The case $p = 1$ was considered in previous works by the author where the notion of immersion had to be weakened.

Definition **I.1.** — *Let* Σ *be a closed Riemann surface and* M^m *be a closed sub-manifold* $M^m \subset \mathbf{R}^Q$. *A map* $\vec{\Phi} \in W^{1,2}(\Sigma, M^m)$ *together with an* L^∞ *bounded integer multiplicity* N_x *is called "integer target harmonic" if for almost every*[2] *domain* $\Omega \subset \Sigma$ *and any smooth function* F *supported in the complement of an open neighborhood of* $\vec{\Phi}(\partial\Omega)$ *we have*

$$(\mathbf{I.2}) \qquad \int_\Omega N_x \left\langle d\left(F(\vec{\Phi})\right), d\vec{\Phi}\right\rangle_{g_0} - N_x\, F(\vec{\Phi})\, A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_{g_0}\, dvol_{g_0}$$

where g_0 *is an arbitrary metric whose conformal structure is the one given by the Riemann surface* Σ, $\langle \cdot, \cdot \rangle_{g_0}$ *denotes the scalar product in* $T^*\Sigma$ *issued from* g_0, $A(\vec{y})$ *denotes the second fundamental form of* $M^m \subset \mathbf{R}^m$ *at the point* $\vec{y}$. *When the function* N *is constant on* Σ *we simply speak about "target harmonic" maps.*

Our main result in the present work is the following convergence theorem.

Theorem **I.1.** — *Let* N^n *be a closed n-dimensional sub-manifold of* $\mathbf{R}^m$ *with* $3 \le n \le m-1$ *being arbitrary. Let* Σ *be an arbitrary closed Riemanian 2-dimensional manifold. Let* $\sigma_k \to 0$ *and let* $\vec{\Phi}_k$ *be a sequence of critical points of*

$$A^{\sigma_k}(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma_k^2 \int_\Sigma \left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}}|^2\right]^p dvol_{g_{\vec{\Phi}}}$$

in the space of $W^{2,2p}$*-immersions of* Σ *and satisfying the entropy condition*

$$(\mathbf{I.3}) \qquad \sigma_k^2 \int_\Sigma \left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}_k}|^2\right]^p dvol_{g_{\vec{\Phi}_k}} = o\left(\frac{1}{\log \sigma_k^{-1}}\right).$$

Then, modulo extraction of a subsequence, there exists a closed Riemann surface (S, h_0) *with*

$$genus(S) \le genus(\Sigma)$$

and a conformal integer target harmonic map $(\vec{\Phi}_\infty, N)$ *from* S *into* N^n *such that*

$$\lim_{k \to +\infty} A^{\sigma_k}(\vec{\Phi}_k) = \frac{1}{2} \int_S N\, |d\vec{\Phi}_\infty|^2_{h_0}\, dvol_{h_0}.$$

Moreover, the oriented varifold associated to $\vec{\Phi}_k$ *converges in the sense of Radon measures towards the stationary integer varifold associated to* $(\vec{\Phi}_\infty, N)$.

The regularity of target harmonic maps is established in [35] and [30]. According to the main results in these works the limit $(\vec{\Phi}_\infty, N)$ is a smooth minimal branched immersion equipped with a smooth integer valued multiplicity.

[2] The notion of *almost every domain* means for every smooth domain Ω and any smooth function f such that $f^{-1}(0) = \partial\Omega$ and $\nabla f \neq 0$ on $\partial\Omega$ then for almost every t close enough to zero and regular value for f one considers the domains contained in Ω or containing Ω and bounded by $f^{-1}(\{t\})$.

Open problem: Assuming $\vec{\Phi}_k$ has a uniformly bounded *Morse index* for the Lagrangian A^{σ_k} one expects that the convergence is a strong $\mathrm{W}^{1,2}$- "bubble tree" convergence (i.e. strong away from finitely many points) which is equivalent to $\mathrm{N} \equiv 1$ on S.

The main difficulty in proving Theorem I.1 in contrast with existing non intrinsic viscous approximations of min-max procedures in the literature is that there is a-priori no ϵ-regularity property *independent* of the viscosity σ available. Indeed the following result is proved in [25].

Proposition **I.1.** — *There exists* $\vec{\Phi}_k \in C^\infty(\mathrm{T}^2, \mathrm{S}^3)$ *and* $\sigma_k \to 0$ *such that* $\vec{\Phi}_k$ *is a sequence of immersions, critical points of* A^{σ_k}*, which is conformal into* S^3 *from a converging sequence of flat tori* $\mathbf{R}^2/\mathbf{Z} + (a_k + i\, b_k)\mathbf{Z}$ *towards* $\mathbf{R}^2/\mathbf{Z} + (a_\infty + i\, b_\infty)\mathbf{Z}$*, for which*

$$\limsup_{k\to+\infty} \mathrm{A}^{\sigma_k}(\vec{\Phi}_k) < +\infty$$

such that also $\vec{\Phi}_k$ *weakly converges to a limiting map* $\vec{\Phi}_\infty$ *in* $\mathrm{W}^{1,2}(\mathbf{R}^2/\mathbf{Z} + (a_\infty + i\, b_\infty)\mathbf{Z}, \mathrm{S}^3)$ *but* $\vec{\Phi}_k$ *nowhere strongly converges: precisely*

$$\forall \mathrm{U} \text{ open set in } \mathbf{R}^2/\mathbf{Z} + (a_\infty + i\, b_\infty)\mathbf{Z}$$
$$\int_{\mathrm{U}} |\nabla \vec{\Phi}_\infty|^2\, dx^2 < \liminf_{k\to+\infty} \int_{\mathrm{U}} |\nabla \vec{\Phi}_k|^2\, dx^2.$$

In order to overcome this major difficulty in the passage to the limit $\sigma_k \to 0$ we prove a quantization result, Lemma III.3, which roughly says that there is a positive number Q_0, depending only on the target $\mathrm{N}^n \subset \mathbf{R}^m$, below which for k large enough, under the entropy condition assumption, there is no critical point of A^{σ_k}. This result is used at several stages in the proof. The main strategy goes as follows. We first establish the stationarity of the limiting varifold. The proof is based on an *almost divergence form* of the Euler Lagrange equation associated to A^σ following the approach introduced in [33] for the Willmore Lagrangian in $\mathbf{R}^m$. The existence of such an *almost divergence form* is due to the symmetry group associated to the same Lagrangian in flat space and the application of Noether theorem (see [2]). As in [25], the exact divergence form in Euclidian space is just an *almost-divergence form* in manifold. Next we choose a conformal parametrization of $\vec{\Phi}_k$ on a possibly degenerating sequence of Riemann surfaces (Σ, h_k) (where h_k denotes the constant curvature metric of volume 1 conformally equivalent to $\vec{\Phi}_k^* g_{\mathrm{N}^n}$). We use Deligne Mumford compactification in order to make converge (Σ, h_k) towards a nodal Riemann surface with punctures (see for instance [13]). We then use the monotonicity formula, deduced from the stationarity, in order to prove that, away from a so called *oscillation set*, the limiting volume density measure on the thick parts of the limiting nodal surface is absolutely continuous with respect to the Lebesgue measure. We then use the monotonicity formula again in order to prove the quantization result Lemma III.3. This quantization result is used in order to show that the limiting volume density measure restricted to the oscillation set is equal to finitely many Dirac masses. The quantization result is again used in order to prove that for the weakly converging sequence $\vec{\Phi}_k$ there is no energy loss neither in the necks in each thick parts of the limiting nodal surface, nor in the collars regions separating possible bubbles, which are possibly formed (see Lemma III.6). The previous results are proved to show the rectifiability of the limiting varifold (see Lemma III.8). We then prove that there is no measure concentrated on the set of points where the rank of the weak limit $\vec{\Phi}_\infty$ on each thick part and on each bubble is not equal to 2. Finally we use all the previous results to prove a "bubble tree convergence" of the sequence $\vec{\Phi}_k$ on each thick part (Lemma III.10) which gives in particular that the limiting rectifiable stationary varifold is integer. The last lemma, Lemma III.13, establishes that the limiting map is a *conformal target harmonic map* on each thick part of the nodal surface and on each bubble.

Theorem I.1 can be used to prove various existence results of optimal varifolds realizing a min-max energy level. We first define the following notion.

Definition **I.2.** — *A family of subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall \mathrm{A} \in \mathcal{A} \quad \Xi(\mathrm{A}) \in \mathcal{A}.$$

Example. — Consider $\mathcal{M} := \mathrm{W}^{2,2p}_{imm}(\Sigma, \mathrm{N}^n)$ for some closed oriented surface Σ and some closed sub-manifold N^n of $\mathbf{R}^m$ and take for any $q \in \mathbf{N}$ and $c \in \pi_q(\mathrm{Imm}(\Sigma), \mathrm{N}^n)$ then the following family is admissible

$$\mathcal{A} := \left\{\vec{\Phi} \in \mathrm{C}^0\left(\mathrm{S}^q, \mathrm{W}^{2,2p}_{imm}\left(\mathrm{S}^2, \mathrm{N}^n\right)\right);\ \text{s.t. } [\vec{\Phi}] = c\right\}.$$

Our second main result is the following.

Theorem **I.2.** — *Let $\mathcal{A}$ be an admissible family in the space of $\mathrm{W}^{2,2p}$-immersions into a closed sub-manifold of an Euclidean space N^n. Assume*

$$\inf_{\mathrm{A} \in \mathcal{A}} \max_{\vec{\Phi} \in \mathrm{A}} \mathrm{Area}(\vec{\Phi}) = \beta^0 > 0, \tag{I.4}$$

then there exists a closed Riemann surface (S, h_0) with genus$(\mathrm{S}) \leq$ genus(Σ) and a conformal integer target harmonic map $(\vec{\Phi}_\infty, \mathrm{N})$ from S into N^n such that

$$\frac{1}{2} \int_{\mathrm{S}} \mathrm{N}\, |d\vec{\Phi}_\infty|^2_{h_0}\, dvol_{h_0} = \beta^0.$$

This general existence result has to be put in perspective with the previous min-max existence results partly discussed above either in *GMT* (see [31], [41], [5], [22], [23], ...) in *harmonic map theory* (see [6], [7], [48], [49]) or using *level set-PDE* approaches (see [15], [47], [11], [10], [42], [43]). Combined with the main regularity results in [35] and [30] Theorem I.2 implies in particular all known results for the realization of arbitrary minmax by minimal surfaces. One technical advantage of the present work over the previous existing literature on minmax theory for surfaces in GMT or harmonic map theory, is that our proof of Theorem I.2 does not require any "replacement argument". The viscosity approach gives moreover, without any additional work, an upper bound of the genus of the optimal surface. Such lower semicontinuity of the genus has been established in the GMT approach in [8] in co-dimension 1 and was not given by the min-max procedure itself. As in the geodesic case studied recently in [25] and where a passage to the limit in the second derivative is proved, the viscosity approach gives under the multiplicity one assumption[3] ($N = 1$ a.e. on S) informations on the limiting index (see [37]). This fact was left open in the *GMT*, the *harmonic map* as well as in the *level set-PDE* approaches in it's full generality (see however partial important results in this direction for the PDE approach in [24]).

The second, and possibly main advantage, of the viscosity method resides in the fact that one can explore min-max within the space of *immersions* of fixed closed surfaces. The spaces $\mathrm{Imm}(\Sigma, N^n)$ offers a richer topology than the space of *integer rectifiable 2-cycles* $\mathcal{Z}_2(N^n)$ considered by Almgren whose homotopy type is more coarse. The author has recently taken advantage of the full strength of Theorem I.2 for introducing new families of minmax problems at the level of immersions called *minmax hierarchies* (see [37]).

In order to simplify the presentation and in particular the computations of the Euler Lagrange equation to A^σ we are presenting the proof of Theorem I.1, in the special case $N^n = S^3$. There is however no argument below which is specific to that case and the proof in the general case follows each step word for word of the S^3 case. Indeed, the almost conservation law in general target manifold is perturbed by lower order terms (see for instance the explicit expression for $p = 1$ and general target in [26]). In arbitrary co-dimension each tensor has it's counterpart which are possibly geometrically more involved but can be treated analytically identically as in the codimension 1 case. As soon as the strong $W^{1,2}$-bubble tree convergence is established, the passage to the limit in the non-linearity of the *harmonic map equation*[4] is totally independent of the type of non linearity the target is producing. We keep from this non linearity, usually denoted

[3] The multiplicity one condition $N \equiv 1$ is expected to hold for finite index minmax problems in general. See the open problem in the first part of the introduction.

[4] Recall that the mean-curvature vector of an immersion $\vec{\Phi}$ into a closed sub-manifold N^n of an Euclidean space is given by

$$2\,\vec{H}_{\vec{\Phi}} = \Delta_{g_{\vec{\Phi}}}\vec{\Phi} + A(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi})_{g_{\vec{\Phi}}}$$

where $\Delta_{g_{\vec{\Phi}}}$ is the *negative Laplace Beltrami* operator with respect to the metric $g_{\vec{\Phi}}$. In conformal coordinates this becomes

$$2\,\vec{H}_{\vec{\Phi}} = e^{-2\lambda}\left[\Delta\vec{\Phi} + A(\vec{\Phi})(\nabla\vec{\Phi}, \nabla\vec{\Phi})\right]$$

where $e^\lambda := |\partial_{x_i}\vec{\Phi}|$.

$A(\vec{\Phi})(\nabla\vec{\Phi},\nabla\vec{\Phi})$ where A is the second fundamental form of the target N^n, exclusively the quadratic dependence in the gradient. The conformal nature of the maps makes moreover the manipulation of the harmonic map equations straightforward independently of the existence or not of symmetries in the target. We took this point of view in order to ease a bit the reading of the proof.

II. The viscous relaxation of the area for surfaces

II.1. *The Finsler manifold of immersions into the spheres with* L^q *bounded second fundamental form.* — For $k \in \mathbf{N}$ and $1 \le q \le +\infty$, we recall the definition of $W^{k,q}$ Sobolev function on a closed smooth surface Σ (i.e. Σ is compact without boundary). To that aim we take some reference smooth metric g_0 on Σ and we set

$$W^{k,q}(\Sigma,\mathbf{R}) := \left\{f \text{ measurable s.t. } \nabla^k_{g_0} f \in L^q(\Sigma, g_0)\right\},$$

where $\nabla^k_{g_0}$ denotes the k-th iteration of the Levi-Civita connection associated to Σ. Since the surface is closed the space defined in this way is independent of g_0. Let N^n be a closed n-dimensional sub-manifold of $\mathbf{R}^m$ with $3 \le n \le m-1$ being arbitrary. The Space of $W^{k,q}$ into N^n is defined as follows

$$W^{k,q}\big(\Sigma, N^n\big) := \left\{\vec{\Phi} \in W^{k,q}\big(\Sigma,\mathbf{R}^m\big);\ \vec{\Phi} \in N^n \text{ almost everywhere}\right\}.$$

We have the following well known proposition

Proposition **II.1.** — *Assuming* $kq > 2$, *the space* $W^{k,q}(\Sigma, N^n)$ *defines a Banach Manifold.*

Proof of Proposition II.1. — This comes mainly from the fact that, under our assumptions,

$$W^{k,q}\big(\Sigma,\mathbf{R}^m\big) \quad\hookrightarrow\quad C^0\big(\Sigma,\mathbf{R}^m\big). \tag{II.1}$$

The Banach manifold structure is then defined as follows. Choose $\delta > 0$ such that each geodesic ball $B^{N^n}_\delta(z)$ for any $z \in N^n$ is strictly convex and the exponential map

$$\exp_z : V_z \subset T_z N^n \longrightarrow B^{N^n}_\delta(z)$$

realizes a C^∞ diffeomorphism for some open neighborhood of the origin in $T_z N^n$ into the geodesic ball $B^{N^n}_\delta(z)$. Because of the embedding (II.1) there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}&\forall\vec{u},\vec{v} \in W^{k,q}\big(\Sigma, N^n\big) \quad \|\vec{u}-\vec{v}\|_{W^{k,q}} < \varepsilon_0\\ &\quad\Longrightarrow\quad \Big\|\mathrm{dist}_N\big(\vec{u}(x),\vec{v}(x)\big)\Big\|_{L^\infty(\Sigma)} < \delta.\end{aligned}$$

We equip now the space $W^{k,q}(\Sigma, N^n)$ with the distance issued from the $W^{k,q}$ norm and for any $\vec{u} \in \mathcal{M} = W^{k,q}(\Sigma, N^n)$ we denote by $B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ the open ball in $\mathcal{M}$ of center $\vec{u}$ and radius ε_0.

As a covering of $\mathcal{M}$ we take $(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}))_{\vec{u}\in\mathcal{M}}$. We denote by

$$E^{\vec{u}} := \Gamma_{W^{k,q}}\left(\vec{u}^{-1}TN\right) := \left\{\vec{w} \in W^{k,q}\left(\Sigma, \mathbf{R}^m\right);\ \vec{w}(x) \in T_{\vec{u}(x)}N^n\ \forall x \in \Sigma\right\}$$

this is the Banach space of $W^{k,q}$-sections of the bundle $\vec{u}^{-1}TN$ and for any $\vec{u} \in \mathcal{M}$ and $\vec{v} \in B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ we define $\vec{w}^{\,\vec{u}}(\vec{v})$ to be the following element of $E^{\vec{u}}$

$$\forall x \in \Sigma \quad \vec{w}^{\,\vec{u}}(\vec{v})(x) := \exp_{\vec{u}(x)}^{-1}\left(\vec{v}(x)\right).$$

It is not difficult to see that

$$\vec{w}^{\,\vec{v}} \circ \left(\vec{w}^{\,\vec{u}}\right)^{-1} : \vec{w}^{\,u}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right) \longrightarrow \vec{w}^{\,v}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right)$$

defines a C^∞ diffeomorphism. □

For $p > 1$ we define

$$\mathcal{E}_{\Sigma,p} = W_{imm}^{2,2p}\left(\Sigma^2, N^n\right) := \left\{\vec{\Phi} \in W^{2,2p}\left(\Sigma^2, N^n\right);\ \mathrm{rank}\,(d\Phi_x) = 2\ \forall x \in \Sigma^2\right\}.$$

The set $W_{imm}^{2,2p}(\Sigma^2, N^n)$ as an open subset of the normal Banach Manifold $W^{2,2p}(\Sigma^2, N^n)$ inherits a Banach Manifold structure.

We equip now the space $W_{imm}^{2,2p}(\Sigma, N^n)$ with a *Finsler manifold structure* on it's tangent bundle (see the definition of Banach bundle space and Tangent bundle to a Banach manifold in [19]). For the convenience of the reader we recall the notion of Finsler structure.

Definition **II.3.** — *Let $\mathcal{M}$ be a normal*[5] *and let $\mathcal{V}$ be a Banach bundle space over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| : \mathcal{V} \longrightarrow \mathbf{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \quad \textit{is a norm on } \mathcal{V}_x.$$

Moreover for any local trivialization τ_i over U_i and for any $x_0 \in U_i$ we define on $\mathcal{V}_x$ the following norm

$$\forall \vec{w} \in \pi^{-1}\left(\{x\}\right) \quad \|\vec{w}\|_{x_0} := \left\|\tau_i^{-1}\left(x_0, \rho\left(\tau_i(\vec{w})\right)\right)\right\|_{x_0},$$

[5] The assumption to be *normal* is a relatively strong separation axiom which ensures that the defined *Finsler structure* generates a distance which makes the topology of the Banach manifold metrizable (see [29], pages 201–202). This assumption can be weakened to *regular* but not to *Hausdorff* only.

where ρ is the canonical projection $\rho\ :\ U_i \times E \to E$ and there exists $C_{x_0} > 1$ such that

$$\forall x \in U_i \quad C_{x_0}^{-1}\, \|\cdot\|_x \le \|\cdot\|_{x_0} \le C_{x_0}\, \|\cdot\|_x .$$

In a C^q Banach bundle, the Finsler structure is said to be C^l for $l \le q$ if, in local charts, the dependence of $\|\cdot\|_x$ is C^l with respect to x.

Definition **II.4.** — *Let $\mathcal{M}$ be a normal C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a* **Finsler Manifold**.

Remark **II.1.** — A Finsler structure on $T\mathcal{M}$ defines in a canonical way a dual Finsler structure on $T^*\mathcal{M}$.

The tangent space to $\mathcal{E}_{\Sigma,p}$ at a point $\vec{\Phi}$ is the space $\Gamma_{W^{2,2p}}(\vec{\Phi}^{-1}TN^n)$ of $W^{2,2p}$-sections of the bundle $\vec{\Phi}^{-1}TN^n$, i.e.

$$T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p} = \left\{\vec{w} \in W^{2,2p}\big(\Sigma^2, \mathbf{R}^m\big);\ \vec{w}(x) \in T_{\vec{\Phi}(x)}N^n\ \forall x \in \Sigma^2\right\}.$$

We equip $T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[\left|\nabla^2\vec{v}\right|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^p dvol_{g_{\vec{\Phi}}}\right]^{1/2p} + \|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)},$$

where we keep denoting, for any $j \in \mathbf{N}$, ∇ to be the connection on $(T^*\Sigma)^{\otimes^j} \otimes \vec{\Phi}^{-1}TN$ over Σ defined by $\nabla := \nabla^{g_{\vec{\Phi}}} \otimes \vec{\Phi}^*\nabla^h$ and $\nabla^{g_{\vec{\Phi}}}$ is the Levi Civita connection on $(\Sigma, g_{\vec{\Phi}})$ and ∇^h is the Levi-Civita connection on N^n.

We check for instance that $\nabla\vec{v}$, resp. $\nabla^2\vec{v}$ defines a C^0, resp. L^{2p}, section of $(T^*\Sigma)\otimes\vec{\Phi}^{-1}TN$, resp. $(T^*\Sigma)^2 \otimes \vec{\Phi}^{-1}TN$.

The fact that we are adding to the $W^{2,2p}$ norm of $\vec{v}$ with respect to $g_{\vec{\Phi}}$ the L^∞ norm of $|\nabla\vec{v}|_{g_{\vec{\Phi}}}$ could look redundant since $W^{2,2p}$ embeds in $W^{1,\infty}$. We are doing it in order to ease the proof of the completeness of the Finsler Space equipped with the Palais distance below.

Observe that, using Sobolev embedding and in particular due to the fact $W^{2,q}(\Sigma, \mathbf{R}^m) \hookrightarrow C^1(\Sigma, \mathbf{R}^m)$ for $q > 2$, the norm $\|\cdot\|_{\vec{\Phi}}$ as a function on the Banach tangent bundle $T\mathcal{E}_{\Sigma,p}$ is obviously continuous.

Proposition **II.2.** — *The norms $\|\cdot\|_{\vec{\Phi}}$ defines a C^2-Finsler structure on the space $\mathcal{E}_{\Sigma,p}$.*

Proof of Proposition II.2. — We introduce the following trivialization of the Banach bundle. For any $\vec{\Phi} \in \mathcal{E}_{\Sigma,p}$ we denote $P_{\vec{\Phi}(x)}$ the orthonormal projection in $\mathbf{R}^m$ onto the n-dimensional vector subspace of $\mathbf{R}^m$ given by $T_{\vec{\Phi}(x)}N^n$ and for any $\vec{\xi}$ in the ball $B^{\mathcal{E}_{\Sigma,p}}_{\varepsilon_1}(\vec{\Phi})$ for some $\varepsilon_1 > 0$ and any $\vec{v} \in T_{\vec{\xi}}\mathcal{E}_{\Sigma,p} = \Gamma_{W^{2,2p}}(\vec{\xi}^{-1}TN)$ we assign the map $\vec{w}(x) := P_{\vec{\Phi}(x)}\vec{v}(x)$. It is straightforward to check that for $\varepsilon_1 > 0$ chosen small enough the map which to $\vec{v}$

assigns $\vec{w}$ is an isomorphism from $T_{\vec{\xi}}\mathcal{E}_{\Sigma,p}$ into $T_{\vec{\Phi}}\mathcal{E}_{\Sigma,p}$ and that there exists $k_{\vec{\Phi}} > 1$ such that $\forall \vec{v} \in TB_{\varepsilon_1}^{\mathcal{E}_{\Sigma,p}}(\vec{\Phi})$

$$k_{\vec{\Phi}}^{-1}\,\|\vec{v}\|_{\vec{\xi}} \le \|\vec{w}\|_{\vec{\Phi}} \le k_{\vec{\Phi}}\,\|\vec{v}\|_{\vec{\xi}}.$$

The C^2-dependence of $\|\cdot\|_{\vec{\xi}}$ with respect to $\vec{\xi}$ in the chart above is left to the reader. This concludes the proof of Proposition II.2. □

II.2. *Palais deformation theory applied to the space of* $W^{2,2p}$*-immersions.*

Theorem **II.1.** — **[Palais 1970]** *Let* $(\mathcal{M}, \|\cdot\|)$ *be a Finsler Manifold. Define on* $\mathcal{M}\times\mathcal{M}$

$$d(p,q) := \inf_{\omega\in\Omega_{p,q}} \int_0^1 \left\|\frac{d\omega}{dt}\right\|_{\omega(t)} dt,$$

where

$$\Omega_{p,q} := \left\{\omega \in C^1\big([0,1],\mathcal{M}\big);\ \omega(0) = p\ \omega(1) = q\right\}.$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold. d is called **Palais distance** *of the Finsler manifold* $(\mathcal{M}, \|\cdot\|)$.

Contrary to the first appearance the non degeneracy of d is not straightforward and requires a proof (see [29]). This last result combined with the famous result of Stones (see [44]) on the paracompactness of metric spaces gives the following corollary.

Corollary **II.1.** — *Let* $(\mathcal{M}, \|\cdot\|)$ *be a Finsler Manifold then* $\mathcal{M}$ *is paracompact.*

The following result[6] is going to play a central role in adapting Palais deformation theory to our framework of $W^{2,2p}$-immersions.

Proposition **II.3.** — *Let $p > 1$ and $\mathcal{M} := \mathcal{E}_{\Sigma,p}$ be the space of $W^{2,2p}$-immersions of a closed oriented surface Σ into a closed sub-manifold N^n of $\mathbf{R}^m$*

$$\mathcal{E}_{\Sigma,p} = W^{2,2p}_{imm}\big(\Sigma^2, N^n\big) := \left\{\vec{\Phi} \in W^{2,2p}\big(\Sigma^2, N^n\big);\ rank\,(d\Phi_x) = 2\ \forall x \in \Sigma^2\right\}.$$

The Finsler Manifold given by the structure

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^p dvol_{g_{\vec{\Phi}}}\right]^{1/2p} + \big\||\nabla\vec{v}|_{g_{\vec{\Phi}}}\big\|_{L^\infty(\Sigma)}$$

is <u>*complete*</u> *for the Palais distance.*

---

[6] As a matter of fact the proof of the completeness with respect to the Palais distance is skipped in various applications of Palais deformation theory in the literature.

Proof of Proposition II.3. — For any $\vec{\Phi} \in \mathcal{M}$ and $\vec{v} \in T_{\vec{\Phi}}\mathcal{M}$ we introduce the tensor in $(T^*\Sigma)^{\otimes^2}$ given in coordinates by

$$\begin{aligned}\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi} + d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v} &= \sum_{i,j=1}^{2}\left[\nabla_{\partial_{x_i}}\vec{v}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\nabla_{\partial_{x_j}}\vec{v}\right]dx_i\otimes dx_j\\ &= \sum_{i,j=1}^{2}\left[\nabla^h_{\partial_{x_i}\vec{\Phi}}\vec{v}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\nabla^h_{\partial_{x_j}\vec{\Phi}}\vec{v}\right]dx_i\otimes dx_j\end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbf{R}^m$. Observe that we have

$$|\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi} + d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v}|_{g_{\vec{\Phi}}} \le 2\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}.$$

Hence, taking a C^1 path $\vec{\Phi}_s$ in $\mathcal{M}$ one has for $\vec{v} := \partial_s\vec{\Phi}$

$$\text{(II.2)}\qquad \begin{aligned}\left\|\,|d\vec{v}\dot{\otimes}d\vec{\Phi} + d\vec{\Phi}\dot{\otimes}d\vec{v}|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)} &= \left\|\sum_{i,j,k,l=1}^{2} g^{ij}_{\vec{\Phi}}g^{kl}_{\vec{\Phi}}\partial_s(g_{\vec{\Phi}})_{ik}\partial_s(g_{\vec{\Phi}})_{jl}\right\|_{L^\infty(\Sigma)}\\ = \left\|\,\left|\partial_s(g_{ij}dx_i\otimes dx_j)\right|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)} &= \left\|\,|\partial_s g_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}.\end{aligned}$$

Hence

$$\text{(II.3)}\qquad \int_0^1 \left\|\,|\partial_s g_{\vec{\Phi}}|_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}ds \le 2\int_0^1 \|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\,ds.$$

We now use the following lemma

Lemma **II.1.** — *Let* M_s *be a* C^1 *path into the space of positive n by n symmetric matrix then the following inequality holds*

$$Tr\left(M^{-2}(\partial_s M)^2\right) = \|\partial_s \log M\|^2 = Tr\left((\partial_s\log M)^2\right).$$

Proof of Lemma II.1. — We write $M = \exp A$ and we observe that

$$\mathrm{Tr}\left(\exp(-2A)(\partial_s\exp A)^2\right) = \mathrm{Tr}\,(\partial_s A)^2.$$

Then the lemma follows. □

Combining the previous lemma with (II.2) and (II.3) we obtain in a given chart

$$\text{(II.4)}\qquad \int_0^1 \left\|\partial_s\log(g_{ij})\right\|ds = \int_0^1\sqrt{\mathrm{Tr}\left((\partial_s\log g_{ij})^2\right)}\,ds \le 2\int_0^1 \|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\,ds.$$

This implies that in the given chart the log of the matrix $(g_{ij}(s))$ is uniformly bounded for $s \in [0,1]$ and hence $\vec{\Phi}_1$ is an immersion. It remains to show that it has a controlled $W^{2,q}$ norm. We denote

$$\mathrm{Hess}_p(\vec{\Phi}) := \int_\Sigma \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}$$

and we compute

$$\text{(II.5)} \qquad \begin{aligned} \frac{d}{ds}\big(\mathrm{Hess}_p(\vec{\Phi})\big) &= p\int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}} \\ &\quad + \int_\Sigma \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p \partial_s(dvol_{g_{\vec{\Phi}}}). \end{aligned}$$

Classical computations give

$$\partial_s(dvol_{g_{\vec{\Phi}}}) = \langle \nabla \partial_s \vec{\Phi}, d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\, dvol_{g_{\vec{\Phi}}}.$$

So we have

$$\text{(II.6)} \qquad \begin{aligned} \left|\int_\Sigma \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p \partial_s(dvol_{g_{\vec{\Phi}}})\right| &\le \Big\| |\nabla \partial_s \vec{\Phi}|_{g_{\vec{\Phi}}} \Big\|_{L^\infty(\Sigma)} \int_\Sigma \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}} \\ &\le \|\partial_s \vec{\Phi}\|_{\vec{\Phi}} \int_\Sigma \left[1 + |\nabla d\vec{\Phi}\big|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}. \end{aligned}$$

In local charts we have

$$|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} = \sum_{i,j,k,l=1}^2 g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi}\right\rangle_h.$$

Thus in bounding $\int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\, dvol_{g_{\vec{\Phi}}}$ we first have to control terms of the form

$$\text{(II.7)} \qquad \left|\int_\Sigma \sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi}\right\rangle_h \left[1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}}\right|.$$

We write

$$\begin{aligned} &\sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}} \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi}\right\rangle_h \\ &\quad = \sum_{i,j,k,l,t,r=1}^2 \partial_s g^{ij}_{\vec{\Phi}}\, g_{jt}\, g^{tr}\, g^{kl}_{\vec{\Phi}} \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi}\right\rangle_h \\ &\quad = -\sum_{i,j,k,l,=1}^2 \left(\sum_{t,r=1}^2 \partial_s g_{jt}\, g^{tr}\right) g^{ij}_{\vec{\Phi}} g^{kl}_{\vec{\Phi}} \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi}\right\rangle_h. \end{aligned}$$

Hence, using (II.2),

$$\begin{aligned}(\mathbf{II.8})\qquad &\left|\int_\Sigma \sum_{i,j,k,l=1}^{2} \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}}\right|\\ &\leq \left\| |\partial_s g_{\vec{\Phi}}|_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)} \int_\Sigma \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}\\ &\leq 2\|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\int_\Sigma \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}.\end{aligned}$$

We have also

$$\begin{aligned}&\partial_s\left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h\\ &\quad = \left\langle \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right),\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h + \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi},\, \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right)\right\rangle_h.\end{aligned}$$

By definition we have

$$\nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) = \nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_s\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) + R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi},$$

where we have used the fact that $[\partial_s\vec{\Phi},\partial_{x_i}\vec{\Phi}] = \vec{\Phi}_*[\partial_s,\partial_{x_i}] = 0$. Using also that $[\partial_s\vec{\Phi},\partial_{x_k}\vec{\Phi}] = 0$, since ∇^h is torsion free, we have finally

$$(\mathbf{II.9})\qquad \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) = \nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}\right) + R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi},$$

where R^h is the Riemann tensor associated to the Levi-Civita connection ∇^h. We have

$$(\mathbf{II.10})\qquad \nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}\right) = \left(\nabla^h\right)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi} + \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}.$$

Hence

$$\begin{aligned}(\mathbf{II.11})\qquad &\left\langle \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right),\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h\\ &= \left\langle \left(\nabla^h\right)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h\\ &\quad + \left\langle \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h + \left\langle R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h.\end{aligned}$$

Combining all the previous and observing that

$$(\mathbf{II.12})\qquad \left|\sum_{i,j,k,l=1}^{2} g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\left\langle \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h\right| \leq C\,|\nabla\partial_s\vec{\Phi}|_{g_{\vec{\Phi}}}|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}$$

gives then

$$
\begin{aligned}
&\left|\int_\Sigma \sum_{i,j,k,l=1}^{2} g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\, \partial_s\left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi},\, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}}\right| \\
&\quad\leq C\int_\Sigma \left|\left\langle \nabla^2\partial_s\vec{\Phi},\, \nabla d\vec{\Phi}\right\rangle_{g_{\vec{\Phi}}}\right| \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}} \\
&\qquad + C\int_\Sigma |\nabla\partial_s\vec{\Phi}|_{g_{\vec{\Phi}}}\, |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}} \\
&\qquad + C\left\|R^h\right\|_{L^\infty(N^n)} \int_\Sigma |\partial_s\vec{\Phi}|_h\, |\nabla d\vec{\Phi}|_{g_{\vec{\Phi}}} \left[1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\right]^{p-1} dvol_{g_{\vec{\Phi}}}.
\end{aligned}
\tag{II.13}
$$

Combining all the above we finally obtain that

$$
\left|\partial_s \mathrm{Hess}_p(\vec{\Phi})\right| \leq C\|\partial_s\vec{\Phi}\|_{\vec{\Phi}}\left[\mathrm{Hess}_p(\vec{\Phi}) + \mathrm{Hess}_p(\vec{\Phi})^{1-1/2p}\right]. \tag{II.14}
$$

Combining (II.4) and (II.14) we deduce using Gromwall lemma that if we take a C^1 path from $[0,1)$ into $\mathcal{E}_{\Sigma,p}$ with finite length for the Palais distance d, the limiting map $\vec{\Phi}_1$ is still a $W^{2,2p}$-immersion of Σ into N^n, which proves the completeness of $(\mathcal{E}_{\Sigma,p}, d)$. □

The following definition is central in Palais deformation theory.

Definition **II.5.** — *Let* E *be a* C^1 *function on a Finsler manifold* $(\mathcal{M}, \|\cdot\|)$ *and* $\beta \in E(\mathcal{M})$. *On says that* E *fulfills the* **Palais–Smale condition** *at the level* β *if for any sequence* u_n *satisfying*

$$
E(u_n) \longrightarrow \beta \quad \text{and} \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0,
$$

then there exists a subsequence $u_{n'}$ *and* $u_\infty \in \mathcal{M}$ *such that*

$$
d(u_{n'}, u_\infty) \longrightarrow 0,
$$

and hence $E(u_\infty) = \beta$ *and* $DE_{u_\infty} = 0$.

The following result is the Palais Smale condition for the functional

$$
A^\sigma_p(\vec{\Phi}) := \mathrm{Area}(\vec{\Phi}) + \sigma^2\int_\Sigma \left[1+|\vec{\mathbf{I}}_{\vec{\Phi}}|^2\right]^p dvol_{g_{\vec{\Phi}}}.
$$

Theorem **II.2.** — *Let* $p>1$ *and* $\vec{\Phi}_k$ *such that*

$$
\limsup_{k\to+\infty} A^\sigma_p(\vec{\Phi}_k) < +\infty,
$$

and satisfying

$$
\lim_{k\to+\infty}\ \sup_{\|\vec{w}\|_{\vec{\Phi}_k}\leq 1} DA^\sigma_p(\vec{\Phi}_k)\cdot\vec{w} = 0. \tag{II.15}
$$

Then, modulo extraction of a subsequence, there exists a sequence of $W^{2,2p}$*-diffeomorphisms* Ψ_k *such that* $\vec{\Phi}_k \circ \Psi_k$ *converges strongly in* $\mathcal{E}_{\Sigma,p}$ *for the Palais distance to a critical point of* A_p^σ*. Moreover, if one assume that* $\vec{\Phi}_k$ *stays inside a fixed ball of the Palais distance one can take* $\Psi_k(x) = x$*.*

Remark **II.2.** — The first part of this theorem has been proved in [18] (Theorem 5.1) in the flat framework which does not differ much from our case of $W^{2,2p}$-immersions into N^n. See also [4] for a proof making use of the underlying conservation laws. The second part is a direct consequence of the proof of Proposition II.3 above and is being used below since in the main Palais theorem II.3 the flow issued by the pseudo-gradient maintains the image at a finite Palais distance.

Definition **II.6.** — *A family of subsets* $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ *of a Banach manifold* $\mathcal{M}$ *is called* **admissible family** *if for every homeomorphism* Ξ *of* $\mathcal{M}$ *isotopic to the identity we have*

$$\forall A \in \mathcal{A} \quad \Xi(A) \in \mathcal{A}.$$

Example. — Consider $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbf{R}^3)$ and take[7] $c \in \pi_1(\mathrm{Imm}(S^2, \mathbf{R}^3)) = \mathbf{Z}_2 \times \mathbf{Z}$ then the following family is admissible

$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0\big([0,1], W^{2,q}_{imm}(S^2, \mathbf{R}^3)\big);\ \vec{\Phi}(0,\cdot) = \vec{\Phi}(1,\cdot) \text{ and } [\vec{\Phi}] = c \right\}.$$

We recall the main theorem of Palais deformation theory.

Theorem **II.3.** — **[Palais 1970]** *Let* $(\mathcal{M}, \|\cdot\|)$ *be a Banach manifold together with a* $C^{1,1}$*-Finsler structure. Assume* $\mathcal{M}$ *is* <u>*complete*</u> *for the induced Palais distance* d *and let* $E \in C^1(\mathcal{M})$ *satisfying the Palais–Smale condition* $(PS)_\beta$ *for the level set* β*. Let* $\mathcal{A}$ *be an admissible family in* $\mathcal{P}(\mathcal{M})$ *such that*

$$\inf_{A \in \mathcal{A}} \sup_{u \in A} E(u) = \beta,$$

then there exists $u \in \mathcal{M}$ *satisfying*

$$\text{(II.16)} \qquad \begin{cases} DE_u = 0 \\ E(u) = \beta \end{cases}$$

[7] It is proved in [40] and [12] that

$$\mathrm{Imm}\big(S^2, \mathbf{R}^3\big) \simeq_{homotop.} SO(3) \times \Omega^2\big(SO(3)\big).$$

II.3. *Struwe's monotonicity trick.* — Because of Theorem II.2, Theorem II.3 can be applied to each of the Lagrangian A^σ_p for any admissible family $\mathcal{A}$ of $\mathcal{E}_{\Sigma,p}$ satisfying

$$\inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} \operatorname{Area}(\vec{\Phi}) = \beta^0 > 0. \tag{II.17}$$

However, beside the difficulty of establishing a convergence of any nature to the corresponding sequence of critical points $\vec{\Phi}_\sigma$ given by Theorem II.3, although it is clear that

$$\lim_{\sigma\to 0} \inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} A^\sigma_p(\vec{\Phi}_\sigma) = \beta^0,$$

nothing excludes *a-priori* that

$$\lim_{\sigma\to 0} \inf_{A\in\mathcal{A}} \max_{\vec{\Phi}\in A} \operatorname{Area}(\vec{\Phi}_\sigma) < \beta^0,$$

and it could be that the smoothing part of the Lagrangian $\sigma^2 \int_\Sigma [1 + |\vec{\mathbf{I}}_{\vec{\Phi}_\sigma}|^2]^p \, dvol_{g_{\vec{\Phi}_\sigma}}$ does not go to zero. In order to prevent this unpleasant situation where the smoothed min-max procedure is not approximating properly the limiting min-max procedure, M. Struwe invented a technic—called sometimes "Struwe's monotonicity trick"—consisting in localizing the action of the pseudo-gradient close to the level set $\operatorname{Area}(\vec{\Phi}) = \beta_0$ exclusively (see [45] and [46]). Precisely we have the following result.

Theorem **II.4.** — *Let* $(\mathcal{M}, \|\cdot\|)$ *be a complete Finsler manifold. Let* E^σ *be a family of* C^1 *functions for* $\sigma\in[0,1]$ *on* $\mathcal{M}$ *such that for every* $\vec{\gamma}\in\mathcal{M}$

$$\sigma \longrightarrow E^\sigma(\vec{\gamma}) \quad \text{and} \quad \sigma \longrightarrow \partial_\sigma E^\sigma(\vec{\gamma}), \tag{II.18}$$

are increasing and continuous functions with respect to σ. *Assume moreover that*

$$\left\| DE^\sigma_{\vec{\gamma}} - DE^\tau_{\vec{\gamma}} \right\|_{\vec{\gamma}} \le C(\sigma)\,\delta\big(|\sigma-\tau|\big)\,f\big(E^\sigma(\vec{\gamma})\big), \tag{II.19}$$

where $C(\sigma)\in L^\infty_{loc}((0,1))$, $\delta\in L^\infty_{loc}(\mathbf{R}_+)$ *and goes to zero at* 0 *and* $f\in L^\infty_{loc}(\mathbf{R})$. *Assume that for every* $\sigma>0$ *the functional* E^σ *satisfies the Palais Smale condition. Let* $\mathcal{A}$ *be an admissible family of* $\mathcal{M}$ *and denote*

$$\beta(\sigma) := \inf_{A\in\mathcal{A}} \sup_{\vec{\gamma}\in A} E^\sigma(\vec{\gamma}).$$

Then there exists a sequence $\sigma_k\to 0$ *and* $\vec{\gamma}_k\in\mathcal{M}$ *such that*

$$E^{\sigma_k}(\vec{\gamma}_k) = \beta(\sigma_k), \qquad DE^{\sigma_k}(\vec{\gamma}_k) = 0.$$

Moreover $\vec{\gamma}_k$ *satisfies the so called "entropy condition"*

$$\partial_{\sigma_k} E^{\sigma_k}(\vec{\gamma}_k) = o\left(\frac{1}{\sigma_k\ \log(\frac{1}{\sigma_k})}\right).$$

A proof of this theorem is given for instance in [36]. Applying Theorem II.4 to our framework gives.

Theorem **II.5.** — *Let $p>1$ and $\mathcal{A}$ be an admissible family in $\mathcal{E}_{\Sigma,p}(\mathrm{N}^n)$ such that*

$$\inf_{A\in\mathcal{A}}\max_{\vec{\Phi}\in A}\mathrm{Area}(\vec{\Phi})=\beta^0>0. \tag{II.20}$$

Then there exists $\sigma_k\to 0$ and a family $\vec{\Phi}_k$ of critical points of $\mathrm{A}_p^{\sigma_k}$ satisfying

$$\lim_{k\to+\infty}\mathrm{Area}(\vec{\Phi}_k)=\beta^0\quad\text{and}\quad\sigma_k^2\int_\Sigma\left[1+|\vec{\mathbf{I}}_{\vec{\Phi}_k}|^2\right]^p dvol_{g_{\vec{\Phi}_k}}=o\left(\frac{1}{\log\sigma_k^{-1}}\right).$$

II.4. *The first variation of the viscous energies* A_p^σ. — Let $\vec{\Phi}$ be a smooth immersion from a closed 2-dimensional manifold Σ into the unit sphere $S^3\subset\mathbf{R}^4$, let $\vec{w}$ be an infinitesimal immersion satisfying $\vec{w}\cdot\vec{\Phi}\equiv 0$ and denote $\vec{\Phi}_t$: a sequence of immersions into S^3 such that $d\vec{\Phi}/dt(0)=\vec{w}$. The Gauss map of the immersion is given in local coordinates by

$$\vec{n}_t=\star_{\mathbf{R}^4}\left(\vec{\Phi}_t\wedge\frac{\partial_{x_1}\vec{\Phi}_t\wedge\partial_{x_2}\vec{\Phi}_t}{|\partial_{x_1}\vec{\Phi}_t\wedge\partial_{x_2}\vec{\Phi}_t|}\right). \tag{II.21}$$

Assuming $\vec{\Phi}$ is expressed locally in conformal coordinates and denote $e^\lambda=|\partial_{x_1}\vec{\Phi}|=|\partial_{x_2}\vec{\Phi}|$. We have

$$\vec{n}_t=\vec{n}+t\,(a_1\,\vec{e}_1+a_2\,\vec{e}_2+b\,\vec{\Phi})+o(t), \tag{II.22}$$

where $\vec{e}_i=e^{-\lambda}\,\partial_{x_i}\vec{\Phi}$. Since $\vec{n}_t\cdot\partial_{x_i}\vec{\Phi}_t\equiv 0$ and $\vec{n}_t\cdot\vec{\Phi}_t\equiv 0$ we have

$$\begin{aligned}\frac{d\vec{n}}{dt}(0)&=-\vec{n}\cdot\vec{w}\;\vec{\Phi}-\sum_{i=1}^2\vec{n}\cdot\partial_{x_i}\vec{w}\;e^{-\lambda}\,\vec{e}_i\\&=-\vec{n}\cdot\vec{w}\;\vec{\Phi}-\langle\vec{n}\cdot d\vec{w}\,,\,d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}.\end{aligned} \tag{II.23}$$

Since $g_{ij}:=\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{\Phi}$, we have

$$\frac{dg_{ij}}{dt}(0)=\partial_{x_i}\vec{w}\cdot\partial_{x_j}\vec{\Phi}+\partial_{x_j}\vec{w}\cdot\partial_{x_i}\vec{\Phi}. \tag{II.24}$$

Since $\sum_i g_{ki}\,g^{ij}=\delta_{kj}$ and $g_{ki}=e^{2\lambda}\,\delta_{ki}$, we have

$$\frac{dg^{ij}}{dt}(0)=-e^{-4\lambda}\,[\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}+\partial_{x_j}\vec{\Phi}\cdot\partial_{x_i}\vec{w}]. \tag{II.25}$$

We have also using (II.23) and (II.25)

$$\textbf{(II.26)}\qquad \frac{d|d\vec{n}|^2_{g_{\vec{\Phi}}}}{dt}=\frac{d}{dt}\left(\sum_{i,j=1}^{2}g^{ij}\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{n}\right)=-2\langle d\vec{\Phi}\dot{\otimes}d\vec{w},d\vec{n}\dot{\otimes}d\vec{n}\rangle_{g_{\vec{\Phi}}}+2\left\langle d\frac{d\vec{n}}{dt};d\vec{n}\right\rangle_{g_{\vec{\Phi}}}$$
$$=-2\langle d\vec{\Phi}\dot{\otimes}d\vec{w},d\vec{n}\dot{\otimes}d\vec{n}\rangle_{g_{\vec{\Phi}}}+4\vec{\mathrm{H}}\cdot\vec{w}-2\big\langle d\langle\vec{n}\cdot d\vec{w},d\vec{\Phi}\rangle_{g_{\vec{\Phi}}};d\vec{n}\big\rangle_{g_{\vec{\Phi}}},$$

where $\vec{\mathrm{H}}$ is the mean-curvature vector given by

$$\vec{\mathrm{H}}:=\frac{1}{2}\sum_{i,j=1}^{2}g^{ij}\vec{\mathbf{I}}_{ij},$$

and $\vec{\mathbf{I}}_{\vec{\Phi}}$ denotes the second fundamental form

$$\vec{\mathbf{I}}_{\vec{\Phi}}=\sum_{i,j=1}^{2}\vec{\mathbf{I}}_{ij}\,dx_i\otimes dx_j=-\sum_{i,j=1}^{2}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{n}\,\vec{n}\,dx_i\otimes dx_j.$$

Finally, we have $dvol_{g_{\vec{\Phi}}}=\sqrt{g_{11}g_{22}-g_{12}^2}\,dx_1\wedge dx_2$, hence

$$\textbf{(II.27)}\qquad \frac{d}{dt}(dvol_{g_{\vec{\Phi}}})(0)=\left[\sum_{i=1}^{2}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}\right]dx_1\wedge dx_2=\langle d\vec{\Phi};d\vec{w}\rangle_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}.$$

Using (II.26) and (II.27) we obtain

$$\textbf{(II.28)}\qquad \frac{d}{dt}\mathrm{Area}(\vec{\Phi}_t)|_{t=0}=\int_\Sigma\langle d\vec{\Phi}\,;d\vec{w}\rangle_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}.$$

For any $p>1$ we denote

$$\mathrm{F}_p(\vec{\Phi}):=\int_\Sigma\left[1+|\vec{\mathbf{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}.$$

Using (II.23) and (II.26) we have

$$\textbf{(II.29)}\qquad \begin{aligned}\frac{d}{dt}\mathrm{F}_p(\vec{\Phi}_t)\Big|_{t=0}&=\int_\Sigma f^p\langle d\vec{\Phi};d\vec{w}\rangle_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\\&\quad-2p\int_\Sigma f^{p-1}\langle d\vec{\Phi}\dot{\otimes}d\vec{w},d\vec{n}\dot{\otimes}d\vec{n}\rangle_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\\&\quad-2p\int_\Sigma f^{p-1}\big\langle d\langle\vec{n}\cdot d\vec{w},d\vec{\Phi}\rangle_{g_{\vec{\Phi}}};d\vec{n}\big\rangle_{g_{\vec{\Phi}}}\,dvol_{g_{\vec{\Phi}}}\\&\quad+4p\int_\Sigma f^{p-1}\vec{\mathrm{H}}\cdot\vec{w}\,dvol_{g_{\vec{\Phi}}},\end{aligned}$$

where $f:=[1+|\vec{\mathbf{I}}_{\vec{\Phi}}|^2]$.

II.5. *The almost conservation laws satisfied by the critical points of* $\mathrm{A}_p^\sigma(\vec{\Phi})$. — The fact that A_p^σ is C^1 in $\mathcal{E}_{\Sigma,p}$ is quite standard for $p>1$. Indeed, in local coordinates the functional has the form

$$\int_\Sigma e\big(\vec{\Phi},\nabla\vec{\Phi},\nabla^2\vec{\Phi}\big)\,dx^2,$$

where e is a C^∞ function. Let $\vec{\Phi}$ be a critical point in $\mathcal{E}_{\Sigma,p}$ of A_p^σ. We then have

$$\begin{aligned}&\vec{\Phi}\wedge d^{*_{g_{\vec{\Phi}}}}\big[\big[1+\sigma^2 f^p\big]\,d\vec{\Phi}\big]-2p\sigma^2\,\vec{\Phi}\wedge\, d^{*_{g_{\vec{\Phi}}}}\big[d^{*_{g_{\vec{\Phi}}}}\big[f^{p-1}\,d\vec{n}\big]\cdot d\vec{\Phi}\,\vec{n}\big]\\ &-2p\sigma^2\,\vec{\Phi}\wedge d^{*_{g_{\vec{\Phi}}}}\big[f^{p-1}(d\vec{n}\dot{\otimes}d\vec{n})\llcorner_{g_{\vec{\Phi}}}d\vec{\Phi}\big]+4p\sigma^2 f^{p-1}\,\vec{\Phi}\wedge\vec{\mathrm{H}}=0\quad\text{in }\mathcal{D}'(\Sigma),\end{aligned}\tag{II.30}$$

where $f:=[1+|\vec{\mathbf{I}}_{\vec{\Phi}}|^2]$ as above, $(d\vec{n}\dot{\otimes}d\vec{n})\llcorner_{g_{\vec{\Phi}}}d\vec{\Phi}$ is the contraction given in local conformal coordinates by

$$(d\vec{n}\dot{\otimes}d\vec{n})\llcorner_{g_{\vec{\Phi}}}d\vec{\Phi}:=e^{-2\lambda}\sum_{i,j=1}^2\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{n}\,\partial_{x_j}\vec{\Phi}\,dx_i,$$

and $d^{*_{g_{\vec{\Phi}}}}$ is the adjoint of d for the L^2 norm on Σ with respect to the metric $g_{\vec{\Phi}}$ induced by the immersion $\vec{\Phi}$. It coincides with $-e^{-2\lambda}\,\mathrm{div}\cdot$ in conformal coordinates. In conformal coordinates again the equation becomes then

$$\begin{aligned}&\vec{\Phi}\wedge\mathrm{div}\big[\big[1+\sigma^2 f^p\big]\nabla\vec{\Phi}-2p\sigma^2e^{-2\lambda}f^{p-1}\big\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\big\rangle\\ &\quad+2p\sigma^2e^{-2\lambda}\,\mathrm{div}\big[f^{p-1}\nabla\vec{n}\big]\cdot\nabla\vec{\Phi}\,\vec{n}\big]-4p\sigma^2f^{p-1}\,\vec{\Phi}\wedge\vec{\mathrm{H}}=0.\end{aligned}\tag{II.31}$$

We rewrite the first term in the second line.

$$\begin{aligned}&2p\sigma^2\,e^{-2\lambda}\,\mathrm{div}\big[f^{p-1}\nabla\vec{n}\big]\cdot\nabla\vec{\Phi}\,\vec{n}\\ &\quad=2p\sigma^2e^{-2\lambda}\,\mathrm{div}\big[f^{p-1}[\nabla\vec{n}+\mathrm{H}\nabla\vec{\Phi}]\big]\cdot\nabla\vec{\Phi}\,\vec{n}-2p\sigma^2\nabla\big[f^{p-1}\mathrm{H}\big]\,\vec{n}.\end{aligned}\tag{II.32}$$

The *trace free part* of the second fundamental form is denoted

$$\vec{\mathbf{I}}^0:=\vec{\mathbf{I}}-\vec{\mathrm{H}}\,g.$$

In coordinates and in codimension 1 one has

$$\vec{\mathbf{I}}^0=\mathbf{I}^0\,\vec{n}=-\sum_{i,j=1}^2[\partial_{x_i}\vec{n}\cdot\partial_{x_j}\vec{\Phi}+\mathrm{H}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{\Phi}]\,dx_i\otimes dx_j.$$

For any $k=1,2$ after some computations we obtain

$$\sum_{i=1}^{2}\partial_{x_i}\Big[f^{p-1}\big[\partial_{x_i}\vec{n}+\mathrm{H}\partial_{x_i}\vec{\Phi}\big]\Big]\cdot\partial_{x_k}\vec{\Phi}\,\vec{n} = -\partial_{x_k}\Big[f^{p-1}\mathbf{I}^0_{k,k}\Big]\vec{n}-\partial_{x_{k+1}}\Big[f^{p-1}\mathbf{I}^0_{k+1,k}\Big]\vec{n}.$$

Denoting $\overline{\nabla}\cdot:=(\partial_{x_1}\cdot,-\partial_{x_2}\cdot)$ and $(\overline{\nabla})^{\perp}\cdot:=(\partial_{x_2}\cdot,\partial_{x_1}\cdot)$, we have then

$$\text{(II.33)}\quad \begin{aligned}&2\,p\,\sigma^2\,e^{-2\lambda}\,\mathrm{div}\Big[f^{p-1}\,[\nabla\vec{n}+\mathrm{H}\,\nabla\vec{\Phi}]\Big]\cdot\nabla\vec{\Phi}\\ &\quad=-2\,p\,\sigma^2\,e^{-2\lambda}\,\Big[\overline{\nabla}\Big[f^{p-1}\,\mathbf{I}^0_{11}\Big]+(\overline{\nabla})^{\perp}\Big[f^{p-1}\,\mathbf{I}^0_{12}\Big]\Big].\end{aligned}$$

Combining (II.32) and (II.33) gives

$$\text{(II.34)}\quad \begin{aligned}&2\,p\,\sigma^2\,e^{-2\lambda}\,\mathrm{div}\Big[f^{p-1}\,\nabla\vec{n}\Big]\cdot\nabla\vec{\Phi}\,\vec{n}\\ &\quad=-2\,p\,\sigma^2\,\nabla\Big[f^{p-1}\,\vec{\mathrm{H}}\,\Big]+2\,p\,\sigma^2 f^{p-1}\mathrm{H}\,\nabla\vec{n}\\ &\qquad-2\,p\,\sigma^2\,e^{-2\lambda}\,\Big[\overline{\nabla}\Big[f^{p-1}\,\mathbf{I}^0_{11}\Big]+(\overline{\nabla})^{\perp}\Big[f^{p-1}\,\mathbf{I}^0_{12}\Big]\Big]\vec{n}.\end{aligned}$$

So the equation (II.31) becomes

$$\text{(II.35)}\quad \begin{aligned}&\vec{\Phi}\wedge\mathrm{div}\Big[\big[1+\sigma^2 f^p\big]\nabla\vec{\Phi}-2\,p\,\sigma^2\nabla\Big[f^{p-1}\,\vec{\mathrm{H}}\,\Big]\\ &\qquad-2\,p\,\sigma^2\,e^{-2\lambda}\,f^{p-1}\,\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\rangle\\ &\qquad+2\,p\,\sigma^2 f^{p-1}\,\mathrm{H}\,\nabla\vec{n}-2\,p\,\sigma^2\,e^{-2\lambda}\,\Big[\overline{\nabla}\Big[f^{p-1}\,\mathbf{I}^0_{11}\Big]+(\overline{\nabla})^{\perp}\Big[f^{p-1}\,\mathbf{I}^0_{12}\Big]\Big]\vec{n}\Big]\\ &\quad=4\,p\,\sigma^2 f^{p-1}\,\vec{\Phi}\wedge\vec{\mathrm{H}}.\end{aligned}$$

The equation (II.35) can be rewritten in an exact divergence free equation of the form $\mathrm{div}(\vec{\Phi}\wedge\cdots)=0$, that is in an exact conservation law which is coming from the SO(4) invariance of the problem in the target. However, since we are interested in general targets, we don't want to take advantage of the "roundness" of S^3 and we shall rewrite (II.35) in an "almost conservation law" which is more generic and which holds in $\mathcal{D}'(\Sigma)$. It is due this time to the translation invariance of the integrand of F_p in $\mathbf{R}^4$ in relation with the Noether theorem as observed in [2]. However the fact that we don't get exactly get a conservation law is coming from the fact that the constraint to take values into the closed sub-manifold S^3 is not translation invariant. This pointwise constraint is "generating" additional terms (i.e. the last term in the l.h.s. and the full r.h.s. of (II.36)) in comparison to the identity we would get if we would release this constraint. Nevertheless these additional terms happen to be of much lower degree and are not going to perturb the arguments in the section below as if we would be dealing with an exact conservation law. This is why we are speaking about an “almost conservation law”.

$$
\begin{aligned}
&-\mathrm{div}\Big[\big[1+\sigma^2 f^p\big]\nabla\vec{\Phi}-2p\sigma^2\,\nabla\big[f^{p-1}\vec{\mathrm{H}}\big]-2p\sigma^2 e^{-2\lambda}f^{p-1}\,\langle\nabla\vec{n}\,\dot{\otimes}\,\nabla\vec{n};\nabla\vec{\Phi}\rangle\\
&+2p\sigma^2 f^{p-1}\,\mathrm{H}\,\nabla\vec{n}-2p\sigma^2\,e^{-2\lambda}\left[\overline{\nabla}\big[f^{p-1}\,\mathbf{I}^0_{11}\big]+(\overline{\nabla})^\perp\big[f^{p-1}\,\mathbf{I}^0_{12}\big]\right]\vec{n}\Big]\\
&+4p\sigma^2 f^{p-1}\,\vec{\mathrm{H}}=\big[1+\sigma^2(1-p)f^p+p\sigma^2 f^{p-1}\big]|\nabla\vec{\Phi}|^2\,\vec{\Phi}.
\end{aligned}
\tag{II.36}
$$

Finally we end up this section by quoting the following theorem

Theorem **II.6.** — *Let $p\geq 1$ and $\vec{\Phi}$ be an element in the space $\mathcal{E}_{\Sigma,p}$ of $\mathrm{W}^{2,2p}$-immersions of a closed surface Σ. Assume $\vec{\Phi}$ is a critical point of $\mathrm{A}^\sigma_p(\vec{\Phi})$ then $\vec{\Phi}$ is C^∞ in any conformal parametrization.*

Remark **II.3.** — A proof of Theorem II.6 has been given in [18] and for C^1 into the Euclidean space. The method of proof in [18] relies on the work of J. Langer with the decomposition of the immersion into the union of graphs. See also [4] for a proof making use of the underlying conservation laws.

II.6. *Proof of Theorem I.2.* — Combine Theorem II.5 and Theorem I.1, this gives Theorem I.2. □

III. The passage to the limit σ → 0 with controlled conformal class

The goal of the present section is to prove the following theorem

Theorem **III.1.** — *Let $p>1$ and let $\vec{\Phi}_k$ be a sequence of critical points of $\mathrm{A}^{\sigma_k}_p$ in the class $\mathcal{E}_{\Sigma,p}$ where $\sigma_k\to 0$ and satisfying*

$$
0<\limsup_{k\to+\infty}\mathrm{Area}(\vec{\Phi}_k)<+\infty,
\tag{III.1}
$$

and

$$
\sigma_k^2\,\mathrm{F}_p(\vec{\Phi}_k)=\sigma_k^2\int_\Sigma\big[1+|\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\big]^p\,dvol_{g_{\vec{\Phi}_k}}=o\left(\frac{1}{\log(1/\sigma_k)}\right).
\tag{III.2}
$$

Assume moreover that the conformal class associated to $(\Sigma,g_{\vec{\Phi}_k})$ is precompact in the moduli space, then, modulo extraction of a subsequence, there exists a closed Riemann surface (S,h_0) with genus$(\mathrm{S})\leq$ genus(Σ), a weakly conformal map $\vec{\Phi}_\infty$ from S into N^n and an integer valued map $\mathrm{N}\in\mathrm{L}^\infty(\mathrm{S},\mathbf{N})$ such that

$$
\lim_{k\to+\infty}\mathrm{A}^{\sigma_k}(\vec{\Phi}_k)=\frac{1}{2}\int_{\mathrm{S}}\mathrm{N}\,|d\vec{\Phi}_\infty|^2_{h_0}\,dvol_{h_0}.
$$

Moreover the push forward of S *by* $\vec{\Phi}_\infty$ *together with the multiplicity* N *defines an oriented stationary integer varifold and the oriented varifold* $|\mathrm{T}_k|$ *equal to the push-forward by* $\vec{\Phi}_k$ *of* Σ *converges in the sense of Radon measures towards the oriented stationary integer varifold associated to* $\vec{\Phi}_\infty$*. The surface* S *is moreover either equal to the union of* Σ *with finitely many copies of* S^2 *or is equal to finitely many copies of* S^2.

Remark **III.1.** — Observe that in Theorem III.1, due to the assumption about the controlled conformal class, there can be a genus jump $genus(\mathrm{S}) < genus(\Sigma)$ only if the area vanishes on the main part of the Riemann surface and $\Phi_\infty(\mathrm{S})$ is going to be a bouquet of minimal sphere. This cannot be excluded a priori

In this section we shall then assume that $\vec{\Phi}_k$ is conformal from a sequence of Riemannian surfaces (Σ, g_k) into S^3 for which the underlying Riemann structure is pre-compact in the moduli space of Σ.

In order to prove Theorem III.1 we shall need several lemma.

Lemma **III.1.** — **[Monotonicity formula]** *Under the assumptions of Theorem III.1 the sequence of varifolds* $|\mathrm{T}_k|$ *equal to the push forward of* Σ *by* $\vec{\Phi}_k$ *converges, modulo extraction of a subsequence, towards a stationary varifold. In particular, introducing the Radon measure in* S^3 *given by*

$$\langle \mu_k, \varphi\rangle := \int_\Sigma \varphi(\vec{\Phi}_k)\, dvol_{g_{\vec{\Phi}_k}}, \tag{III.3}$$

μ_k *converges modulo extraction of a subsequence to a limiting Radon measure* μ_∞ *satisfying the following monotonicity formula*

$$\forall \vec{q} \in supp(\mu_\infty)\ \forall r > 0 \quad \frac{d}{dr}\left[\frac{e^{\mathrm{C}\,r}\mu_\infty(\mathrm{B}_r(\vec{q}))}{r^2}\right] \geq 0 \tag{III.4}$$

for some $\mathrm{C} > 0$ *independent of* $\vec{q}$ *and* r.

Proof of Lemma III.1. — The monotonicity formula for the limiting measure μ_∞ is a direct consequence of the fact that $|\mathrm{T}_k|$ converges towards a stationary varifold (see [1] and [39]). So it would suffices to prove this last fact in order to get (III.4). However the proof of both statements (that can be proven independently of each other) are very similar. In the first case it suffices to prove that for any vector field $\vec{\mathrm{X}}$ we have

$$\begin{aligned}&\lim_{k\to+\infty}\int_{\mathrm{M}_k} \mathrm{div}_{\mathrm{M}_k}\vec{\mathrm{X}}\ , d\mathcal{H}^2\\ &= \lim_{k\to+\infty}\int_\Sigma\left[\sum_{i=1}^4 \left\langle\partial_{y_i}\vec{\mathrm{X}}(\vec{\Phi}_k)\,\nabla\Phi_k^i, \nabla\vec{\Phi}_k\right\rangle - \vec{\mathrm{X}}(\vec{\Phi}_k)\cdot\vec{\Phi}_k\,|\nabla\vec{\Phi}_k|^2\right] dx^2 = 0,\end{aligned} \tag{III.5}$$

where $M_k := \vec{\Phi}_k(\Sigma)$ and $\vec{\Phi}_k = (\Phi_k^1, \ldots, \Phi_k^4)$. The computations for proving (III.5) are more or less the same as the one for proving (III.4) and we shall only present the later since we shall revisit them in the forthcoming Lemma III.3.

The explicit mention of the indices σ_k and k can be deleted when there is no possible confusion. For any $\vec{q} \in S^3$ and any radius r small enough, *Simon's monotonicity formula* (see [39], Chapter 4) applied to $\vec{\Phi}(\Sigma)$ (which is smooth immersion for any k) which is seen as a *varifold* from $\mathbf{R}^4$ gives

$$
\begin{aligned}
\frac{d}{dr}\left[\frac{1}{r^2}\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} dvol_{g_{\vec{\Phi}}}\right] &= \frac{d}{dr}\left[\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} \frac{|(\vec{n}\wedge\vec{\Phi})\llcorner(\vec{\Phi}-\vec{q})|^2}{|\vec{\Phi}-\vec{q}|^4}\, dvol_{g_{\vec{\Phi}}}\right] \\
&\quad -\frac{1}{2\,r^3}\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot d^{*_g} d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}} \\
&\geq -\frac{1}{2\,r^3}\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot d^{*_g} d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}},
\end{aligned}
\tag{III.6}
$$

where we have used that the first term in the r.h.s. of (III.6) is non negative.[8] Thanks to equation (II.36) we obtain

$$
\begin{aligned}
-\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot d^{*_g} d\vec{\Phi}\; dvol_{g_{\vec{\Phi}}} &= \int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot \Delta\vec{\Phi}\, dx^2 \\
= -\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} & (\vec{\Phi}-\vec{q})\cdot \operatorname{div}\Big[\sigma^2 f^{p-1}\Big[f\,\nabla\vec{\Phi} - 2\,p\,\big(\mathrm{H}\,\nabla\vec{n} \\
&- e^{-2\lambda}\langle\nabla\vec{n}\,\dot{\otimes}\,\nabla\vec{n};\nabla\vec{\Phi}\rangle\big)\Big]\Big]\, dx^2 \\
&+ 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot \operatorname{div}\Big[e^{-2\lambda}\Big[\,\overline{\nabla}\big[f^{p-1}\,\mathbf{I}^0_{11}\big] + (\overline{\nabla})^{\perp}\big[f^{p-1}\,\mathbf{I}^0_{12}\big]\Big]\,\vec{n}\Big]\, dx^2 \\
&+ 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot \Delta\big[f^{p-1}\,\vec{\mathrm{H}}\big]\, dx^2 \\
&- \int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} \big[1+(1-p)\sigma^2 f^p + p\,\sigma^2 f^{p-1}\big](\vec{\Phi}-\vec{q})\cdot\vec{\Phi}\,|\nabla\vec{\Phi}|^2\, dx^2.
\end{aligned}
\tag{III.7}
$$

Regarding the last line, observe in one hand that $(\vec{\Phi}-\vec{q})\cdot\vec{\Phi} = 1-\cos(\vec{\Phi},\vec{q}) = O(r^2)$ hence

$$
\left|\frac{1}{r^3}\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} (\vec{\Phi}-\vec{q})\cdot\vec{\Phi}\,|\nabla\vec{\Phi}|^2\, dx^2\right| \leq \frac{C}{r}\int_{\vec{\Phi}^{-1}(B_r^4(\vec{q}))} dvol_{g_{\vec{\Phi}}}
\tag{III.8}
$$

[8] Indeed we are taking the derivative of an integral of a positive integrand over a bigger and bigger set.

and in the other hand, again for fixed r and $\vec{q}$, as $k \to +\infty$

$$\textbf{(III.9)} \quad \begin{aligned} &\left| \int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} \left[(1-p)\sigma^2 f^p + p\,\sigma^2 f^{p-1}\right](\vec{\Phi}-\vec{q}) \cdot \vec{\Phi}\, |\nabla\vec{\Phi}|^2 \, dx^2 \right| \\ &\quad \le C\,\sigma^2\, F_p(\vec{\Phi}) + C\,\sigma^2\, M(T)^{1/p}\left[F_p(\vec{\Phi})\right]^{1-1/p} \to 0. \end{aligned}$$

Integrating by parts each of the two first lines in the r.h.s. of (III.7) gives

$$\textbf{(III.10)} \quad \begin{aligned} &-\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} (\vec{\Phi}-\vec{q}) \cdot \operatorname{div}\left[\sigma^2 f^{p-1}\left[f\,\nabla\vec{\Phi} - 2\,p\big(H\,\nabla\vec{n}\right.\right. \\ &\qquad \left.\left.\left. - e^{-2\lambda}\langle\nabla\vec{n}\,\dot{\otimes}\,\nabla\vec{n};\nabla\vec{\Phi}\rangle\big)\right]\right]dx^2 \\ &\qquad + 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} (\vec{\Phi}-\vec{q}) \cdot \operatorname{div}\left[e^{-2\lambda}\left[\overline{\nabla}\left[f^{p-1}\,\mathbf{I}^0_{11}\right] + (\overline{\nabla})^{\perp}\left[f^{p-1}\,\mathbf{I}^0_{12}\right]\right]\vec{n}\right]dx^2 \\ &= \sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} f^{p-1}\left[f\,|\nabla\vec{\Phi}|^2 - 2\,p\,H\,\nabla\vec{n}\cdot\nabla\vec{\Phi} + 2\,p\,(f-1)\,e^{2\lambda}\right]dx^2 \\ &\qquad -\sigma^2\int_{\vec{\Phi}^{-1}\left(\partial B^4_r(\vec{q})\right)} f^p\,\partial_\nu\vec{\Phi}\cdot(\vec{\Phi}-\vec{q}) - 2\,p f^{p-1}\,H\,\partial_\nu\,\vec{n}\cdot(\vec{\Phi}-\vec{q})\,dl \\ &\qquad + 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))} f^{p-1}\big\langle\partial_\nu\vec{n}\cdot\nabla\vec{n},\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\big\rangle\,dl \\ &\qquad + 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}\left(\partial B^4_r(\vec{q})\right)} e^{-2\lambda}(\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_1\,\partial_{x_1}\left[f^{p-1}\,\mathbf{I}^0_{11}\right] - \nu_2\,\partial_{x_2}\left[f^{p-1}\,\mathbf{I}^0_{11}\right]\right]dl \\ &\qquad + 2\,p\,\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))} e^{-2\lambda}(\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_1\,\partial_{x_2}\left[f^{p-1}\,\mathbf{I}^0_{12}\right] + \nu_2\,\partial_{x_1}\left[f^{p-1}\,\mathbf{I}^0_{12}\right]\right]dl, \end{aligned}$$

where ν is the outward unit (in the coordinates) normal to $\vec{\Phi}^{-1}(B^4_r(\vec{q}))$ and is given explicitly by

$$\nu = \big(\partial_{x_1}|\vec{\Phi}-\vec{q}|\,,\,\partial_{x_2}|\vec{\Phi}-\vec{q}|\big)/\big|\nabla|\vec{\Phi}-\vec{q}|\big|.$$

This is nothing but the normalized gradient of the function distance to $\vec{q}$. We clearly have

$$\textbf{(III.11)} \quad \lim_{k\to+\infty}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))} f^{p-1}\left[f\,|\nabla\vec{\Phi}|^2 - 2\,p\,H\,\nabla\vec{n}\cdot\nabla\vec{\Phi} + 2\,p\,e^{2\lambda}(f-1)\right]dx^2 = 0.$$

Multiplying (III.10) by $\chi(r)/r^3$ where χ is an arbitrary compactly supported function in $\mathbf{R}_+^*$ and integrating over $\mathbf{R}_+^*$ gives successively

$$\begin{aligned}
&\sigma^2\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B_r^4(\vec{q}))}\left[f^p\,\partial_\nu\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})-2pf^{p-1}\,\mathrm{H}\,\partial_\nu\vec{n}\cdot(\vec{\Phi}-\vec{q})\right]dl\\
&\quad+2p\sigma^2\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B_r^4(\vec{q}))}f^{p-1}\left\langle\partial_\nu\vec{n}\cdot\nabla\vec{n},\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\right\rangle dl\\
&=\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\Bigg[f^p\,\frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|^2}\\
&\quad-2pf^{p-1}\,\mathrm{H}\,\frac{\nabla|\vec{\Phi}-\vec{q}|}{|\vec{\Phi}-\vec{q}|^3}\cdot\left\langle\nabla\vec{n}\cdot(\vec{\Phi}-\vec{q})\right\rangle\Bigg]\,dx^2\\
&\quad+2p\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)f^{p-1}\left\langle\frac{\nabla|\vec{\Phi}-\vec{q}|}{|\vec{\Phi}-\vec{q}|^3}\cdot\nabla\vec{n},\nabla\vec{n}\,\nabla\vec{\Phi}\cdot(\vec{\Phi}-\vec{q})\right\rangle dx^2\\
&\longrightarrow 0\quad\text{as } k\to+\infty,
\end{aligned}\tag{III.12}$$

where we have bound the r.h.s. of (III.12) by a constant depending on χ times $\sigma^2\mathrm{F}_p(\vec{\Phi})$. We also obtain

$$\begin{aligned}
&-2p\sigma^2\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B_r^4(\vec{q}))}e^{-2\lambda}(\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_1\,\partial_{x_1}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dl\\
&\quad+2p\sigma^2\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\partial B_r^4(\vec{q}))}e^{-2\lambda}\,(\vec{\Phi}-\vec{q})\cdot\vec{n}\left[\nu_2\,\partial_{x_2}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dl\\
&=-p\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\left[e^{-2\lambda}\partial_{x_1}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_1}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dx^2\\
&\quad+p\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\left[e^{-2\lambda}\,\partial_{x_2}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_2}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dx^2.
\end{aligned}\tag{III.13}$$

Integrating by parts the r.h.s of (III.13), we have

$$\begin{aligned}
&-p\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\left[e^{-2\lambda}\,\partial_{x_1}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_1}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dx^2\\
&\quad+p\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\vec{n}\left[e^{-2\lambda}\,\partial_{x_2}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_2}\left[f^{p-1}\,\mathbf{I}_{11}^0\right]\right]dx^2\\
&=p\sigma^2\int_\Sigma f^{p-1}\,\mathbf{I}_{11}^0\,\overline{\nabla}\left[\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})\cdot\vec{n}}{|\vec{\Phi}-\vec{q}|^4}\right]e^{-2\lambda}\,\nabla|\vec{\Phi}-\vec{q}|^2\,dx^2\\
&\quad+p\sigma^2\int_\Sigma f^{p-1}\,\mathbf{I}_{11}^0\cdot\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})\cdot\vec{n}}{|\vec{\Phi}-\vec{q}|^4}\,\overline{\nabla}\left[e^{-2\lambda}\,\nabla|\vec{\Phi}-\vec{q}|^2\right]dx^2.
\end{aligned}\tag{III.14}$$

We recall that we have respectively

$$\textbf{(III.15)}\qquad \overline{\nabla}\cdot\left(e^{-2\lambda}\,\nabla\vec{\Phi}\right)=2\,e^{-2\lambda}\,\vec{\mathbf{I}}^0_{11}\quad\text{and}\quad(\overline{\nabla})^{\perp}\cdot\left(e^{-2\lambda}\,\nabla\vec{\Phi}\right)=2\,e^{-2\lambda}\,\vec{\mathbf{I}}^0_{12}.$$

Combining these identities with the fact that $\vec{\Phi}$ is conformal we deduce that

$$\textbf{(III.16)}\qquad \begin{aligned}&\overline{\nabla}\left[e^{-2\lambda}\,\nabla|\vec{\Phi}-\vec{q}|^2\right]\\&=2\,\overline{\nabla}\left[e^{-2\lambda}\,\nabla(\vec{\Phi}-\vec{q})\right]\cdot(\vec{\Phi}-\vec{q})+2\,e^{-2\lambda}\,\nabla(\vec{\Phi}-\vec{q})\cdot\overline{\nabla}(\vec{\Phi}-\vec{q})\\&=4\,e^{-2\lambda}\,\vec{\mathbf{I}}^0_{11}\cdot(\vec{\Phi}-\vec{q}).\end{aligned}$$

Combining (III.14) and (III.16) and observing that we have the following pointwise upper bound

$$\begin{aligned}&\left|\overline{\nabla}\left[\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})\cdot\vec{n}}{|\vec{\Phi}-\vec{q}|^4}\right]\right|\\&\quad\le \mathrm{C}\,\left[\|\chi'\|_\infty\,d_\chi^{-3}+\|\chi\|_\infty d_\chi^{-4}\right]|\nabla\vec{\Phi}|(x)+\|\chi\|_\infty\,d_\chi^{-3}\,|\nabla\vec{n}|(x),\end{aligned}$$

where d_χ is the distance of the support of χ to zero we deduce

$$\textbf{(III.17)}\qquad \begin{aligned}&\left|-p\,\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\left[e^{-2\lambda}\,\partial_{x_1}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_1}\left[f^{p-1}\,\vec{\mathbf{I}}^0_{11}\right]\right]dx^2\right.\\&\qquad\left.+p\,\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\frac{(\vec{\Phi}-\vec{q})}{|\vec{\Phi}-\vec{q}|^4}\cdot\left[e^{-2\lambda}\,\partial_{x_2}|\vec{\Phi}-\vec{q}|^2\,\partial_{x_2}\left[f^{p-1}\,\vec{\mathbf{I}}^0_{11}\right]\right]dx^2\right|\\&\le \mathrm{C}_\chi\,\sigma^2\,\mathrm{F}_p(\vec{\Phi})+\mathrm{C}_\chi\,\sigma^2\,\mathrm{M}(\mathrm{T})^{1/p}\left[\mathrm{F}_p(\vec{\Phi})\right]^{1-1/p}\to 0.\end{aligned}$$

The control of the last term of the r.h.s. of (III.10) is performed similarly to the preceding one following each step between (III.13) and (III.17). So finally deduce that for any χ compactly supported in $\mathbf{R}^*_+$ we have

$$\textbf{(III.18)}\qquad \begin{aligned}&-\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\Big[\sigma^2 f^{p-1}\Big[f\,\nabla\vec{\Phi}\\&\qquad-2\,p\left(\mathrm{H}\,\nabla\vec{n}-e^{-2\lambda}\langle\nabla\vec{n}\dot{\otimes}\nabla\vec{n};\nabla\vec{\Phi}\rangle\right)\Big]\Big]\,dx^2\\&\qquad+2\,p\,\sigma^2\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\mathrm{div}\Big[e^{-2\lambda}\left[\overline{\nabla}\left[f^{p-1}\,\mathbf{I}^0_{11}\right]\right.\\&\qquad\left.+(\overline{\nabla})^{\perp}\left[f^{p-1}\,\mathbf{I}^0_{12}\right]\right]\vec{n}\Big]\,dx^2\\&\to 0.\end{aligned}$$

It remains to bound

$$
\begin{aligned}
&-\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\Delta\left[f^{p-1}\,\vec{\mathrm{H}}\right]dx^2\\
&\quad=\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\,\vec{\mathrm{H}}\right]dx^2\\
&\qquad-\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\,\vec{\mathrm{H}}\right]dl.
\end{aligned}
\tag{III.19}
$$

The last integral in the r.h.s. of (III.19) is equal to

$$
\begin{aligned}
&-\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\,\vec{\mathrm{H}}\right]dl\\
&\quad=-\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\nabla|\vec{\Phi}-\vec{q}|\cdot\left\langle\nabla\left[f^{p-1}\vec{\mathrm{H}}\right],\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle dx^2.
\end{aligned}
\tag{III.20}
$$

We observe that since $\vec{\mathrm{H}}\cdot\nabla\vec{\Phi}=0$

$$
\begin{aligned}
\left\langle\nabla\left[f^{p-1}\vec{\mathrm{H}}\right],\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle&=\nabla\left\langle f^{p-1}\vec{\mathrm{H}},\tfrac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle\\
&\quad+3\left\langle f^{p-1}\vec{\mathrm{H}},\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^4}\right\rangle\nabla|\vec{\Phi}-\vec{q}|.
\end{aligned}
$$

Hence, after integrating by parts we obtain from (III.20)

$$
\begin{aligned}
&-\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\,\vec{\mathrm{H}}\right]dl\\
&\quad=\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\Delta|\vec{\Phi}-\vec{q}|\left\langle f^{p-1}\vec{\mathrm{H}},\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle dx^2\\
&\qquad+\sigma^2\int_\Sigma\left[\chi'\left(|\vec{\Phi}-\vec{q}|\right)-3\,\frac{\chi(|\vec{\Phi}-\vec{q}|)}{|\vec{\Phi}-\vec{q}|}\right]\left|\nabla|\vec{\Phi}-\vec{q}|\right|^2\\
&\qquad\times\left\langle f^{p-1}\vec{\mathrm{H}},\frac{\vec{\Phi}-\vec{q}}{|\vec{\Phi}-\vec{q}|^3}\right\rangle dx^2.
\end{aligned}
\tag{III.21}
$$

We observe that in the domain where $\chi(|\vec{\Phi}-\vec{q}|)\neq 0$ we have

$$
\Delta|\vec{\Phi}-\vec{q}|=\frac{(\vec{\Phi}-\vec{q})\cdot\Delta\vec{\Phi}}{|\vec{\Phi}-\vec{q}|}+\frac{|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|}-\frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|},
$$

and using the fact that $\Delta\vec{\Phi} = -\vec{\Phi}\,|\nabla\vec{\Phi}|^2 + \vec{H}\,|\nabla\vec{\Phi}|^2$ we finally obtain

$$\textbf{(III.22)}\qquad \Delta|\vec{\Phi}-\vec{q}| = -\frac{1-\vec{q}\cdot\vec{\Phi}}{|\vec{\Phi}-\vec{q}|} + \frac{(\vec{\Phi}-\vec{q})\cdot\vec{H}\,|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|} + \frac{|\nabla\vec{\Phi}|^2}{|\vec{\Phi}-\vec{q}|} - \frac{|\nabla|\vec{\Phi}-\vec{q}||^2}{|\vec{\Phi}-\vec{q}|}.$$

Hence combining (III.20), (III.21) and (III.22) we obtain

$$\textbf{(III.23)}\qquad \begin{aligned} &\left|\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(\partial B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\partial_\nu\left[f^{p-1}\,\vec{H}\right]dl\right|\\ &\quad\le C_\chi\,\sigma^2\,F_p(\vec{\Phi}) + C_\chi\,\sigma^2\,M(T)^{1/p}\left[F_p(\vec{\Phi})\right]^{1-1/p}\to 0.\end{aligned}$$

Taking now the first integral in the r.h.s. of (III.19) we have

$$\textbf{(III.24)}\qquad \begin{aligned} &\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\,\vec{H}\right]dx^2\\ &\quad= -\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\Delta\vec{\Phi}\cdot f^{p-1}\,\vec{H}\,dx^2\\ &\qquad+\sigma^2\int_\Sigma\chi\left(|\vec{\Phi}-\vec{q}|\right)\nabla|\vec{\Phi}-\vec{q}|\cdot\left\langle\nabla(\vec{\Phi}-\vec{q}),f^{p-1}\,\vec{H}\right\rangle dx^2.\end{aligned}$$

So we have also

$$\textbf{(III.25)}\qquad \begin{aligned} &\left|\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}\nabla(\vec{\Phi}-\vec{q})\cdot\nabla\left[f^{p-1}\,\vec{H}\right]dx^2\right|\\ &\quad\le C_\chi\,\sigma^2\,F_p(\vec{\Phi}) + C_\chi\,\sigma^2\,M(T)^{1/p}\left[F_p(\vec{\Phi})\right]^{1-1/p}\to 0.\end{aligned}$$

Combining (III.19) and (III.23) and (III.25) we have

$$\textbf{(III.26)}\qquad \left|\int_{\mathbf{R}_+}\chi(r)\,\frac{dr}{r^3}\sigma^2\int_{\vec{\Phi}^{-1}(B^4_r(\vec{q}))}(\vec{\Phi}-\vec{q})\cdot\Delta\left[f^{p-1}\,\vec{H}\right]dx^2\right|\to 0.$$

Combining now (III.7), (III.8), (III.9), (III.18) and (III.26) we have that for any fixed non negative $\chi(r)$ compactly supported in $\mathbf{R}^*_+$ and any $\vec{q}\in\mathbf{R}^4$

$$\textbf{(III.27)}\qquad \begin{aligned} &-\int_0^\infty\chi'(r)\,dr\,\frac{1}{r^2}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))}dvol_{g_{\vec{\Phi}_k}}\\ &\quad\ge -C\int_0^\infty\chi(r)\,dr\,\frac{1}{r}\int_{\vec{\Phi}_k^{-1}(B^4_r(\vec{q}))}dvol_{g_{\vec{\Phi}_k}} - C_\chi\,\sigma^2\,F_p(\vec{\Phi})\\ &\qquad- C_\chi\,\sigma^2\,M(T)^{1/p}\left[F_p(\vec{\Phi})\right]^{1-1/p},\end{aligned}$$

for some constant C_χ depending on χ. Taking μ_k the Radon measure on $\mathbf{R}^4$ given by (III.3) this can be rewritten as

$$-\int_0^\infty \chi'(r)\, dr\, \frac{1}{r^2}\mu_k\big(B^4_r(\vec q)\big) \geq -C\int_0^\infty \chi(r)\, dr\, \frac{1}{r}\mu_k\big(B^4_r(\vec q)\big) + o_k(1).$$

We extract a subsequence such that μ_k converges weakly in Radon measure and we finally obtain that for any fixed non negative $\chi(r)$ compactly supported in $\mathbf{R}^*_+$ and any $\vec q \in \mathbf{R}^4$

$$-\int_0^\infty \chi'(r)\, dr\, \frac{1}{r^2}\mu_\infty\big(B^4_r(\vec q)\big) \geq -C\int_0^\infty \chi(r)\, dr\, \frac{1}{r}\mu_\infty\big(B^4_r(\vec q)\big),$$

which classically implies (III.4) and Lemma III.1 is proved. □

A rather direct consequence of the proof of the limiting monotonicity formula is given by the following non concentration result.

Lemma **III.2.** — **[Non collapsing lemma]** *Let $p > 1$ and $0 < \sigma < 1$. There exists $\delta > 0$ and $\varepsilon > 0$ such that for any critical point $\vec\Phi$ of A^σ satisfying*

$$\sigma^2 F_p(\vec\Phi) \leq \varepsilon\, \mathrm{Area}(\vec\Phi), \tag{III.28}$$

then

$$\mathrm{diam}\big(\vec\Phi(\Sigma)\big) > \delta. \tag{III.29}$$

Proof of Lemma III.2. — Assume (III.28) for some ε fixed later. Let $1 > \delta > 0$ and choose $\chi_\delta = (r-\delta)^+$ on $[0, 1+\delta]$ identically equal to 1 on $[1+\delta, 2+\delta]$ and equal to $(3+\delta-r)^+$ for $r > 2+\delta$. Assuming that the whole immersed surface is included in a ball $B^4_\delta(\vec q)$, the inequality (III.27) gives then

$$-\mathrm{Area}(\vec\Phi)\left[\int_\delta^{1+\delta}\frac{dr}{r^2} - \int_{2+\delta}^{3+\delta}\frac{dr}{r^2}\right] \geq -C\mathrm{Area}(\vec\Phi)\int_\delta^{3+\delta}\frac{dr}{r} - C_\delta\, \varepsilon^{1-1/p}\mathrm{Area}(\vec\Phi). \tag{III.30}$$

Dividing by $\mathrm{Area}(\vec\Phi)$ we obtain

$$C\log\frac{1}{\delta} \geq \frac{1}{\delta} - \frac{1}{4} - C_\delta\, \varepsilon^{1-1/p}.$$

Assume that δ is small enough in such a way that $C\log\frac{1}{\delta} < \frac{1}{\delta} - 1$, choosing $\varepsilon > 0$ such that $C_\delta\, \varepsilon^{1-1/p} < 3/4$ we obtain a contradiction. This proves Lemma III.2. □

The next result establishes a uniform lower bound of the limiting area for any sequence of immersions satisfying the assumptions of Theorem III.1. This result is the "work-horse" in our proof of the main theorem and shall be used crucially at several steps. Precisely we have the following result

Lemma **III.3.** — **[Global energy quantization]** *Let $p > 1$. For every $\Lambda > 0$ there exists $Q_0(\Lambda) > 0$ and $\sigma(\Lambda) > 0$ such that the following holds. Let Σ be a closed surface and let $\vec{\Phi}$ be a critical points of A^σ_p for $\sigma < \sigma(\Lambda)$ and satisfying*

$$\textbf{(III.31)}\qquad \sigma^2\, F_p(\vec{\Phi}) = \sigma^2 \int_\Sigma \left[1 + |\mathbf{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}} \leq \frac{\Lambda}{\log(1/\sigma)} \mathrm{Area}(\vec{\Phi}),$$

then,

$$\textbf{(III.32)}\qquad \mathrm{Area}(\vec{\Phi}) \geq Q_0(\Lambda).$$

Proof of Lemma III.3. — We denote as usual

$$f(\sigma) = \frac{\sigma^2\, F_p(\vec{\Phi})}{\mathrm{Area}(\vec{\Phi})}.$$

Let $\eta > 0$ to be fixed later. For any $\vec{q} \in \vec{\Phi}(\Sigma)$ we consider the 4-dimensional ball in $\mathbf{R}^4$, $B^4_\sigma(\vec{q})$ centered at $\vec{q}$ with radius σ. We consider the subset E_η of $\vec{\Phi}(\Sigma)$ given by

$$E_\eta := \left\{ \vec{q} \in \vec{\Phi}(\Sigma) \subset S^3 ;\, \sigma^{-2} \int_{B^4_\sigma(\vec{q}) \cap \vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}} < \eta \right\}.$$

From the covering $(B^4_\sigma(\vec{q}))_{\vec{q} \in E_\eta}$ we extract a Besicovitch sub-covering $(B^4_\sigma(\vec{q}_i))_{i \in I}$ such that each point in $\mathbf{R}^4$ is covered by at most N balls where N is a universal number. A corollary of Simon's monotonicity formula (see Corollary 5.12 [32] and take $T = \sigma$) gives for each $i \in I$

$$\textbf{(III.33)}\qquad \sigma^{-2} \int_{B^4_\sigma(\vec{q}) \cap \vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}} \geq \frac{2\pi}{3} - \frac{1}{2} \int_{B^4_\sigma(\vec{q}_i)} \left|\vec{H}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}}.$$

Considering $\eta = \pi/3$ this imposes

$$\textbf{(III.34)}\qquad \int_{B^4_\sigma(\vec{q}_i)} \left|\vec{H}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}} > \frac{2\pi}{3}.$$

Hence

$$\textbf{(III.35)}\qquad \int_{\cup_{i\in I} B^4_\sigma(\vec{q}_i)} \left|\vec{H}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}} \geq \frac{1}{N} \sum_{i \in I} \int_{B^4_\sigma(\vec{q}_i)} \left|\vec{H}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}} \geq \frac{2\pi}{3N} \mathrm{card}\, I.$$

Combining (III.31) and (III.35) we obtain

$$\textbf{(III.36)}\qquad \sigma^2 \frac{2\pi}{3N} \mathrm{card}\, I \leq f(\sigma)\, \mathrm{Area}(\vec{\Phi}).$$

So we have

$$\text{(III.37)} \qquad \int_{E_{\pi/3}} dvol_{g_{\vec{\Phi}}} \le \int_{\cup_{i\in I} B^4_\sigma(\vec{q}_i)} dvol_{g_{\vec{\Phi}}} \le \frac{\pi}{3}\,\sigma^2\,\mathrm{card}\,I \le f(\sigma)\,\mathrm{Area}(\vec{\Phi}).$$

Let $1>\delta>0$ to be fixed later . Consider now for $j\in\{1,2,\ldots,\log_2\sigma^{-1}\}$. We use the notation

$$A(j,\vec{q}) := \int_{B^4_{2^j\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}} \quad \text{and}$$

$$F(j,\vec{q}) := \sigma^2 \int_{B^4_{2^j\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)} \left[1+|\mathbf{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}},$$

$$G^j_\delta := \left\{ \begin{array}{l} \vec{q}\in\vec{\Phi}(\Sigma)\setminus E_{\pi/3}; \\ \dfrac{(2^{-2j}A(j+1,\vec{q}))^{1/p}\,F(j+1,\vec{q})^{1-1/p}+F(j,\vec{q})}{A(j,\vec{q})} \ge \dfrac{f(\sigma)}{\delta} \\ \text{and} \quad A(j+1,\vec{q}) \le 3\,\pi\,2^{2j+2}\,\sigma^2. \end{array} \right\}.$$

For each $j\in\{1,2,\ldots,\log_2\sigma^{-1}\}$ and for any $\vec{q}\in G^j_\delta$ we consider the closed balls $B^4_{2^j\sigma}(\vec{q})$. The following covering of $G_\delta := \cup_{j\in\{1,2,\ldots,\log_2\sigma^{-1}-1\}} G^j_\delta$

$$\left(\left(B^4_{2^j\sigma}(\vec{q})\right)_{\vec{q}\in G^j_\delta}\right)_{j=1,2,\ldots,\log_2\sigma^{-1}}$$

realizes a Besicovitch covering of G_δ. By the mean of Besicovitch theorem, we extract a Besicovitch sub-covering

$$\left(\left(B^4_{2^j\sigma}(\vec{q}_i)\right)_{i\in I_j}\right)_{j=1,\ldots,\log_2\sigma^{-1}}$$

of G_δ such that each point in $\mathbf{R}^4$ is covered by at most $\mathcal{N}$ balls where $\mathcal{N}$ is a universal number.[9] In other words we have

$$\text{(III.38)} \qquad \left\| \sum_{j=1}^{\log_2\sigma^{-1}-1} \sum_{i\in I_j} \mathbf{1}_{B^4_{2^j\sigma}(\vec{q}_i)} \right\|_{L^\infty(\mathbf{R}^4)} \le \mathcal{N}.$$

For any $j=1,\ldots,\log_2\sigma^{-1}$ the balls $B^4_{2^j\sigma}(\vec{q}_i)$ for $i\in I_j$ have all the same radius, moreover each point of $\mathbf{R}^4$ is covered by at most $\mathcal{N}$ of such balls. Hence by doubling each of these

[9] Observe that it is not clear whether for each j the sub-familly $(B^4_{2^j\sigma}(\vec{q}_i))_{i\in I_j}$ covers the whole G^j_δ but at least the union of these families cover G_δ.

balls and considering $B^4_{2^{j+1}\sigma}(\vec{q}_i)$, since they all have the same radius there exists a universal number[10] $\mathfrak{N}$ such that

$$\sup_{j=1,\ldots,\log_2\sigma^{-1}}\left\|\sum_{i\in I_j}\mathbf{1}_{B^4_{2^{j+1}\sigma}(\vec{q}_i)}\right\|_{L^\infty(\mathbf{R}^4)}\le\mathfrak{N},$$

where $\mathbf{1}_{B^4_{2^{j+1}\sigma}(\vec{q}_i)}$ is the characteristic function of the ball $B^4_{2^{j+1}\sigma}(\vec{q}_i)$. We have for any $\alpha>0$ that

$$\textbf{(III.39)}\qquad \left\|\sum_{j=1}^{\log_2\sigma^{-1}-1}\ \sum_{i\in I_j}\mathbf{1}_{B^4_{2^{j+1}\sigma}(\vec{q}_i)}\,2^{\alpha j}\right\|_{L^\infty(\mathbf{R}^4)}\le C\,\mathfrak{N}\sum_{j=0}^{\log_2\sigma^{-1}}2^{\alpha j}\le C\,\mathfrak{N}\,\sigma^{-\alpha}.$$

For any $j\in\{1,2,\ldots,\log_2\sigma^{-1}\}$ and $\vec{q}\in G^j_\delta$, the whole support of $\vec{\Phi}(\Sigma)$ cannot be included in $B^4_{2^j\sigma}(\vec{q})$ otherwise we would contradict the non collapsing lemma III.2 for σ small enough. Hence, since $\vec{q}\in\vec{\Phi}(\Sigma)$ for any radius $r\in(2^j\sigma,2^{j+1}\sigma)$ we have $\vec{\Phi}(\Sigma)\cap\partial B_r(\vec{q})\neq\emptyset$ and we can apply Lemma A.1. Hence we deduce

$$\textbf{(III.40)}\qquad 0<\varepsilon_0(4)<\int_{B^4_{2^{j+1}\sigma}(\vec{q})}\left|\vec{\mathbf{I}}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}}.$$

Since $A(j+1,\vec{q})\le 3\pi\,2^{2j+2}\,\sigma^2$ inequality (III.40) implies

$$\textbf{(III.41)}\qquad \begin{aligned}\frac{A(j+1,\vec{q})}{2^{2j+2}}&\le\frac{3\pi\,\sigma^2}{\varepsilon_0(4)}\int_{B^4_{2^{j+1}\sigma}(\vec{q})}\left|\vec{\mathbf{I}}^{\mathbf{R}^4}_{\vec{\Phi}}\right|^2 dvol_{g_{\vec{\Phi}}}\\&\le\frac{3\pi\,\sigma^2}{\varepsilon_0(4)}A(j+1,\vec{q})^{1-1/p}\left(\int_{B^4_{2^{j+1}\sigma}(\vec{q})}\left[1+|\mathbf{I}_{\vec{\Phi}}|^2\right]^p dvol_{g_{\vec{\Phi}}}\right)^{1/p}\end{aligned}$$

and we deduce that

$$\textbf{(III.42)}\qquad \frac{A(j+1,\vec{q})}{2^{2j}}\le C\left(2^{j+1}\sigma\right)^{2p-2}F(j+1,\vec{q}).$$

So for $\vec{q}\in G^j_\delta$ we have combining the definition of G^j_δ with (III.42)

$$\textbf{(III.43)}\qquad \frac{f(\sigma)}{\delta}A(j,\vec{q})\le F(j,\vec{q})+C\left(2^{j+1}\sigma\right)^{2-2/p}F(j+1,\vec{q})$$

[10] Observe that a-priori each point of $\mathbf{R}^4$ can be covered by at most $\mathfrak{N}\log_2\sigma^{-1}$ of the double balls $((B^4_{2^{j+1}\sigma}(\vec{q}_i))_{i\in I_j})_{j=1,\ldots,\log_2\sigma^{-1}}$.

summing this identity with respect to $j\in J$ we obtain

$$
\begin{aligned}
\frac{f(\sigma)}{\delta}\int_{G_\delta} dvol_{g_{\vec\Phi}} &\le \frac{f(\sigma)}{\delta}\sum_{j=1}^{\log_2\sigma^{-1}}\int_{G_\delta^j} dvol_{g_{\vec\Phi}}\\
&\le \frac{f(\sigma)}{\delta}\sum_{j=1}^{\log_2\sigma^{-1}}\sum_{i\in I_j}\int_{B^4_{2^j\sigma}(\vec q_i)} dvol_{g_{\vec\Phi}}\\
&\le \sum_{j=1}^{\log_2\sigma^{-1}}\sum_{i\in I_j}\sigma^2\int_{B^4_{2^j i\sigma}(\vec q_i)}\left[1+|\mathbf{I}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right]^p dvol_{g_{\vec\Phi}}\\
&\quad+\sigma^2\int_\Sigma\sum_{j=1}^{\log_2\sigma^{-1}}\sum_{i\in I_j}\mathbf{1}_{B^4_{2^{j+1}\sigma}(\vec q_i)}\,2^{\alpha j}\,\sigma^\alpha\left[1+|\mathbf{I}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right]^p dvol_{g_{\vec\Phi}},
\end{aligned}
\tag{III.44}
$$

where $\alpha:=2-2p$. Using now (III.38) and (III.39), we then deduce

$$
\frac{f(\sigma)}{\delta}\int_{G_\delta} dvol_{g_{\vec\Phi}}\le C\,\sigma^2\int_\Sigma\left[1+|\mathbf{I}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right]^p dvol_{g_{\vec\Phi}}=C f(\sigma)\int_\Sigma dvol_{g_{\vec\Phi}}.
\tag{III.45}
$$

We deduce from (III.37) and (III.45)

$$
\int_{E_{\pi/3}\cup G_\delta} dvol_{g_{\vec\Phi}}\le\left(C\,\delta+f(\sigma)\right)\int_\Sigma dvol_{g_{\vec\Phi}}.
\tag{III.46}
$$

Since $f(\sigma)\to0$ as $\sigma\to0$, by taking any $0<\delta<1/C$ we have that for σ small enough $\vec\Phi(\Sigma)\setminus(E_{\pi/3}\cup G_\delta)\ne\emptyset$. Let now $\vec q\in\vec\Phi(\Sigma)\setminus(E_{\pi/3}\cup G_\delta)$. Take $j_0=j(\vec q)$ the largest index such that

$$
\int_{B^4_{2^{j_0}\sigma}(\vec q)} dvol_{g_{\vec\Phi}}\ge 2^{2j_0}\sigma^2\,\pi/3.
$$

Since $\vec q\in\vec\Phi(\Sigma)\setminus(E_{\pi/3}\cup G_\delta)$ we must have

$$
\left\{
\begin{aligned}
&\int_{B^4_{2^{j_0}\sigma}(\vec q)} dvol_{g_{\vec\Phi}}\ge 2^{2j_0}\sigma^2\,\pi/3\quad\text{and}\\
&\forall j\ge j_0\quad \frac{f(\sigma)}{\delta}\int_{B^4_{2^j\sigma}(\vec q)\cap\vec\Phi(\Sigma)} dvol_{g_{\vec\Phi}}\ge\sigma^2\int_{B^4_{2^j\sigma}(\vec q)\cap\vec\Phi(\Sigma)}\left[1+|\mathbf{I}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right]^p dvol_{g_{\vec\Phi}}\\
&\qquad+\left[\sigma^2\int_{B^4_{2^{j+1}\sigma}(\vec q)\cap\vec\Phi(\Sigma)}\left[1+|\mathbf{I}_{\vec\Phi}|^2_{g_{\vec\Phi}}\right]^p dvol_{g_{\vec\Phi}}\right]^{1-1/p}\\
&\qquad\times\left[2^{-2j}\int_{B^4_{2^{j+1}\sigma}(\vec q)\cap\vec\Phi(\Sigma)} dvol_{g_{\vec\Phi}}\right]^{1/p}.
\end{aligned}
\right.
\tag{III.47}
$$

Let $j \in \{j_0, \ldots, \log_2 \sigma^{-1} - 1\}$ and let χ be an arbitrary smooth function, bounded by 1, supported in $[2^{j-2}\sigma, 2^{j+1}\sigma]$ and such that $|\chi'| \le \mathrm{C}\, 2^{-j}\sigma^{-1}$. We can estimate each error terms between (III.6) and (III.27) in the computations of the monotonicity formula at fixed k between (III.6) and (III.27) by the mean of the area we obtain

$$
\begin{aligned}
&-\int_0^{+\infty} \chi'(r)\frac{dr}{r^2}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}\\
&\ge -\mathrm{C}\int_0^{+\infty} \chi(r)\, dr\left[\frac{1}{r}+\frac{o_\sigma(1)}{r^2}\right]\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}\\
&\quad -\mathrm{C}\int_0^{+\infty}\chi(r)\frac{dr}{r^3}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))}\sigma^2\left[1+|\mathbf{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}} \qquad \text{(III.48)}\\
&\quad -\mathrm{C}\, 2^{-3j}\sigma^{-3}\left[\sigma^2\int_{\mathrm{B}^4_{2^{j+1}\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)}\left[1+|\mathbf{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p dvol_{g_{\vec{\Phi}}}\right]^{1-1/p}\\
&\quad \times\left[2^{-2j}\int_{\mathrm{B}^4_{2^{j+1}\sigma}(\vec{q})\cap\vec{\Phi}(\Sigma)} dvol_{g_{\vec{\Phi}}}\right]^{1/p}.
\end{aligned}
$$

Using (III.47) we deduce that for any $r \in [2^{j_0}\sigma, 1/2]$

$$
\begin{aligned}
\frac{d}{dr}\left[\frac{1}{r^2}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}\right] &\ge -\left[\frac{\mathrm{C}}{r}+\frac{o_\sigma(1)}{r^2}\right]\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}\\
&\quad -\mathrm{C}\frac{f(\sigma)}{\delta}\frac{1}{r^3}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}.
\end{aligned} \qquad \text{(III.49)}
$$

Let $\mathrm{Y}(r) := \frac{1}{r^2}\int_{\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}))} dvol_{g_{\vec{\Phi}}}$, this ordinary differential inequality gives, for σ small enough, the existence of $\mathrm{C} > 0$ independent of r and σ and δ, such that for $r \in [2^{j_0}\sigma, 1/2]$

$$
\frac{d}{dr}\left[e^{\mathrm{C}\, r}\, r^{\frac{\mathrm{C}f(\sigma)}{\delta}}\,\mathrm{Y}\right] \ge 0. \qquad \text{(III.50)}
$$

Integrating between $2^{j_0}\sigma$ and $1/2$ gives

$$
e^{\mathrm{C}/2}\,\mathrm{Y}(1/2)\, 2^{-\frac{\mathrm{C}f(\sigma)}{\delta}} \ge e^{\mathrm{C}\,2^{j_0}\sigma}\left(2^{j_0}\sigma\right)^{\frac{\mathrm{C}f(\sigma)}{\delta}}\mathrm{Y}\left(2^{j_0}\sigma\right).
$$

Using the fact that $\vec{q} \in \vec{\Phi}(\Sigma)\setminus \mathrm{E}_{\pi/3}$ we have then using the first line in (III.47)

$$
\mathrm{Y}(1/2) \ge e^{-\mathrm{C}/2}\, 2^{\frac{3\mathrm{C}f(\sigma)}{\pi}}\, e^{\mathrm{C}\,2^{j_0}\sigma}\left(2^{j_0}\sigma\right)^{\frac{3\mathrm{C}f(\sigma)}{\pi}}\frac{\pi}{3}. \qquad \text{(III.51)}
$$

Since $f(\sigma)\log_2\sigma^{-1}) \le \Lambda$ we have $(2^{j_0}\sigma)^{\frac{\mathrm{C}f(\sigma)}{\delta}} = 2^{\mathrm{C}f(\sigma)\,\delta^{-1}\log_2(2^{j_0}\sigma)} \ge 2^{\mathrm{C}f(\sigma)\,\delta^{-1}\log_2\sigma} \ge 2^{-\mathrm{C}\,\delta^{-1}\Lambda}$. So $\mathrm{Q}_0 := 2^{-\mathrm{C}\,\delta^{-1}\Lambda}\, e^{-\mathrm{C}/2}\,\pi/3$ satisfies (III.32) and the Lemma III.3 is proved. □

We now introduce two definitions. First we define the *Oscillation set.*

Definition **III.7.** — *Let $\vec{\Phi}_k$ be a sequence of conformal smooth immersions from*[11] (Σ, g_k), *critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := \mathrm{Area}(\vec{\Phi}) + \sigma_k^2\, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2\left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p\right] dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 *and for* $\sigma_k \to 0$. *Assume*

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty \quad \textit{weakly in } W^{1,2}\big(\Sigma, S^3\big),$$

where Σ *is equipped with a reference metric* g_0. *Assume the sequence of Riemann surfaces* $(\Sigma, g_{\vec{\Phi}_k})$ *is pre-compact in the moduli space of conformal structures on* Σ *and assume*

$$\nu_k := |d\vec{\Phi}_k|^2_{h_k}\, dvol_{h_k} = |\nabla\vec{\Phi}_k|^2\, dx^2 \rightharpoonup \nu_\infty \quad \textit{in Radon measures}.$$

The oscillation set $\mathcal{O} \subset \Sigma$ *is the set of points* $x \in \Sigma$ *such that*

$$\textbf{(III.52)} \qquad \mathcal{O} := \left\{ \begin{array}{l} x \in \Sigma; \quad \nu_\infty\big(B_\rho(x)\big) \neq 0 \quad \forall \rho > 0 \\[2mm] \quad \textit{and} \quad \liminf_{\rho\to 0} \dfrac{\int_{B_{2\rho}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}}{\nu_\infty(\overline{B_\rho(x)})} = 0 \end{array} \right\}.$$

Now we define the *vanishing set* $\mathcal{V}$.

Definition **III.8.** — *Let $\vec{\Phi}_k$ be a sequence of conformal smooth immersions from* (Σ, g_k), *critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := \mathrm{Area}(\vec{\Phi}) + \sigma_k^2\, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2\left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p\right] dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 *and for* $\sigma_k \to 0$. *We assume* (Σ, g_k) *to be pre-compact in the moduli space of conformal structures on* Σ. *Denote*

$$\textbf{(III.53)} \qquad f(\sigma_k) := \frac{\sigma_k^2 \displaystyle\int_\Sigma [1 + |\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}]^p\, dvol_{g_{\vec{\Phi}_k}}}{\displaystyle\int_\Sigma dvol_{g_{\vec{\Phi}_k}}}.$$

We call the "vanishing set" the subset Σ_0 *of* Σ *given by*

$$\textbf{(III.54)} \qquad \Sigma_0 := \left\{ x \in \Sigma; \quad \liminf_{r\to 0} \limsup_{k\to+\infty} \frac{f(\sigma_k)\int_{B_r(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \int_{B_r(x)} [1 + |\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}]^p\, dvol_{g_{\vec{\Phi}_k}}} = 0 \right\}.$$

[11] Recall that in this section we are assuming that the underlying conformal class to (Σ, g_k) is precompact in the moduli space.

We will need later on the following lemma which justifies the denomination *vanishing set*.

Lemma **III.4.** — **[No limiting measure on the vanishing set]** *Let* $\vec{\Phi}_k$ *be a sequence of conformal smooth immersions from* (Σ, g_k) *into* S^3, *critical points of*

$$A_p^{\sigma_k}(\vec{\Phi}) := \text{Area}(\vec{\Phi}) + \sigma_k^2 \, F_p(\vec{\Phi}) = \int_\Sigma \left[1 + \sigma_k^2 \left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p\right] dvol_{g_{\vec{\Phi}}}$$

in the space of weak immersions into S^3 *for* $\sigma_k \to 0$. *We assume* (Σ, g_k) *is strongly pre-compact in the Moduli space of* Σ. *Assume*

$$\vec{\Phi}_k \rightharpoonup \vec{\Phi}_\infty \quad \text{weakly in } W^{1,2}(\Sigma, S^3),$$

and assume the following sequence of Radon measure weakly converges

$$\nu_k := |d\vec{\Phi}_k|^2_{g_k} \, dvol_{g_k} \rightharpoonup \nu_\infty,$$

then we have

$$\textbf{(III.55)} \qquad \nu_\infty(\Sigma_0) = 0.$$

Proof of Lemma III.4. — We have

$$\forall x \in \Sigma_0 \quad \forall \delta > 0 \quad \exists\, k_{x,\delta} \in \mathbf{N} \quad \exists\, r_{x,\delta} > 0$$

$$\textbf{(III.56)} \qquad \text{s.t.} \quad \forall k \geq k_{x,\delta} \quad \frac{f(\sigma_k) \displaystyle\int_{B_{r_x}(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \displaystyle\int_{B_{r_x}(x)} [1 + |\vec{\mathbf{I}}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}] \, dvol_{g_{\vec{\Phi}_k}}} < \delta.$$

For any $\delta > 0$ and $j \in \mathbf{N}$ we denote

$$\Sigma_0^j(\delta) := \{x \in \Sigma_0;\ k_{x,\delta} \leq j\}.$$

We have clearly $\Sigma_0 = \cup_{j \in \mathbf{N}} \Sigma_0^j(\delta)$. From the covering $(\overline{B_{r_{x,\delta}}(x)})_{x \in \Sigma_0^j(\delta)}$ we extract a Besicovitch sub-covering of $\Sigma_0^j(\delta)$ that we denote $(\overline{B_{r_{x_i,\delta}}(x_i)})_{i \in I}$ in such a way that any point of Σ is covered by at most N balls from this sub-covering. We have for all $k \geq j$

$$\int_{B_{r_{x_i}}(x_i)} dvol_{g_{\vec{\Phi}_k}} \leq \frac{\delta}{f(\sigma_k)} \, \sigma_k^2 \int_{B_{r_{x_i}}(x_i)} \left[1 + |\vec{\mathbf{I}}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^p dvol_{g_{\vec{\Phi}_k}}.$$

Summing over $i \in \mathrm{I}$ gives

$$\nu_k\Big(\bigcup_{i\in \mathrm{I}} \overline{\mathrm{B}_{r_{x_i}(x_i)}}\Big) \le \sum_{i\in \mathrm{I}} \int_{\mathrm{B}_{r_{x_i}}(x_i)} dvol_{g_{\vec{\Phi}_k}} \le \frac{\delta}{f(\sigma_k)}\sigma_k^2 \sum_{i\in \mathrm{I}} \int_{\mathrm{B}_{r_{x_i}}(x_i)} \left[1+|\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^p dvol_{g_{\vec{\Phi}_k}}$$
$$\le \mathrm{N}\,\frac{\delta}{f(\sigma_k)}\,\sigma_k^2 \int_{\cup_{i\in \mathrm{I}} \mathrm{B}_{r_{x_i}}(x_i)} \left[1+|\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^p dvol_{g_{\vec{\Phi}_k}} \le \mathrm{N}\,\delta \int_\Sigma dvol_{g_{\vec{\Phi}_k}}. \qquad \text{(III.57)}$$

This implies that

$$\nu_\infty\big(\Sigma_0^j(\delta)\big) \le \limsup_{k\to+\infty} \nu_k\Big(\bigcup_{i\in \mathrm{I}} \overline{\mathrm{B}_{r_{x_i}(x_i)}}\Big) \le \mathrm{N}\,\delta\,\nu_\infty(\Sigma). \qquad \text{(III.58)}$$

This inequality is independent of j and since $\Sigma_0^j(\delta) \subset \Sigma_0^{j+1}(\delta)$ we deduce that

$$\nu_\infty(\Sigma_0) \le \mathrm{N}\,\delta\,\nu_\infty(\Sigma). \qquad \text{(III.59)}$$

Since this holds for any $\delta > 0$ we have proven

$$\nu_\infty(\Sigma_0) = 0. \qquad \text{(III.60)}$$

This completes the proof of Lemma III.4. □

The next goal is to prove the following orthogonal decomposition of the limiting measure ν_∞.

Lemma **III.5.** — **[Structure of the limiting measure]** *Under the assumptions of Theorem III.1, we have the existence of finitely many points $a_1, \ldots, a_n$ in Σ such that the measure ν_∞ decomposes orthogonally as follows*

$$\nu_\infty = m(x)\mathcal{L}^2 + \sum_{i=1}^n \alpha_i\,\delta_{a_i}, \qquad \text{(III.61)}$$

where $\mathcal{L}^2$ is the Lebesgue measure on Σ equipped with the reference metric g_0, m is an L^1 function with respect to the Lebesgue measure and α_i are positive numbers bounded from below by the universal positive number $\mathrm{Q}_0 = \lim_{\Lambda\to 0} \mathrm{Q}_0(\Lambda)$ given by Lemma III.3.

Proof of Lemma III.5. — **Step 1**: We prove that

$$\int_{\mathcal{O}} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} = 0. \qquad \text{(III.62)}$$

Indeed, for any $\varepsilon > 0$ to any $x \in \mathcal{O}$ we assign r_x such that

$$\int_{\mathrm{B}_{r_x}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le \int_{\mathrm{B}_{2r_x}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le \varepsilon\,\nu_\infty\big(\overline{\mathrm{B}_{r_x}(x)}\big). \qquad \text{(III.63)}$$

Extracting a Besicovitch covering $(\overline{B_{r_i}(x_i)})_{i\in I})$ such that each point of Σ is covered by at most N balls from the covering. We obtain that

$$\textbf{(III.64)} \qquad \int_{\cup_{i\in I} B_{r_i}(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le \varepsilon \sum_{i\in I} \nu_\infty\left(\overline{B_{r_i}(x_i)}\right) \le \varepsilon\, N\, \nu_\infty(\Sigma),$$

and since this holds for any $\varepsilon > 0$ we obtain (III.62).

Step 2: Proof of the **absolute continuity of** ν_∞ with respect to the Lebesgue measure away from the oscillation set $\mathcal{O}$. Precisely we prove in this step

$$\textbf{(III.65)} \qquad \nu_\infty \llcorner (\Sigma \setminus \mathcal{O}) = m\, d\mathcal{L}^2,$$

where $m \in L^1(\Sigma)$.

Let $\varepsilon > 0$. Following (III.64), we first include $\mathcal{O}$ in an open subset $\mathcal{O}^\varepsilon$ such that

$$\textbf{(III.66)} \qquad \int_{\mathcal{O}^\varepsilon} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le \varepsilon.$$

Let $x \in \Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ then there exists $\delta_x > 0$ such that

$$\inf_{\rho>0} \frac{\int_{B_{2\rho}(x)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}}{\nu_\infty(\overline{B_\rho(x)})} \ge \delta_x.$$

We denote $F_j := \{x \in \Sigma \setminus \mathcal{O} \quad ;\ \delta_x > 2^{-j}\}$. We then have

$$\Sigma \setminus \mathcal{O} = \bigcup_{j\in\mathbf{N}} F_j.$$

Let G be a closed subset of $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ such that $\mathcal{H}^2(G) = 0$. We claim that

$$\textbf{(III.67)} \qquad \nu_\infty(G) = 0.$$

Since $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$ is closed G is compact. Let $\alpha > 0$ to be fixed later on. Since $\mathcal{H}^2(G) = 0$ and since G is compact

$$\exists\, \beta > 0 \quad \text{s.t. } \mathcal{H}^2(G_\beta) \le \alpha \quad \text{where } G_\beta := \left\{x \in \Sigma;\ \operatorname{dist}(x, G) < \beta\right\}.$$

Indeed the closeness of G implies $G := \cap_{n\in\mathbf{N}} G_{1/n}$, $G_{1/n}$ is decreasing for the inclusion and fundamental properties of Hausdorff measures give then $\mathcal{H}^2(G) = \lim_{n\to+\infty} \mathcal{H}^2(\cap_{n\in\mathbf{N}} G_{1/n})$. Let $j \in \mathbf{N}$. From the covering $(\overline{B_{\beta/2}(x)})_{x\in G\cap F_j}$ we extract a Vitalli covering $(\overline{B_{\beta/2}(x_i)})_{i\in I}$ in such a way that the balls $\overline{B_{\beta/6}(x_i)}$ are disjoint. Since all the balls have the same radius $\beta/2$ with centers at distances at least $\beta/3$ each point of Σ is covered by at most N balls $B_\beta(x_i)$ where N is a universal number. Since each $x_j \in F_j$

$$\nu_\infty\left(\overline{B_{\beta/2}(x)}\right) \le 2^{j+1} \int_{B_\beta(x)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}. \tag{III.68}$$

Since all the balls $B_\beta(x_i)$ are included in G_β we have

$$\mathcal{H}^2\left(\bigcup_{i\in I} B_\beta(x_i)\right) \le \alpha. \tag{III.69}$$

We have moreover

$$\begin{aligned}\nu_\infty(G\cap F_j) &\le \sum_{i\in I} \nu_\infty\left(\bigcup \overline{B_{\beta/2}(x_i)}\right) \le 2^{j+1} \sum_{i\in I} \int_{B_\beta(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}\\ &\le 2^{j+1}\, N \int_{\cup_{i\in I} B_\beta(x_i)} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le 2^{j+1}\, N \int_{G_\beta} |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}.\end{aligned} \tag{III.70}$$

Since $|d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0}$ is absolutely continuous with respect to the Lebesgue measure, for any $\eta > 0$ there exists $\alpha > 0$ such that

$$\forall E \text{ measurable} \quad \mathcal{H}^2(E) \le \alpha \quad \Longrightarrow \quad \int_E |d\vec{\Phi}_\infty|^2_{g_0}\, dvol_{g_0} \le \eta. \tag{III.71}$$

Hence we finally get combining (III.69), (III.70) and (III.71)

$$\nu_\infty(G\cap F_j) \le 2^{j+1}\, N\, \eta. \tag{III.72}$$

For any $j \in \mathbf{N}$ the inequality (III.72) holds for any $\eta > 0$ thus $\nu_\infty(G\cap F_j) = 0$ and we deduce (III.67). Since (III.67) holds true for any closed measurable subset of $\Sigma^\varepsilon := \Sigma \setminus \mathcal{O}^\varepsilon$, then using the fundamental property of Radon measures saying that

$$\forall G \text{ measurable} \quad \nu_\infty(G) = \sup\left\{\nu_\infty(K);\ K \subset G;\ K \text{ compact}\right\},$$

we obtain that ν_∞ for any measurable subset G of $\Sigma \setminus \mathcal{O}^\varepsilon$ satisfying on $\Sigma \setminus \mathcal{O}^\varepsilon$ is absolutely continuous with respect to the Lebesgue measure. By making ε go to zero this implies (III.65).

Step 3: Detecting the "**bubbles**". In this step we are just splitting the **oscillation set** $\mathcal{O}$ into it's **vanishing part** $\mathcal{O}_0 := \Sigma_0 \cap \mathcal{O}$ and the **bubble part** $\mathcal{B}$ where we recall that the Σ_0 is the so called *vanishing set* defined in Definition III.8:

$$\mathcal{B} := \mathcal{O} \setminus \left(\mathcal{O} \bigcap \Sigma_0\right).$$

Recall that we have proved in Lemma III.4 $\nu_\infty(\Sigma_0) = 0$ hence

$$\nu_\infty(\mathcal{O}_0) = 0. \tag{III.73}$$

Step 4: **Finiteness of the bubble set** $\mathcal{B}$. Precisely in this step we are proving that for the constant $Q_0 > 0$ given by Lemma III.3 then

(**III.74**) $$\forall x \in \mathcal{B}\ \forall r > 0 \quad \nu_\infty\big(B_r(x)\big) \geq Q_0.$$

Once (III.74) will be established we can then deduce that $\mathcal{B}$ is made of finitely many points. Let then $x \in \mathcal{B}$, then there exists $\delta_x > 0$ and $r_x > 0$ that can be taken as small as one wants such that

(**III.75**) $$\forall r < r_x \quad \limsup_{k\to+\infty} \frac{f(\sigma_k)\int_{B_r(x)} dvol_{g_{\vec{\Phi}_k}}}{\sigma_k^2 \int_{B_r(x)} [1+|\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}]^p\, dvol_{g_{\vec{\Phi}_k}}} \geq \delta_x > 0.$$

Let $0 < r_c < r_x$ to be fixed later, let $\vec{\Phi}_{k'}$ a sequence for which

(**III.76**) $$\forall k' \in \mathbf{N} \quad \frac{f(\sigma_{k'})\int_{B_{r_c}(x)} dvol_{g_{\vec{\Phi}_{k'}}}}{\sigma_{k'}^2 \displaystyle\int_{B_{r_c}(x)} [1+|\mathbf{I}_{\vec{\Phi}_{k'}}|^2_{g_{\vec{\Phi}_{k'}}}]^p\, dvol_{g_{\vec{\Phi}_{k'}}}} \geq \frac{\delta_x}{2}.$$

By assumption (III.2) from Theorem III.1 we have that $f(\sigma) = o(1/\log\sigma^{-1})$ we are "almost" fulfilling the assumptions of Lemma III.3 except that we have a surface with boundary $B_r(x)$ and not a closed surface. So we have to choose a "nice" cut r_c in such a way to be able to apply the arguments of Lemma III.3.

Since $x \in \mathcal{O}$, by definition, for any $\eta > 0$ there exists $\rho < r_x$ such that

(**III.77**) $$\eta\, \nu_\infty\big(B_\rho(x)\big) \geq \int_{B_{2\rho}} |\nabla\vec{\Phi}_\infty|^2\, dx^2.$$

Using Fubini and the mean-value theorem we can find $r \in [\rho, 2\rho]$ such that

(**III.78**) $$\begin{aligned}\lim_{k\to+\infty} \left\|\vec{\Phi}_k(x) - \vec{\Phi}_k(y)\right\|^2_{(L^\infty(\partial B_r(x_1)))^2} &= \left\|\vec{\Phi}_\infty(x) - \vec{\Phi}_\infty(y)\right\|^2_{(L^\infty(\partial B_r(x_1)))^2}\\ &\leq \left[\int_{\partial B_r(x_1)} |\nabla\vec{\Phi}_\infty|\, dl \leq\right]^2 \leq 8\pi \int_{B_{2\rho}(x_1)} |\nabla\vec{\Phi}_\infty|^2\, dx^2.\end{aligned}$$

We take this $r = r_c$ to be our "nice cut". We can assume

$$s := \sqrt{8\pi \int_{B_{2\rho}(x_1)} |\nabla\vec{\Phi}_\infty|^2\, dx^2} > 0,$$

the case $s = 0$ could be treated in a similar way but we would have to introduce a new small parameter... Let $\vec{q}_0 := \vec{\Phi}_\infty(x_2)$ for some fixed arbitrary $x_2 \in \partial B_r(x_1)$. For k large enough we have that

(**III.79**) $$\vec{\Phi}_k\big(\partial B_{r_c}(x_1)\big) \subset B^4_{2s}(\vec{q}_0).$$

Let $\mathrm{R}>4$ to be fixed later. The monotonicity formula (III.4) and (III.77) imply that

$$\mu_\infty\left(\mathrm{B}^4_{\mathrm{R}\,s}(\vec{q}_0)\right)\le \mathrm{C}\,\mathrm{R}^2\,s^2\le \mathrm{C}\,\mathrm{R}^2\,\eta\,\nu_\infty\left(\mathrm{B}_\rho(x)\right). \tag{III.80}$$

Hence for η chosen in such a way that $\mathrm{C}\,\mathrm{R}^2\,\eta<1/2$ we have that for k' large enough (recall that k' is the sequence satisfying (III.76) for our "nice cut" r_c which is fixed now)

$$\int_{\mathrm{B}_{r_c}(x)\setminus(\vec{\Phi}_{k'})^{-1}(\mathrm{B}^4_{\mathrm{R}\,s}(\vec{q}_0))} dvol_{g_{\vec{\Phi}_{k'}}}\ge 4^{-1}\int_{\mathrm{B}_{r_c}(x)} dvol_{g_{\vec{\Phi}_{k'}}}.$$

Taking the same notations of the proof of Lemma III.3 where Σ is replaced by $\mathrm{B}_{r_c}(x)$ we can then find $\vec{q}_1\in\vec{\Phi}_{k'}(\mathrm{B}_{r_c}(x))\setminus(\mathrm{E}_{\pi/3}\cup\mathrm{F}_\delta\cup\mathrm{B}^4_{\mathrm{R}\,s}(\vec{q}_0))$. As in the proof of Lemma III.3 we shall apply the monotonicity formula centered at this point $\vec{q}_1$ but we will remove from $\vec{\Phi}_{k'}(\mathrm{B}_{r_c}(x))$ the balls $\mathrm{B}^4_{t\,s}(\vec{q}_0)$ for $t\in[2,4]$. The monotonicity formula with boundary (see for instance [34]) gives for all $r>0$

$$\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^2}\int_{\mathrm{B}_{r_c}(x)\cap\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}_1)\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0))} dvol_{g_{\vec{\Phi}}}\right]\\
&\quad=\frac{d}{dr}\left[\int_{\mathrm{B}_{r_c}(x)\cap\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}_1)\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0))}\frac{|(\vec{n}\wedge\vec{\Phi})\llcorner(\vec{\Phi}-\vec{q}_1)|^2}{|\vec{\Phi}-\vec{q}_1|^4}\,dvol_{g_{\vec{\Phi}}}\right]\\
&\qquad-\frac{1}{2\,r^3}\int_{\mathrm{B}_{r_c}(x)\cap\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}_1)\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0)}(\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\,dvol_{g_{\vec{\Phi}}}\\
&\qquad-\frac{1}{r^3}\int_{\mathbf{R}^4}<\vec{q}-\vec{q}_1,\vec{\nu}>d\mathcal{H}^1\llcorner\left[\vec{\Phi}\left(\mathrm{B}_{r_c}(x)\right)\cap\mathrm{B}^4_r(\vec{q}_1)\cap\partial\mathrm{B}^4_{t\,s}(\vec{q}_0)\right]\\
&\quad\ge-\frac{1}{2\,r^3}\int_{\mathrm{B}_{r_c}(x)\cap\vec{\Phi}^{-1}(\mathrm{B}^4_r(\vec{q}_1)\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0))}(\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\,dvol_{g_{\vec{\Phi}}}\\
&\qquad-\frac{1}{r^3}\int_{\mathbf{R}^4}\langle\vec{q}-\vec{q}_1,\vec{\nu}\rangle\,d\mathcal{H}^1\llcorner\left[\vec{\Phi}\left(\mathrm{B}_{r_c}(x)\right)\cap\mathrm{B}^4_r(\vec{q}_1)\cap\partial\mathrm{B}^4_{t\,s}(\vec{q}_0)\right],
\end{aligned}\tag{III.81}$$

where $\vec{\nu}$ is the outward unit tangent to the surface $\vec{\Phi}_k(\mathrm{B}_{r_c}(x))\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0)$ along the boundary

$$\partial\left(\vec{\Phi}_k\left(\mathrm{B}_{r_c}(x)\right)\setminus\mathrm{B}^4_{t\,s}(\vec{q}_0)\right)=\vec{\Phi}_k\left(\mathrm{B}_{r_c}(x)\right)\cap\partial\mathrm{B}^4_{t\,s}(\vec{q}_0)$$

and perpendicular to this boundary.[12] We consider $\chi(t)$ a smooth non negative function supported in $[1,2]$ satisfying $\int_2^4\chi(t)\,dt=1$, $\chi\le 1$ and $|\chi'|\le 1$. We multiply the inequality (III.81) by $\chi(t)$ and we integrate between 2 and 4 this gives, after observing that the

[12] Observe that $\vec{\Phi}_k(\partial\mathrm{B}_{r_c}(x))\subset\mathrm{B}^4_{t\,s}(\vec{q}_0)$ so there is no contribution from $\vec{\Phi}_k(\partial\mathrm{B}_{r_c}(x))$ outside $\mathrm{B}^4_{t\,s}(\vec{q}_0)$.

first term in the r.h.s. of (III.81) is non negative,[13]

$$
\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^2}\int_2^4 \chi(t)\,dt\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{ts}(\vec{q}_0))} dvol_{g_{\vec{\Phi}}}\right]\\
&\geq -\frac{1}{2\,r^3}\int_2^4 \chi(t)\,dt\int_{B_{r_c}(x)\cap\vec{\Phi}^{-1}(B^4_r(\vec{q}_1)\setminus B^4_{ts}(\vec{q}_0))} (\vec{\Phi}-\vec{q}_1)\cdot d^{*_g}d\vec{\Phi}\,dvol_{g_{\vec{\Phi}}}\\
&\quad-\frac{1}{r^3}\int_2^4 \chi(t)\,dt\int_{\mathbf{R}^4}\langle\vec{q}-\vec{q}_1,\vec{\nu}\rangle\,d\mathcal{H}^1\llcorner\left[\vec{\Phi}\big(B_{r_c}(x)\big)\cap B^4_r(\vec{q}_1)\cap\partial B^4_{ts}(\vec{q}_0)\right].
\end{aligned}
\tag{III.82}
$$

By substituting $d^{*_g}d\vec{\Phi}$ with it's expression deduced from (II.36), exactly as in the proof of the monotonicity formula (III.4) and as in the proof of Lemma III.3 the new terms involving σ coming from the boundaries $\partial B^4_{ts}(\vec{q}_0)$ in the first integral of the r-h-s of (III.82) tend to zero as k tends to infinity since the distance between the center $\vec{q}_1$ and this boundary is bounded from below by $R\,s>0$ independently of σ. So it remains then to estimate the last term in (III.82). This is done as follows

$$
\begin{aligned}
&\left|\frac{1}{r^3}\int_2^4 \chi(t)\,dt\int_{\mathbf{R}^4}\langle\vec{q}-\vec{q}_1,\vec{\nu}\rangle\,d\mathcal{H}^1\llcorner\left[B_{r_c}(x)\cap\vec{\Phi}^{-1}\big(B^4_r(\vec{q}_1)\big)\cap\vec{\Phi}^{-1}\big(\partial B^4_{ts}(\vec{q}_0)\big)\right]\right|\\
&\leq\frac{2\,|\vec{q}_1-\vec{q}_0|}{r^3}\int_2^4 dt\,\mathcal{H}^1\big(\partial B^4_{ts}(\vec{q}_0)\big)\\
&\leq\frac{2\,|\vec{q}_1-\vec{q}_0|}{r^3}\int_{\vec{\Phi}^{-1}(B^4_{4s}\setminus B^4_{2s})}\frac{|d|\vec{\Phi}-\vec{q}_0||_{g_{\vec{\Phi}}}}{s}\,dvol_{g_{\vec{\Phi}}}\leq C\,\frac{|\vec{q}_1-\vec{q}_0|}{r^3}\,s,
\end{aligned}
\tag{III.83}
$$

where we used successively the coarea formula for the function $|\vec{\Phi}-\vec{q}_0|/s$ and the monotonicity formula (III.4) in the last inequality. Observe that this term appears only for $r>\text{dist}(B^4_{4s}(\vec{q}_0),\vec{q}_1)>|\vec{q}_0-\vec{q}_1|/2$. Hence the integral with respect to r between σ and $1/2$ gives

$$
\begin{aligned}
&\left|\int_\sigma^{1/2}\frac{dr}{r^3}\int_2^4 \chi(t)\,dt\int_{\mathbf{R}^4}\langle\vec{q}-\vec{q}_1,\vec{\nu}\rangle\,d\mathcal{H}^1\llcorner\left[\vec{\Phi}(B_{r_c}(x))\cap B^4_r(\vec{q}_1)\cap\partial B^4_{ts}(\vec{q}_0)\right]\right|\\
&\quad\leq\frac{C}{|\vec{q}_1-\vec{q}_0|^2}\,|\vec{q}_1-\vec{q}_0|\,s\leq\frac{C}{R}.
\end{aligned}
\tag{III.84}
$$

The rest of the argument of the proof of Lemma III.3 carries through and we get that

$$
\nu_\infty\big(B_{r_c}(x)\big)\geq Q_0-C/R.
$$

[13] Indeed we are taking the derivative of an integral of a positive integrand over a bigger and bigger set.

Since we can take R as large as we want, we obtain (III.74). Hence ν_∞ restricted to $\mathcal{O}$ is equal to a finite sum of Dirac masses and this last step concludes the proof of Lemma III.5. □

We shall now prove the following lemma

Lemma **III.6.** — **[Absence of energy in the necks]** *Let $\vec{\Phi}_k$ satisfying the assumptions of Theorem III.1. Let $1 > \eta_k > 0$, $1 > \delta_k > 0$ and $x_k \in \Sigma$ satisfying*

$$\lim_{k\to+\infty} \log\frac{\eta_k}{\delta_k} = +\infty, \tag{III.85}$$

and such that

$$\lim_{k\to 0} \sup_{j\in\{1,\ldots,\log_2(\eta_k/\delta_k)\}} \nu_k\big(B_{2^{j+1}\delta_k}(x_k)\setminus B_{2^j\delta_k}(x_k)\big) = 0. \tag{III.86}$$

Then

$$\lim_{k\to 0} \nu_k\big(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k)\big) = 0. \tag{III.87}$$

Proof of Lemma III.6. — We argue by contradiction. If (III.87) does not hold we can then find a subsequence that we denote still $\vec{\Phi}_k$ such that

$$\lim_{k\to 0} \nu_k\big(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k)\big) = A > 0. \tag{III.88}$$

Let Q_0 be the universal constant in the Lemma III.3. We can assume without loss of generality that

$$A < Q_0. \tag{III.89}$$

Indeed, if this would not be the case we would replace δ_k by a larger number that we keep denoting δ_k and since (III.86) holds we necessarily have (III.85) for this new δ_k. We have for k large enough

$$\frac{\sigma_k^2\int_\Sigma[1+|\mathbf{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}]^p\, dvol_{g_{\vec{\Phi}_k}}}{\int_{B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k)} dvol_{g_{\vec{\Phi}_k}}} \le \frac{2\,\nu_\infty(\Sigma)}{A} f(\sigma_k). \tag{III.90}$$

Following the approach of step 5 of the proof of Lemma III.5, we first select 2 "good cuts" at the two ends of the annulus. So we choose respectively $\delta_{k,c}\in[\delta_k, 2\delta_k]$ and $\eta_{k,c}\in[\eta_k/2,\eta_k]$ such that we have respectively

$$s_k^2 := \left[\int_{\partial B_{\delta_{k,c}}(x_k)} |\nabla\vec{\Phi}_k|\, dl\right]^2 \le \pi\;\nu_k\big(B_{2\,\delta_k}(x_k)\setminus B_{\delta_k}(x_k)\big) \longrightarrow 0,$$

and

$$t_k^2 := \left[\int_{\partial B_{\eta_{k,c}}(x_k)} |\nabla\vec{\Phi}_k|\, dl\right]^2 \le \pi\, \nu_k\big(B_{2\,\eta_k}(x_k)\setminus B_{\eta_k}(x_k)\big) \longrightarrow 0.$$

Let $x_{1,k}\in\partial B_{\delta_{k,c}}(x_k)$ and $x_{2,k}\in\partial B_{\delta_{k,c}}(x_k)$ arbitrary. We have respectively

$$\textbf{(III.91)}\qquad \vec{\Phi}_k\big(\partial B_{\delta_{k,c}}(x_k)\big)\subset B^4_{s_k}\big(\vec{\Phi}_k(x_{1,k})\big) \quad\text{and}\quad \vec{\Phi}_k\big(\partial B_{\eta_{k,c}}(x_k)\big)\subset B^4_{t_k}\big(\vec{\Phi}_k(x_{2,k})\big).$$

Arguing as in the proof of the non collapsing lemma III.2, which is a corollary of the monotonicity formula, there exists $s>0$ fixed such that

$$\max_{\vec{q}\in\mathbf{R}^4}\mu_\infty\big(B^4_s(\vec{q})\big) < A/4.$$

We then have for k large enough

$$\mu_k\Big(\vec{\Phi}_k\big(B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k)\big)\setminus\Big(B^4_s\big(\vec{\Phi}_k(x_{1,k})\big)\cup B^4_s\big(\vec{\Phi}_k(x_{2,k})\big)\Big)\Big)\ge A/2.$$

As in the step 5 of the proof of Lemma III.5, we adopt the notations from the proof of Lemma III.3 and replacing Σ by the annulus $B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k)$, we can find $\vec{q}_k$ such that

$$\vec{q}_k\in\vec{\Phi}_k\big(B_{\eta_{k,c}}(x_k)\setminus B_{\delta_{k,c}}(x_k)\big)\setminus\big(E_{\pi/3}\cup G_\delta\cup B^4_s\big(\vec{\Phi}_k(x_{1,k})\big)\cup B^4_s\big(\vec{\Phi}_k(x_{2,k})\big).$$

We can carry over one by one the computation of the monotonicity formula centered at $\vec{q}$, controlling the boundary terms induced by the two cuts $\vec{\Phi}_k(\partial B_{\eta_{k,c}}(x_k))$ and $\vec{\Phi}_k(\partial B_{\eta_{k,c}}(x_k))$ which stay at a distance bounded from bellow with respect to $\vec{q}_k$, following the approach of the end of the step 5 of the proof of Lemma III.5. It is here even simpler since the lengths of the cuts s_k and t_k shrink to zero in the present case. Hence we obtain

$$A=\lim_{k\to 0}\nu_k\big(B_{\eta_k}(x_k)\setminus B_{\delta_k}(x_k)\big)\ge Q_0,$$

which contradicts (III.89). This concludes the proof of Lemma III.6. □

Defining the bubble tree Because of the previous quantization property, together with the no-neck energy property, following a classical combinatorics argument (in the style of Proposition III.1 in [3]—see also [27]), after extracting an ad-hoc subsequence, one can construct a family of sequences of smooth conformal injections $(\psi_k^i)_{i=1,\dots,L}$ from $S^i\setminus\bigcup_{j=1}^{n_i}B_\varepsilon(a_j^i)$ (for any ε for k large enough) into $(\Sigma, g_{\vec{\Phi}_k})$, equipped with a strongly converging constant curvature metric h_k^i, in such a way that

$$\nu_k^j := dvol_{g_{\vec{\Phi}_k\circ\psi_k^j}} \rightharpoonup \nu_\infty^j = m^j\, dvol_{h_\infty^j}\quad\text{as Radon measures on } S^i\setminus\bigcup_{j=1}^{n_i}B_\varepsilon\big(a_j^i\big)$$

for any ε and

$$\sum_{i=1}^{L} \nu_\infty^i(S^i) = \mu_\infty(S^3).$$

In the case for instance when the conformal class of $(\Sigma, g_{\vec{\Phi}_k})$ is controlled, the first bubble is given by Σ itself and the others are S^2. Except for the next lemma where we are working in the junction regions between several bubbles, the so called *neck regions*, we shall be working on a single bubble that we shall generically denote Σ.

Lemma **III.7.** — **[Construction of an approximating sequence]** *Assume the hypothesis of Theorem III.1 are fulfilled and that we have extracted subsequences such that* $\vec{\Phi}_k$ *converges weakly towards* $\vec{\Phi}_\infty$ *in* $W^{1,2}(\Sigma)$ *and* ν_k *converges towards* ν_∞ *satisfying (III.61) where* $\mathcal{B} := \{a_1, \ldots, a_l\}$ *the blow-up set. Let* ϕ *be a function in* $C_0^\infty(B_1^2(0))$ *satisfying* $\int_{B_1^2(0)} \phi(x)\, dx^2 = 1$ *and denote* $\phi_t(x) := t^{-2}\phi(x/t)$*. Then, modulo extraction of a subsequence, the family of smooth maps* $\phi_r \star \vec{\Phi}_\infty$*, converging* <u>*strongly*</u> *in* $W^{1,2}_{loc}(\Sigma \setminus \mathcal{B})$ *to* $\vec{\Phi}_\infty$ *as r goes to zero, satisfies*

$$\text{(III.92)} \qquad \lim_{r\to 0} \limsup_{k\to+\infty} \int_{\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)} |\vec{\Phi}_k - \phi_r \star \vec{\Phi}_\infty|\, dvol_{g_{\Phi_k}} = 0.$$

Proof of Lemma III.7. — Let $\varepsilon > 0$. Let $x \in \Sigma \setminus \cup_{l=1}^n B_\varepsilon(a_l)$ arbitrary and $r > 0$ such that there exists $k_{x,r}$ such that

$$\text{(III.93)} \qquad \forall k \geq k_{x,r} \quad \int_{B_{4r}(x)} |\nabla \vec{\Phi}_k|^2\, dx^2 < \varepsilon.$$

As before, we use Fubini and the mean value theorem to extract a slice $r_k \in (r, 2r)$ such that

$$\left\| \vec{\Phi}_\infty(x) - \vec{\Phi}_\infty(y) \right\|^2_{L^\infty(\partial B_{r_k}(x))^2} \leq C \int_{B_{2\rho}} |\nabla \vec{\Phi}_\infty|^2\, dx^2 \leq \varepsilon$$

$$\text{and} \quad \int_{\partial B_{r_k}(x)} |\nabla \vec{\Phi}_k|^2 \leq \frac{C}{r} \int_{B_{2r}(x)} |\nabla \vec{\Phi}_k|^2\, dx^2 < \frac{C\,\varepsilon}{r}$$

$$\left\| \vec{\Phi}_\infty(x) - \vec{\Phi}^\rho_\infty(x) \right\|_{L^\infty(\partial B_{r_k}(x))} \leq \varepsilon \quad \text{where } \vec{\Phi}^\rho_\infty(x) := \frac{1}{|B_{2r}|} \int_{B_{2r}(x)} \vec{\Phi}_\infty\, dx^2.$$

Because of the weak $W^{1,2}$ convergence of $\vec{\Phi}_k$ towards $\vec{\Phi}_\infty$, and because of the uniform $W^{1,2}$-bound on $\partial B_{r_k}(x)$ of $\vec{\Phi}_k(r_k, \theta)$, by *Rellich Kondrachov* compact embedding theorem, $\vec{\Phi}_k(r_k, \theta) - \vec{\Phi}_\infty(r_k, \theta)$ converges to zero in L^∞ norm. We then choose $k_{x,r}$ such that

$$\forall k \geq k_{x,r} \quad \|\vec{\Phi}_k - \vec{\Phi}_\infty\|_{L^\infty(\partial B_{r_k}(x))} \leq \sqrt{\varepsilon}.$$

Denote $\Sigma_k^r(x) := B_{r_k}(x) \setminus \vec{\Phi}_k^{-1}(B^4_{R\sqrt{\varepsilon}}(\vec{\Phi}^r_\infty(x)))$ and assume that

$$\frac{\sigma_k^2 \int_{\Sigma_k^r(x)} (1+|\mathbf{I}_{\vec{\Phi}_k}|^2)^p \, dvol_{g_{\vec{\Phi}_k}}}{\int_{\Sigma_k^r(x)} dvol_{g_{\vec{\Phi}_k}}} \leq \frac{1}{\log \sigma_k^{-1}}.$$

Again we can then argue word by word as in the proof of Lemma III.3 for the surface $\Sigma_k^r(x)$ until (III.47) in order to find a point $\vec{q}$ in $\vec{\Phi}_k(\Sigma_k^r(x)) \setminus (E_{\pi/3} \cup G_\delta)$. Once we have this point we perform the rest of the argument of Lemma III.3 but for the surface with boundary $\vec{\Phi}_k(B_{r_k}(x_k) \setminus \vec{\Phi}_k^{-1}(B^4_{R\varepsilon}(\vec{\Phi}^r_\infty(x))))$. The boundary is going to generate a new term in the monotonicity formula

$$-\frac{1}{r^3} \int_{\mathbf{R}^4} \langle \vec{q} - \vec{q}_1, \vec{\nu} \rangle \, d\mathcal{H}^1 \llcorner \left[\vec{\Phi}_k\big(B_{r_k}(x)\big) \cap \partial B^4_{t\varepsilon}(\vec{q})\right],$$

for $t \in [2, 4]$ that we treat exactly as in (III.83) in order to get that for k large enough $\int_{B_{r_k}(x_k)} |\nabla \vec{\Phi}_k|^2 \, dx^2 \geq Q_0 - C/R$ which is a contradiction for R large enough. Hence we have

$$\text{(III.94)} \qquad \frac{\sigma_k^2 \int_{\Sigma_k^\rho(x)} (1+|\mathbf{I}_{\vec{\Phi}_k}|^2)^p \, dvol_{g_{\vec{\Phi}_k}}}{\int_{\Sigma_k^r(x)} dvol_{g_{\vec{\Phi}_k}}} > \frac{1}{\log \sigma_k^{-1}},$$

and then

$$\text{(III.95)} \qquad \begin{aligned} &\int_{B_{r_k}(x)} |\vec{\Phi}_k(y) - \vec{\Phi}^\rho_\infty(x)| \, |\nabla \vec{\Phi}_k|^2(y) \, dy^2 \\ &\quad \leq R\sqrt{\varepsilon} \int_{B_{r_k}(x)} |\nabla \vec{\Phi}_k|^2(y) \, dy^2 + C \log \sigma_k^{-1} \, \sigma_k^2 \int_{B_{r_k}(x)} \left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}}. \end{aligned}$$

Let ϕ be a function in $C^\infty_0(B^2_1(0))$ satisfying $\int_{B^2_1(0)} \phi(x) \, dx^2 = 1$ and denote $\phi_t(x) := t^{-2}\phi(x/t)$ we have for all $y \in B_r(x)$

$$\begin{aligned} &\phi_r \star \vec{\Phi}_\infty(y) - \vec{\Phi}^r_\infty(x) \\ &\quad = \int_{z \in B_{2r}(y)} \phi_r(y-z) \, \vec{\Phi}_\infty(z) \, dz^2 - \int_{z \in B_{2r}(y)} \phi_r(y-z) \, \vec{\Phi}^r_\infty(x) \, dz^2. \end{aligned}$$

Hence

$$\begin{aligned} &\left|\phi_r \star \vec{\Phi}_\infty(y) - \vec{\Phi}^r_\infty(x)\right| \\ &\quad \leq \frac{C}{r^4} \int_{z \in B_{2r}(y)} \int_{v \in B_{2r}(x)} \left|\phi\left(\frac{y-z}{r}\right)\right| \left|\vec{\Phi}_\infty(z) - \vec{\Phi}_\infty(v)\right| dz^2 \, dv^2 \\ &\quad \leq \frac{C}{r^4} \int_{z \in B_{4r}(x)} \int_{v \in B_{4r}(x)} |\vec{\Phi}_\infty(z) - \vec{\Phi}_\infty(v)| \, dz^2 \, dv^2. \end{aligned}$$

Thus, using Poincaré inequality on $B_{4r}(x)$

(**III.96**)
$$\forall x\in\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)\quad \left\|\phi_r\star\vec{\Phi}_\infty(y)-\vec{\Phi}^r_\infty(x)\right\|^2_{L^\infty(B_r(x))}\le C\int_{B_{4r}(x)}|\nabla\vec{\Phi}_\infty|^2(y)\,dy^2 .$$

Let r such that

$$\sup_{x\in\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)}\nu_\infty\big(B_{4r}(x)\big)\le\varepsilon/2 .$$

One takes a finite covering $(B_r(x_i))_{i\in I}$ of $\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)$ by balls of fixed radius r such that each point is covered by at most a universal number $\mathfrak{N}$ of balls of size $2r$. Summing (III.95) gives for k large enough

$$\begin{aligned}\sum_{i\in I}\int_{B_r(x_i)}&\left|\vec{\Phi}_k(y)-\vec{\Phi}^r_\infty(x_i)\right|\,|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\le R\,\mathfrak{N}\sqrt{\varepsilon}\int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)}|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\quad+C\,\mathfrak{N}\,\log\sigma_k^{-1}\,\sigma_k^2\int_\Sigma\left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}} .\end{aligned}$$

Combining this inequality with (III.96) gives then

(**III.97**)
$$\begin{aligned}\int_{\Sigma\setminus\cup_{l=1}^n B_{2\varepsilon_0}(a_l)}&|\vec{\Phi}_k-\phi_r\star\vec{\Phi}_\infty|(y)\,|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\le C\sqrt{\varepsilon}\int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)}|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\quad+C\,\mathfrak{N}\log\sigma_k^{-1}\,\sigma_k^2\int_{\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)}\left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}} .\end{aligned}$$

This concludes the proof of the lemma. □

Lemma **III.8. — [Rectifiability of the limit]** *Let $\vec{\Phi}_k$ satisfying the assumptions of Theorem III.1. Then the limiting measure μ_∞ is supported by a rectifiable 2-dimensional subset* K *of* S^3 *given by the image of the different bubbles by the* $W^{1,2}$ *map* $\vec{\Phi}_\infty$. *Precisely there exists a uniformly bounded $\mathcal{H}^2$ measurable function θ on* K *such that*

(**III.98**)
$$\mu_\infty=\theta\,d\mathcal{H}^2\llcorner K .$$

Moreover if we decompose $\mu_\infty = \sum_{i=1}^{L} \mu_\infty^i$ where each μ_∞^i is the limiting measure produced by one bubble we have for each bubble

$$\mu_\infty^i(\phi) = \int_\Sigma \phi(\vec{\Phi}_\infty)\, d\nu_\infty^i = \int_\Sigma \phi(\vec{\Phi}_\infty) m^i(x)\, dx^2, \tag{III.99}$$

where $\nu_\infty^i = m^i\, d\mathcal{L}^2$.

Proof of Lemma III.8. — We first prove (III.99). Let $\varepsilon > 0$. Using (III.97) we have the existence of r such that, for k large enough

$$\begin{aligned}\int_{\Sigma\setminus\cup_{l=1}^n B_{2\varepsilon_0}(a_l)} &|\vec{\Phi}_k - \phi_r \star \vec{\Phi}_\infty|(y)\, |\nabla\vec{\Phi}_k|^2(y)\, dy^2 \\ &\le C\sqrt{\varepsilon} \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} |\nabla\vec{\Phi}_k|^2(y)\, dy^2 \\ &\quad + C\mathfrak{N} \log\sigma_k^{-1}\, \sigma_k^2 \int_{\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)} \left(1 + |\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}}.\end{aligned} \tag{III.100}$$

Let $\varphi \in C^1(\mathbf{R}^4)$ we have

$$\begin{aligned}\mu_k^1(\varphi) = \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\vec{\Phi}_k)\, dvol_{g_{\vec{\Phi}_k}} &= \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\phi_r \star \vec{\Phi}_\infty)\, dvol_{g_{\vec{\Phi}_k}} \\ &+ \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\vec{\Phi}_k) - \varphi(\phi_r \star \vec{\Phi}_\infty)\, dvol_{g_{\vec{\Phi}_k}},\end{aligned} \tag{III.101}$$

where μ_k^1 is the measure issued from $\vec{\Phi}_k$ restricted to $\Sigma \setminus \mathcal{B}$. We have in one hand by the convergence of Radon measures

$$\begin{aligned}\lim_{k\to+\infty} &\int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\phi_r \star \vec{\Phi}_\infty)\, dvol_{g_{\vec{\Phi}_k}} \\ &= \nu_\infty(\varphi(\phi_r \star \vec{\Phi}_\infty)) = \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\phi_r \star \vec{\Phi}_\infty)\, m^1(x)\, dx^2,\end{aligned} \tag{III.102}$$

and in the other hand we have

$$\begin{aligned}&\left| \int_{\Sigma\setminus\cup_{l=1}^n B_{\varepsilon_0}(a_l)} \varphi(\vec{\Phi}_k) - \varphi(\phi_r \star \vec{\Phi}_\infty)\, dvol_{g_{\vec{\Phi}_k}} \right| \\ &\quad \le \|\nabla\varphi\|_\infty \int_{\Sigma\setminus\cup_{l=1}^n B_{2\varepsilon_0}(a_l)} |\vec{\Phi}_k - \phi_r \star \vec{\Phi}_\infty|(y)\, |\nabla\vec{\Phi}_k|^2(y)\, dy^2.\end{aligned} \tag{III.103}$$

Combining (III.100)–(III.103) we obtain

$$\limsup_{k\to+\infty} \left|\mu_k^1(\varphi) - \nu_\infty\big(\varphi(\phi_r \star \vec{\Phi}_\infty)\big)\right| \le C_\varphi\, \varepsilon. \tag{III.104}$$

By taking ε smaller and smaller as well as ρ gets smaller and smaller we obtain (III.99). It remains to prove (III.98). Because of the monotonicity formula μ^∞ vanishes on any measurable set of $\mathcal{H}^2$ measure zero in S^3. Using the quantitative Lusin type property for Sobolev maps of F.C. Liu (see [21]) we deduce that for any $\alpha > 0$ there exists a C^1 map $\vec{\Xi}^\alpha$ from Σ into[14] S^3 and an open subset B^α of Σ such that

$$\textbf{(III.105)} \qquad \begin{cases} \mathcal{H}^2(B^\alpha) \le \alpha, \\ \vec{\Phi}_\infty = \vec{\Xi}^\alpha \quad \text{on } \Sigma \setminus B^\alpha \quad \text{and} \quad d\vec{\Phi}_\infty = d\vec{\Xi}^\alpha \quad \text{on } \Sigma \setminus B^\alpha, \\ \|\vec{\Phi}_\infty - \vec{\Xi}^\alpha\|^2_{W^{1,2}(\Sigma)} \le \alpha. \end{cases}$$

The identity (III.99) implies then

$$\mu^i_\infty(\varphi) = \int_{\Sigma\setminus B^\alpha} \varphi\big(\vec{\Xi}^\alpha\big)\, d\nu^i_\infty + \int \varphi(\vec{\Phi}_\infty)\, d\nu^i_\infty \llcorner B^\alpha.$$

Since $\vec{\Xi}^\alpha$ is C^1 on Σ the measurable set $K^\alpha := \vec{\Xi}^\alpha(\Sigma)$ is 2 rectifiable and there exists a measure τ^α supported on K^α such that

$$\mu^i_\infty(\varphi) = \int_{K^\alpha} \varphi(\vec{q})\, d\tau^\alpha(\vec{q}) + \int \varphi(\vec{\Phi}_\infty)\, d\nu^i_\infty \llcorner B^\alpha.$$

Observe that since ν^i_∞ is absolutely continuous with respect to the Lebesgue measure on Σ we have

$$\lim_{\alpha\to 0} \sup_{|E|<\alpha} \nu^i_\infty(E) = 0.$$

Hence, by taking $K := \cup_{n\in\mathbf{N}^*} K^{1/n}$, there exists a measure τ on K such that $\mu_\infty := \tau \llcorner K$. Because of the monotonicity formula μ^∞ vanishes on any measurable set of $\mathcal{H}^2$ measure zero in S^3 and hence τ is absolutely continuous with respect to $d\mathcal{H}^2 \llcorner K$ and there exist an $\mathcal{H}^2$ measurable function θ on K such that (III.98) holds and this concludes the proof of Lemma III.8. □

Lemma **III.9. — [Vanishing of the limiting measure on the degenerating set]** *Let $\mathfrak{L}_{\nabla\vec{\Phi}_\infty}$ be the subset of $\Sigma \setminus \mathcal{B}$ of Lebesgue points for $\nabla\vec{\Phi}_\infty$. We denote by $\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}$ the measurable subset of $\mathfrak{L}_{\nabla\vec{\Phi}_\infty}$ of points where the Lebesgue representative of $\nabla\vec{\Phi}_\infty$ has rank strictly less than 2. Then we have*

$$\textbf{(III.106)} \qquad \nu_\infty\big(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}\big) = 0.$$

[14] The fact that we can apply Liu's result for maps into $W^{1,2}(\Sigma, S^3)$ comes from the fact that smooth maps in $C^1(\Sigma, S^3)$ are dense in $W^{1,2}(\Sigma, S^3)$ for the $W^{1,2}$-topology.

Proof of Lemma III.9. — Let K be a compact subset of $\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}$ such that

$$\nu_\infty(\mathrm{K}) \geq 2^{-1}\nu_\infty\big(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}\big).$$

Let $\alpha > 0$ and consider B^α and $\vec{\Xi}^\alpha$ satisfying (III.105). We choose α small enough in such a way that

$$\nu_\infty\big(\mathrm{K}\setminus\mathrm{B}^\alpha\big) \geq 2^{-1}\nu_\infty(\mathrm{K}).$$

Since $\int_{\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}} |\partial_{x_1}\vec{\Phi}_\infty \times \partial_{x_2}\vec{\Phi}_\infty|\, dx^2 = 0$ and since $\nabla\vec{\Phi}_\infty = \nabla\vec{\Xi}^\alpha$ on $\Sigma\setminus\mathrm{B}^\alpha$ we have that $\mathcal{H}^2(\vec{\Xi}^\alpha(\mathrm{K}\setminus\mathrm{B}^\alpha)) = 0$. Observe that $\Omega^\alpha := \vec{\Xi}^\alpha(\mathrm{K}\setminus\mathrm{B}^\alpha)$ is compact in $\mathbf{R}^4$. Let $\mathrm{B}_{\rho_i}(\vec{q}_i)$ be a finite covering of Ω^α such that $\sum_{i\in\mathrm{I}}\rho_i^2 \leq \alpha$. Let φ^α be a C^1 non negative function in $\mathbf{R}^4$, identically equal to one on Ω^α, less than one and supported in $\cup_{i\in\mathrm{I}}\mathrm{B}_{\rho_i}(\vec{q}_i)$. Because of the monotonicity formula we have

$$\text{(III.107)} \qquad \int_{\mathbf{R}^4} \varphi^\alpha(\vec{q})\, d\mu^\infty(\vec{q}) \leq \mathrm{C}\,\alpha.$$

The formula (III.99) and the fact that $\varphi^\alpha(\vec{\Xi}^\alpha)$ is identically equal to one on $\mathrm{K}\setminus\mathrm{B}^\alpha$ gives

$$\nu_\infty\big(\mathrm{K}\setminus\mathrm{B}^\alpha\big) \leq \int_{\mathbf{R}^4} \varphi^\alpha(\vec{q})\, d\mu^\infty(\vec{q}),$$

hence we obtain that $\nu_\infty(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}) \leq 4\,\mathrm{C}\,\alpha$ for any α and this concludes the proof of Lemma III.9. □

Lemma **III.10.** — **[Convergence to an integer rectifiable varifold]** *Under the assumptions of Theorem III.1, we have that one we can extract a subsequence such that the integer varifold* $\mathbf{v}_k$ *associated to the current* $(\vec{\Phi}_k)_*[\Sigma]$ *converges to an integer rectifiable varifold supported by a finite union of the images by* $\mathrm{W}^{1,2}$*-maps of surfaces. More precisely we have that on each bubble there exists a function* $\mathrm{N}^i \in \mathrm{L}^\infty(\mathrm{S}^i, \mathbf{N})$ *such that*

$$\text{(III.108)} \qquad \nu^i_\infty = \mathrm{N}^i\, |\partial_{x_1}\vec{\Phi}_\infty \times \partial_{x_2}\vec{\Phi}_\infty|\, dx^2.$$

Proof of Lemma III.10. — Since we have proved that the necks contain no energy at the limit, it suffices to prove the convergence for $\vec{\Phi}_k$ restricted to $\Sigma\setminus\cup_{l=1}^n\mathrm{B}_\varepsilon(a_l)$. We denote by $\mathbf{v}_{\varepsilon,k}$ the integer varifold associated to the current $(\vec{\Phi}_k)_*[\Sigma\setminus\cup_{l=1}^n\mathrm{B}_\varepsilon(a_l)]$.

The proof of Lemma III.10 is a bit long and is therefore decomposed into two main parts. In the first part we establish the varifold convergence of $\mathbf{v}_{\varepsilon,k}$ towards a limiting varifold $\mathbf{v}_{\varepsilon,\infty}$ which is—as a Radon measure on the Grassman bundle of TN^n—absolutely continuous with respect to $(\vec{\Phi}_\infty)_*\delta_{\mathrm{T}\Sigma\setminus\cup_{l=1}^n\mathrm{B}_\varepsilon(a_l)}$. The second step consists in proving the integrality of $\mathbf{v}_{\varepsilon,\infty}$.

Step 1: **The convergence of $\mathbf{v}_{\varepsilon,k}$ towards $\mathbf{v}_{\varepsilon,\infty} \ll (\vec{\Phi}_\infty)_*\delta_{T\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)}$.**

We fix $\alpha > 0$ and we consider the map $\vec{\Xi}^\alpha$ and the open set B^α given by (III.105). We choose a Lebesgue point x for $\nabla\vec{\Phi}_\infty$ in $\Sigma \setminus B^\alpha$ such that

$$\textbf{(III.109)} \qquad \lim_{r\to 0} \frac{|B_r(x) \setminus B^\alpha|}{|B_r(x)|} = 1.$$

We assume that x is not in the vanishing set Σ_0.We also assume that x is not in the degenerating set $\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}$. These restrictions have no consequences since we have respectively $\nu_\infty \ll \mathcal{L}^2$, $\nu_\infty(\Sigma_0) = 0$ and $\nu_\infty(\mathfrak{L}^0_{\nabla\vec{\Phi}_\infty}) = 0$. Such a point is a Lebesgue point for x and one has

$$\textbf{(III.110)} \qquad \lim_{r\to 0} \phi_r \star \vec{\Phi}_\infty(x) = \vec{\Xi}^\alpha(x) = \vec{\Phi}_\infty(x).$$

Without loss of generality, modulo the action of rotations, we assume that $\vec{\Xi}^\alpha(x) = \vec{\Phi}_\infty(x) = (0,0,1,0)$, that $\partial_{x_1}\vec{\Xi}^\alpha(x) = \partial_{x_1}\vec{\Phi}_\infty(x) = (a,0,0,0)$ and $\partial_{x_2}\vec{\Xi}^\alpha(x) = \partial_{x_2}\vec{\Phi}_\infty(x) = (b,c,0,0)$. We have $a\,c \neq 0$ since $\nabla\vec{\Phi}_\infty$ has rank 2. Moreover the approximate tangent plane at $\vec{\Phi}_\infty(x)$ coincides with $\text{Span}\{(1,0,0,0),(0,1,0,0)\}$. Observe that the existence of this approximate tangent plane and the fact that $\vec{\Xi}^\alpha(x)$ is a regular point for $\vec{\Xi}^\alpha$ forces $\text{Span}\{\partial_{x_1}\vec{\Xi}^\alpha, \partial_{x_2}\vec{\Xi}^\alpha\} = \{(1,0,0,0),(0,1,0,0)\}$ at any point in $(\vec{\Xi}^\alpha)^{-1}(\vec{\Xi}^\alpha(x))$.

We recall that we adopt the notation $\vec{\Phi} = (\Phi^1,\Phi^2,\Phi^3,\Phi^4)$. We first have for the third coordinate

$$\textbf{(III.111)} \qquad \begin{aligned} \int_{B_r(x)} \left|\nabla\Phi^3_k\right|^2 dy^2 &= \int_{B_r(x)} \left|\Phi^3_k\,\nabla\Phi^3_k\right|^2 dy^2 + \int_{B_r(x)} \left(1 - \left|\Phi^3_k\right|^2\right) \left|\nabla\Phi^3_k\right|^2 dy^2 \\ &= \int_{B_r(x)} \left|\Phi^3_k\,\nabla\Phi^3_k\right|^2 dy^2 + \int_{B_r(x)} \left(\left|\Phi^3_\infty(x)\right|^2 - \left|\Phi^3_k\right|^2\right) \left|\nabla\Phi^3_k\right|^2 dy^2. \end{aligned}$$

We have $\Phi^3_k\,\nabla\Phi^3_k = -\Phi^1_k\,\nabla\Phi^1_k - \Phi^2_k\,\nabla\Phi^2_k - \Phi^4_k\,\nabla\Phi^4_k$ and since also for any $i = 1,\ldots,4$ we have

$$\textbf{(III.112)} \qquad \left|\nabla\Phi^i_k\right|^2 dy_1 \wedge dy_2 \leq 2\, dvol_{g_{\vec{\Phi}_k}},$$

and keeping in mind also $|\Phi^i_k| \leq 1$, we deduce that (III.111) gives

$$\begin{aligned} \int_{B_r(x)} \left|\nabla\Phi^3_k\right|^2 dy^2 \leq 2 \int_{B_r(x)} &\left[\left|\Phi^1_k(y)\right|^2 + \left|\Phi^2_k(y)\right|^2 + \left|\Phi^4_k(y)\right|^2\right] dvol_{g_{\vec{\Phi}_k}} \\ &+ 4 \int_{B_r(x)} \left|\Phi^3_k(y) - \Phi^3_\infty(x)\right| dvol_{g_{\vec{\Phi}_k}}. \end{aligned}$$

Since $\Phi^i_\infty(x)=0$ for $i\neq 3$ we have then

(III.113) $$\int_{B_r(x)}\left|\nabla\Phi^3_k\right|^2\,dx^2\le 10\int_{B_r(x)}\left|\vec{\Phi}_k-\vec{\Phi}_\infty(x)\right|\,dvol_{g_{\vec{\Phi}_k}}.$$

We have

(III.114) $$\begin{aligned}&\int_{B_r(x)}\left|\vec{\Phi}_k-\vec{\Phi}_\infty(x)\right|\,dvol_{g_{\vec{\Phi}_k}}\\&\quad\le\int_{B_r(x)}\left|\vec{\Phi}_k-\phi_r\star\vec{\Phi}_\infty(x)\right|\,dvol_{g_{\vec{\Phi}_k}}+\left|\vec{\Phi}_\infty(x)-\phi_r\star\vec{\Phi}_\infty(x)\right|\,\nu_k\big(B_r(x)\big).\end{aligned}$$

For any $\varepsilon>0$, for r small enough, using (III.95) and (III.96) we have the existence of a radius $r_k\in(\rho/2,\rho)$ such that

(III.115) $$\begin{aligned}&\int_{B_{r_k}(x)}\left|\vec{\Phi}_k(y)-\phi_r\star\vec{\Phi}_\infty(y)\right|\,|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\&\quad\le C\sqrt{\varepsilon}\int_{B_{r_k}(x)}|\nabla\vec{\Phi}_k|^2(y)\,dy^2+C\,\log\sigma_k^{-1}\,\sigma_k^2\int_{B_{r_k}(x)}\left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p\,dvol_{g_{\vec{\Phi}_k}}.\end{aligned}$$

Since we are at a point which does not belong to the vanishing set we obtain, modulo extraction of a subsequence

(III.116) $$\lim_{r\to 0}\limsup_{k\to+\infty}\frac{\int_{B_r(x)}|\vec{\Phi}_k(y)-\vec{\Phi}_\infty(x)|\,|\nabla\vec{\Phi}_k|^2(y)\,dy^2}{\int_{B_r(x)}|\nabla\vec{\Phi}_k|^2(y)\,dy^2}=0.$$

Combining (III.113) with (III.116) we obtain

(III.117) $$\lim_{r\to 0}\limsup_{k\to+\infty}\frac{\int_{B_r(x)}|\nabla\Phi^3_k|^2\,dx^2}{\int_{B_r(x)}dvol_{g_{\vec{\Phi}_k}}}=0.$$

Since x is a Lebesgue point for $\nabla\vec{\Phi}_\infty$ one has

$$\int_{B_r(x)}\left|\nabla\vec{\Phi}(y)-\nabla\vec{\Xi}^\alpha(x)\right|^2=o\big(r^2\big).$$

Then, using Fubini theorem together with the mean value theorem, for any $r>0$ and for each k one can find a "good slice" $r_k(r)\in[2r,4r]$ such that

$$\text{(III.118)}\qquad \left\{\begin{array}{l}\displaystyle\int_0^{2\pi}\left|\partial_\theta\left(\vec{\Phi}_\infty\left(r_k(r),\theta\right)-r_k(r)\,\cos\theta\,\partial_{x_1}\vec{\Xi}^\alpha(x)-r_k(r)\,\sin\theta\,\partial_{x_2}\vec{\Xi}^\alpha(x)\right)\right|\,d\theta\\[2mm] \quad=o(r),\\[3mm] \mathcal{H}^1\left(\partial B_{r_k(r)}(x)\cap B^\alpha\right)=o(r),\\[3mm] \displaystyle\int_{\partial B_{r_k(r)}(x)}|\nabla\vec{\Phi}_k|^2\,dl_{\partial B_{r_k(r)}}\le\frac{2}{r}\int_{B_{4r}(x)}|\nabla\vec{\Phi}_k|^2\,dx^2.\end{array}\right.$$

Since

$$\text{(III.119)}\qquad \begin{array}{l}\left\|\vec{\Xi}^\alpha\left(r_k(r),\theta\right)-\vec{\Xi}^\alpha(x)-r_k(r)\,\cos\theta\,\partial_{x_1}\vec{\Xi}^\alpha(x)-r_k(r)\,\sin\theta\,\partial_{x_2}\vec{\Xi}^\alpha(x)\right\|_{L^\infty([0,2\pi])}\\[2mm] \quad=o(r),\end{array}$$

from (III.118) and (III.119) we deduce

$$\text{(III.120)}\qquad \begin{array}{l}\left\|\vec{\Phi}_\infty\left(r_k(r),\theta\right)-\vec{\Xi}^\alpha(x)-r_k(r)\,\cos\theta\,\partial_{x_1}\vec{\Xi}^\alpha(x)-r_k(r)\,\sin\theta\,\partial_{x_2}\vec{\Xi}^\alpha(x)\right\|_{L^\infty([0,2\pi])}\\[2mm] \quad=o(r).\end{array}$$

Moreover since $\vec{\Phi}_k(r_k,\theta)\rightharpoonup\vec{\Phi}_\infty(r_k,\theta)$ weakly in $H^{1/2}([0,2\pi])$ because of the last condition of (III.118), there exists $k_{x,r}\in\mathbf{N}$ such that

$$\text{(III.121)}\qquad \begin{array}{l}\forall k\ge k_{x,r}\quad\left\|\vec{\Phi}_k\left(r_k(r),\theta\right)-\vec{\Xi}^\alpha(x)-r_k(r)\,\cos\theta\,\partial_{x_1}\vec{\Xi}^\alpha(x)\right.\\[2mm] \qquad\qquad\left.-\,r_k(r)\,\sin\theta\,\partial_{x_2}\vec{\Xi}^\alpha(x)\right\|_{L^\infty([0,2\pi])}=o(r).\end{array}$$

Because of (III.121), there exists $k_{x,r}$ such that

$$\forall k\ge k_{x,r}\quad\vec{\Phi}_k\left(\partial B_{r_k(r)}(x)\right)\subset B^4_{3\,|\nabla\vec{\Xi}^\alpha(x)|\,r}\left(\vec{\Phi}_\infty(x)\right)\setminus B^4_{\gamma r}\left(\vec{\Phi}_\infty(x)\right),$$

where $\gamma:=\inf\{|\partial_{x_1}\vec{\Xi}^\alpha(x),\partial_{x_2}\vec{\Xi}^\alpha(x)\}$. For any $\tau>2\,|\nabla\vec{\Xi}^\alpha(x)|\,r$ we denote by $\omega_k(\tau)$ the component of $\vec{\Phi}_k^{-1}(B^4_\tau(\vec{\Phi}_\infty(x)))$ containing $\partial B_{r_k(r)}(x)$. Let

$$\Omega_k(\tau):=\omega_k(\tau)\cup B_{r_k(r)}(x).$$

Replacing r by $\gamma^{-1}r/4\,|\nabla\vec{\Xi}^\alpha(x)|$ the corresponding "good cut" at $r_k(\gamma^{-1}r/4\,|\nabla\vec{\Xi}^\alpha(x)|)$ is sent by $\vec{\Phi}_k$ outside $B^4_{4\,|\nabla\vec{\Xi}^\alpha(x)|\,r}(\vec{\Phi}_\infty(x))$ hence, since $\partial\Omega_k(\tau)\subset\vec{\Phi}_k^{-1}(\partial B^4_\tau(\vec{\Phi}_\infty))$

$$\text{(III.122)}\qquad \forall\,\tau\in\left[2\left|\nabla\vec{\Xi}^\alpha(x)\right|r,\,4\left|\nabla\vec{\Xi}^\alpha(x)\right|r\right]\quad\Omega_k(\tau)\subset B_{\gamma^{-1}r/2\,|\nabla\vec{\Xi}^\alpha(x)|}(x).$$

We denote

$$\Sigma_{k,r} := \Omega_k\big(4\,\big|\nabla\vec{\Xi}^\alpha(x)\big|\,r\big).$$

Let $\chi^\alpha_{r,x}$ be a smooth non negative function on $\mathbf{R}^4$ supported in the ball $B^4_{4\,|\nabla\vec{\Xi}^\alpha(x)|\,r}(\vec{\Phi}_\infty(x))$, identically equal to one on $B^4_{3\,|\nabla\vec{\Xi}^\alpha(x)|\,r}(\vec{\Phi}_\infty(x))$ and such that $\|d^l\chi^\alpha_{r,x}\|_{L^\infty(\mathbf{R}^4)} \le r^{-l}\,|\nabla\vec{\Xi}^\alpha(x)|^{-l}_\infty$ for $l = 0, 1, 2$. We have in particular for $j = 1, \dots, 4$

$$\textbf{(III.123)}\qquad \int_{B_{\gamma^{-1}r/2\,|\nabla\vec{\Xi}^\alpha(x)|}(x)} \big|\nabla\Phi^j_k\big|^2\,dx^2 \ge \int_{\Sigma_{k,r}} \chi^\alpha_{r,x}(\vec{\Phi}_k)\big|\nabla\Phi^j_k\big|^2\,dx^2 \ge \int_{B_r(x)} \big|\nabla\Phi^j_k\big|^2\,dx^2.$$

Multiplying the 4th coordinate of equation (II.36) by $\chi^\alpha_{r,x}(\vec{\Phi}_k)\Phi^4_k$ and integrating over Σ gives, arguing exactly as in the proof of Lemma III.1,

$$\textbf{(III.124)}\qquad \begin{aligned}\int_{\Sigma_{k,r}} \chi^\alpha_{r,x}(\vec{\Phi}_k)\big|\nabla\Phi^4_k\big|^2\,dx^2 &= \int_{\Sigma_{k,r}} \chi^\alpha_{r,x}(\vec{\Phi}_k)\big|\Phi^4_k\big|^2\,\big|\nabla\Phi^4_k\big|^2\,dx^2\\ &\quad - \int_{\Sigma_{k,r}} \Phi^4_k\,\nabla\big(\chi^\alpha_{r,x}(\vec{\Phi}_k)\big)\cdot\nabla\Phi^4_k + o_k(1).\end{aligned}$$

We shall now define a radius $s_r = \delta(r)\,r$ where $\delta(r) = o_r(1)$ in the following way. Using Poincaré inequality as for proving (III.96) we have

$$\textbf{(III.125)}\qquad \|\phi_{s_r}\star\vec{\Phi}_\infty - \vec{\Phi}^r_\infty\|^2_{L^\infty(\Sigma_{r,k})} \le \frac{C_x}{\delta^2_r}\int_{B_{\gamma^{-1}r/2\,|\nabla\vec{\Xi}^\alpha(x)|}(\vec{\Phi}_\infty)} |\nabla\vec{\Phi}_\infty|^2\,dx^2,$$

where C_x does not depend on r but on x only. Using the fact that, since $\vec{\Xi}^\alpha$ is C^1,

$$\textbf{(III.126)}\qquad \begin{aligned}& r^{-2}\int_{B_{\gamma^{-1}r/2\,|\nabla\vec{\Xi}^\alpha(x)|}(x)} \big|\nabla\vec{\Phi}_\infty - \nabla\vec{\Xi}^\alpha\big|^2\,dy^2 = \varepsilon(r)\\ &\quad\text{and}\quad \big|\nabla\Xi^{\alpha,4}\big|(x) = 0 \quad\Rightarrow\quad \big\|\nabla\Xi^{\alpha,4}\big\|_{L^\infty(B_r(x))} = o_r(1),\end{aligned}$$

where $\varepsilon(r) = o_r(1)$ by choosing $\delta^2(r) := \max\{\|\nabla\Xi^{\alpha,4}\|_{L^\infty(B_r(x))}, \varepsilon(r)^{1/2}\}$ we deduce from (III.125)

$$\textbf{(III.127)}\qquad \begin{aligned}&\left\|\phi_{s_r}\star\Phi^4_\infty - \frac{1}{B_r(x)}\int_{B_r(x)}\Phi^4_\infty\right\|^2_{L^\infty(\Sigma_{r,k})}\\ &\quad\le C_x\Big[\sqrt{\varepsilon(r)} + \|\nabla\Xi^{\alpha,4}\|_{L^\infty(B_r(x))}\Big]\,r^2 = o\big(r^2\big).\end{aligned}$$

On $B_{r_k}(x)$ we decompose $\vec{\Phi}_\infty - \vec{\Xi}^\alpha = v + \psi$ such that $\Delta v = 0$ in $B_{r_k}(x)$ and $\psi = 0$ on $\partial B_{r_k}(x)$. Because of (III.120) one has, using respectively the *maximum principle* and the

Dirichlet Principle,

$$\|v\|_{L^\infty(B_{r_k}(x))} = o(r) \quad \text{and}$$

(**III.128**)
$$\int_{B_{r_k}(x)} |\nabla\psi|^2 \le \int_{B_{r_k}(x)} \left|\nabla\vec{\Phi}_\infty - \nabla\vec{\Xi}^\alpha\right|^2 dy^2 = \varepsilon(r)\, r^2.$$

Sobolev–Poincaré inequality gives

$$\frac{1}{|B_{r_k}(x)|}\int_{B_{r_k}(x)} |\psi|^2 \le C \int_{B_{r_k}(x)} \left|\nabla\vec{\Phi}_\infty - \nabla\vec{\Xi}^\alpha\right|^2 dx^2.$$

Combining this last fact with (III.128) gives

$$\left|\frac{1}{|B_{r_k}(x)|}\int_{B_{r_k}(x)} \left[\vec{\Phi}_\infty(y) - \vec{\Xi}^\alpha(y)\right] dy^2\right|^2 = o\left(r^2\right).$$

This implies $\frac{1}{|B_{r_k}(x)|}\int_{B_{r_k}(x)} \vec{\Phi}^4_\infty(y) = o(r)$. Observe that similarly to the proof of (III.96) by the mean again of Poincaré inequality one has

$$\left|\frac{1}{|B_{r_k}(x)|}\int_{B_{r_k}(x)} \Phi^4_\infty(y)\, dy^2 - \frac{1}{|B_r(x)|}\int_{B_r(x)} \Phi^4_\infty(y)\, dy^2\right|^2$$
$$\le C\int_{B_{2r}(x)} \left|\nabla\Phi^4_\infty\right|^2 dy^2 = o\left(r^2\right).$$

Combining these two last estimates with (III.127) we finally obtain

(**III.129**)
$$\left\|\phi_{s_r} \star \Phi^4_\infty\right\|_{L^\infty(\Sigma_{r,k})} = o(r).$$

We shall denote simply $\vec{\Phi}_{s_r} = \phi_{s_r} \star \vec{\Phi}_\infty$. Arguing now exactly as in the proof of Lemma III.1, we have

(**III.130**)
$$\int_{\Sigma_{k,r}} \chi^\alpha_{r,x}(\vec{\Phi}_k)\left|\nabla\Phi^4_k\right|^2 dx^2$$
$$= \int_{\hat{\Sigma}_\varepsilon} \chi^\alpha_{r,x}(\vec{\Phi}_k)\left|\Phi^4_k\right|^2 \left|\nabla\Phi^4_k\right|^2 dx^2 - \int_{\Sigma_{k,r}} \Phi^4_k\, \nabla\left(\chi^\alpha_{r,x}(\vec{\Phi}_k)\right)\cdot\nabla\Phi^4_k + o_k(1).$$

Observe that from (III.129) one has $|\Phi^4_{s_r}| = o(r)$ hence $|\Phi^4_{s_r}\,\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_{s_r}))| \le o(r)\, r^{-1} = o(1)$. Thus we have

$$\text{(III.131)}\quad \begin{aligned}&\int_{\Sigma_{k,r}} \left[\chi^\alpha_{r,x}(\vec{\Phi}_k) - o_r(1)\right]\left|\nabla\Phi^4_k\right|^2 dx^2\\ &\quad = \int_{\Sigma_{k,r}} \chi^\alpha_{r,x}(\vec{\Phi}_k)\left[\left|\Phi^4_k\right|^2 - |\Phi_{s_r}|^2\right]\left|\nabla\Phi^4_k\right|^2 dx^2\\ &\qquad - \sum_{j=1}^4 \int_{\Sigma_{k,r}} \left[\Phi^4_k\left(\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_k)\right) - \Phi^4_{s_r}\,\partial_{z_j}\chi^\alpha_{r,x}(\vec{\Phi}_{s_r})\right)\right]\nabla\Phi^j_k\cdot\nabla\Phi^4_k + o_k(1).\end{aligned}$$

Because of the first line in (III.126) one has

$$\sup_{y\in\Sigma_{k,r}} \int_{B_{s_r}(y)} |\nabla\vec{\Phi}_\infty|^2(z)dz^2 \le \varepsilon(r) r^2 + C_x s_r^2 \le C_x s_r^2.$$

Replacing r by s_r and ε by s_r^2 and $\Sigma\setminus\cup_{l=1}^n B_\varepsilon(a_l)$ by $\Sigma_{k,r}$, one can transpose word by word the arguments from equation (III.93) until equation (III.97) in order to obtain

$$\text{(III.132)}\quad \begin{aligned}&\int_{\Sigma_{k,r}} |\vec{\Phi}_k - \phi_{s_r}\star\vec{\Phi}_\infty|(y)\,|\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\quad\le C\,s_r\int_{\Sigma_{k,r}} |\nabla\vec{\Phi}_k|^2(y)\,dy^2 + C\,\mathfrak{N}\log\sigma_k^{-1}\,\sigma_k^2\int_{\Sigma_{k,r}}\left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}}.\end{aligned}$$

Combining (III.131) with (III.132) gives then

$$\text{(III.133)}\quad \begin{aligned}&\int_{\Sigma_{k,r}} \left[\chi^\alpha_{r,x}(\vec{\Phi}_k) - o_r(1)\right]\left|\nabla\Phi^4_k\right|^2 dx^2\\ &\quad\le C\,s_r\int_{\Sigma_{k,r}} |\nabla\vec{\Phi}_k|^2(y)\,dy^2 + C\,\mathfrak{N}\log\sigma_k^{-1}\,\sigma_k^2\int_{\Sigma_{k,r}}\left(1+|\mathbf{I}_{\vec{\Phi}_k}|^2\right)^p dvol_{g_{\vec{\Phi}_k}}\\ &\qquad + C\int_{\Sigma_{k,r}} \left|\vec{\Phi}^4_k - \phi_{s_r}\star\vec{\Phi}^4_\infty\right|(y)\,\left|\partial_z\chi^\alpha_{r,x}(\vec{\Phi}_k)\right||\nabla\vec{\Phi}_k|^2(y)\,dy^2\\ &\qquad + C\int_{\Sigma_{k,r}} \left|\vec{\Phi}^4_{s_r}\right|(y)\,\left|\partial_z\chi^\alpha_{r,x}(\vec{\Phi}_k) - \partial_z\chi^\alpha_{r,x}(\vec{\Phi}_{s_r})\right|\,\|\nabla\vec{\Phi}_k|^2(y)\,dy^2.\end{aligned}$$

Using the fact that $|\partial_z\chi^\alpha_{r,x}| \le C\,r^{-1}$, that $|\partial_z\chi^\alpha_{r,x}| \le C\,r^{-2}$ together with (III.129) and (III.132) again we finally obtain

$$\text{(III.134)}\quad \limsup_{k\to 0}\frac{\int_{\Sigma_{k,r}}[\chi^\alpha_{r,x}(\vec{\Phi}_k) - o_r(1)]|\nabla\Phi^4_k|^2\,dx^2}{\int_{\Sigma_{k,r}}|\nabla\vec{\Phi}_k|^2(y)\,dy^2} \le C\left[r^{-1}\,s_r + r^{-2}\,s_r^2\right].$$

Combining this fact with (III.123) and the fact that $s_r\, r^{-1} = o(1)$ we finally obtain

$$\textbf{(III.135)} \qquad \limsup_{k\to 0} \frac{\int_{B_r(x)} |\nabla \Phi^4_k|^2\, dx^2}{\int_{B_r(x)} |\nabla \vec{\Phi}_k|^2(y)\, dy^2} = o_r(1).$$

Combining (III.117) and (III.135) we have then

$$\textbf{(III.136)} \qquad \lim_{r\to 0} \lim_{k\to +\infty} \frac{\int_{B_r(x)} [|\nabla \vec{\Phi}^1_k|^2 + |\nabla \vec{\Phi}^2_k|^2]\, dx^2}{\int_{B_r(x)} |\nabla \vec{\Phi}_k|^2\, dx^2} = 1$$

as well as

$$\textbf{(III.137)} \qquad \lim_{\rho\to 0} \limsup_{k\to +\infty} \frac{\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))} [|\nabla \vec{\Phi}^1_k|^2 + |\nabla \vec{\Phi}^2_k|^2]\, dx^2}{\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))} |\nabla \vec{\Phi}_k|^2\, dx^2} = 1.$$

Since $\vec{\Phi}_k$ is conformal we have then

$$\textbf{(III.138)} \qquad \lim_{r\to 0} \lim_{k\to +\infty} \frac{\int_{B_r(x)} 2\, |\partial_{x_1} \vec{\zeta}_k \wedge \partial_{x_2} \vec{\zeta}_k|\, dx^2}{\int_{B_r(x)} |\nabla \vec{\Phi}_k|^2\, dx^2} = 1,$$

where $\vec{\zeta}_k := (\Phi^1_k, \Phi^2_k)$ and, combining (III.136) with (III.138)

$$\textbf{(III.139)} \qquad \lim_{r\to 0} \lim_{k\to +\infty} \frac{\int_{B_r(x)} 2\, |\partial_{x_1} \vec{\zeta}_k \wedge \partial_{x_2} \vec{\zeta}_k|\, dx^2}{\int_{B_r(x)} |\nabla \vec{\zeta}_k|^2\, dx^2} = 1.$$

One difficulty at this stage is that we can not remove the absolute values inside the upper integral of (III.139). If we would be able to do so, we would be proving the strong convergence for $\nabla\vec{\Phi}_k$ towards $\nabla\vec{\Phi}_\infty$ and the lemma would be proven.[15] The rest of the argument consists in proving that the limiting *un-oriented* varifold associated to the current $(\vec{\Phi}_k)_*[B_r(x)]$ is going to be equal, asymptotically as r goes to zero, to an integer times $\vec{\Xi}^\alpha_* T_x \Sigma$. We formulate that differently. Denote by $\tilde{G}_2(S^3)$ to be the Grassmanian of <u>oriented</u> 2 dimensional planes of the tangent bundle to S^3, TS^3. The image by $\vec{\Phi}_k$ of $\Sigma^\alpha_\varepsilon$, induces an <u>oriented integer rectifiable varifold</u> (see [14]) $\tilde{\mathbf{v}}^\alpha_{\varepsilon,k}$, where the choice of orientation of the tangent plane is taken to be the one induced by the push forward by the immersion $\vec{\Phi}_k$ of the one fixed on Σ. The sequence of oriented varifolds $\tilde{\mathbf{v}}_k$ converges to a limiting oriented varifold $\tilde{\mathbf{v}}_\infty$ which is a limiting measure on the oriented 2-Grassmanian $\tilde{G}_2(S^3)$. Denote by $T^+\Sigma$ the tangent bundle to Σ with the positive orientation and $T^-\Sigma$ the same tangent bundle but with the opposite orientation. We see $\vec{\Xi}^\alpha_*(T^+\hat{\Sigma}^\alpha_\varepsilon \cup T^-\hat{\Sigma}^\alpha_\varepsilon)$

[15] Unfortunately we still don't know whether we can exchange the integration and the absolute values in (III.139) at this stage of our study of the viscosity method.

as a measurable subset of $\tilde{G}_2(S^3)$. With these notations, the identity (III.136) is in fact equivalent to

$$\text{(III.140)} \qquad \tilde{\mathbf{v}}^\alpha_{\varepsilon,\infty}\big(\tilde{G}_2(S^3)\setminus \vec{\Xi}^\alpha_*\big(T^+\hat{\Sigma}^\alpha_\varepsilon \cup T^-\hat{\Sigma}^\alpha_\varepsilon\big)\big) = 0.$$

The goal is now to prove

$$\text{(III.141)} \qquad \mathbf{v}^\alpha_{\varepsilon,\infty} = N_x\, \delta_{\vec{\Xi}^\alpha_*(T_x\hat{\Sigma}^\alpha_\varepsilon)} \quad \text{where } N_x \in \mathbf{N}^*,$$

where $\mathbf{v}^\alpha_{\varepsilon,\infty}$ is the *un-oriented* varifold associated to $\tilde{\mathbf{v}}^\alpha_{\varepsilon,\infty}$ and $\delta_{\vec{\Xi}^\alpha_*(T\hat{\Sigma}^\alpha_\varepsilon)}$ is the Dirac mass at the *un-oriented* tangent plane $\vec{\Xi}^\alpha_*(T_x\hat{\Sigma}^\alpha_\varepsilon)$.

Step 2: The integrality of $\mathbf{v}_{\varepsilon,\infty}$: the proof of (III.141).

To simplify the presentation, in order not to have to localize in the domain that would make the notations heavier, we shall assume that

$$\text{(III.142)} \qquad \big(\vec{\Xi}^\alpha\big)^{-1}\big(\vec{\Xi}^\alpha(x)\big) = \{x\}.$$

For $i=1,\ldots,4$ we denote by $\nabla^{\Sigma_k}y^i$ the vector-field tangent to $\Phi_k(\Sigma)$ given by the projection of the i-th canonical vector of $\mathbf{R}^4$ onto $(\vec{\Phi}_k)_*T\Sigma$. We also denote $*_k\nabla^{\Sigma_k}y^i$ the rotation by $\pi/2$ of this vector in the tangent plane to $\Phi_k(\Sigma)$, taking into account the orientation given by the push-forward by $\vec{\Phi}_k$ of the one we fixed on Σ. Denote by $(\vec{\varepsilon}_i)_{i=1,\ldots,4}$ the canonical basis of $\mathbf{R}^4$. The identity (III.137) implies that

$$\text{(III.143)} \qquad \limsup_{k\to+\infty}\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))} \mathrm{dist}\left(\frac{\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k}{|\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k|}, \pm\,\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2\right)|\nabla\vec{\Phi}_k|^2\,dx^2 = o\big(\rho^2\big),$$

recall $\mu_\infty(B^4_\rho(\vec{\Phi}_\infty(x)))\simeq\rho^2$. This also implies

$$\text{(III.144)} \qquad \forall i=1,2 \quad \limsup_{k\to+\infty}\int_{B^4_\rho(\vec{\Phi}_\infty(x))}\left|\nabla^{\Sigma_k}y_i-\vec{\varepsilon}_i\right|d\mathcal{H}^2\llcorner\,\vec{\Phi}_k(\Sigma) = o\big(\rho^2\big).$$

For $(\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k)\cdot(\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2)\neq 0$ we denote $J_k=\mathrm{sign}((\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k)\cdot(\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2))$ otherwise we simply take $J_k=0$. Identity (III.143) and (III.144) imply

$$\text{(III.145)} \qquad \begin{aligned}&\limsup_{k\to+\infty}\int_{B^4_\rho(\vec{\Phi}_\infty(x))}\Big[\left|*_k\nabla^{\Sigma_k}y_1-J_k\,\varepsilon_2\right|+\left|*_k\nabla^{\Sigma_k}y_2+J_k\,\varepsilon_1\right|\Big]\,d\mathcal{H}^2\llcorner\,\vec{\Phi}_k(\Sigma)\\ &\quad = o\big(\rho^2\big).\end{aligned}$$

Let $\vec{T}^\rho_k$ be the following vector-valued one dimensional currents

$$\forall\alpha\in\Omega^1\big(\mathbf{R}^4\big)\quad \big\langle\vec{\mathrm{T}}^\rho_k,\alpha\big\rangle:=\int_{\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}\alpha\wedge *_k\,d\vec{y}$$

$$=\int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))}\vec{\Phi}_k^*\alpha\wedge *\,d\vec{\Phi}_k.$$

Let φ be a smooth function in $C^\infty_0(B^4_1(0))$ such that $\int_{\mathbf{R}^4}\varphi(y)\;dy^4=1$. Denote $\varphi_{\sigma_k}:=\sigma_k^{-4/p}\varphi(\cdot/\sigma_k^{1/p})$. We recall the definition of the σ_k-*smoothing* $\varphi_{\sigma_k}\star\vec{\mathrm{T}}^\rho_k$ of the current $\vec{\mathrm{T}}^\rho_k$ (see [9], 4.1.2)

$$\forall\alpha\in\Omega^1\big(\mathbf{R}^4\big)\quad \big\langle\varphi_{\sigma_k}\star\vec{\mathrm{T}}^\rho_k,\alpha\big\rangle:=\int_{\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}(\varphi_{\sigma_k}\star\alpha)\wedge *_k d\vec{y},$$

where $\alpha_{\sigma_k}:=\varphi_{\sigma_k}\star\alpha$ denotes the following convolution operation

$$\alpha_{\sigma_k}=\varphi_{\sigma_k}\star\alpha:=\int_{\mathbf{R}^4}\varphi_{\sigma_k}(-z)\,\tau_z^*\alpha\,dz^4$$

where $\tau_z(y)=y+z$. We shall use the following lemma

Lemma **III.11.** — **[Convergence of the σ_k-approximation of $\vec{\mathrm{T}}^\rho_k$]** *Under the previous notations we have*

(**III.146**) $$\limsup_{k\to+\infty}\ \sup_{supp(\phi)\subset\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x));\ \|d\phi\|_\infty\le 1}\big\langle\vec{\mathrm{T}}^\rho_k-\varphi_{\sigma_k}\star\vec{\mathrm{T}}^\rho_k,d\phi\big\rangle=0.$$

Proof of Lemma III.11. — Let ϕ be a Lipschitz function supported in $\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))$ with $\|d\phi\|_\infty\le 1$. We have

$$\big\langle\vec{\mathrm{T}}^\rho_k-\varphi_{\sigma_k}\star\vec{\mathrm{T}}^\rho_k,d\phi\big\rangle$$

$$=\int_{\mathbf{R}^4}dz\,\varphi_{\sigma_k}(-z)\int_{\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}\big(d\phi-\tau_z^*d\phi\big)\wedge *_k d\vec{y}$$

$$=-\int_{\mathbf{R}^4}dz\,\varphi_{\sigma_k}(-z)\int_{\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)}\big(\phi(y)-\phi(y+z)\big)\wedge d *_k d\vec{y}.$$

Using the fact that $\|d\phi\|_\infty\le 1$ and that φ_{σ_k} is supported in $\mathrm{B}^4_{\sigma_k^{1/p}}(0)$, we have

$$\Big|\big\langle\vec{\mathrm{T}}^\rho_k-\varphi_{\sigma_k}\star\vec{\mathrm{T}}^\rho_k,d\phi\big\rangle\Big|\le\sigma_k^{1/p}\int_\Sigma\big[|\vec{\mathrm{H}}_k|+1\big]\,dvol_{g_{\vec{\Phi}_k}}$$

$$\le\Big[\sigma_k^2\int_\Sigma\big[|\vec{\mathrm{H}}_k|^{2p}+1\big]\,dvol_{g_{\vec{\Phi}_k}}\Big]^{1/2p}\mathrm{Area}\big(\vec{\Phi}_k(\Sigma)\big)^{1-1/2p}$$

$$=o(1).$$

This concludes the proof of Lemma III.11. ☐

Lemma **III.12.** — **[Asymptotic vanishing of the boundary of $\vec{\mathrm{T}}_k^\rho$ in $\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))$]** *Under the previous notations we have*

$$\limsup_{k\to+\infty}\ \sup_{\mathrm{supp}(\phi)\subset \mathrm{B}^4_\rho(\vec{\Phi}_\infty(x));\ \|d\phi\|_\infty\le 1} \left\langle \vec{\mathrm{T}}_k^\rho, d\phi\right\rangle = o\left(\rho^2\right), \tag{III.147}$$

and for the two first directions $i=1,2$ we have

$$\limsup_{k\to+\infty}\ \sup_{\mathrm{supp}(\phi)\subset \mathrm{B}^4_\rho(\vec{\Phi}_\infty(x));\ \|d\phi\|_\infty\le 1} \vec{\varepsilon}_i\cdot\left\langle \vec{\mathrm{T}}_k^\rho, d\phi\right\rangle = \mathrm{O}\left(\rho^4\right). \tag{III.148}$$

Proof of Lemma III.12. — Because of (III.137) it suffices to prove (III.148). Because of the previous lemma it suffices to prove (III.147) where $\vec{\varepsilon}_i\cdot\vec{\mathrm{T}}_k^\rho$ for $i=1,2$ is replaced by $\vec{\varepsilon}_i\cdot\varphi_{\sigma_k}\star\vec{\mathrm{T}}_k^\rho$. We assume $\phi(\vec{\Phi}_\infty(x))=0$ in such a way that $\|\phi\|_\infty\le\rho$. We have

$$\left\langle \varphi_{\sigma_k}\star\vec{\mathrm{T}}_k^\rho, d\phi\right\rangle = \int_{\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x))\cap\vec{\Phi}_k(\Sigma)} d(\varphi_{\sigma_k}\star\phi)\wedge *_k d\vec{y}. \tag{III.149}$$

Integrating by parts and using (II.36) we have, omitting to write explicitly the subscript k,

$$\begin{aligned}
&\left\langle \varphi_\sigma\star\vec{\mathrm{T}}^\rho, d\phi\right\rangle\\
&= \int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))} \nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\sigma^2 f^p\,\nabla\vec{\Phi}\,dx^2\\
&\quad -2p\sigma^2\int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))} e^{-2\lambda}\,\nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\Big[\overline{\nabla}\big[f^{p-1}\,\mathbf{I}^0_{11}\big]\\
&\quad +(\overline{\nabla})^\perp\big[f^{p-1}\,\mathbf{I}^0_{12}\big]\Big]\vec{n}\,dx^2\\
&\quad -2p\sigma^2\int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))} \nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\nabla\big[f^{p-1}\,\vec{\mathrm{H}}\big]\,dx^2\\
&\quad +2p\sigma^2\int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))} \nabla\left(\varphi_\sigma\star\phi(\vec{\Phi})\right)\cdot\Big[f^{p-1}\,\mathrm{H}\,\nabla\vec{n}\\
&\quad - e^{-2\lambda}f^{p-1}\,\langle\nabla\vec{n}\,\dot{\otimes}\,\nabla\vec{n};\nabla\vec{\Phi}\rangle\Big]\,dx^2\\
&\quad -\int_{\vec{\Phi}_k^{-1}(\mathrm{B}^4_\rho(\vec{\Phi}_\infty(x)))} \varphi_\sigma\star\phi(\vec{\Phi})\Big(\big[1+\sigma^2(1-p)f^p+p\sigma^2 f^{p-1}\big]\,\vec{\Phi}\,|\nabla\vec{\Phi}|^2\\
&\quad -4p\sigma^2 f^{p-1}\,\vec{\mathrm{H}}\Big)\,dx^2.
\end{aligned} \tag{III.150}$$

Observe that $\|\partial^2_{y_iy_j}(\varphi_\sigma\star\phi)\|_\infty\le\sigma^{-1/p}$ hence integrating by parts $\overline{\nabla}$ and $(\overline{\nabla})^\perp$ in the second line of (III.150) as well as integrating by parts ∇ in the fourth line of (III.150) and using (III.15) as in the proof of the monotonicity formula, we obtain that all the terms in the first, second, third and fourth lines of the r.h.s. of (III.150) vanish as k goes to $+\infty$. In the fifth line only the term $\int_{\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\varphi_\sigma\star\phi(\vec{\Phi})\ \vec{\Phi}\,|\nabla\vec{\Phi}|^2dx^2$ is not necessarily converging towards 0. Since we are considering the first and second canonical directions and since Φ^1 and Φ^2 are $O(\rho)$ in $\vec{\Phi}_k^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))$ and since $\|\phi\|_\infty\le\rho$ we obtain (III.148) and Lemma III.12 is proved. □

Proof of Lemma III.10 continued. — Denote $\vec{\Phi}'_k:=(\Phi^3_k,\Phi^4_k)$. By taking $\phi(y):=h(y_1,y_2)\ \chi_\rho(y_3,y_4)$ where χ_ρ is identically equal to ρ on $B^2_\rho(1,0)$, is non negative, supported in $B^2_{2\rho}(1,0)$, we have for $i=1,2$

$$(\textbf{III.151})\quad \limsup_{k\rightarrow+\infty}\sup_{\text{supp}(h)\subset B^2_\rho(\vec{\Phi}_\infty(x));\ \|dh\|_\infty\le\rho^{-1}}\vec{\varepsilon}_i\cdot\int_{B^4_{4\rho}(\vec{\Phi}_\infty(x))}*_k d\vec{y}\wedge(\chi_\rho\ dh+h\,d\chi_\rho)=O\big(\rho^4\big).$$

Because of the existence of an approximate tangent plane at $\vec{\Phi}_\infty(x)$, which is equal to $\text{Span}\{\vec{\varepsilon}_1,\vec{\varepsilon}_2\}$, the asymptotic mass of the current in $B^4_{4\rho}(\vec{\Phi}_\infty(x))$ contained in the support of $d\chi_\rho$ which is included in $B^2_{4\rho}(0,0)\times(B^2_{2\rho}(1,0)\setminus B^2_\rho(1,0))$ is a $o(\rho^2)$. Hence we deduce for $i=1,2$

$$(\textbf{III.152})\qquad \limsup_{k\rightarrow+\infty}\sup_{\text{supp}(h)\subset B^2_\rho(0,0);\ \|dh\|_\infty\le\rho^{-1}}\int_{B^2_\rho(0,0)\times B^2_\rho(1,0)}\partial_{y_i}h\,d\mathcal{H}^2\llcorner\ \vec{\Phi}_k(\Sigma)=o\big(\rho^2\big).$$

This implies, using (III.136),

$$(\textbf{III.153})\qquad \limsup_{k\rightarrow+\infty}\sup_{\text{supp}(h)\subset B^2_\rho(0,0);\ \|dh\|_\infty\le\rho^{-1}}\int_{B^2_\rho(0,0)}N_k(y)\ \partial_{y_i}h\ d\mathcal{L}^2=o\big(\rho^2\big),$$

where $N_k(y)$ is the number of pre-images of $y=(y_1,y_2)$ by $\vec{\zeta}_k$. Since $M(B^2_\rho(0,0)\cap\vec{\zeta}_k(\Sigma))\simeq\rho^2$ we then have

$$(\textbf{III.154})\qquad \limsup_{k\rightarrow+\infty}\sum_{i=1}^2\sup_{\text{supp}(h)\subset B^2_\rho(0,0);\ \|dh\|_\infty\le\rho^{-1}}\frac{\int_{B^2_\rho(0,0)}N_k(y)\ \partial_{y_i}h\ dy^2}{\int_{B^2_\rho(0,0)}N_k(y)\ dy^2}=o_\rho(1).$$

The quantity on the numerator of (III.154) is almost but not quite the *Flat Norm*[16] of the relative boundary in $B^2_\rho(0,0)$ of the 2 dimensional *integer rectifiable current* given by $C_k(\rho):=[N_k(y)\ dy^2]\llcorner\ B^2_\rho(0,0)$ while the denominator equals it's total mass.

[16] The flat norm would have been

$$\sup_{\text{supp}(X)\subset B^2_\rho(0,0);\ \|\text{div}X\|_\infty\le\rho^{-1}}\int_{B^2_\rho(0,0)}N_k(y)\ \text{div}(X)\ dy^2$$

and cannot a-priori be controlled by the numerator of (III.154).

In [30] the following inequality is proved. For any measurable function f on the 2 dimensional unit ball $B_1(0)$ the following inequality holds

$$\text{(III.155)}\qquad \begin{aligned}&\left\| f-\frac{1}{|B_{1/2}(0)|}\int_{B_{1/2}(0)} f(y)\,dy^2\right\|_{L^{1,\infty}(B_{1/2}(0))}\\ &\quad\le C\,\sup\left\{\int_{B_1(0)} f(y)\,\nabla\phi(y)\,dy^2;\ \phi\in C^\infty_0(B_1(0))\quad \|\nabla\phi\|_\infty\le 1\right\}.\end{aligned}$$

Combining (III.154) and (III.155) gives that

$$\text{(III.156)}\qquad \limsup_{k\to+\infty}\left\|N_k(\rho\, x)-\frac{1}{|B_{1/2}(0)|}\int_{B_{1/2}(0)} N_k(\rho\, y)\,dy^2\right\|_{L^{1,\infty}(B_{1/2}(0))}=o_\rho(1).$$

This shows that the average $\frac{1}{|B_{1/2}(0)|}\int_{B_{1/2}(0)} N_k(\rho\, y)\,dy^2$ is $o_\rho(1)$ close to an integer $n_k^\rho\in\mathbf{N}^*$ as $k$tends to infinity and that

$$\text{(III.157)}\qquad \limsup_{k\to+\infty}\left\|N_k(\rho\, x)-n_k^\rho\right\|_{L^{1,\infty}(B_{1/2}(0))}=o_\rho(1).$$

Since this integer is bounded and bounded away from zero, modulo extraction of a subsequence we can assume that $n^\rho_k=n^\rho$ is independent of k and, taking a sequence of radii $\rho_j\to 0$ we can also assume that n^{ρ_j} is independent of j and we have the existence of $n\in\mathbf{N}^*$ such that

$$\text{(III.158)}\qquad \lim_{j\to+\infty}\limsup_{k\to+\infty}\left\|N_k(\rho_j\, x)-n\right\|_{L^{1,\infty}(B_{1/2}(0))}=0,$$

this proves (III.141) and this concludes the proof of Lemma III.10. □

Lemma **III.13.** — **[Convergence to a bubble tree of conformal "integer target harmonic" maps]** *Under the assumptions of Theorem III.1, we have that one we can extract a subsequence such that the integer varifold* $|\vec{\Phi}_k(\Sigma)|$ *converges to an integer rectifiable varifold supported by a finite union of the images by target harmonic conformal* $W^{1,2}$*-maps of Riemann surfaces .*

We adopt the same notations as in the proof of Lemma III.10 and assume to simplify the presentation that (III.142) holds where we recall among other things that x is chosen also to be a Lebesgue point for $\nabla\vec{\Phi}_\infty(x)$. One has

$$\text{(III.159)}\qquad \lim_{\rho\to 0}\lim_{k\to+\infty}\frac{\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}|\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k|\,dx^2}{\vec{\varepsilon}_1\wedge\vec{\varepsilon}_2\cdot\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}\partial_{x_1}\vec{\Phi}_\infty\wedge\partial_{x_2}\vec{\Phi}_\infty\,dx^2}=N_x.$$

Observe also that[17] The lower semicontinuity of the norm gives

(**III.160**)
$$\begin{aligned}\liminf_{k\to+\infty}\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}|\partial_{x_1}\vec{\Phi}_k\wedge\partial_{x_2}\vec{\Phi}_k|\,dx^2&=\liminf_{k\to+\infty}\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}2^{-1}\,|\nabla\vec{\Phi}_k|^2\,dx^2\\&\geq\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}2^{-1}\,|\nabla\vec{\Phi}_\infty|^2\,dx^2.\end{aligned}$$

Hence combining (III.159) and (III.160) one gets

$$\begin{aligned}&\lim_{\rho\to 0}\frac{\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}2^{-1}\,|\nabla\vec{\Phi}_\infty|^2\,dx^2}{\int_{\vec{\Phi}_\infty^{-1}(B^4_\rho(\vec{\Phi}_\infty(x)))}|\partial_{x_1}\vec{\Phi}_\infty\wedge\partial_{x_2}\vec{\Phi}_\infty|\,dx^2}\\&\qquad\leq N_x=\pi^{-1}\lim_{\rho\to 0}\rho^{-2}\,\mu_\infty\big(B^4_\rho(\vec{\Phi}_\infty(x))\big).\end{aligned}$$

This gives, using the *Monotonicity Formula*, we have

(**III.161**)
$$\text{for } \nu_\infty \text{ a.e. } x\in D^2\setminus\mathcal{B}\qquad 1\leq\frac{|\nabla\vec{\Phi}_\infty|^2(x)}{2\,|\partial_{x_1}\vec{\Phi}_\infty\wedge\partial_{x_2}\vec{\Phi}_\infty|(x)}\leq\pi^{-1}\,e^{2C}\,\mu_\infty(S^3)=K.$$

Take $g_{ij}:=\partial_{x_i}\vec{\Phi}_\infty\cdot\partial_{x_j}\vec{\Phi}_\infty$ and introduce

$$\text{for a.e. } x\in D^2\setminus\mathcal{L}^0_{\nabla\vec{\Phi}_\infty}\qquad \mu(x):=\frac{g_{11}-g_{22}+2ig_{12}}{g_{11}+g_{22}+2\sqrt{g_{11}g_{22}-g_{12}^2}}$$

on $D^2\setminus\mathcal{L}^0_{\nabla\vec{\Phi}_\infty}$, with the above notations (III.161) can be recast in the following way

$$4\leq\frac{(g_{11}+g_{22})^2}{g_{11}g_{22}-g_{12}^2}\leq\frac{4}{\pi^2}\,e^{4C}\,\mu_\infty^2(S^3)=4\,K^2.$$

Extend μ by zero on the whole $\mathbf{C}$. Observe that we have

$$\|\mu\|^2_\infty\leq\left\|\frac{(g_{11}+g_{22})^2-4(g_{11}g_{22}-g_{12}^2)}{(g_{11}+g_{22})^2+4(g_{11}g_{22}-g_{12}^2)}\right\|_{L^\infty(D^2\setminus\mathcal{L}^0_{\nabla\vec{\Phi}_\infty})}\leq\frac{K^2-1}{K^2+1}<1.$$

Hence μ defines a compactly supported *Beltrami coefficient*. Consider the *normal solution* of the *Beltrami equation* given by Theorem 4.24 of [16]

$$\partial_{\bar z}\varphi=\mu\,\partial_z\varphi.$$

[17] We recall among other things that x is chosen also to be a Lebesgue point for $\nabla\vec{\Phi}_\infty$ and that $\nabla\vec{\Phi}_\infty(x)=\nabla\vec{\Xi}^\alpha(x)$.

The quasiconformal map φ realizes in particular an homeomorphism whose inverse φ^{-1} is also quasiconformal in $W^{1,p}_{loc}(\mathbf{C})$ for some $p > 2$ and one has

$$\partial_{\overline{w}}\varphi^{-1} = \omega\partial_w\varphi^{-1},$$

where $\omega = -(\mu\partial_z\varphi/\overline{\partial_z\varphi}) \circ \varphi^{-1}$. Being an homeomorphic map of *bounded distortion* in $W^{1,2}(\varphi(D^2))$ it is quasi-regular, the chain rule applies with $\vec{\Phi}_\infty$ (see Theorem 16.13.3 of [17]) and $\vec{\Phi}_\infty \circ \varphi^{-1} \in W^{1,2}(\varphi(D^2))$. A classical computation gives

$$\partial_w\left(\vec{\Phi}_\infty \circ \varphi^{-1}\right) \cdot \partial_w\left(\vec{\Phi}_\infty \circ \varphi^{-1}\right) = 0 \quad \text{a.e. on } \varphi\left(D^2\right).$$

"Pasting" together all these conformal charts gives a smooth conformal structure on Σ and a global quasi-conformal homeomorphism ψ of Σ such that $\vec{\Phi}_\infty \circ \psi$ is weakly conformal. Moreover, the condition for the image of Σ by $\vec{\Phi}_\infty$ equipped with the integer multiplicity N to be stationary is equivalent to (I.2). It remains to show that $(N, \vec{\Phi}_\infty \circ \psi)$ defines an integer target harmonic map.

We omit to mention the composition by ψ and we simply write $\vec{\Phi}_\infty$ for $\vec{\Phi}_\infty \circ \psi$. We can apply Lemma III.1 to $\Sigma \setminus \bigcup_{l=1}^n B_{r_k}(a_l)$ where r_k are "nice cuts" taken between $\varepsilon/2$ and ε on which $\vec{\Phi}_k$ converges in C^0 to deduce, using because of (III.108), that there exists n points $\vec{q}_{l,\rho}$ such that

$$\left|(\vec{\Phi}_\infty)_*\big(N[\Sigma]\big) \llcorner \left(\mathbf{R}^4 \setminus \bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho})\right)\right|$$

realizes an integer rectifiable stationary varifold in $S^3 \setminus \bigcup_{l=1}^n B^4_{s_\rho}(\vec{q}_{l,\rho})$. This is equivalent to

$$\text{(III.162)} \quad \int_{\Sigma\setminus\bigcup_{l=1}^n B_r(a_l)} N\left[\sum_{i=1}^4 \left\langle \partial_{y_i}\vec{X}(\vec{\Phi}_\infty)\,\nabla\Phi^i_\infty; \nabla\vec{\Phi}_\infty\right\rangle - N\,\vec{X}(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\,|\nabla\vec{\Phi}_\infty|^2\right] dx^2 = 0.$$

We chose a sequence of radii $\rho_k \to 0$ such that

$$\forall l = 1, \ldots, n \quad \vec{q}_{l,\rho_k} \to \vec{q}_{l,0} \in S^3.$$

Since $s_{\rho_k} \to 0$, $(\vec{\Phi}_\infty)_*(N[\Sigma])$ is stationary in $S^3 \setminus \{\vec{q}_{1,0}, \ldots, \vec{q}_{n,0}\}$. Let $\chi_\delta(t) = \chi(t/\delta)$ where $\chi \in C^\infty_0([0, 2], \mathbf{R}_+)$, χ is identically equal to one on $[0, 1]$. For any arbitrary smooth vector field $\vec{X}$ from $\Gamma(TS^3)$ we proceed to the following decomposition:

$$\vec{X}(\vec{q}) = \sum_{l=1}^n \chi_\delta\left(|\vec{q} - \vec{q}_{l,0}|\right)\vec{X} + \vec{X}_\delta(\vec{q}) \quad \text{where}$$

$$\vec{X}_\delta(\vec{q}) := \left[1 - \sum_{l=1}^n \chi_\delta\left(|\vec{q} - \vec{q}_{l,0}|\right)\right]\vec{X}$$

Since $\mathrm{Supp}(\vec{X}_\delta) \subset \mathbf{R}^4 \setminus \bigcup_{l=1}^n B^4_\delta(\vec{q}_{l,0})$ we have

$$\text{(III.163)} \qquad \int_\Sigma N\left[\sum_{i=1}^4 \left\langle \partial_{y_i}\vec{X}_\delta(\vec{\Phi}_\infty)\, \nabla\Phi^i_\infty ; \nabla\vec{\Phi}_\infty\right\rangle - \vec{X}_\delta(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\, |\nabla\vec{\Phi}_\infty|^2\right] dx^2 = 0$$

and we have

$$\text{(III.164)} \qquad \begin{aligned} &\left|\int_\Sigma N\left[\sum_{i=1}^4 \left\langle \partial_{y_i}(\vec{X}-\vec{X}_\delta)(\vec{\Phi}_\infty)\, \nabla\Phi^i_\infty ; \nabla\vec{\Phi}_\infty\right\rangle \right.\right.\\ &\qquad \left.\left. - (\vec{X}-\vec{X}_\delta)(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\, |\nabla\vec{\Phi}_\infty|^2\right] dx^2\right| \\ &\le \|\vec{X}\|_\infty \frac{1}{\delta}\sum_{l=1}^n \mu_\infty\big(B^4_{2\delta}(\vec{q}_{l,0})\big) + \|\nabla\vec{X}\|_\infty \sum_{l=1}^n \mu_\infty\big(B^4_{2\delta}(\vec{q}_{l,0})\big) = O(\delta) \end{aligned}$$

where we are using the monotonicity formula. Combining (III.163) and (III.164) with $\delta \to 0$ we obtain that

$$\text{(III.165)} \qquad \int_\Sigma N\left[\sum_{i=1}^4 \left\langle \partial_{y_i}\vec{X}(\vec{\Phi}_\infty)\, \nabla\Phi^i_\infty ; \nabla\vec{\Phi}_\infty\right\rangle - \vec{X}(\vec{\Phi}_\infty)\cdot\vec{\Phi}_\infty\, |\nabla\vec{\Phi}_\infty|^2\right] dx^2 = 0.$$

What we have done for the whole Σ can be done for any subdomain Ω assuming that the support of $\vec{X}$ is contained in a complement of an open neighborhood of $\vec{\Phi}_\infty(\partial\Omega)$. We deduce that $\vec{\Phi}_\infty$ is integer target harmonic from Σ into S^3. This concludes the proof of the Lemma III.13. □

IV. The proof of Theorem I.1

We consider the general case where $(\Sigma, g_{\vec{\Phi}_k})$ possibly degenerate in the moduli space. Modulo extraction of a subsequence, following Deligne–Mumford compactification described in section II of [34] we have a "splitting" of the original surface into collars, called also "thin parts" and a Nodal Riemann surface $\tilde{\Sigma}$ called also "thick part". The parts of the collars that contain no bubbles can be treated exactly as the necks in Lemma III.6, indeed a collar has the conformal type of a degenerating annulus and, if such a collar contains no bubble, by definition, it means that on each sub-annulus of controlled conformal type (in each dyadic annulus in particular) there is no concentration measure ν_∞. Hence in a collar region containing no bubble the statement of Lemma III.6 applies word by word. The "thick parts" as well as the "bubbles" formed either in the thick parts or in the collars can be treated exactly as the surface Σ in the compact case presented in the previous section. So we deduce Theorem I.1.

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Appendix A

Lemma **A.1.** — *There exists a universal number* $\varepsilon_0(m) > 0$ *such that, for any* $\vec{\Phi}$ *smooth immersion of* Σ*, a smooth surface with boundary, into* $B^m_2(0) \setminus B^m_1(0)$ *and satisfying*

$$\text{Area}\big(\vec{\Phi}(\Sigma)\big) < 3\pi, \tag{A.1}$$

and

$$\forall r \in (1,2) \quad \vec{\Phi}(\Sigma) \cap \partial B^m_r(0) \neq \emptyset \quad \text{and} \quad \vec{\Phi}(\partial\Sigma) \subset \partial\big(B^m_2(0) \setminus B^m_1(0)\big), \tag{A.2}$$

then

$$\int_\Sigma |d\vec{n}|^2_{g_{\vec{\Phi}}}\, dvol_{\vec{\Phi}} \geq \varepsilon_0(m). \tag{A.3}$$

Proof of Lemma A.1. — We argue by contradiction. We consider a sequence Σ_k and $\vec{\Phi}_k$ such that

$$\text{Area}\big(\vec{\Phi}_k(\Sigma_k)\big) < 3\pi, \tag{A.4}$$

such that

$$\forall r \in (1,2) \quad \vec{\Phi}_k(\Sigma_k) \cap \partial B^m_r(0) \neq \emptyset \quad \text{and} \quad \vec{\Phi}_k(\partial\Sigma_k) \subset \partial\big(B^m_2(0) \setminus B^m_1(0)\big), \tag{A.5}$$

and

$$\lim_{k\to+\infty} \int_{\Sigma_k} |d\vec{n}|^2_{g_{\vec{\Phi}_k}}\, dvol_{\vec{\Phi}_k} = 0. \tag{A.6}$$

Let V_k be the oriented varifold associated to the immersion of $\vec{\Phi}_k$ with L^2-bounded second fundamental form (see [14]). Using Theorem 3.1 and 5.3.2 of [14], modulo extraction of a subsequence V_k varifold converges to an integer oriented varifold V_∞ with generalized second fundamental form equal to zero and without boundary in $B_2(0) \setminus B_1(0)$. V_∞ is then stationary and included in an at most countable union of 2-planes. Using the constancy theorem [39] we deduce that V_∞ is an oriented varifold given by at most countably many intersections of 2-planes with the annulus $B_2(0) \setminus B_1(0)$ with locally constant integer multiplicities. We claim that the intersection between the closed set given by the support of V_∞ and $\partial B_r(0) \times G_2(\mathbf{R}^m)$ is non empty for any $r \in (1,2)$. Indeed, from the assumption (A.5), using Simon's monotonicity formula, for any $r \in (1,2)$ and $0 < \rho < \min\{2-r, r-1\}$, there exists $x_k^r \in \partial B_r(0)$ such that

$$\frac{2\pi}{3}\rho^2 \leq M\big(\vec{\Phi}_k(\Sigma_k) \cap B^m_\rho\big(x_k^r\big)\big) + \frac{\rho^2}{2}\int_{\Sigma_k} |\vec{H}_{\vec{\Phi}_k}|^2 \, dvol_{g_{\vec{\Phi}_k}}.$$

Using (A.6) we deduce that for any $\rho < \min\{2-r, r-1\}$

$$\mu_{V_\infty}\big(B_{r+\rho}(0) \setminus B_{r-\rho}(0)\big) \geq \frac{2\pi}{3}\rho^2.$$

Hence the support of V_∞ intersects all the $\partial B_r(0) \times G_2(\mathbf{R}^m)$ for any $r \in (1,2)$. We consider a sequence of radii $r_i > 1$ and converging to 1. The 2-planes belonging to the support of V_∞ and intersecting $\partial B_{r_i}(0) \times G_2(\mathbf{R}^m)$ has to be constant for i large enough. This implies that the support of V_∞ contains the intersection between the annulus $B_2(0) \setminus B_1(0)$ and a plane touching $\overline{B_1(0)}$. This imposes

$$\mu_{V_\infty}\big(B_2(0) \setminus B_1(0)\big) \geq 3\,\pi.$$

The later contradicts (A.4) and Lemma A.1 is proved. □

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