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PERMANENCE UNDER STRONG AGGRESSIONS IS POSSIBLE

LA PERMANENCE D'ESPÈCES QUI S'ENTRE-DÉVORENT EST POSSIBLE

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ABSTRACT. – This paper analyzes the limiting behavior of the positive solutions of a general class of sublinear elliptic weighted mixed boundary value problems as the amplitude of the positive part of the lower order terms of the differential operator blows up to infinity. The main result establishes that the positive solutions approximate zero within the support of the positive part of the potential, whereas they stabilize to the positive solution of a certain elliptic mixed boundary value problem on its complement. Further, we use this result for deriving some general principles in competing species dynamics. Precisely, we shall show that in the presence of a refuge region two competing species must coexist if their reproduction rates are sufficiently large, independently of the strength of the competition. It should be emphasized that the abstract theory developed here allows measuring how large the reproduction rates should be for being permanent, providing us, simultaneously, with the limiting behavior of each of the species separately. Basically, when the pressure from the competitor grows the tackled species concentrates within its refuge. The results of this paper are substantial extensions of some pioneer results found by one of the authors in [16, Section 4]. The main ingredients in deriving the main results of this paper are the continuous dependence of the principal eigenvalue with respect to a general class of perturbations of the domain around its Dirichlet boundary – very recent result coming from [6] – and the continuous dependence of the positive solutions of the sublinear problem – coming from [7].
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RÉSUMÉ. – On étudie le comportement asymptotique des solutions positives d'une classe très générale de problèmes aux limites non linéaires elliptiques lorsque l'amplitude du potentiel

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d'ordre zéro de l'opérateur différentiel tend vers l'infini. En particulier, on verra que la solution tend vers zéro sur le support de la partie positive du potentiel, tandis qu'elle converge vers la solution positive d'un certain problème aux limites elliptiques auxiliaires sur la région ou le potentiel est nul. De plus, on va tirer de ce résultat de convergence un postulat biologique concernant la lutte pour la vie des espèces qui s'entredévorent : indépendamment de l'intensité des agressions, en présence d'un refuge pour chaque compétiteur les espèces coexistent si leurs coefficients d'accroissement sont assez longs. Il faut préciser qu'avec la théorie développée ici on peut mesurer le coefficient d'accroissement critique des espèces pour avoir la coexistence. Naturellement, l'espèce agressée va se concentrer sur les refuges correspondants si l'intensité des agressions croît. Tous les résultats obtenus ici sont des généralisations substantielles des résultats [16, Section 4]. Pour démontrer ces résultats on utilise la dépendance continue de la première valeur propre, et de la solution positive même du problème aux limites non linéaire, par rapport aux perturbations du domaine (cf. [6] et [7]).
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1. Introduction

In this paper we analyze the limiting behavior as $\gamma \nearrow \infty$ of the positive solutions of the following elliptic boundary value problem

$$\begin{cases} \mathfrak{L}u + \gamma V(x)u = \lambda W(x)u - \mathcal{X}(x) f(x,u)u & \text{in } \Omega, \\ \mathfrak{B}(b)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{X}, V, W \in L_\infty(\Omega)$, $\mathcal{X}$ and V belong to a certain class of nonnegative potentials to be introduced later, and we assume the following:

(a) Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, of class $\mathcal{C}^2$, i.e., $\overline{\Omega}$ is an N-dimensional compact connected $\mathcal{C}^2$-submanifold of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$.

(b) $\gamma, \lambda \in \mathbb{R}$, and

$$\mathfrak{L} := -\sum_{i,j=1}^{N} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i(x)\frac{\partial}{\partial x_i} + \alpha_0(x)$$

is an uniformly strongly elliptic second order differential operator in Ω with

$$\alpha_{ij} = \alpha_{ji} \in \mathcal{C}^1(\overline{\Omega}), \quad \alpha_i \in \mathcal{C}(\overline{\Omega}), \quad \alpha_0 \in L_\infty(\Omega), \quad 1 \leqslant i, j \leqslant N.$$

Subsequently, we denote by $\mu > 0$ the ellipticity constant of $\mathfrak{L}$ in Ω. Then, for any $\xi \in \mathbb{R}^N \setminus \{0\}$ and $x \in \overline{\Omega}$ we have that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x)\xi_i\xi_j \geqslant \mu|\xi|^2.$$

(c) $\mathfrak{B}(b)$ stands for the boundary operator

$$\mathfrak{B}(b)u := \begin{cases} u & \text{on } \Gamma_0, \\ \partial_\nu u + bu & \text{on } \Gamma_1, \end{cases}$$

where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$ with

$$\Gamma_0 \cup \Gamma_1 = \partial\Omega,$$

$b \in \mathcal{C}(\Gamma_1)$, and

$$\nu = (\nu_1, \ldots, \nu_N) \in \mathcal{C}^1(\Gamma_1; \mathbb{R}^N)$$

is an outward pointing nowhere tangent vector field. Necessarily, Γ_0 and Γ_1 possess finitely many components. Note that, $\mathfrak{B}(b)$ is the Dirichlet boundary operator on Γ_0, denoted in the sequel by $\mathfrak{D}$, and the Neumann or a first order regular oblique derivative boundary operator on Γ_1. It should be pointed out that either Γ_0 or Γ_1 might be empty.

(d) The function $f : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ satisfies

$$f \in \mathcal{C}^1(\bar{\Omega} \times [0, \infty); \mathbb{R}), \qquad \lim_{u \nearrow \infty} f(x, u) = \infty \quad \text{uniformly in } \bar{\Omega},$$

and

$$\partial_u f(\cdot, u) > 0 \quad \text{for all } u \geqslant 0. \tag{1.2}$$

It should be noted that

$$f(\cdot, 0) \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R})$$

and that there is no sign restriction on $f(\cdot, 0)$ in Ω. Moreover, (1.2) implies

$$f(\cdot, 0) = \inf_{\xi > 0} f(\cdot, \xi). \tag{1.3}$$

As far as the weight functions $\mathcal{X}, V \in L_\infty(\Omega)$ are concerned, it is assumed that

$$\mathcal{X}, V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega),$$

where $\mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ is the class of nonnegative potentials introduced by the following definition.

DEFINITION 1.1. – *Given $a \in L^+_\infty(\Omega)$ ($a \in L_\infty(\Omega)$ such that $a \geqslant 0$), it is said that*

$$a \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$$

if an open subset Ω^0_a of Ω and a compact subset K_a of $\bar{\Omega}$ with Lebesgue measure zero exist for which

$$K_a \cap (\bar{\Omega}^0_a \cup \Gamma_1) = \emptyset, \tag{1.4}$$

$$\Omega^+_a := \{x \in \Omega:\ a(x) > 0\} = \Omega \setminus (\bar{\Omega}^0_a \cup K_a), \tag{1.5}$$

and each of the following four conditions is satisfied:

($\mathcal{A}_1$) *Ω^0_a possesses finitely many components of class $\mathcal{C}^2$, say $\Omega^{0,j}_a$, $1 \leqslant j \leqslant m$, such that*

$$\bar{\Omega}^{0,i}_a \cap \bar{\Omega}^{0,j}_a = \emptyset \quad \textit{if } i \neq j$$

and

$$\operatorname{dist}\big(\Gamma_1, \partial\Omega_a^0 \cap \Omega\big) > 0. \tag{1.6}$$

Thus, if we denote by Γ_1^i, $1 \leqslant i \leqslant n_1$, *the components of* Γ_1, *then for each* $1 \leqslant i \leqslant n_1$ *either* $\Gamma_1^i \subset \partial\Omega_a^0$ *or else* $\Gamma_1^i \cap \partial\Omega_a^0 = \emptyset$. *Moreover, if* $\Gamma_1^i \subset \partial\Omega_a^0$, *then* Γ_1^i *must be a component of* $\partial\Omega_a^0$. *Indeed, if* $\Gamma_1^i \cap \partial\Omega_a^0 \neq \emptyset$ *but* Γ_1^i *is not a component of* $\partial\Omega_a^0$, *then* $\operatorname{dist}(\Gamma_1^i, \partial\Omega_a^0 \cap \Omega) = 0$.

$(\mathcal{A}_2)$ *Let* $\{i_1, \ldots, i_p\}$ *denote the subset of* $\{1, \ldots, n_1\}$ *for which*

$$\Gamma_1^j \cap \partial\Omega_a^0 = \emptyset \quad \Longleftrightarrow \quad j \in \{i_1, \ldots, i_p\}.$$

Then, a *is bounded away from zero in any compact subset of*

$$\Omega_a^+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}.$$

Note that if $\Gamma_1 \subset \partial\Omega_a^0$, *then we are only imposing* a *to be bounded away from zero in any compact subset of* Ω_a^+.

$(\mathcal{A}_3)$ *Let* Γ_0^i, $1 \leqslant i \leqslant n_0$, *denote the components of* Γ_0, *and let* $\{i_1, \ldots, i_q\}$ *be the subset of* $\{1, \ldots, n_0\}$ *for which*

$$\big(\partial\Omega_a^0 \cup K_a\big) \cap \Gamma_0^j \neq \emptyset \quad \Longleftrightarrow \quad j \in \{i_1, \ldots, i_q\}.$$

Then, a *is bounded away from zero on any compact subset of*

$$\Omega_a^+ \cup \left[\bigcup_{j=1}^{q} \Gamma_0^{i_j} \setminus \big(\partial\Omega_a^0 \cup K_a\big)\right].$$

Note that if $(\partial\Omega_a^0 \cup K_a) \cap \Gamma_0 = \emptyset$, *then we are only imposing that* a *is bounded away from zero on any compact subset of* Ω_a^+.

$(\mathcal{A}_4)$ *For any* $\eta > 0$ *there exist a natural number* $\ell_a(\eta) \geqslant 1$ *and* $\ell_a(\eta)$ *open subsets of* $\mathbb{R}^N$, G_j^η, $1 \leqslant j \leqslant \ell_a(\eta)$, *with* $|G_j^\eta| < \eta$, $1 \leqslant j \leqslant \ell_a(\eta)$, *such that*

$$\overline{G}_i^\eta \cap \overline{G}_j^\eta = \emptyset \quad \text{if } i \neq j, \qquad K_a \subset \bigcup_{j=1}^{\ell_a(\eta)} G_j^\eta,$$

and, for each $1 \leqslant j \leqslant \ell_a(\eta)$, *the open set* $G_j^\eta \cap \Omega$ *is connected and of class* $\mathcal{C}^2$.

Subsequently, it will be said that $a \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$ *if* $a \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ *and* $\Omega_a^0 = \emptyset$.

Remark 1.2. – When $a \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$, we have that

$$K_a \cap \Gamma_1 = \emptyset \quad \wedge \quad \Omega_a^+ := \big\{x \in \Omega\colon\ a(x) > 0\big\} = \Omega \setminus K_a.$$

Moreover, if we denote by Γ_0^i, $1 \leqslant i \leqslant n_0$, the components of Γ_0 and by $\{i_1, \ldots, i_q\}$ the subset of $\{1, \ldots, n_0\}$ for which

$$K_a \cap \Gamma_0^j \neq \emptyset \quad \Longleftrightarrow \quad j \in \{i_1, \ldots, i_q\},$$

then, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ are satisfied if, and only if, a is bounded away from zero on compact subsets of

$$\Omega_a^+ \cup \Gamma_1 \cup \left(\bigcup_{j=1}^q \Gamma_0^{i_j} \setminus K_a \right).$$

If, in addition, $K_a \cap \Gamma_0 = \emptyset$, then $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ are satisfied if, and only if, a is bounded away from zero on compact subsets of $\Omega_a^+ \cup \Gamma_1$.

Also, this paper assumes that Ω_V^0 is connected and

$$\Gamma_0^0 := \partial\Omega_V^0 \setminus \Gamma_1 \subset \Omega, \quad \operatorname{dist}(\Gamma_1, \Gamma_0^0) > 0. \tag{1.7}$$

Note that, since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, the second relation of (1.7) follows from (1.6). As an immediate consequence from (1.7), for each $1 \leqslant i \leqslant n_1$ either $\Gamma_1^i \subset \partial\Omega_V^0$ or else $\Gamma_1^i \cap \partial\Omega_V^0 = \emptyset$. Moreover, Γ_1^i must be a component of $\partial\Omega_V^0$ if $\Gamma_1^i \subset \partial\Omega_V^0$. Assumption (1.7) allow us to apply [7, Theorem 4.2] (cf. Theorem 2.19 of Section 2 here in). Subsequently, for any $\delta \geqslant 0$ sufficiently small, Ω_V^δ will stand for the open set

$$\Omega_V^\delta := \Omega_V^0 \cup \{x \in \Omega\colon \operatorname{dist}(x, \Gamma_0^0) < \delta\}$$

and we assume that there is a sequence υ_n, $n \geqslant 1$, such that $\lim_{n\to\infty} \upsilon_n = 0$ for which some of the general assumptions (a)–(d) or (e) of Theorem 2.19 of Section 2 with

$$(a, \Omega_0, \Omega_n) = (\mathcal{X}, \Omega_V^0, \Omega_V^{\upsilon_n}), \quad n \geqslant 1,$$

are satisfied. Moreover, we also assume that, for each $\delta \geqslant 0$ sufficiently small,

$$\mathcal{X} \in \mathcal{A}_{\partial\Omega_V^\delta \setminus \Gamma_1, \partial\Omega_V^\delta \cap \Gamma_1}(\Omega_V^\delta). \tag{1.8}$$

Throughout this paper, (1.1) will be refereed to as problem $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$, and $\Lambda[\gamma, \Omega, \mathfrak{B}(b)]$ will stand for the set of values of $\lambda \in \mathbb{R}$ for which $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ possesses a positive solution. Thanks to the main result of [5], $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ has a unique positive solution if $\lambda \in \Lambda[\gamma, \Omega, \mathfrak{B}(b)]$. Throughout this paper such a solution will be denoted by

$$u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}.$$

To state our main result we need to introduce some notation. Given any proper subdomain Ω_0 of Ω of class $\mathcal{C}^2$ satisfying

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0$$

we shall denote by $\mathfrak{B}(b,\Omega_0)$ the boundary operator defined from $\mathfrak{B}(b)$ through

$$\mathfrak{B}(b,\Omega_0) := \begin{cases} \mathfrak{D} & \text{on } \partial\Omega_0 \cap \Omega, \\ \mathfrak{B}(b) & \text{on } \partial\Omega_0 \cap \partial\Omega. \end{cases}$$

The main result of this paper reads as follows.

THEOREM 1.3. – *Beside all previous general assumptions, suppose the following conditions hold*:

(1) $\lambda \in \Lambda[0,\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]$,

(2) $\gamma_0 > 0$ *exists for which*

$$\lambda \in \bigcap_{\gamma \geqslant \gamma_0} \Lambda\big[\gamma,\Omega,\mathfrak{B}(b)\big].$$

(3) *For each* $1 \leqslant i \leqslant N$,

$$\nu_i := \sum_{j=1}^{N} \alpha_{ij} n_j \quad \textit{on } \Gamma_1 \cap \partial\Omega_V^0.$$

Then, for each $p \in [1,\infty)$,

$$\lim_{\gamma \nearrow \infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]}\|_{L_p(\Omega_V^0)} = 0 \tag{1.9}$$

and

$$\lim_{\gamma \nearrow \infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}\|_{L_\infty(K)} = 0 \tag{1.10}$$

in any compact subset K *of* $\overline{\Omega} \setminus \overline{\Omega}_V^0$. *In particular,*

$$\lim_{\gamma \nearrow \infty} u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} = \begin{cases} u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]} & \textit{in } \Omega_V^0, \\ 0 & \textit{in } \Omega \setminus \Omega_V^0 \end{cases} \quad \textit{a.e. in } \Omega.$$

This theorem provides us with a substantially sharper version of [16, Theorem 4.1], where a very special case was treated. No other result of this nature seems to available in the mathematical literature. The proof of Theorem 1.3 is based upon the construction of an adequate supersolution of problem $P[\gamma,\lambda,\Omega,\mathfrak{B}(b)]$ for $\gamma > 0$ sufficiently large. Such construction is extremely delicate, since it contains a number of very fine technical details. In constructing these supersolutions one should slightly enlarge the vanishing set Ω_V^0 of the potential V in the domain Ω and it is in this precise moment when we need to use the results on continuous dependence with respect to the underlying domain of the positive solutions of $P[\gamma,\lambda,\Omega,\mathfrak{B}(b)]$. Those results, coming from [7], will be collected in Section 2.

An outline of this paper is as follows. In Section 2 we fix the main notations and give some previous results – more or less known – that are going to be used throughout this paper. In Section 3 we prove Theorem 1.3. In Section 4 we give some sufficient conditions so that Theorem 1.3 can be applied. Finally, in Section 5 we use Theorem 1.3

to show that in the presence of refuge areas two competing species will coexist provided their reproduction rates are sufficiently large. How large those rates must be will be ascertained in terms of the principal eigenvalues of some elliptic operators supported in the refuges of the species. Quite strikingly, the critical reproduction rates are independent of the aggression caused by competition. Further, it will be shown that as soon as the competition level grows the corresponding species must concentrate in its refuge area. Actually, they must segregate toward their respective refuges as the "amplitude" of the competitive interaction becomes large. So, in the presence of a refuge, the stress caused by competition forces the species to concentrate in its refuge area. In obtaining all those sort of biological principles we will use a general class of Lotka–Volterra competing species models with diffusion and transports effects.

Competition, as most ecologists employ the word, means the active demand by a number of individuals of the same species – intraspecific competition – or members of a number of species at the same trophic level – interspecific competition – for a common resource or requirement that is actually, or potentially, limiting, [4,27]. It is commonly agreed that this definition is consistent with the assumptions of the Lotka–Volterra equations, which still seems to conform the basis of the mathematical theory of competition. So, our results might have a significant value from the point of view of mathematical biology. Actually, the model is providing us with an idealized behavior, apparently described for the first time, against which reality can be judged and measured.

The weakest part of those models from the modeling perspective is the diffusion term. Nevertheless, although filled with hard to justify (or even doubtful) hypothesis, the competition Lotka–Volterra model does not suffer so much faults from the point of view of population dynamics.

It should be noted that the *concentration principle* described in Section 5 cannot occur in *homogeneous models*, but exclusively in heterogeneous ones. The mathematical difficulties that one must overcome to deal with degenerate spatially heterogeneous problems might explain the lack of mathematical results in that direction (cf., e.g., [20,22], and the references there in).

2. Preliminaries, notations and previous results

This section fixes some notations and collects some of the main results of [1,3,5,6] and [7]; those results will be used in subsequent sections.

For each $p>1$ we consider

$$W^2_{p,\mathfrak{B}(b)}(\Omega):=\{u\in W^2_p(\Omega)\colon\ \mathfrak{B}(b)u=0\},$$
$$W^2_{\mathfrak{B}(b)}(\Omega):=\bigcap_{p>1}W^2_{p,\mathfrak{B}(b)}(\Omega)\subset H^2(\Omega),$$

and use the natural product order in $L_p(\Omega)\times L_p(\partial\Omega)$,

$$(f_1,g_1)\geqslant(f_2,g_2)\quad\Longleftrightarrow\quad f_1\geqslant f_2\ \wedge\ g_1\geqslant g_2.$$

It will be said that $(f_1,g_1)>(f_2,g_2)$ if $(f_1,g_1)\geqslant(f_2,g_2)$ and $(f_1,g_1)\neq(f_2,g_2)$.

Since $b \in \mathcal{C}(\Gamma_1)$, it follows from [21] that, for each $p > 1$,

$$\mathfrak{B}(b) \in \mathcal{L}\big(W_p^2(\Omega); W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1)\big).$$

Moreover, for any $\mathcal{P} \in L_\infty(\Omega)$ the linear eigenvalue problem

$$\begin{cases} (\mathfrak{L} + \mathcal{P})\varphi = \lambda\varphi & \text{in } \Omega, \\ \mathfrak{B}(b)\varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

possesses a least real eigenvalue, denoted in the sequel by $\sigma[\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b)]$ and called the *principal eigenvalue of* $(\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{B}(b))$. The principal eigenvalue is simple and associated with it there is a positive eigenfunction, unique up to multiplicative constants; this eigenfunction is called the *principal eigenfunction* of $(\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{B}(b))$. Thanks to [1, Theorem 12.1], the principal eigenfunction, subsequently denoted by φ, satisfies

$$\varphi \in W^2_{\mathfrak{B}(b)}(\Omega) \subset H^2(\Omega)$$

and it is *strongly positive in* Ω in the sense that $\varphi(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial_\nu\varphi(x) < 0$ for each $x \in \Gamma_0$. Moreover, $\sigma[\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{B}(b)]$ is the unique eigenvalue of (2.1) possessing a positive eigenfunction, and it is dominant in the sense that

$$\mathrm{Re}\,\sigma > \sigma\big[\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{B}(b)\big]$$

for any other eigenvalue σ of (2.1). Furthermore, setting

$$(\mathfrak{L} + \mathcal{P})_p := (\mathfrak{L} + \mathcal{P})|_{W^2_{p,\mathfrak{B}(b)}(\Omega)},$$

we have that, for each $\omega > -\sigma[\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{B}(b)]$ and $p > N$, the operator

$$\big[\omega + (\mathfrak{L} + \mathcal{P})_p\big]^{-1} \in \mathcal{L}\big(L_p(\Omega)\big)$$

is a positive, compact and irreducible (cf. [23, V.7.7]).

Throughout this paper, given any proper subdomain Ω_0 of Ω of class $\mathcal{C}^2$ with

$$\mathrm{dist}\big(\Gamma_1, \partial\Omega_0 \cap \Omega\big) > 0, \tag{2.2}$$

we shall denote by $\mathfrak{B}(b, \Omega_0)$ the boundary operator defined from $\mathfrak{B}(b)$ through

$$\mathfrak{B}(b, \Omega_0) := \begin{cases} \mathfrak{D} & \text{on } \partial\Omega_0 \cap \Omega, \\ \mathfrak{B}(b) & \text{on } \partial\Omega_0 \cap \partial\Omega. \end{cases} \tag{2.3}$$

When $\Omega_0 = \Omega$ we set

$$\mathfrak{B}(b, \Omega) := \mathfrak{B}(b).$$

It should be noted that if $\overline{\Omega}_0 \subset \Omega$, then $\partial\Omega_0 \subset \Omega$ and, hence, $\mathfrak{B}(b, \Omega_0) = \mathfrak{D}$, by definition. Also, we will denote by $\sigma[\mathfrak{L} + \mathcal{P}, \Omega_0, \mathfrak{B}(b, \Omega_0)]$ the principal eigenvalue of the linear boundary value problem

$$\begin{cases} (\mathfrak{L}+\mathcal{P})\psi = \lambda\psi & \text{in } \Omega_0, \\ \mathfrak{B}(b,\Omega_0)\psi = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{2.4}$$

We now recall the concept of *principal eigenvalue* for a domain with several components.

DEFINITION 2.1. – *Suppose Ω_0 is an open subset of Ω with a finite number of components of class $\mathcal{C}^2$, say Ω_0^j, $1 \leqslant j \leqslant m$, such that $\overline{\Omega}_0^i \cap \overline{\Omega}_0^j = \emptyset$ if $i \neq j$ and*

$$\operatorname{dist}\big(\Gamma_1, \partial\Omega_0 \cap \Omega\big) > 0. \tag{2.5}$$

Then, the principal eigenvalue of $(\mathfrak{L}+\mathcal{P}, \Omega_0, \mathfrak{B}(b,\Omega_0))$ is defined through

$$\sigma\big[\mathfrak{L}+\mathcal{P}, \Omega_0, \mathfrak{B}(b,\Omega_0)\big] := \min_{1\leqslant j\leqslant m} \sigma\big[\mathfrak{L}+\mathcal{P}, \Omega_0^j, \mathfrak{B}\big(b,\Omega_0^j\big)\big]. \tag{2.6}$$

Remark 2.2. – Since Ω_0 is of class $\mathcal{C}^2$, it follows from (2.5) that each of the principal eigenvalues $\sigma[\mathfrak{L}+\mathcal{P}, \Omega_0^j, \mathfrak{B}(b,\Omega_0^j)]$, $1 \leqslant j \leqslant m$, is well defined. This shows the consistency of Definition 2.1.

Suppose $p > N$ and $\mathcal{P} \in L_\infty(\Omega)$. Then, a function $\bar{u} \in W_p^2(\Omega)$ is said to be a *positive strict supersolution* of $(\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b))$ if

$$\bar{u} \geqslant 0 \quad \wedge \quad \big((\mathfrak{L}+\mathcal{P})\bar{u}, \mathfrak{B}(b)\bar{u}\big) > 0.$$

A function $u \in W_p^2(\Omega)$ is said to be *strongly positive* if $u(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial_\beta u(x) < 0$ for each $x \in \Gamma_0$ satisfying $u(x) = 0$ and any outward pointing nowhere tangent vector field $\beta \in \mathcal{C}^1(\Gamma_0; \mathbb{R}^N)$. Finally, $(\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b))$ is said to satisfy the *strong maximum principle* if $p > N$, $u \in W_p^2(\Omega)$, and $((\mathfrak{L}+\mathcal{P})u, \mathfrak{B}(b)u) > 0$ imply that u is strongly positive. It should be recalled that for any $p > N$

$$W_p^2(\Omega) \hookrightarrow \mathcal{C}^{2-N/p}\big(\overline{\Omega}\big) \tag{2.7}$$

and that any function $u \in W_p^2(\Omega)$ is a.e. in Ω twice differentiable (cf., e.g., [25, Theorem VIII.1]).

The following characterization of the strong maximum principle provides us with one of the main technical tools to make most of the comparisons of this paper. It goes back to [17,18], thought the version given here comes from [3].

THEOREM 2.3. – *For any $\mathcal{P} \in L_\infty(\Omega)$ the following assertions are equivalent*:

- $\sigma[\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b)] > 0$;
- $(\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b))$ *possesses a positive strict supersolution*;
- $(\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b))$ *satisfies the strong maximum principle.*

Now, we collect some of the main properties of $\sigma[\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b)]$; they are taken from [6, Proposition 3.2, 3.3].

PROPOSITION 2.4. – *Let Ω_0 be a proper subdomain of Ω of class $\mathcal{C}^2$ satisfying* (2.2). *Then,*

$$\sigma\left[\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b)\right] < \sigma\left[\mathfrak{L}+\mathcal{P}, \Omega_0, \mathfrak{B}(b, \Omega_0)\right],$$

where $\mathfrak{B}(b, \Omega_0)$ is the boundary operator defined by (2.3).

PROPOSITION 2.5. – *Let $\mathcal{P}_1, \mathcal{P}_2 \in L_\infty(\Omega)$ such that $\mathcal{P}_1 < \mathcal{P}_2$ in a set of positive Lebesgue measure. Then,*

$$\sigma\left[\mathfrak{L}+\mathcal{P}_1, \Omega, \mathfrak{B}(b)\right] < \sigma\left[\mathfrak{L}+\mathcal{P}_2, \Omega, \mathfrak{B}(b)\right].$$

A crucial result for the mathematical analysis carried out in the next sections is the continuous dependence of the principal eigenvalue $\sigma[\mathfrak{L}+\mathcal{P}, \Omega, \mathfrak{B}(b)]$ with respect to the perturbations of the domain around its Dirichlet boundary. To state it we need introducing the following concepts.

DEFINITION 2.6. – *Let Ω_0 be a bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ such that $\Gamma_0^0 \cap \Gamma_1 = \emptyset$, where Γ_0^0 satisfies the same requirements as Γ_0, and $\Omega_n,\ n \geqslant 1$, a sequence of bounded domains of $\mathbb{R}^N$ with boundaries $\partial\Omega_n = \Gamma_0^n \cup \Gamma_1$ of class $\mathcal{C}^2$ such that*

$$\Gamma_0^n \cap \Gamma_1 = \emptyset, \quad n \geqslant 1,$$

and $\Gamma_0^n,\ n \geqslant 1$, satisfies the same requirements as Γ_0. Then, it is said that Ω_n converges to Ω_0 from the exterior if, for each $n \geqslant 1$,

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n \quad \textit{and} \quad \bigcap_{n=1}^{\infty} \overline{\Omega}_n = \overline{\Omega}_0.$$

Throughout the remaining of this paper it is said that $\nu = (\nu_1, \ldots, \nu_N)$ is the *conormal vector field* if

$$\nu_i := \sum_{j=1}^{N} \alpha_{ij} n_j, \quad 1 \leqslant i \leqslant N, \tag{2.8}$$

where $n = (n_1, \ldots, n_N)$ is the outward unit normal to Ω on Γ_1. In this case ∂_ν will be called the *conormal derivative*. Let $\mu > 0$ denote the ellipticity constant of $\mathfrak{L}$ and assume that (2.8) is satisfied. Then,

$$\langle \nu, n\rangle = \sum_{i,j=1}^{N} \alpha_{ij} n_j n_i \geqslant \mu |n|^2 = \mu > 0$$

and, therefore, ν is an outward pointing nowhere tangent vector field. It should be noted that $\nu \in \mathcal{C}^1(\Gamma_1; \mathbb{R}^N)$, since $\alpha_{ij} \in \mathcal{C}^1(\overline{\Omega}),\ 1 \leqslant i, j \leqslant N$, and Γ_1 is of class $\mathcal{C}^2$. It is time for establishing the main result about the continuous dependence of the principal eigenvalue with respect to the perturbations of the domain around its Dirichlet boundary; it goes back to [6, Theorem 7.1].

THEOREM 2.7 (Exterior Continuous Dependence). – *Suppose* (2.8) *and consider* $\mathcal{P} \in L_\infty(\Omega)$. *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathcal{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1, \qquad \Gamma_0^0 \cap \Gamma_1 = \emptyset,$$

where Γ_0^0 *satisfies the same requirements as* Γ_0, *and let* $\Omega_n \subset \Omega$, $n \geqslant 1$, *be a sequence of bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the exterior. For each* $n \geqslant 0$, *let* $\mathfrak{B}_n(b)$ *denote the boundary operator defined through*

$$\mathfrak{B}_n(b)u := \begin{cases} u & \text{on } \Gamma_0^n, \\ \partial_\nu u + bu & \text{on } \Gamma_1, \end{cases}$$

where

$$\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1, \quad n \geqslant 0,$$

and denote by $(\sigma[\mathfrak{L} + \mathcal{P}, \Omega_n, \mathfrak{B}_n(b)], \varphi_n)$ *the principal eigen-pair associated with* $(\mathfrak{L} + \mathcal{P}, \Omega_n, \mathfrak{B}_n(b))$, *where the principal eigenfunction* φ_n *is assumed to be normalized so that*

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1, \quad n \geqslant 0.$$

Then, $\varphi_0 \in W^2_{\mathfrak{B}_0(b)}(\Omega_0)$ *and*

$$\lim_{n\to\infty} \sigma\left[\mathfrak{L} + \mathcal{P}, \Omega_n, \mathfrak{B}_n(b)\right] = \sigma\left[\mathfrak{L} + \mathcal{P}, \Omega_0, \mathfrak{B}_0(b)\right], \qquad \lim_{n\to\infty} \|\varphi_n|_{\Omega_0} - \varphi_0\|_{H^1(\Omega_0)} = 0.$$

The following result establishes that $(\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{D})$ satisfies the strong maximum principle if $|\Omega|$ is sufficiently small. It goes back to [17, Theorem 5.1] and [6, Theorem 10.1]. Hereafter, $|\cdot|$ will stand for the Lebesgue measure in $\mathbb{R}^N$.

THEOREM 2.8. – *Suppose* $\mathcal{P} \in L_\infty(\Omega)$ *and*

$$\alpha_{ij} \in \mathcal{C}(\overline{\Omega}) \cap W^1_\infty(\Omega), \quad 1 \leqslant i, j \leqslant N. \tag{2.9}$$

Then

$$\liminf_{|\Omega|\searrow 0} \sigma[\mathfrak{L} + \mathcal{P}, \Omega, \mathfrak{D}]|\Omega|^{2/N} \geqslant \mu\Sigma_1|B_1|^{2/N},$$

where

$$B_1 := \{x \in \mathbb{R}^N\colon\ |x| < 1\}, \qquad \Sigma_1 := \sigma[-\Delta, B_1, \mathfrak{D}], \tag{2.10}$$

and $\mu > 0$ *is the ellipticity constant of* $\mathfrak{L}$ *in* Ω.

Another fundamental result for the mathematical analysis carried out in the subsequent sections is the next one; it goes back to [17, Theorem 6.2] and [6, Theorem 11.4].

THEOREM 2.9. – *Assume that* (2.8) *is satisfied on* $\Gamma_1 \cap \partial\Omega_V^0$. *Then*

$$\lim_{\gamma\nearrow\infty} \sigma\left[\mathfrak{L} + \gamma V, \Omega, \mathfrak{B}(b)\right] = \sigma\left[\mathfrak{L}, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)\right].$$

The proof of this result can be easily adapted to show that in the particular case when $\Omega_V^0 = \emptyset$ the next theorem follows; it should be noted that condition (2.8) is not required and that the regularity of the coefficients of $\mathfrak{L}$ is weaker than the regularity required in Theorem 2.9.

THEOREM 2.10. – *Suppose* (2.9) *and* $V \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$. *Then*

$$\lim_{\gamma \nearrow \infty} \sigma\left[\mathfrak{L} + \gamma V, \Omega, \mathfrak{B}(b)\right] = \infty.$$

Now, we shall state the concept of *strong solution* for problem $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ and collect the results of [5] that characterize the existence of positive solutions. A function u is said to be a *strong solution* of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ if $u \in W_p^2(\Omega)$ for some $p > N$ and it satisfies (1.1). A function u is said to be a *positive solution* of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ if it is a strong solution and $u > 0$ in Ω. The solutions of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ will be regarded as couples (λ, u). Accordingly, it will be said that (λ_0, u_0) is a solution of (1.1) if u_0 is a solution of $P[\gamma, \lambda_0, \Omega, \mathfrak{B}(b)]$. The following result is [7, Lemma 2.12].

LEMMA 2.11. – *Suppose* (λ, u) *is a positive solution of* (1.1). *Then,* u *is strongly positive in* Ω *and* $u \in W^2_{\mathfrak{B}(b)}(\Omega)$. *Moreover,*

$$\sigma\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, u), \Omega, \mathfrak{B}(b)\right] = 0. \tag{2.11}$$

In particular, $u \in \mathcal{C}^{1,\vartheta}(\bar{\Omega})$ *for each* $\vartheta \in (0, 1)$, *and it is a.e. in* Ω *twice differentiable.*

The following result characterizes the existence of positive solutions for (1.1); it goes back to [5, Theorem 4.2].

THEOREM 2.12. – *Suppose* (2.8) *on* $\Gamma_1 \cap \partial\Omega^0_{\mathcal{X}}$. *Then,* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ *possesses a positive solution if, and only if,*

$$\sigma\left[\mathfrak{L} + \gamma V + \mathcal{X} f(\cdot, 0) - \lambda W, \Omega, \mathfrak{B}(b)\right] < 0 < \sigma\left[\mathfrak{L} + \gamma V - \lambda W, \Omega^0_{\mathcal{X}}, \mathfrak{B}\left(b, \Omega^0_{\mathcal{X}}\right)\right].$$

Moreover, the positive solution is unique if it exists. Subsequently, it will be denoted by

$$u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}.$$

Furthermore, for any $u_0 \in L_p(\Omega)$, $p > N/2$, *the evolutionary problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\mathfrak{L} + \gamma V)u = \lambda W u - \mathcal{X} f(\cdot, u)u & \text{in } \Omega \times (0, \infty), \\ \mathfrak{B}(b)u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases} \tag{2.12}$$

possesses a unique strong solution and, if we denote it by $\Phi_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}(x, t; u_0)$, *one has that*

$$\lim_{t \nearrow \infty} \left\| \Phi_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}(\cdot, t; u_0) - u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]} \right\|_{C^1_0(\bar{\Omega})} = 0.$$

Remark 2.13. – Imposing (2.8) on $\Gamma_1 \cap \partial\Omega^0_{\mathcal{X}}$ is not needed for the uniqueness.

Arguing as in the proof of Theorem 2.12 (cf. [5, Theorem 4.2]) the following result is easily obtained.

THEOREM 2.14. – *Suppose* $\mathcal{X} \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$. *Then,* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ *has a positive solution if, and only if,*

$$\sigma\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\right] < 0.$$

Moreover, the positive solution is unique if it exists; *subsequently denoted by*

$$u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}.$$

Furthermore, for any $u_0 \in L_p(\Omega)$, $p > \frac{N}{2}$, *the evolutionary problem* (2.12) *possesses a unique strong solution and, if we denote it by* $\Phi_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}(x, t; u_0)$,

$$\lim_{t \nearrow \infty} \left\| \Phi_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}(\cdot\,, t; u_0) - u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]} \right\|_{C^1_0(\bar{\Omega})} = 0.$$

In Theorem 2.14 condition (2.8) on $\Gamma_1 \cap \partial\Omega^0_{\mathcal{X}}$ is not required, since $\Omega^0_{\mathcal{X}} = \emptyset$. Now, we introduce the concept of *positive supersolution*.

DEFINITION 2.15. – *Given* $p > N$, *it is said that* $u \in W^2_p(\Omega)$ *is a positive supersolution* (*resp. positive subsolution*) *of* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ *if* $u > 0$ *and*

$$\left(\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, u)\right]u, \mathfrak{B}(b)u\right) \geqslant 0$$
$$\left(\textit{resp.} \left(\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, u)\right]u, \mathfrak{B}(b)u\right) \leqslant 0\right).$$

The following comparison result is crucial in our mathematical analysis; it is [7, Theorem 2.15].

THEOREM 2.16. – *Suppose* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ *possesses a positive solution,* $p > N$, *and let* $u \in W^2_p(\Omega)$ *be a positive supersolution* (*resp. subsolution*) *of* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$. *Then,*

$$u \geqslant u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]} \quad (\textit{resp. } u \leqslant u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]}).$$

The following results are [7, Theorem 3.1] and [7, Corollary 3.2], respectively. They collect some crucial properties of the families of potentials $\mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ and $\mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$. Subsequently, if $a \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, $\tilde{\Omega} \subset \Omega$ is an open subset satisfying

$$\operatorname{dist}(\partial\Omega, \partial\tilde{\Omega} \cap \Omega) > 0$$

and

$$a \in \mathcal{A}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega}), \quad \tilde{\Gamma}_1 := \Gamma_1 \cap \partial\tilde{\Omega}, \; \tilde{\Gamma}_0 := \partial\tilde{\Omega} \setminus \tilde{\Gamma}_1,$$

we will denote by $[\tilde{\Omega}]^0_a$ the maximal open subset of $\tilde{\Omega}$ where the potential a vanishes.

THEOREM 2.17. – *Suppose* $a \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ *and let* $\tilde{\Omega}$ *be an open subdomain of* Ω *of class* $\mathcal{C}^2$ *such that*

$$\operatorname{dist}(\partial\Omega, \partial\tilde{\Omega} \cap \Omega) > 0. \tag{2.13}$$

Then, each of the following sets

$$\tilde{\Gamma}_0 := \partial\tilde{\Omega} \cap (\Gamma_0 \cup \Omega), \qquad \tilde{\Gamma}_1 := \partial\tilde{\Omega} \setminus \tilde{\Gamma}_0 = \partial\tilde{\Omega} \cap \Gamma_1,$$

is closed and open in $\partial\tilde{\Omega}$*. Moreover, the following assertions are true*:

(a) *If* $\Omega_a^0 \cap \tilde{\Omega} \neq \emptyset$ *is of class* $\mathcal{C}^2$ *and*

$$\partial\tilde{\Omega} \cap \Omega \cap \partial\big(\Omega_a^0 \cap \tilde{\Omega}\big) = \partial\tilde{\Omega} \cap \Omega \cap \bar{\Omega}_a^0, \tag{2.14}$$

then $a \in \mathcal{A}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$ *and*

$$[\tilde{\Omega}]_a^0 = \Omega_a^0 \cap \tilde{\Omega}.$$

(b) *Suppose* $\Omega_a^0 \cap \tilde{\Omega} = \emptyset$ *and*

$$\Gamma \cap K_a \neq \emptyset \quad \Longrightarrow \quad \Gamma \setminus K_a \subset \Omega_a^+$$

for any component Γ *of* $\partial\tilde{\Omega} \cap \Omega$*. Then,* $a \in \mathcal{A}^{\pm}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\tilde{\Omega})$*. In particular,*

$$a \in \mathcal{A}^{+}_{\Gamma_0,\Gamma_1}(\Omega) \quad \Longrightarrow \quad a \in \mathcal{A}^{\pm}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}\big(\tilde{\Omega}\big).$$

COROLLARY 2.18. – *Suppose* $a, b \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ *with* Ω_b^0 *connected and*

$$\operatorname{dist}\big(\Gamma_0, \partial\Omega_b^0 \cap \Omega\big) > 0.$$

Then,

$$\tilde{\Gamma}_0 := \partial\Omega_b^0 \cap (\Gamma_0 \cup \Omega) \quad \textit{and} \quad \tilde{\Gamma}_1 := \partial\Omega_b^0 \setminus \tilde{\Gamma}_0 = \partial\Omega_b^0 \cap \Gamma_1$$

are closed and open sets of class $\mathcal{C}^2$*, and each of the following assertions is true*:

(a) *If* $\Omega_a^0 \cap \Omega_b^0 \neq \emptyset$ *is of class* $\mathcal{C}^2$ *and*

$$\partial\Omega_b^0 \cap \Omega \cap \partial\big(\Omega_a^0 \cap \Omega_b^0\big) = \partial\Omega_b^0 \cap \Omega \cap \overline{\Omega}_a^0, \tag{2.15}$$

then $a \in \mathcal{A}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\Omega_b^0)$ *and*

$$\big[\Omega_b^0\big]_a^0 = \Omega_a^0 \cap \Omega_b^0.$$

(b) *Suppose* $\Omega_a^0 \cap \Omega_b^0 = \emptyset$ *and*

$$\Gamma \cap K_a \neq \emptyset \quad \Longrightarrow \quad \Gamma \setminus K_a \subset \Omega_a^+$$

for any component Γ *of* $\partial\Omega_b^0 \cap \Omega$*. Then,* $a \in \mathcal{A}^{\pm}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\Omega_b^0)$*. In particular,*

$$a \in \mathcal{A}^{+}_{\Gamma_0,\Gamma_1}(\Omega) \quad \Longrightarrow \quad a \in \mathcal{A}^{\pm}_{\tilde{\Gamma}_0,\tilde{\Gamma}_1}(\Omega_b^0).$$

Another crucial result in obtaining Theorem 1.3 is the following theorem; it is [7, Theorem 4.2].

THEOREM 2.19. – *Suppose $a \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, let Ω_0 be a proper subdomain of Ω with boundary of class $\mathcal{C}^2$ such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 satisfies the same requirements as Γ_0, and let $\Omega_n \subset \Omega$, $n \geqslant 1$, be a sequence of bounded domains of $\mathbb{R}^N$ of class $\mathcal{C}^2$ converging to Ω_0 from the exterior such that

$$\operatorname{dist}(\partial\Omega, \partial\Omega_n \cap \Omega) > 0, \quad n \geqslant 0. \tag{2.16}$$

For each natural number $n \geqslant 0$ let $\mathfrak{B}_n(b)$ be the boundary operator defined by

$$\mathfrak{B}_n(\Omega) := \begin{cases} \mathfrak{D} & \text{on } \Gamma_0^n, \\ \mathfrak{B}(b) & \text{on } \Gamma_1, \end{cases}$$

where

$$\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1.$$

Then, the following assertions are true:

(a) *Suppose* (2.8) *on $\Gamma_1 \cap \partial\Omega_a^0$ and $\emptyset \neq \Omega_a^0 \subset \Omega_0$. Then, for each $n \geqslant 0$,*

$$a \in \bigcap_{n=0}^{\infty} \mathcal{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n) \quad \text{and} \quad [\Omega_n]_a^0 = \Omega_a^0,$$

where $[\Omega_n]_a^0$ is the corresponding open set of the definition of the class $\mathcal{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)$, $n \geqslant 0$. Suppose, in addition, that $a = \mathcal{X}$ and $\lambda \in \Lambda[\gamma, \Omega_0, \mathfrak{B}_0(b)]$. Then,

$$\lambda \in \bigcap_{n=0}^{\infty} \Lambda\big[\gamma, \Omega_n, \mathfrak{B}_n(b)\big].$$

(b) *Suppose $\overline{\Omega}_0 \cap \overline{\Omega}_a^0 = \emptyset$. Then, $a \in \mathcal{A}^+_{\Gamma_0^0,\Gamma_1}(\Omega_0)$. Moreover, $n_0 \in \mathbb{N}$ exists for which*

$$a \in \bigcap_{n=n_0}^{\infty} \mathcal{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n).$$

Furthermore,

$$\lambda \in \bigcap_{n=n_0}^{\infty} \Lambda\big[\gamma, \Omega_n, \mathfrak{B}_n(b)\big]$$

if $a = \mathcal{X}$ and $\lambda \in \Lambda[\gamma, \Omega_0, \mathfrak{B}_0(b)]$.

(c) *Suppose $\overline{\Omega}_a^0 \cap \overline{\Omega}_0 \neq \emptyset$, $\Omega_0 \cap \Omega_a^0 = \emptyset$, and $n_0 \in \mathbb{N}$ exists for which $\Omega_n \cap \Omega_a^0$ is of class $\mathcal{C}^2$ and*

$$\partial\Omega_n \cap \Omega \cap \partial\big(\Omega_a^0 \cap \Omega_n\big) = \partial\Omega_n \cap \Omega \cap \overline{\Omega}_a^0, \quad n \geqslant n_0. \tag{2.17}$$

Suppose, in addition, that

$$\Gamma \cap K_a \neq \emptyset \quad \Longrightarrow \quad \Gamma \setminus K_a \subset \Omega_a^+$$

for any component Γ *of* Γ_0^0. *Then,* $a \in \mathcal{A}^+_{\Gamma_0^0,\Gamma_1}(\Omega_0)$ *and*

$$a \in \bigcap_{n=n_0}^{\infty} \mathcal{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n), \quad [\Omega_n]_a^0 = \Omega_a^0 \cap \Omega_n, \quad n \geqslant n_0.$$

Suppose, in addition, that $a = \mathcal{X}$ *and* $\lambda \in \Lambda[\gamma, \Omega_0, \mathfrak{B}_0(b)]$. *Then,* $m_0 \in \mathbb{N}$, $m_0 \geqslant n_0$, *exists for which*

$$\lambda \in \bigcap_{n=m_0}^{\infty} \Lambda\big[\gamma, \Omega_n, \mathfrak{B}_n(b)\big].$$

(d) *Suppose* (2.8) *on* $\Gamma_1 \cap \partial[\Omega_0]_a^0$ *and*
(1) $\Omega_a^0 \cap \Omega_0 \neq \emptyset$ *is of class* $\mathcal{C}^2$,
(2) $\Omega_a^0 \cap (\Omega \setminus \Omega_0) \neq \emptyset$,
(3) $n_0 \in \mathbb{N}$ *exists such that* $\Omega_a^0 \cap \Omega_n$ *is a proper subdomain of* Ω *of class* $\mathcal{C}^2$ *if* $n \geqslant n_0$,
(4) (2.14) *is satisfied for any* $\tilde{\Omega} \in \{\Omega_0, \Omega_{n_0+j}\colon\ j \geqslant 0\}$.
Then, $m_0 \geqslant n_0$ *exists for which*

$$a \in \bigcap_{n=m_0}^{\infty} \mathcal{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n) \quad \wedge \quad [\Omega_n]_a^0 = \Omega_n \cap \Omega_a^0 \text{ if } n \in \{0, m_0 + j\colon\ j \geqslant 0\}.$$

Moreover, if, in addition, $a = \mathcal{X}$ *and* $\lambda \in \Lambda[\gamma, \Omega_0, \mathfrak{B}_0(b)]$, *then, for some* $\ell_0 \geqslant m_0$,

$$\lambda \in \bigcap_{n=\ell_0}^{\infty} \Lambda\big[\gamma, \Omega_n, \mathfrak{B}_n(b)\big].$$

(e) *Suppose* $a \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$, *i.e.,* $\Omega_a^0 = \emptyset$. *Then,*

$$a \in \bigcap_{n=0}^{\infty} \mathcal{A}^+_{\Gamma_0^n,\Gamma_1}(\Omega_n),$$

i.e., $a \in \mathcal{A}_{\Gamma_0^n,\Gamma_1}(\Omega_n)$ *and* $[\Omega_n]_a^0 = \emptyset$ *for each* $n \geqslant 0$. *Moreover,*

$$a = \mathcal{X}\ \wedge\ \lambda \in \Lambda\big[\gamma, \Omega_0, \mathfrak{B}_0(b)\big] \quad \Longrightarrow \quad \lambda \in \bigcap_{n=0}^{\infty} \Lambda\big[\gamma, \Omega_n, \mathfrak{B}_n(b)\big].$$

Furthermore, in any of the five previous cases, if $a = \mathcal{X}$, *then*

$$\lim_{n\to\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega_n,\mathfrak{B}_n(b)]}|_{\Omega_0} - u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega_0,\mathfrak{B}_0(b)]}\|_{H^1(\Omega_0)} = 0 \tag{2.18}$$

if

$$\lambda \in \Lambda\big[\gamma, \Omega_0, \mathfrak{B}_0(b)\big].$$

3. Proof of Theorem 1.3

This section proves Theorem 1.3. Subsequently, for any

$$a \in \mathcal{A}_{\partial\Omega_V^0 \setminus \Gamma_1, \partial\Omega_V^0 \cap \Gamma_1}(\Omega_V^0) \cap \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$$

satisfying $\Omega_a^0 \cap \Omega_V^0 \neq \emptyset$ we set

$$\Omega_{a,V}^0 := [\Omega_V^0]_a^0 = [\Omega_a^0]_V^0 = \Omega_a^0 \cap \Omega_V^0.$$

Our proof of Theorem 1.3 is based upon the following proposition.

PROPOSITION 3.1. – *Suppose*

$$\lambda \in \Lambda\left[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)\right] \tag{3.1}$$

and (2.8) *holds on* $\Gamma_1 \cap \partial\Omega_V^0$.

Then, $\delta_0 > 0$ *exists such that for each* $\delta \in (0, \delta_0)$ *there are a real number* $\Lambda(\delta) > 0$ *and a positive function* $\bar{u}_\delta$ *satisfying the following conditions*:

(i) $\bar{u}_\delta$ *is a positive strict supersolution of* $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ *for each* $\gamma > \Lambda(\delta)$.

(ii) *One has that*

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta - u_{[\mathfrak{L}, \lambda W, \mathcal{X}, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]}\|_{H^1(\Omega_V^0)} = 0 \tag{3.2}$$

and, for any compact subset $K \subset \overline{\Omega} \setminus \overline{\Omega}_V^0$,

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta\|_{L_\infty(K)} = 0. \tag{3.3}$$

In particular,

$$\lim_{\delta \searrow 0} \bar{u}_\delta = \begin{cases} u_{[\mathfrak{L}, \lambda W, \mathcal{X}, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]} & \text{in } \Omega_V^0, \\ 0 & \text{in } \Omega \setminus \Omega_V^0 \end{cases} \quad \text{a.e. in } \Omega. \tag{3.4}$$

Proof. – Firstly, we shall prove part (i) in case

$$\Gamma_0 \cap K_V = \emptyset. \tag{3.5}$$

Then, since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, we have that

$$K_V \cap \left(\overline{\Omega}_V^0 \cup \Gamma_1\right) = \emptyset$$

and, hence, (3.5) gives

$$K_V \cap \partial\Omega = \emptyset.$$

Thus, taking into account (1.7), we have that

$$K_V \subset \Omega, \qquad K_V \cap \overline{\Omega}_V^0 = \emptyset, \qquad \overline{\Omega}_V^0 \subset \Omega \cup \Gamma_1, \tag{3.6}$$

and, in particular,

$$\operatorname{dist}\big(\Gamma_0, \overline{\Omega}_V^0 \cup K_V\big) > 0, \qquad \operatorname{dist}(\Gamma_1, K_V) > 0, \qquad \operatorname{dist}\big(K_V, \overline{\Omega}_V^0\big) > 0. \tag{3.7}$$

Fix $\eta > 0$. Since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, it follows from $(\mathcal{A}_4)$, that there exist a natural number $\ell_V(\eta) \geqslant 1$ and $\ell_V(\eta)$ open sets

$$G_j^\eta \subset \mathbb{R}^N, \quad 1 \leqslant j \leqslant \ell_V(\eta),$$

such that

$$\begin{gathered} \big|G_j^\eta\big| < \eta, \quad 1 \leqslant j \leqslant \ell_V(\eta), \\ K_V \subset \bigcup_{j=1}^{\ell_V(\eta)} G_j^\eta \quad \wedge \quad \overline{G}_i^\eta \cap \overline{G}_j^\eta = \emptyset \quad \text{if } i \neq j \end{gathered}$$

and for each $1 \leqslant j \leqslant \ell_V(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathcal{C}^2$. Thanks to (3.6), the G_j^η's can be chosen so that

$$K_V \subset \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \subset \Omega, \qquad \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \cap \overline{\Omega}_V^0 = \emptyset. \tag{3.8}$$

Indeed, since

$$\operatorname{dist}\big(K_V, \overline{\Omega}_V^0 \cup \Gamma_0 \cup \Gamma_1\big) > 0,$$

an open set G exists such that

$$K_V \subset G, \qquad \overline{G} \subset \Omega, \qquad \overline{G} \cap \overline{\Omega}_V^0 = \emptyset,$$

and, hence, in order to have (3.8), it suffices considering $G \cap G_j^\eta$ instead of G_j^η, $1 \leqslant j \leqslant \ell_V(\eta)$.

Thanks to (3.8), there exist $\varepsilon := \varepsilon(\eta) > 0$ and $\ell_V(\eta)$ open sets $G_j^{\eta,\varepsilon}$, $1 \leqslant j \leqslant \ell_V(\eta)$, of class $\mathcal{C}^2$ such that

$$\overline{G}_j^\eta \subset G_j^{\eta,\varepsilon} \subset G_j^\eta + B_\varepsilon, \quad \big|G_j^{\eta,\varepsilon}\big| < 2\eta, \quad 1 \leqslant j \leqslant \ell_V(\eta), \tag{3.9}$$

and

$$K_V \subset \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \subset \bigcup_{j=1}^{\ell_V(\eta)} G_j^{\eta,\varepsilon} \subset \Omega, \qquad \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^{\eta,\varepsilon} \cap \overline{\Omega}_V^0 = \emptyset, \tag{3.10}$$

where, for any $\varrho > 0$, B_ϱ stands for the ball of radius ϱ centered at zero. Since

$$\lim_{\eta \searrow 0} \big|G_j^{\eta,\varepsilon}\big| = 0, \quad 1 \leqslant j \leqslant \ell_V(\eta),$$

it follows from Theorem 2.8 that $\eta_0 > 0$ exists such that for each $\eta \in (0, \eta_0)$ and $1 \leqslant j \leqslant \ell_V(\eta)$ we have that

$$\min_{1 \leqslant j \leqslant \ell_V(\eta)} \sigma\big[\mathfrak{L} + \mathcal{X} f(\cdot, 0) - \lambda W, G_j^{\eta,\varepsilon(\eta)}, \mathfrak{D}\big] > 0. \tag{3.11}$$

Subsequently we consider $\eta \in (0, \eta_0)$ fixed. For each $k \in \{0, 1\}$, let Γ_k^j, $1 \leqslant j \leqslant n_k$, denote the components of Γ_k. Let $\{i_1, \ldots, i_p\}$ denote the subset of $\{1, \ldots, n_1\}$ for which

$$\Gamma_1^j \cap \partial\Omega_V^0 = \emptyset \quad \Longleftrightarrow \quad j \in \{i_1, \ldots, i_p\}.$$

Since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, we find from $(\mathcal{A}_1)$, that Γ_1^j is a component of $\partial\Omega_V^0$ for each $j \in \{1, \ldots, n_1\} \setminus \{i_1, \ldots, i_p\}$. Then,

$$\Gamma_1 \cap \partial\Omega_V^0 = \bigcup_{j \in \{1,\ldots,n_1\}\setminus\{i_1,\ldots,i_p\}} \Gamma_1^j \quad \wedge \quad \bigcup_{j=1}^{p} \Gamma_1^{i_j} \cap \partial\Omega_V^0 = \emptyset. \tag{3.12}$$

In particular,

$$\operatorname{dist}\left(\bigcup_{j=1}^{p} \Gamma_1^{i_j}, \partial\Omega_V^0\right) > 0. \tag{3.13}$$

Subsequently, for each

$$\delta \in \big(0, \operatorname{dist}\big(\Gamma_0^0, \partial\Omega\big)\big)$$

we consider the open δ-neighborhood

$$\Omega_V^\delta := \Omega_V^0 \cup \big\{x \in \Omega\colon \operatorname{dist}\big(x, \Gamma_0^0\big) < \delta\big\}.$$

By definition, for any sequence δ_n, $n \geqslant 1$, such that

$$\lim_{n\to\infty} \delta_n = 0,$$

we have that the sequence

$$\Omega_n := \Omega_V^{\delta_n}, \quad n \geqslant 1,$$

converges to Ω_V^0 from the exterior as $n \to \infty$; it will be simply said that

$$\lim_{\delta \searrow 0} \Omega_V^\delta = \Omega_V^0 \quad \text{from the exterior.}$$

By construction, for each $\delta > 0$ sufficiently small, we have

$$\Gamma_0^0 \cup \Omega_V^0 \subset \Omega_V^\delta \subset \Omega \tag{3.14}$$

and

$$\partial\Omega_V^0 \cap \Gamma_1 = \partial\Omega_V^\delta \cap \Gamma_1 = \bigcup_{j \in \{1,\ldots,n_1\}\setminus\{i_1,\ldots,i_p\}} \Gamma_1^j. \tag{3.15}$$

Now, for each $\delta > 0$ sufficiently small, we consider the δ-neighborhoods

$$\begin{aligned} \mathcal{N}_\delta^{0,j} &:= \big(\Gamma_0^j + B_\delta\big) \cap \Omega, \quad 1 \leqslant j \leqslant n_0, \\ \mathcal{N}_\delta^{1,j} &:= \big(\Gamma_1^j + B_\delta\big) \cap \Omega, \quad j \in \{i_1, \ldots, i_p\}. \end{aligned} \tag{3.16}$$

Since

$$K_V \cap \left(\overline{\Omega}_V^0 \cup \Gamma_1\right) = \emptyset, \qquad \Omega_V^+ = \Omega \setminus \left(\overline{\Omega}_V^0 \cup K_V\right),$$

it follows from (1.7), (3.5), (3.10) and (3.12) that $\delta_1 > 0$ exists such that, for any $\delta \in (0, \delta_1)$,

$$\bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^{\eta,\varepsilon} \cap \overline{\Omega}_V^\delta = \emptyset, \qquad \partial\Omega_V^\delta \setminus \Gamma_1 \subset \Omega_V^+, \qquad \bigcup_{j=1}^{n_0} \overline{\mathcal{N}}_\delta^{0,j} \setminus \Gamma_0 \subset \Omega_V^+ \tag{3.17}$$

and

$$\left(\bigcup_{j=1}^{p} \overline{\mathcal{N}}_\delta^{1,i_j} \cup \bigcup_{j=1}^{n_0} \overline{\mathcal{N}}_\delta^{0,j}\right) \cap \left(\overline{\Omega}_V^\delta \cup \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^{\eta,\varepsilon}\right) = \emptyset. \tag{3.18}$$

Moreover, since $\Gamma_k^j \cap \Gamma_\ell^i = \emptyset$ if $(i,\ell) \neq (j,k)$, $\delta_2 \in (0, \delta_1)$ exists such that for each $0 < \delta < \delta_2$

$$\overline{\mathcal{N}}_\delta^{k,j} \cap \overline{\mathcal{N}}_\delta^{\ell,i} = \emptyset \quad \text{if } (i,\ell) \neq (j,k),\ k,\ell \in \{0,1\}. \tag{3.19}$$

Furthermore, since

$$\lim_{\delta \searrow 0} \left|\mathcal{N}_\delta^{0,j}\right| = 0, \quad 1 \leqslant j \leqslant n_0,$$

it follows from Theorem 2.8 that $\delta_3 \in (0, \delta_2)$ exists such that for each $0 < \delta < \delta_3$

$$\sigma\left[\mathfrak{L} + \mathcal{X} f(\cdot, 0) - \lambda W, \mathcal{N}_\delta^{0,j}, \mathfrak{D}\right] > 0, \quad 1 \leqslant j \leqslant n_0. \tag{3.20}$$

Since

$$\lambda \in \Lambda\left[0, \Omega_V^0, \mathfrak{B}\left(b, \Omega_V^0\right)\right],$$

taking into account the general assumptions in Section 1, it follows from Theorem 2.19 that there exists $\delta_4 \in (0, \delta_3)$ such that

$$\lambda \in \Lambda\left[0, \Omega_V^{\delta_4}, \mathfrak{B}\left(b, \Omega_V^{\delta_4}\right)\right]. \tag{3.21}$$

For each $\delta \geqslant 0$ sufficiently small let

$$u_\delta := u_{[\mathfrak{L}, \lambda W, \mathcal{X}, \Omega_V^\delta, \mathfrak{B}(b, \Omega_V^\delta)]}$$

denote the unique positive solution of $P[0, \lambda, \Omega_V^\delta, \mathfrak{B}(b, \Omega_V^\delta)]$, if it exists; the uniqueness is a consequence from Theorem 2.12, since we are assuming that (2.8) holds on $\Gamma_1 \cap \partial\Omega_V^0$ and, by construction, we have (1.8) and

$$\Gamma_1 \cap \partial\left[\Omega_V^\delta\right]_{\mathcal{X}}^0 \subset \Gamma_1 \cap \partial\Omega_V^0.$$

It is easy to see that u_{δ_4} is a positive supersolution of $P[0, \lambda, \Omega_V^\delta, \mathfrak{B}(b, \Omega_V^\delta)]$ if $\delta \in (0, \delta_4)$. In fact, thanks to the monotone structure of the nonlinearity, for any $\kappa \geqslant 1$, κu_{δ_4}

is a positive supersolution of $P[0,\lambda,\Omega_V^\delta,\mathfrak{B}(b,\Omega_V^\delta)]$. On the other hand, it is easy to see that, for any $\delta\in(0,\delta_4)$,

$$v_\delta := \begin{cases} u_0 & \text{in } \Omega_V^0, \\ 0 & \text{in } \Omega_V^\delta \setminus \Omega_V^0 \end{cases}$$

is a subsolution of $P[0,\lambda,\Omega_V^\delta,\mathfrak{B}(b,\Omega_V^\delta)]$ such that

$$v_\delta \leqslant \kappa u_\delta$$

for each $\kappa>1$ sufficiently large. Therefore,

$$\lambda \in \bigcap_{0<\delta<\delta_4} \Lambda\big[0,\Omega_V^\delta,\mathfrak{B}\big(b,\Omega_V^\delta\big)\big]. \tag{3.22}$$

In other words, u_δ exists – and it is unique – for any $\delta\in(0,\delta_4)$.

Subsequently, we fix $\delta\in(0,\delta_4)$ and set

$$H_{\delta/2}^\eta := \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \cup \bigcup_{j=1}^{p} \overline{\mathcal{N}}_{\delta/2}^{1,i_j} \cup \bigcup_{j=1}^{n_0} \overline{\mathcal{N}}_{\delta/2}^{0,j}.$$

Let ψ_δ^i, $i\in\{i_1,\dots,i_p\}$, and ξ_δ^j, $1\leqslant j\leqslant n_0$, denote the principal eigenfunctions associated with

$$\sigma\big[\mathfrak{L}+\mathcal{X}f(\cdot,0)-\lambda W,\mathcal{N}_\delta^{1,i},\mathfrak{B}\big(b,\mathcal{N}_\delta^{1,i}\big)\big], \quad i\in\{i_1,\dots,i_p\},$$

and

$$\sigma\big[\mathfrak{L}+\mathcal{X}f(\cdot,0)-\lambda W,\mathcal{N}_\delta^{0,j},\mathfrak{D}\big], \quad 1\leqslant j\leqslant n_0,$$

respectively, normalized so that

$$\big\|\psi_\delta^i\big\|_{L_\infty(\mathcal{N}_\delta^{1,i})}=1, \quad \big\|\xi_\delta^j\big\|_{L_\infty(\mathcal{N}_\delta^{0,j})}=1, \quad i\in\{i_1,\dots,i_p\},\ 1\leqslant j\leqslant n_0, \tag{3.23}$$

and let ϑ_δ^j, $1\leqslant j\leqslant \ell_V(\eta)$, denote the principal eigenfunctions associated with

$$\sigma\big[\mathfrak{L}+\mathcal{X}f(\cdot,0)-\lambda W,G_j^{\eta,\varepsilon},\mathfrak{D}\big]$$

normalized so that

$$\big\|\vartheta_\delta^j\big\|_{L_\infty(G_j^{\eta,\varepsilon})}=1, \quad 1\leqslant j\leqslant \ell_V(\eta). \tag{3.24}$$

Now, consider the positive function

$$\bar u_\delta : \overline{\Omega} \to [0,\infty)$$

defined by

$$\bar{u}_\delta := \begin{cases} u_\delta & \text{in } \overline{\Omega}_V^{\delta/2}, \\ \delta\vartheta_\delta^j & \text{in } \overline{G}_j^\eta, \qquad 1 \leqslant j \leqslant \ell_V(\eta), \\ \delta\psi_\delta^{i_j} & \text{in } \overline{\mathcal{N}}_{\delta/2}^{1,i_j}, \qquad 1 \leqslant j \leqslant p, \\ \delta\xi_\delta^j & \text{in } \overline{\mathcal{N}}_{\delta/2}^{0,j}, \qquad 1 \leqslant j \leqslant n_0, \\ \zeta_\delta & \text{in } \overline{\Omega} \setminus \big(\overline{\Omega}_V^{\delta/2} \cup H_{\delta/2}^\eta\big), \end{cases} \tag{3.25}$$

where ζ_δ is any positive regular extension of the function

$$u_\delta \cup \bigcup_{j=1}^{\ell_V(\eta)} \delta\vartheta_\delta^j \cup \bigcup_{j=1}^{p} \delta\psi_\delta^{i_j} \cup \bigcup_{j=1}^{n_0} \delta\xi_\delta^j$$

from $\overline{\Omega}_V^{\delta/2} \cup H_{\delta/2}^\eta$ to $\overline{\Omega}$ with the property of being bounded away from zero in $\overline{\Omega} \setminus (\overline{\Omega}_V^{\delta/2} \cup H_{\delta/2}^\eta)$; ζ_δ exists since each of the functions

$$u_\delta\big|_{\partial\Omega_V^{\delta/2}\setminus\Gamma_1}, \qquad \vartheta_\delta^j\big|_{\partial G_j^\eta}, \quad 1 \leqslant j \leqslant \ell_V(\eta),$$
$$\psi_\delta^{i_j}\big|_{\partial\mathcal{N}_{\delta/2}^{1,i_j}\setminus\Gamma_1}, \quad \xi_\delta^i\big|_{\partial\mathcal{N}_{\delta/2}^{0,i}\setminus\Gamma_0}, \quad 1 \leqslant j \leqslant p,\ 1 \leqslant i \leqslant n_0,$$

is positive and bounded away from zero. When $\Gamma_1 \subset \partial\Omega_V^0$, one should remove the $\psi_\delta^{i_j}$'s, $1 \leqslant j \leqslant p$, from the definition of $\bar{u}_\delta$. It should be noted that, thanks to (3.17), (3.18) and (3.19), the function $\bar{u}_\delta$ is well defined. Moreover,

$$\bar{u}_\delta(x) > 0 \quad \text{for each } x \in \Omega.$$

To complete the proof of part (i) when (3.5) occurs it remains to show that $\Lambda = \Lambda(\delta) > 0$ exists such that $\bar{u}_\delta$ provides us with a strict supersolution of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ for each $\gamma > \Lambda(\delta)$. Indeed, since

$$V \geqslant 0 \quad \wedge \quad \Omega_V^0 \subset \Omega_V^{\delta/2},$$

we find that, in $\Omega_V^{\delta/2}$, the following estimate is satisfied for any $\gamma > 0$

$$\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, \bar{u}_\delta)\big]\bar{u}_\delta = \big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, u_\delta)\big]u_\delta = \gamma V u_\delta > 0,$$

by construction. Also, since $\delta\vartheta_\delta^j > 0$ in G_j^η for each $1 \leqslant j \leqslant \ell_V(\eta)$, it follows from (1.2) that

$$f\big(\cdot, \delta\vartheta_\delta^j\big) > f(\cdot, 0) \quad \text{in } G_j^\eta,$$

and, hence, for each $1 \leqslant j \leqslant \ell_V(\eta)$ and $\gamma > 0$ the following estimate holds in G_j^η

$$\begin{aligned} \big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, \bar{u}_\delta)\big]\bar{u}_\delta &= \delta\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f\big(\cdot, \delta\vartheta_\delta^j\big)\big]\vartheta_\delta^j \\ &= \delta\big\{\sigma\big[\mathfrak{L} + \mathcal{X} f(\cdot, 0) - \lambda W, G_j^{\eta,\varepsilon}, \mathfrak{D}\big] + \gamma V + \mathcal{X}\big[f\big(\cdot, \delta\vartheta_\delta^j\big) - f(\cdot, 0)\big]\big\}\vartheta_\delta^j \\ &\geqslant \delta\sigma\big[\mathfrak{L} + \mathcal{X} f(\cdot, 0) - \lambda W, G_j^{\eta,\varepsilon}, \mathfrak{D}\big]\vartheta_\delta^j. \end{aligned}$$

Thus, it follows from (3.11) that, for any $\gamma > 0$,

$$\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta > 0 \quad \text{in } G_j^\eta,\ 1 \leqslant j \leqslant \ell_V(\eta).$$

Similarly, since $\delta\xi_\delta^j > 0$ in $\mathcal{N}_{\delta/2}^{0,j}$ for each $1 \leqslant j \leqslant n_0$, (1.2) implies

$$f\big(\cdot\,,\delta\xi_\delta^j\big) > f(\cdot\,,0) \quad \text{in } \mathcal{N}_{\delta/2}^{0,j},\ 1 \leqslant j \leqslant n_0,$$

and, hence, for each $1 \leqslant j \leqslant n_0$ and $\gamma > 0$ the following estimate is satisfied in $\mathcal{N}_{\delta/2}^{0,j}$,

$$\begin{aligned}\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta &= \delta\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f\big(\cdot\,,\delta\xi_\delta^j\big)\big]\xi_\delta^j \\ &= \delta\big\{\sigma\big[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot\,,0), \mathcal{N}_\delta^{0,j}, \mathfrak{D}\big] + \gamma V + \mathcal{X}\big[f\big(\cdot\,,\delta\xi_\delta^j\big) - f(\cdot\,,0)\big]\big\}\xi_\delta^j \\ &\geqslant \delta\sigma\big[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot\,,0), \mathcal{N}_\delta^{0,j}, \mathfrak{D}\big]\xi_\delta^j.\end{aligned}$$

Thus, thanks to (3.20), for each $\gamma > 0$ the following estimate is satisfied

$$\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta > 0 \quad \text{in } \mathcal{N}_{\delta/2}^{0,j},\ 1 \leqslant j \leqslant n_0.$$

Summarizing, up to now we have shown that, for each $\delta \in (0, \delta_4)$ and $\gamma > 0$,

$$\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta > 0 \quad \text{in } \overline{\Omega}_V^{\delta/2} \cup \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \cup \bigcup_{j=1}^{n_0} \overline{\mathcal{N}}_{\delta/2}^{0,j}. \tag{3.26}$$

Now, since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$, due to $(\mathcal{A}_2)$ a constant $\omega > 0$ exists such that

$$V > \omega > 0 \quad \text{in any compact subset of } \Omega_V^+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j} \tag{3.27}$$

and, hence,

$$V > \omega > 0 \quad \text{in } \big[\,\overline{\Omega} \setminus \big(\overline{\Omega}_V^{\delta/2} \cup H_{\delta/2}^\eta\big)\big] \cup \bigcup_{j=1}^{p} \mathcal{N}_{\delta/2}^{1,i_j} \subset \Omega_V^+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}. \tag{3.28}$$

Thus, since $\delta\psi_\delta^{i_j} > 0$ in $\mathcal{N}_{\delta/2}^{1,i_j}$ for each $1 \leqslant j \leqslant p$, we find from (1.2) that

$$f\big(\cdot\,,\delta\psi_\delta^{i_j}\big) > f(\cdot\,,0) \quad \text{in } \mathcal{N}_{\delta/2}^{1,i_j},\ 1 \leqslant j \leqslant p.$$

Hence, thanks to (3.28), the following estimate is satisfied in $\mathcal{N}_{\delta/2}^{1,i_j}$ for each $1 \leqslant j \leqslant p$

$$\begin{aligned}\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta &= \delta\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f\big(\cdot\,,\delta\psi_\delta^{i_j}\big)\big]\psi_\delta^{i_j} \\ &= \delta\big\{\sigma\big[\mathfrak{L} + \mathcal{X} f(\cdot\,,0) - \lambda W, \mathcal{N}_\delta^{1,i_j}, \mathfrak{B}\big(b, \mathcal{N}_\delta^{1,i_j}\big)\big] \\ &\quad + \gamma V + \mathcal{X}\big[f\big(\cdot\,,\delta\psi_\delta^{i_j}\big) - f(\cdot\,,0)\big]\big\}\psi_\delta^{i_j} \\ &> \delta\big\{\sigma\big[\mathfrak{L} + \mathcal{X} f(\cdot\,,0) - \lambda W, \mathcal{N}_\delta^{1,i_j}, \mathfrak{B}\big(b, \mathcal{N}_\delta^{1,i_j}\big)\big] + \gamma\omega\big\}\psi_\delta^{i_j}\end{aligned}$$

and, therefore,

$$\big[\mathfrak{L}+\gamma V-\lambda W+\mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta>0\quad\text{in }\bigcup_{j=1}^{p}\mathcal{N}_{\delta/2}^{1,i_j}$$

if

$$\gamma>\Lambda_1(\delta):=\omega^{-1}\max_{1\leqslant j\leqslant p}\big\{\big|\sigma\big[\mathfrak{L}+\mathcal{X} f(\cdot\,,0)-\lambda W,\mathcal{N}_\delta^{1,i_j},\mathfrak{B}\big(b,\mathcal{N}_\delta^{1,i_j}\big)\big]\big|\big\}\geqslant 0.$$

Moreover, since

$$\bar\Omega\setminus\big(\bar\Omega_V^{\delta/2}\cup H_{\delta/2}^{\eta}\big)\subset\Omega_V^{+},$$

it follows from (3.28) that there exists

$$\Lambda(\delta)>\max\big\{\Lambda_1(\delta),0\big\}$$

such that for each $\gamma>\Lambda(\delta)$ the following estimates are satisfied in $\bar\Omega\setminus(\bar\Omega_V^{\delta/2}\cup H_{\delta/2}^{\eta})$

$$\begin{aligned}\big[\mathfrak{L}+\gamma V-\lambda W+\mathcal{X} f(\cdot\,,\bar u_\delta)\big]\bar u_\delta&=\big[\mathfrak{L}+\gamma V-\lambda W+\mathcal{X} f(\cdot\,,\zeta_\delta)\big]\zeta_\delta\\&>\big[\mathfrak{L}-\lambda W+\mathcal{X} f(\cdot\,,\zeta_\delta)+\gamma\omega\big]\zeta_\delta>0,\end{aligned}$$

because $\omega>0$, ζ_δ is bounded away from zero in $\bar\Omega\setminus(\bar\Omega_V^{\delta/2}\cup H_{\delta/2}^{\eta})$ and the function

$$\big[\mathfrak{L}-\lambda W+\mathcal{X} f(\cdot\,,\zeta_\delta)\big]\zeta_\delta$$

is independent of γ.

On the other hand, by construction, we have that

$$\begin{aligned}&\mathfrak{B}(b)\bar u_\delta=\delta\mathfrak{D}\xi_\delta^{j}=0\quad\text{on }\Gamma_0^{j},\ 1\leqslant j\leqslant n_0,\\&\mathfrak{B}(b)\bar u_\delta=\delta(\partial_\nu+b)\psi_\delta^{i_j}=0\quad\text{on }\Gamma_1^{i_j},\ 1\leqslant j\leqslant p,\end{aligned}$$

and, thanks to (3.15),

$$\mathfrak{B}(b)\bar u_\delta=(\partial_\nu+b)u_\delta=0\quad\text{on }\partial\Omega_V^{0}\cap\Gamma_1.$$

Therefore,

$$\mathfrak{B}(b)\bar u_\delta=0\quad\text{on }\partial\Omega$$

and, for each $\delta\in(0,\delta_4)$ and $\gamma>\Lambda(\delta)$, the function $\bar u_\delta$ provides us with a positive strict supersolution of $P[\gamma,\lambda,\Omega,\mathfrak{B}(b)]$. This completes the proof of part (i) under condition (3.5).

Now, suppose

$$\Gamma_0\cap K_V\neq\emptyset,\tag{3.29}$$

instead of (3.5), and let $\{i_1,\dots,i_q\}$ be the subset of $\{1,\dots,n_0\}$ for which

$$\Gamma_0^{j}\cap K_V\neq\emptyset\quad\Longleftrightarrow\quad j\in\{i_1,\dots,i_q\}.$$

Subsequently, for any $\varrho > 0$ sufficiently small we will consider the new support domain

$$\mathfrak{D}_\varrho := \Omega \cup \left(\bigcup_{j=1}^{q} \Gamma_0^{i_j} + B_\varrho \right).$$

Fix $\varrho_1 > 0$, let

$$\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji} \in \mathcal{C}^1(\overline{\mathfrak{D}}_\varrho), \quad \tilde{\alpha}_i \in \mathcal{C}(\overline{\mathfrak{D}}_\varrho), \quad \tilde{\alpha}_0, \widetilde{W} \in L_\infty(\mathfrak{D}_\varrho), \quad 1 \leqslant i, j \leqslant N,$$

be any regular extensions from $\overline{\Omega}$ to $\overline{\mathfrak{D}}_\varrho$ of each of the coefficients

$$\alpha_{ij} = \alpha_{ji}, \quad \alpha_i, \quad \alpha_0, \quad W, \quad 1 \leqslant i, j \leqslant N,$$

respectively, and consider the auxiliary differential operator

$$\widetilde{\mathfrak{L}} := -\sum_{i,j=1}^{N} \tilde{\alpha}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \tilde{\alpha}_i \frac{\partial}{\partial x_i} + \tilde{\alpha}_0 \quad \text{in } \mathfrak{D}_{\varrho_1}.$$

Since $\mathfrak{L}$ is strongly uniformly elliptic in Ω with ellipticity constant $\mu > 0$, $\varrho \in (0, \varrho_1)$ exists for which the corresponding $\widetilde{\mathfrak{L}}$ is strongly uniformly elliptic in

$$\widetilde{\Omega} := \mathfrak{D}_\varrho$$

with ellipticity constant $\mu/2$. Now, we will consider the auxiliary potentials

$$\widetilde{\mathcal{X}} := \begin{cases} 1 & \text{in } \widetilde{\Omega} \setminus \Omega, \\ \mathcal{X} & \text{in } \Omega, \end{cases} \qquad \widetilde{V} := \begin{cases} 1 & \text{in } \widetilde{\Omega} \setminus \Omega, \\ V & \text{in } \Omega, \end{cases}$$

the boundary operator

$$\widetilde{\mathfrak{B}}(b) := \begin{cases} \mathfrak{D} & \text{on } \partial\widetilde{\Omega} \setminus \Gamma_1, \\ \partial_\nu + b & \text{on } \Gamma_1, \end{cases}$$

and any regular extension of f, say

$$\tilde{f} \in \mathcal{C}^1(\overline{\widetilde{\Omega}} \times [0, \infty), \mathbb{R}),$$

from $\overline{\Omega} \times [0, \infty)$ to $\overline{\widetilde{\Omega}} \times [0, \infty)$, such that

$$\lim_{u \nearrow \infty} \tilde{f}(x, u) = \infty \quad \text{uniformly in } \overline{\widetilde{\Omega}}.$$

Since $\mathcal{X}, V \in \mathcal{A}_{\Gamma_0, \Gamma_1}(\Omega)$, it is easy to see that

$$\widetilde{\mathcal{X}}, \widetilde{V} \in \mathcal{A}_{\partial\widetilde{\Omega} \setminus \Gamma_1, \Gamma_1}(\widetilde{\Omega}).$$

Moreover, by construction,

$$\bigcup_{j=1}^{q} \Gamma_0^{i_j} \subset \tilde\Omega,$$

and, since $\tilde{\mathcal{X}} = \tilde V = 1$ in $\tilde\Omega \setminus \Omega$,

$$\tilde\Omega^0_{\tilde{\mathcal{X}}} = \Omega^0_{\mathcal{X}}, \qquad \tilde\Omega^0_{\tilde V} = \Omega^0_V \subset \bar\Omega, \qquad \tilde K_{\tilde V} = K_V \subset \tilde\Omega. \tag{3.30}$$

Thus, thanks to (1.7),

$$\bigl(\partial\tilde\Omega \setminus \Gamma_1\bigr) \cap \bigl(\partial\tilde\Omega^0_{\tilde V} \cup \tilde K_{\tilde V}\bigr) = \bigl(\partial\tilde\Omega \setminus \Gamma_1\bigr) \cap \bigl(\partial\Omega^0_V \cup K_V\bigr) = \bigl(\partial\tilde\Omega \setminus \Gamma_1\bigr) \cap K_V$$

and, hence, it follows from the construction of $\Gamma_0^{i_j}$, $1 \leqslant j \leqslant q$, and $\tilde\Omega$ that

$$\bigl(\partial\tilde\Omega \setminus \Gamma_1\bigr) \cap \bigl(\partial\tilde\Omega^0_{\tilde V} \cup \tilde K_{\tilde V}\bigr) = \emptyset. \tag{3.31}$$

Moreover, thanks to (3.30), condition (2.8) is satisfied on

$$\Gamma_1 \cap \partial\tilde\Omega^0_{\tilde V} = \Gamma_1 \cap \partial\Omega^0_V$$

and, due to (1.7),

$$\tilde{\mathfrak{B}}\bigl(b, \tilde\Omega^0_{\tilde V}\bigr) = \tilde{\mathfrak{B}}\bigl(b, \Omega^0_V\bigr) = \mathfrak{B}\bigl(b, \Omega^0_V\bigr).$$

Thus,

$$\lambda \in \Lambda\bigl[0, \Omega^0_V, \mathfrak{B}\bigl(b, \Omega^0_V\bigr)\bigr] = \Lambda\bigl[0, \tilde\Omega^0_{\tilde V}, \tilde{\mathfrak{B}}\bigl(b, \tilde\Omega^0_{\tilde V}\bigr)\bigr]$$

and, thanks to (3.31), condition (3.5) is satisfied for the new problem in $\tilde\Omega$. Therefore, we can apply the result of part (i) in the special case when (3.5) is satisfied to the extended problem

$$\begin{cases} \tilde{\mathfrak{L}}\tilde u + \gamma \tilde V(x)\tilde u = \lambda \tilde W(x)\tilde u - \tilde{\mathcal{X}}(x)\tilde f(x,\tilde u)\tilde u & \text{in } \tilde\Omega, \\ \tilde{\mathfrak{B}}(b)\tilde u = 0 & \text{on } \partial\tilde\Omega. \end{cases} \tag{3.32}$$

As a result, for each $\delta > 0$ sufficiently small there exist $\tilde\Lambda(\delta) > 0$ and a positive function

$$\bar{\tilde u}_\delta : \bar{\tilde\Omega} \to [0,\infty)$$

such that

$$\bar{\tilde u}_\delta(x) > 0 \quad \text{for each } x \in \tilde\Omega,$$

and

$$\begin{cases} \bigl[\tilde{\mathfrak{L}} + \gamma \tilde V - \lambda \tilde W + \tilde{\mathcal{X}}\tilde f(\cdot, \bar{\tilde u}_\delta)\bigr]\bar{\tilde u}_\delta > 0 & \text{in } \tilde\Omega, \\ \tilde{\mathfrak{B}}(b)\bar{\tilde u}_\delta = 0 & \text{on } \partial\tilde\Omega \end{cases} \tag{3.33}$$

for each $\gamma > \tilde\Lambda(\delta)$. Now, set

$$\bar u_\delta := \bar{\tilde u}_\delta|_{\bar\Omega}. \tag{3.34}$$

Then, thanks to (3.33) and (3.34), for each $\gamma > \tilde{\Lambda}(\delta)$ we have that

$$\big[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, \bar{u}_\delta)\big]\bar{u}_\delta = \big[\widetilde{\mathfrak{L}} + \gamma \widetilde{V} - \lambda \widetilde{W} + \widetilde{\mathcal{X}} \tilde{f}(\cdot, \bar{\bar{u}}_\delta)\big]\bar{\bar{u}}_\delta \geqslant 0 \quad \text{in } \Omega. \tag{3.35}$$

Moreover, since $\bigcup_{j=1}^q \Gamma_0^{i_j} \subset \widetilde{\Omega}$,

$$\bar{u}_\delta(x) = \bar{\bar{u}}_\delta(x) > 0 \quad \text{for each } x \in \bigcup_{j=1}^q \Gamma_0^{i_j}. \tag{3.36}$$

Also,

$$\bar{u}_\delta = \bar{\bar{u}}_\delta = 0 \quad \text{on } \Gamma_0 \setminus \bigcup_{j=1}^q \Gamma_0^{i_j}$$

and

$$(\partial_\nu + b)\bar{u}_\delta = (\partial_\nu + b)\bar{\bar{u}}_\delta = 0 \quad \text{on } \Gamma_1.$$

Thus,

$$\mathfrak{B}(b)\bar{u}_\delta > 0 \quad \text{on } \partial\Omega$$

and, therefore, thanks to (3.35), for each $\delta > 0$ sufficiently small the function $\bar{u}_\delta$ defined by (3.34), provides us with a positive strict supersolution of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ if $\gamma > \tilde{\Lambda}(\delta) > 0$. This completes the proof of Part (i).

Now, we shall prove part (ii). As in proving part (i) we will proceed separately distinguishing between the cases when (3.5) or (3.29) is satisfied.

Suppose (3.5). Then, (3.22) is satisfied. For each $\delta \in [0, \delta_4)$, let

$$u_\delta := u_{[\mathfrak{L}, \lambda W, \mathcal{X}, \Omega_V^\delta, \mathfrak{B}(b, \Omega_V^\delta)]}$$

denote the unique positive solution of $P[0, \lambda, \Omega_V^\delta, \mathfrak{B}(b, \Omega_V^\delta)]$. Since (2.8) holds on $\Gamma_1 \cap \partial\Omega_V^0$, it follows from Theorem 2.19 that

$$\lim_{\delta \searrow 0} \|u_\delta - u_0\|_{H^1(\Omega_V^0)} = 0. \tag{3.37}$$

Moreover, the positive strict supersolution $\bar{u}_\delta$ defined through (3.25) satisfies

$$\bar{u}_\delta|_{\Omega_V^0} = u_\delta|_{\Omega_V^0},$$

by definition, and, therefore, (3.37) implies

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta - u_0\|_{H^1(\Omega_V^0)} = 0.$$

This shows (3.2).

We now prove (3.3). By definition of $\bar{u}_\delta$, it follows from (3.23) and (3.24) that, for each $\delta > 0$ sufficiently small,

$$\|\bar{u}_\delta\|_{L_\infty(H^\eta_{\delta/2})} \leqslant \delta. \tag{3.38}$$

Let K be a compact subset of $\overline{\Omega} \setminus \overline{\Omega}_V^0$. By construction, there exists

$$\delta_5 := \delta_5(K) \in (0, \delta_4)$$

such that, for each $\delta \in (0, \delta_5)$,

$$K \subset \overline{\Omega} \setminus \overline{\Omega}_V^{\delta/2}. \tag{3.39}$$

Thus, it follows from (3.38) and (3.39) that

$$\|\bar{u}_\delta\|_{L_\infty(K \cap H^\eta_{\delta/2})} \leqslant \delta, \qquad \|\bar{u}_\delta\|_{L_\infty(K \setminus H^\eta_{\delta/2})} = \|\zeta_\delta\|_{L_\infty(K \setminus H^\eta_{\delta/2})},$$

since, by definition, $\bar{u}_\delta = \zeta_\delta$ on $K \setminus H^\eta_{\delta/2}$, and, hence,

$$\|\bar{u}_\delta\|_{L_\infty(K)} \leqslant \max\bigl\{\delta, \|\zeta_\delta\|_{L_\infty(K \setminus H^\eta_{\delta/2})}\bigr\}. \tag{3.40}$$

Finally, since ζ_δ is an arbitrary regular positive extension of

$$u_\delta \cup \bigcup_{j=1}^{\ell_V(\eta)} \delta \vartheta_\delta^j \cup \bigcup_{j=1}^{p} \delta \psi_\delta^{i_j} \cup \bigcup_{j=1}^{n_0} \delta \xi_\delta^j$$

from

$$\overline{\Omega}_V^{\delta/2} \cup \bigcup_{j=1}^{\ell_V(\eta)} \overline{G}_j^\eta \cup \bigcup_{j=1}^{p} \overline{\mathcal{N}}_{\delta/2}^{1,i_j} \cup \bigcup_{j=1}^{n_0} \overline{\mathcal{N}}_{\delta/2}^{0,j}$$

to $\overline{\Omega}$, and (3.23), (3.24) imply

$$\lim_{\delta \searrow 0} \bigl\|\delta \vartheta_\delta^j\bigr\|_{L_\infty(G_j^\eta)} = \lim_{\delta \searrow 0} \bigl\|\delta \psi_\delta^{i_j}\bigr\|_{L_\infty(\mathcal{N}_{\delta/2}^{1,i_j})} = \lim_{\delta \searrow 0} \bigl\|\delta \xi_\delta^j\bigr\|_{L_\infty(\mathcal{N}_{\delta/2}^{0,j})} = 0,$$

passing to the limit as $\delta \searrow 0$ in (3.40) it is rather clear that ζ_δ can be adjusted so that (3.3) holds; (3.4) is easily obtained from (3.2) and (3.3). This completes the proof of part (ii) under (3.5).

Now, suppose (3.29), instead of (3.5). Then, arguing as in the proof of part (i) under condition (3.29), we have that the positive strict supersolution $\tilde{\bar{u}}_\delta$ built up in $\widetilde{\Omega}$ satisfies

$$\lim_{\delta \searrow 0} \bigl\|\tilde{\bar{u}}_\delta - u_{[\widetilde{\mathfrak{L}}, \lambda \widetilde{W}, \widetilde{\mathcal{X}}, \widetilde{\Omega}^0_{\widetilde{V}}, \widetilde{\mathfrak{B}}(b, \widetilde{\Omega}^0_{\widetilde{V}})]}\bigr\|_{H^1(\widetilde{\Omega}^0_{\widetilde{V}})} = 0. \tag{3.41}$$

On the other hand, by construction, we have that

$$\widetilde{\Omega}^0_{\widetilde{V}} = \Omega^0_V \subset \overline{\Omega}$$

and

$$\tilde{\bar{u}}_\delta|_{\widetilde{\Omega}^0_{\widetilde{V}}} = \bar{u}_\delta, \qquad \widetilde{\mathfrak{B}}\bigl(b, \widetilde{\Omega}^0_{\widetilde{V}}\bigr) = \widetilde{\mathfrak{B}}\bigl(b, \Omega^0_V\bigr) = \mathfrak{B}\bigl(b, \Omega^0_V\bigr),$$

where $\bar{u}_\delta$ is the positive strict supersolution defined by (3.25). Moreover,

$$\widetilde{\mathfrak{L}}|_{\widetilde{\Omega}^0_{\widetilde{V}}} = \mathfrak{L}, \qquad \widetilde{W}|_{\widetilde{\Omega}^0_{\widetilde{V}}} = W, \qquad \widetilde{\mathcal{X}}|_{\widetilde{\Omega}^0_{\widetilde{V}}} = \mathcal{X}.$$

Thus, (3.41) becomes into

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{H^1(\Omega^0_V)} = 0,$$

so proving (3.2) under condition (3.29).

Similarly, adapting the argument given in the case when condition (3.5) is satisfied, we have that

$$\lim_{\delta \searrow 0} \|\bar{\tilde{u}}_\delta\|_{L_\infty(\widetilde{K})} = 0 \tag{3.42}$$

in any compact subset

$$\widetilde{K} \subset \bar{\widetilde{\Omega}} \setminus \bar{\widetilde{\Omega}}^0_{\widetilde{V}}.$$

In particular, (3.42) holds in any compact subset K of $\bar{\Omega} \setminus \bar{\Omega}^0_V$, since

$$\bar{\Omega} \setminus \bar{\Omega}^0_V = \bar{\Omega} \setminus \bar{\widetilde{\Omega}}^0_{\widetilde{V}} \subset \bar{\widetilde{\Omega}} \setminus \bar{\widetilde{\Omega}}^0_{\widetilde{V}}.$$

Therefore, since

$$\bar{\tilde{u}}_\delta|_{\bar{\Omega}\setminus\bar{\Omega}^0_V} = \bar{u}_\delta,$$

(3.42) implies (3.3); (3.4) follows readily from (3.2) and (3.3). This completes the proof of part (ii) and concludes the proof of the proposition. □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. – Suppose $\gamma_0 > 0$ exists such that

$$\lambda \in \Lambda\left[0, \Omega^0_V, \mathfrak{B}\left(b, \Omega^0_V\right)\right] \cap \Lambda\left[\gamma, \Omega, \mathfrak{B}(b)\right]$$

for each $\gamma > \gamma_0$, and (2.8) holds on $\Gamma_1 \cap \partial\Omega^0_V$. For each $\gamma > \gamma_0$, let

$$u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \quad \wedge \quad u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}$$

denote the unique positive solutions of problems $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ and $P[0, \lambda, \Omega^0_V, \mathfrak{B}(b, \Omega^0_V)]$, respectively. Since $u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}$ is strongly positive in Ω, we have that

$$\mathfrak{B}\left(b, \Omega^0_V\right) u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} = u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} > 0 \quad \text{on } \partial\Omega^0_V \cap \Omega. \tag{3.43}$$

Thus, $u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}$ is a positive strict supersolution of $P[0, \lambda, \Omega^0_V, \mathfrak{B}(b, \Omega^0_V)]$ for each $\gamma > \gamma_0$, and, hence, thanks to Theorem 2.16,

$$u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]} < u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \quad \text{in } \Omega^0_V \text{ for each } \gamma > \gamma_0. \tag{3.44}$$

Therefore, the auxiliary function

$$u_* := \begin{cases} u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]} & \text{in } \overline{\Omega}_V^0, \\ 0 & \text{in } \overline{\Omega} \setminus \overline{\Omega}_V^0 \end{cases}$$

satisfies

$$u_* < u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \quad \text{in } \Omega \text{ for each } \gamma > \gamma_0. \tag{3.45}$$

On the other hand, since we are working under the assumptions of Proposition 3.1, $\delta_0 > 0$ exists such that for each $\delta \in (0, \delta_0)$ there are a real number $\Lambda(\delta) > 0$ and a positive function $\bar{u}_\delta$ such that $\bar{u}_\delta$ is a positive strict supersolution of $P[\gamma, \lambda, \Omega, \mathfrak{B}(b)]$ for each $\gamma > \Lambda(\delta)$,

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]}\|_{H^1(\Omega_V^0)} = 0$$

and, for any compact subset $K \subset \overline{\Omega} \setminus \overline{\Omega}_V^0$,

$$\lim_{\delta \searrow 0} \|\bar{u}_\delta\|_{L_\infty(K)} = 0.$$

In particular,

$$\lim_{\delta \searrow 0} \bar{u}_\delta = \begin{cases} u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]} & \text{in } \Omega_V^0, \\ 0 & \text{in } \Omega \setminus \Omega_V^0, \end{cases} \quad \text{a.e. in } \Omega.$$

Thanks to Theorem 2.16,

$$u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \leqslant \bar{u}_\delta \quad \text{in } \Omega \text{ for each } \delta \in (0, \delta_0) \text{ and } \gamma > \Lambda(\delta). \tag{3.46}$$

Therefore, it follows from (3.45) and (3.46) that

$$u_* < u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \leqslant \bar{u}_\delta \quad \text{in } \Omega \text{ for each } \delta \in (0, \delta_0) \text{ and } \gamma > \gamma_\delta, \tag{3.47}$$

where

$$\gamma_\delta := \max\{\Lambda(\delta), \gamma_0\}.$$

Thanks to (3.47), for each $\delta \in (0, \delta_0)$ and $\gamma > \gamma_\delta$, we have that

$$u_*|_{\Omega_V^0} = u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]} \leqslant u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} \leqslant \bar{u}_\delta \quad \text{in } \Omega_V^0 \tag{3.48}$$

and, hence,

$$\begin{aligned} &\|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]}\|_{L_2(\Omega_V^0)} \\ &\quad \leqslant \|\bar{u}_\delta - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega_V^0,\mathfrak{B}(b,\Omega_V^0)]}\|_{L_2(\Omega_V^0)}. \end{aligned} \tag{3.49}$$

Thus, for each $\delta \in (0, \delta_0)$, we have that

$$\limsup_{\gamma\nearrow\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_2(\Omega^0_V)} \leqslant \|\bar u_\delta - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_2(\Omega^0_V)}. \tag{3.50}$$

On the other hand, thanks to Proposition 3.1, we already know that

$$\lim_{\delta\searrow 0} \|\bar u_\delta - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_2(\Omega^0_V)} = 0, \tag{3.51}$$

and, therefore, combining (3.50) and (3.51) gives

$$\lim_{\gamma\nearrow\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_2(\Omega^0_V)} = 0. \tag{3.52}$$

Now, fix $\delta \in (0,\delta_0)$. Then, thanks to (3.48),

$$\|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}\|_{L_\infty(\Omega^0_V)} \leqslant \|\bar u_\delta\|_{L_\infty(\Omega^0_V)} \quad \text{for each } \gamma > \gamma_\delta$$

and, hence, there exists a constant $C > 0$ such that

$$\|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_\infty(\Omega^0_V)} \leqslant C \quad \text{for each } \gamma > \gamma_\delta.$$

Thus, (3.52) implies

$$\lim_{\gamma\nearrow\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_p(\Omega^0_V)} = 0 \quad \text{for each } p \in [2,\infty). \tag{3.53}$$

On the other hand, since $L_2(\Omega^0_V) \hookrightarrow L_p(\Omega^0_V)$ if $p \in [1,2)$, (3.52) gives

$$\lim_{\gamma\nearrow\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]} - u_{[\mathfrak{L},\lambda W,\mathcal{X},\Omega^0_V,\mathfrak{B}(b,\Omega^0_V)]}\|_{L_p(\Omega^0_V)} = 0 \quad \text{for each } p \in [1,2).$$

This concludes the proof of (1.9).

Let K be a compact subset of $\bar\Omega \setminus \bar\Omega^0_V$. Then, thanks to (3.47),

$$0 = u_*|_K \leqslant u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}|_K \leqslant \bar u_\delta|_K \quad \text{for each } \delta \in (0,\delta_0) \text{ and } \gamma > \gamma_\delta,$$

and, hence,

$$\|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}\|_{L_\infty(K)} \leqslant \|\bar u_\delta\|_{L_\infty(K)} \quad \text{for each } \delta \in (0,\delta_0) \text{ and } \gamma > \gamma_\delta. \tag{3.54}$$

Thus, passing to the limit as $\gamma \nearrow \infty$ in (3.54) gives

$$0 \leqslant \limsup_{\gamma\nearrow\infty} \|u_{[\mathfrak{L}+\gamma V,\lambda W,\mathcal{X},\Omega,\mathfrak{B}(b)]}\|_{L_\infty(K)} \leqslant \|\bar u_\delta\|_{L_\infty(K)} \quad \text{for each } \delta \in (0,\delta_0). \tag{3.55}$$

On the other hand, thanks to Proposition 3.1, we have that

$$\lim_{\delta\searrow 0} \|\bar u_\delta\|_{L_\infty(K)} = 0$$

and, therefore, (3.55) implies

$$\lim_{\gamma \nearrow \infty} \| u_{[\mathfrak{L}+\gamma V, \lambda W, \mathcal{X}, \Omega, \mathfrak{B}(b)]} \|_{L_\infty(K)} = 0.$$

This completes the proof of (1.10) and concludes the proof of the theorem. □

4. Some sufficient conditions so that 1 ⇒ 2 in Theorem 1.3

The following results provide us with some sufficient conditions ensuring that $\lambda \in \Lambda[\gamma, \Omega, \mathfrak{B}(b)]$ for any γ sufficiently large whenever $\lambda \in \Lambda[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]$.

THEOREM 4.1. – *Suppose* $\lambda \in \Lambda[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]$, (2.8) *holds on* $\Gamma_1 \cap (\partial \Omega_V^0 \cup \partial \Omega_{\mathcal{X}}^0)$ *and*

$$\mathcal{X} \in \mathcal{A}_{\partial \Omega_V^0 \setminus \Gamma_1, \Gamma_1 \cap \partial \Omega_V^0}(\Omega_V^0) \quad \wedge \quad V \in \mathcal{A}_{\partial \Omega_{\mathcal{X}}^0 \setminus \Gamma_1, \Gamma_1 \cap \partial \Omega_{\mathcal{X}}^0}(\Omega_{\mathcal{X}}^0) \quad \wedge \quad \Omega_V^0 \cap \Omega_{\mathcal{X}}^0 \in \mathcal{C}^2,$$

if $\Omega_V^0 \cap \Omega_{\mathcal{X}}^0 \neq \emptyset$, *or*

$$\mathcal{X} \in \mathcal{A}^+_{\partial \Omega_V^0 \setminus \Gamma_1, \Gamma_1 \cap \partial \Omega_V^0}(\Omega_V^0) \quad \wedge \quad V \in \mathcal{A}^+_{\partial \Omega_{\mathcal{X}}^0 \setminus \Gamma_1, \Gamma_1 \cap \partial \Omega_{\mathcal{X}}^0}(\Omega_{\mathcal{X}}^0),$$

when $\Omega_V^0 \cap \Omega_{\mathcal{X}}^0 = \emptyset$. *Then, there exists* $\gamma_0 > 0$ *such that*

$$\lambda \in \bigcap_{\gamma \geqslant \gamma_0} \Lambda[\gamma, \Omega, \mathfrak{B}(b)] \tag{4.1}$$

and, therefore, Theorem 1.3 *can be applied.*

Proof. – Suppose

$$\left([\Omega_V^0]^0_{\mathcal{X}} = [\Omega_{\mathcal{X}}^0]^0_V =\right) \Omega_V^0 \cap \Omega_{\mathcal{X}}^0 \neq \emptyset.$$

Then, since $\Omega_V^0 \cap \Omega_{\mathcal{X}}^0$ is of class $\mathcal{C}^2$, (2.8) holds on $\Gamma_1 \cap (\partial \Omega_V^0 \cup \partial \Omega_{\mathcal{X}}^0)$, and

$$\partial(\Omega_V^0 \cap \Omega_{\mathcal{X}}^0) \subset \partial \Omega_V^0 \cup \partial \Omega_{\mathcal{X}}^0,$$

necessarily (2.8) holds on $\Gamma_1 \cap \partial(\Omega_V^0 \cap \Omega_{\mathcal{X}}^0)$. Thus, thanks to Theorem 2.12,

$$\begin{gathered} \lambda \in \Lambda[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)] \\ \Updownarrow \\ \sigma[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot, 0), \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)] < 0 < \sigma[\mathfrak{L} - \lambda W, [\Omega_V^0]^0_{\mathcal{X}}, \mathfrak{B}(b, [\Omega_V^0]^0_{\mathcal{X}})]. \end{gathered} \tag{4.2}$$

Similarly, for any $\gamma > 0$,

$$\begin{gathered} \lambda \in \Lambda[\gamma, \Omega, \mathfrak{B}(b)] \\ \Updownarrow \\ \sigma[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, 0), \Omega, \mathfrak{B}(b)] < 0 < \sigma[\mathfrak{L} + \gamma V - \lambda W, \Omega_{\mathcal{X}}^0, \mathfrak{B}(b, \Omega_{\mathcal{X}}^0)]. \end{gathered} \tag{4.3}$$

On the other hand, since (2.8) holds on $\Gamma_1 \cap \partial[\Omega^0_{\mathcal{X}}]^0_V$, it follows from Theorem 2.9 that

$$\lim_{\gamma \nearrow \infty} \sigma\left[\mathfrak{L} + \gamma V - \lambda W, \Omega^0_{\mathcal{X}}, \mathfrak{B}\left(b, \Omega^0_{\mathcal{X}}\right)\right] = \sigma\left[\mathfrak{L} - \lambda W, \left[\Omega^0_{\mathcal{X}}\right]^0_V, \mathfrak{B}\left(b, \left[\Omega^0_{\mathcal{X}}\right]^0_V\right)\right]. \tag{4.4}$$

Similarly,

$$\lim_{\gamma \nearrow \infty} \sigma\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, 0), \Omega, \mathfrak{B}(b)\right] = \sigma\left[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot, 0), \Omega^0_V, \mathfrak{B}\left(b, \Omega^0_V\right)\right]. \tag{4.5}$$

Therefore, thanks to (4.2)–(4.4) and (4.5), condition (4.1) is satisfied for some $\gamma_0 > 0$ if $\lambda \in \Lambda[0, \Omega^0_V, \mathfrak{B}(b, \Omega^0_V)]$. This completes the proof of the theorem in this special case.

Now, suppose

$$\left(\left[\Omega^0_V\right]^0_{\mathcal{X}} = \left[\Omega^0_{\mathcal{X}}\right]^0_V =\right) \Omega^0_V \cap \Omega^0_{\mathcal{X}} = \emptyset.$$

Since

$$\left[\Omega^0_V\right]^0_{\mathcal{X}} = \emptyset \quad \wedge \quad \mathcal{X} \in \mathcal{A}^+_{\partial\Omega^0_V \setminus \Gamma_1, \Gamma_1 \cap \partial\Omega^0_V}\left(\Omega^0_V\right),$$

it follows from Theorem 2.14 that

$$\lambda \in \Lambda\left[0, \Omega^0_V, \mathfrak{B}\left(b, \Omega^0_V\right)\right] \quad \Longleftrightarrow \quad \sigma\left[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot, 0), \Omega^0_V, \mathfrak{B}\left(b, \Omega^0_V\right)\right] < 0. \tag{4.6}$$

Similarly, thanks to Theorem 2.12, the equivalence (4.3) holds.

Since $V \in \mathcal{A}_{\Gamma_0, \Gamma_1}(\Omega)$ and (2.8) is satisfied on $\Gamma_1 \cap \partial\Omega^0_V$, it follows from Theorem 2.9 that (4.5) holds. Thus, since $\lambda \in \Lambda[0, \Omega^0_V, \mathfrak{B}(b, \Omega^0_V)]$, it follows from (4.5) and (4.6) that

$$\lim_{\gamma \nearrow \infty} \sigma\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, 0), \Omega, \mathfrak{B}(b)\right] < 0. \tag{4.7}$$

Similarly, since

$$V \in \mathcal{A}_{\partial\Omega^0_{\mathcal{X}} \setminus \Gamma_1, \Gamma_1 \cap \partial\Omega^0_{\mathcal{X}}}\left(\Omega^0_{\mathcal{X}}\right) \quad \wedge \quad \left[\Omega^0_{\mathcal{X}}\right]^0_V = \emptyset,$$

it follows from Theorem 2.10 that

$$\lim_{\gamma \nearrow \infty} \sigma\left[\mathfrak{L} + \gamma V - \lambda W, \Omega^0_{\mathcal{X}}, \mathfrak{B}\left(b, \Omega^0_{\mathcal{X}}\right)\right] = \infty. \tag{4.8}$$

Then, combining (4.7) and (4.8), we find that, for any $\gamma > 0$ sufficiently large, the following estimate is satisfied

$$\sigma\left[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot, 0), \Omega, \mathfrak{B}(b)\right] < 0 < \sigma\left[\mathfrak{L} + \gamma V - \lambda W, \Omega^0_{\mathcal{X}}, \mathfrak{B}\left(b, \Omega^0_{\mathcal{X}}\right)\right]. \tag{4.9}$$

Therefore, it follows from (4.3) and (4.9) that (4.1) must be satisfied for some $\gamma_0 > 0$. This completes the proof of the theorem. □

It should be noted that Corollary 2.18 provides us some sufficient conditions for having

$$\mathcal{X} \in \mathcal{A}_{\partial\Omega^0_V \setminus \Gamma_1, \Gamma_1 \cap \partial\Omega^0_V}\left(\Omega^0_V\right) \quad \wedge \quad V \in \mathcal{A}_{\partial\Omega^0_{\mathcal{X}} \setminus \Gamma_1, \Gamma_1 \cap \partial\Omega^0_{\mathcal{X}}}\left(\Omega^0_{\mathcal{X}}\right)$$

in case $\Omega_V^0 \cap \Omega_{\mathcal{X}}^0 \neq \emptyset$, and

$$\mathcal{X} \in \mathcal{A}^+_{\partial\Omega_V^0\setminus\Gamma_1,\Gamma_1\cap\partial\Omega_V^0}\bigl(\Omega_V^0\bigr) \quad \wedge \quad V \in \mathcal{A}^+_{\partial\Omega_{\mathcal{X}}^0\setminus\Gamma_1,\Gamma_1\cap\partial\Omega_{\mathcal{X}}^0}\bigl(\Omega_{\mathcal{X}}^0\bigr)$$

when $\Omega_V^0 \cap \Omega_{\mathcal{X}}^0 = \emptyset$.

THEOREM 4.2. – *Assume* $\lambda \in \Lambda[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]$, $\mathcal{X} \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$, *and* (2.8) *holds on* $\Gamma_1 \cap \partial\Omega_V^0$. *Then,* $\gamma_0 > 0$ *exists for which condition* (4.1) *is satisfied. Therefore, Theorem* 1.3 *can be applied.*

Proof. – Since $\mathcal{X} \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$, Corollary 2.18 implies

$$\mathcal{X} \in \mathcal{A}^+_{\partial\Omega_V^0\setminus\Gamma_1,\partial\Omega_V^0\cap\Gamma_1}\bigl(\Omega_V^0\bigr)$$

and, hence, thanks to Theorem 2.14,

$$\Lambda\bigl[0, \Omega_V^0, \mathfrak{B}\bigl(b, \Omega_V^0\bigr)\bigr] = \bigl\{\lambda \in \mathbb{R}\colon \sigma\bigl[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega_V^0, \mathfrak{B}\bigl(b, \Omega_V^0\bigr)\bigr] < 0\bigr\}. \tag{4.10}$$

Similarly, for each $\gamma > 0$, we have that

$$\Lambda\bigl[\gamma, \Omega, \mathfrak{B}(b)\bigr] = \bigl\{\lambda \in \mathbb{R}\colon \sigma\bigl[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\bigr] < 0\bigr\}. \tag{4.11}$$

On the other hand, since $V \in \mathcal{A}_{\Gamma_0,\Gamma_1}(\Omega)$ and (2.8) holds on $\Gamma_1 \cap \partial\Omega_V^0$, Theorem 2.9 gives

$$\begin{aligned}\lim_{\gamma\nearrow\infty} \sigma&\bigl[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\bigr] \\ &= \sigma\bigl[\mathfrak{L} - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega_V^0, \mathfrak{B}\bigl(b, \Omega_V^0\bigr)\bigr].\end{aligned} \tag{4.12}$$

Thus, since $\lambda \in \Lambda[0, \Omega_V^0, \mathfrak{B}(b, \Omega_V^0)]$, we find from (4.10) and (4.12) that

$$\sigma\bigl[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\bigr] < 0 \tag{4.13}$$

for each γ sufficiently large. Therefore, thanks to (4.11), $\lambda \in \Lambda[\gamma, \Omega, \mathfrak{B}(b)]$ for γ large. This completes the proof. □

It should be noted that if $V \in \mathcal{A}^+_{\Gamma_0,\Gamma_1}(\Omega)$, then it does not make sense analyzing the limiting behavior as $\gamma \nearrow \infty$ of the positive solution of (1.1). Indeed, in such case, Theorem 2.10 implies

$$\lim_{\gamma\nearrow\infty} \sigma\bigl[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\bigr] = \infty$$

and, thanks to Theorem 2.12 and Theorem 2.14, for each $\gamma > 0$,

$$\Lambda\bigl[\gamma, \Omega, \mathfrak{B}(b)\bigr] \subseteq \bigl\{\lambda \in \mathbb{R}\colon \sigma\bigl[\mathfrak{L} + \gamma V - \lambda W + \mathcal{X} f(\cdot\,, 0), \Omega, \mathfrak{B}(b)\bigr] < 0\bigr\}.$$

Therefore, there exists $\gamma_0 := \gamma_0(\lambda) > 0$ such that

$$\lambda \notin \bigcup_{\gamma\geqslant\gamma_0} \Lambda\bigl[\gamma, \Omega, \mathfrak{B}(b)\bigr].$$

It turns out that the positive solution of problem (1.1) for small values of γ becomes the zero solution at the unique value of γ for which

$$\sigma\left[\mathfrak{L}+\gamma V-\lambda W+\mathcal{X} f(\cdot\,,0),\Omega,\mathfrak{B}(b)\right]=0.$$

Therefore, it is consistent saying that the limiting profile of the maximal nonnegative solution is zero.

5. Permanence under unlimited aggression is possible

This section applies Theorem 1.3 for obtaining the following biological principle: no species can be driven to extinction by a competitor if it possesses a refuge and its birth rate in the overall habitat is sufficiently large. Moreover, it will be shown how the species concentrates within the refuge when it suffers high level aggressions. For establishing these principles we adopt as a model for competing species the following spatially heterogeneous evolutionary model of Lotka–Volterra type with diffusion and transport effects

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mathfrak{L}_1 u=\lambda u-\mathcal{X}_1(x)u^2-\gamma_1 V_1(x)uv & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial t}+\mathfrak{L}_2 v=\mu v-\mathcal{X}_2(x)v^2-\gamma_2 V_2(x)uv & \text{in } \Omega\times(0,\infty),\\ \mathfrak{B}_1 u=\mathfrak{B}_2 v=0 & \text{on } \partial\Omega\times(0,\infty),\\ \big(u(\cdot\,,0),v(\cdot\,,0)\big)=(u_0,v_0) & \text{in } \Omega \end{cases} \tag{5.1}$$

under the following assumptions:

(1) Ω is a bounded domain in $\mathbb{R}^N$, $N\geqslant 1$, of class $\mathcal{C}^2$.

(2) $\lambda,\mu\in\mathbb{R}$ and, for each $j\in\{1,2\}$, $\gamma_j\in\mathbb{R}$ and $\mathfrak{L}_j$ is an uniformly strongly elliptic second order differential operator in Ω of the same type as $\mathfrak{L}$.

(3) For each $j\in\{1,2\}$, $\mathfrak{B}_j:=\mathfrak{B}_j(b_j)$ stands for the boundary operator

$$\mathfrak{B}_j:=\begin{cases}\mathfrak{D} & \text{on } \Gamma_0^j,\\ \partial_{\nu_j}+b_j & \text{on } \Gamma_1^j,\end{cases}$$

where Γ_0^j and Γ_1^j are two disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0^j\cup\Gamma_1^j=\partial\Omega$, $b_j\in\mathcal{C}(\Gamma_1^j)$, and $\nu_j=(\nu_{j,1},\dots,\nu_{j,N})\in\mathcal{C}^1(\Gamma_1^j;\mathbb{R}^N)$ is an outward pointing nowhere tangent vector field.

(4) Each of the functions V_j and $\mathcal{X}_j$, $j\in\{1,2\}$, is non-negative measurable and bounded in Ω. Moreover,

$$\mathcal{X}_j\in\mathcal{A}^+_{\Gamma_0^j,\Gamma_1^j}(\Omega),\qquad V_j\in\mathcal{A}_{\Gamma_0^j,\Gamma_1^j}(\Omega),\qquad j\in\{1,2\}.$$

(5) For each $j\in\{1,2\}$, $\Omega^0_{V_j}$ is connected and

$$\partial\Omega^0_{V_j}\setminus\Gamma_1^j\subset\Omega,\qquad \operatorname{dist}\big(\Gamma_1^j,\partial\Omega^0_{V_j}\setminus\Gamma_1^j\big)>0.$$

Moreover, setting

$$\Omega^{\delta}_{V_j} := \Omega^0_{V_j} \cup \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega^0_{V_j} \setminus \Gamma^j_1) < \delta\}$$

we assume that there is a sequence υ^j_n, $n \geqslant 1$, such that $\lim_{n\to\infty} \upsilon^j_n = 0$ for which the general assumptions of Theorem 2.19(e) of Section 2 with

$$(a, \Omega_0, \Omega_n) = \big(\mathcal{X}_j, \Omega^0_{V_j}, \Omega^{\upsilon^j_n}_{V_j}\big), \quad n \geqslant 1,$$

is satisfied (note that $\mathcal{X}_j \in \mathcal{A}^+_{\Gamma^j_0,\Gamma^j_1}(\Omega)$). Furthermore, there exists $\delta_0 > 0$ such that, for each $\delta \in [0, \delta_0)$,

$$\mathcal{X}_j \in \mathcal{A}^+_{\partial\Omega^{\delta}_{V_j}\setminus\Gamma^j_1, \partial\Omega^{\delta}_{V_j}\cap\Gamma^j_1}\big(\Omega^{\delta}_{V_j}\big), \quad j \in \{1, 2\}.$$

(6) $(u_0, v_0) \in X_0^2$ where $X_0 := L^+_p(\Omega)$ for some $p > \frac{N}{2}$.

Under these assumptions, for each initial data $(u_0, v_0) \in X_0^2$, (5.1) has a unique global strict solution $(u(x,t;u_0,v_0), v(x,t;u_0,v_0))$ (cf. [2]). In fact, thanks to the parabolic maximum principle, for any $t > 0$ we have that

$$0 \leqslant u(\cdot, t; u_0, v_0) \leqslant T_1(t)u_0 \quad \wedge \quad 0 \leqslant v(\cdot, t; u_0, v_0) \leqslant T_2(t)v_0,$$

where, $T_1(t)$ and $T_2(t)$ stand for the L_p-evolution operators associated with $\mathfrak{L}_1 - \lambda$ and $\mathfrak{L}_2 - \mu$, respectively. It is well known that most of the limiting profiles of the positive solutions of (5.1) as $t \nearrow \infty$ are given by the strong non-negative steady states of (5.1) (cf. [14] and the further developments of [11–13,24,26], and the references therein). The steady-states of (5.1) are the non-negative strong solutions of

$$\begin{cases} \mathfrak{L}_1 u = \lambda u - \mathcal{X}_1(x)u^2 - \gamma_1 V_1(x)uv & \text{in } \Omega,\\ \mathfrak{L}_2 v = \mu v - \mathcal{X}_2(x)v^2 - \gamma_2 V_2(x)uv & \text{in } \Omega,\\ \mathfrak{B}_1 u = \mathfrak{B}_2 v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

As we consider (5.1) to model competition between populations, we shall pay attention only to the component-wise non-negative steady states. Besides $(0,0)$, the problem (5.2) admits three types of component-wise non-negative solution couples. Namely, the solutions having one component vanishing, $(u,0)$ or $(0,v)$, known as the *semi-trivial positive solutions*, and the solutions having both component positive, known as the *coexistence states* of (5.2). Due to Theorem 2.14, (5.2) possesses a semi-trivial positive solution of the form $(u,0)$ if, and only if,

$$\lambda > \sigma[\mathfrak{L}_1, \Omega, \mathfrak{B}_1].$$

Moreover, in this case, $(\Xi_\lambda, 0)$ is the unique of these semi-trivial states, where Ξ_λ stands for the unique positive solution of

$$\begin{cases} \mathfrak{L}_1 u = \lambda u - \mathcal{X}_1(x)u^2 & \text{in } \Omega,\\ \mathfrak{B}_1 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Similarly, (5.2) possesses a semi-trivial positive solution of the form $(0, v)$ if, and only if,

$$\mu > \sigma[\mathfrak{L}_2, \Omega, \mathfrak{B}_2],$$

and, in such case, $(0, \Upsilon_\mu)$ is the unique of these semi-trivial states, where Υ_μ stands for the unique positive solution of

$$\begin{cases} \mathfrak{L}_2 v = \mu v - \mathcal{X}_2(x) v^2 & \text{in } \Omega, \\ \mathfrak{B}_2 v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

The following result characterizes the stability of each of the semi-trivial positive solutions of (5.2).

PROPOSITION 5.1. – *Suppose* $\lambda > \sigma[\mathfrak{L}_1, \Omega, \mathfrak{B}_1]$. *Then,* $(\Xi_\lambda, 0)$ *is linearly asymptotically stable* (*l.a.s.*) *if, and only if,*

$$\mu < \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2],$$

linearly neutrally stable (*l.n.s.*) *if, and only if,*

$$\mu = \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2],$$

and linearly unstable (*l.u.*) *if, and only if,*

$$\mu > \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2].$$

By symmetry, in case $\mu > \sigma[\mathfrak{L}_2, \Omega, \mathfrak{B}_2]$, *the state* $(0, \Upsilon_\mu)$ *is l.a.s. if* $\lambda < \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1]$, *l.n.s. if* $\lambda = \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1]$ *and l.u. if* $\lambda > \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1]$.

Proof. – Suppose $\lambda > \sigma[\mathfrak{L}_1, \Omega, \mathfrak{B}_1]$. By definition, the linear stability of $(\Xi_\lambda, 0)$ is given by the sign of the real parts of the eigenvalues of the linearizations of (5.2) at $(\Xi_\lambda, 0)$, i.e., by the signs of the real parts of the τ's for which the following linear problem possesses a solution $(u, v) \neq (0, 0)$:

$$\begin{cases} \mathfrak{L}_1 u = (\lambda - 2\mathcal{X}_1 \Xi_\lambda) u - \gamma_1 V_1 \Xi_\lambda v + \tau u & \text{in } \Omega, \\ \mathfrak{L}_2 v = (\mu - \gamma_2 V_2 \Xi_\lambda) v + \tau v & \text{in } \Omega, \\ \mathfrak{B}_1 u = \mathfrak{B}_2 v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

If $v = 0$, then (5.5) becomes

$$\begin{cases} \mathfrak{L}_1 u = (\lambda - 2\mathcal{X}_1 \Xi_\lambda) u + \tau u & \text{in } \Omega, \\ \mathfrak{B}_1 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Moreover, thanks to Proposition 2.5 and Lemma 2.11, we have that

$$\sigma[\mathfrak{L}_1 + 2\mathcal{X}_1 \Xi_\lambda - \lambda, \Omega, \mathfrak{B}_1] > \sigma[\mathfrak{L}_1 + \mathcal{X}_1 \Xi_\lambda - \lambda, \Omega, \mathfrak{B}_1] = 0. \tag{5.7}$$

Thus, since the principal eigenvalue is dominant, we find that

$$\operatorname{Re}\tau > \sigma[\mathfrak{L}_1 + 2\mathcal{X}_1\Xi_\lambda - \lambda, \Omega, \mathfrak{B}_1] > 0$$

and, therefore, the linear stability of $(\Xi_\lambda, 0)$ is determined from the signs of the eigenvalues of (5.5) possessing an eigenfunction of the form (u, v) with $v \neq 0$, i.e., by the sign of

$$\delta_\mu := \mu - \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2] = -\sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda - \mu, \Omega, \mathfrak{B}_2].$$

Indeed, if $\delta_\mu < 0$, then

$$-\delta_\mu = \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda - \mu, \Omega, \mathfrak{B}_2] > 0$$

and, since the principal eigenvalue is dominant, any eigenvalue of (5.5) satisfies

$$\operatorname{Re}\tau \geqslant \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda - \mu, \Omega, \mathfrak{B}_2] > 0.$$

Therefore, in this case $(\Xi_\lambda, 0)$ is linearly asymptotically stable.

Now, suppose $\delta_\mu = 0$. Then,

$$-\delta_\mu = \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda - \mu, \Omega, \mathfrak{B}_2] = 0$$

and, hence, the pair $(\tau, v) = (0, \varphi_v)$, where $\varphi_v > 0$ is the principal eigenfunction associated with $(\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda - \mu, \Omega, \mathfrak{B}_2)$, solves the v-equation of (5.5). Moreover, thanks to (5.7), the u-equation of (5.5) possesses a unique solution – in u – for $(\tau, v) = (0, \varphi_v)$. Namely,

$$u = -\gamma_1(\mathfrak{L}_1 + 2\mathcal{X}_1\Xi_\lambda - \lambda)^{-1}(V_1\Xi_\lambda\varphi_v).$$

Thus, $\tau = 0$ is an eigenvalue of (5.5). As any other value of τ for which the v-equation of (5.5) can be solved must be positive, we obtain that $(\Xi_\lambda, 0)$ is linearly neutrally stable.

Finally, suppose $\delta_\mu > 0$. Then, adapting the argument of the previous case, one readily gets that $-\delta_\mu < 0$ is an eigenvalue of (5.5) and, therefore, $(\Xi_\lambda, 0)$ is linearly unstable.

By symmetry, one obtains the corresponding characterizations for $(0, \Upsilon_\mu)$. □

Thanks to the linearized stability principle, $(\Xi_\lambda, 0)$ (resp. $(0, \Upsilon_\mu)$) is exponentially asymptotically stable if it is linearly asymptotically stable, and it is unstable if it is linearly unstable – as steady states of (5.1). The following concept is very important in mathematical biology. Subsequently, $P_{C_0^1(\overline{\Omega})}$ stands for the cone of positive functions of $C_0^1(\overline{\Omega})$.

DEFINITION 5.2. – *The problem* (5.1) *is permanent – or, equivalently, compressive – if there is a subdomain*

$$\mathfrak{R} \subset (\operatorname{Int} P_{C_0^1(\overline{\Omega})})^2$$

such that

$$\big(u(\cdot, t; u_0, v_0), v(\cdot, t; u_0, v_0)\big) \in \mathfrak{R}$$

for each $(u_0, v_0) \in X_0^2,\ u_0 > 0,\ v_0 > 0$, *after some time* $t_0 := t(u_0, v_0)$.

The abstract theory developed in [13] and the results of [10] and [19] are easily adapted to show that global extinction of some of the species occurs if (5.2) does not admit a coexistence state (cf. [15] for a further general version of that result).

Thus, (5.2) must possess a coexistence state if (5.1) is permanent. Let (u_0, v_0) be a coexistence state of (5.2). Then, thanks to Proposition 2.5, we have

$$\sigma[\mathfrak{L}_1 - \lambda, \Omega, \mathfrak{B}_1] < \sigma[\mathfrak{L}_1 - \lambda + \mathcal{X}_1 u_0 + \gamma_1 V_1 v_0, \Omega, \mathfrak{B}_1] = 0$$

and

$$\sigma[\mathfrak{L}_2 - \mu, \Omega, \mathfrak{B}_2] < \sigma[\mathfrak{L}_2 - \mu + \mathcal{X}_2 v_0 + \gamma_2 V_2 u_0, \Omega, \mathfrak{B}_2] = 0.$$

Thus,

$$\lambda > \sigma[\mathfrak{L}_1, \Omega, \mathfrak{B}_1] \quad \wedge \quad \mu > \sigma[\mathfrak{L}_2, \Omega, \mathfrak{B}_2]. \tag{5.8}$$

In particular, (5.2) exhibits the two possible semi-trivial positive solutions, $(\Xi_\lambda, 0)$ and $(0, \Upsilon_\mu)$. Obviously, (5.1) cannot be permanent if some of these semi-trivial states is linearly asymptotically stable. Therefore, thanks to Proposition 5.1, the following estimates are necessary for permanence

$$\lambda \geqslant \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1] \quad \wedge \quad \mu \geqslant \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2].$$

Conversely, the following result is satisfied (cf. [13,16] and [8] for some previous results in this direction).

THEOREM 5.3. – *Suppose* (5.8) *and* $(\Xi_\lambda, 0)$, $(0, \Upsilon_\mu)$ *are linearly unstable, i.e., thanks to Proposition* 5.1,

$$\lambda > \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1] \quad \wedge \quad \mu > \sigma[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2]. \tag{5.9}$$

Then, (5.1) *possesses a coexistence state and it is compressive.*

Proof. – We shall use a practical persistence argument based upon the arguments of [16] and [9]. The existence of the coexistence state can be obtained by using the theory of [13]. The permanence can be obtained arguing as follows. Pick $u_0,\ v_0 \in X_0$ such that $u_0 > 0$ and $v_0 > 0$. Then, thanks to the parabolic maximum principle,

$$\begin{aligned} 0 < u(\cdot\,, t; u_0, v_0) < \Phi_{[\mathfrak{L}_1, \lambda, \mathcal{X}_1, \mathfrak{B}_1]}(\cdot\,, t; u_0),\\ 0 < v(\cdot\,, t; u_0, v_0) < \Phi_{[\mathfrak{L}_2, \mu, \mathcal{X}_2, \mathfrak{B}_2]}(\cdot\,, t; v_0), \end{aligned} \tag{5.10}$$

where $\Phi_{[\mathfrak{L}, \gamma, \mathcal{X}, \mathfrak{B}]}(x, t; w_0)$ stands for the unique solution of the parabolic problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + \mathfrak{L} w = \gamma w - \mathcal{X} w^2 & \text{in } \Omega \times (0, \infty),\\ \mathfrak{B} w = 0 & \text{on } \partial\Omega \times (0, \infty),\\ w(\cdot\,, 0) = w_0 & \text{in } \Omega. \end{cases} \tag{5.11}$$

On the other hand, thanks to Theorem 2.14, we have

$$\lim_{t\nearrow\infty}\big\|\Phi_{[\mathfrak{L}_1,\lambda,\mathcal{X}_1,\mathfrak{B}_1]}(\cdot\,,t;u_0)-\Xi_\lambda\big\|_{C_0^1(\bar\Omega)}=0,$$
$$\lim_{t\nearrow\infty}\big\|\Phi_{[\mathfrak{L}_2,\mu,\mathcal{X}_2,\mathfrak{B}_2]}(\cdot\,,t;v_0)-\Upsilon_\mu\big\|_{C_0^1(\bar\Omega)}=0,$$

and, hence, for any $\varepsilon>0$ there exists $t_\varepsilon>0$ such that

$$\begin{cases}0<u(\cdot\,,t;u_0,v_0)<\Xi_\lambda+\varepsilon,\\ 0<v(\cdot\,,t;u_0,v_0)<\Upsilon_\mu+\varepsilon\end{cases}\quad \text{in }\Omega\text{ for each }t\geqslant t_\varepsilon. \tag{5.12}$$

Choose $\varepsilon>0$ sufficiently small so that

$$\lambda>\sigma\big[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\Omega,\mathfrak{B}_1\big]\quad\wedge\quad\mu>\sigma\big[\mathfrak{L}_2+\gamma_2V_2(\Xi_\lambda+\varepsilon),\Omega,\mathfrak{B}_2\big]. \tag{5.13}$$

Substituting the second estimate of (5.12) into the u-equation of (5.1), gives

$$\frac{\partial u}{\partial t}+\mathfrak{L}_1u=\lambda u-\mathcal{X}_1u^2-\gamma_1V_1uv>\lambda u-\mathcal{X}_1u^2-\gamma_1V_1(\Upsilon_\mu+\varepsilon)u$$

for each $t\geqslant t_\varepsilon$, and, hence, thanks to the parabolic maximum principle,

$$u(\cdot\,,t;u_0,v_0)\geqslant\Phi_{[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\lambda,\mathcal{X}_1,\mathfrak{B}_1]}\big(\cdot\,,t;u(\cdot\,,t_\varepsilon;u_0,v_0)\big)\quad\text{in }\Omega\text{ for each }t\geqslant t_\varepsilon.$$

On the other hand, thanks to Theorem 2.14,

$$\lim_{t\nearrow\infty}\Phi_{[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\lambda,\mathcal{X}_1,\mathfrak{B}_1]}\big(\cdot\,,t;u(\cdot\,,t_\varepsilon;u_0,v_0)\big)=u_{[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\lambda,\mathcal{X}_1,\Omega,\mathfrak{B}_1]}$$

uniformly in $\bar\Omega$, where $u_{[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\lambda,\mathcal{X}_1,\Omega,\mathfrak{B}_1]}$ stands for the unique positive solution of

$$\begin{cases}\big[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon)\big]u=\lambda u-\mathcal{X}_1u^2 & \text{in }\Omega,\\ \mathfrak{B}_1u=0 & \text{on }\partial\Omega\end{cases}$$

whose existence is guaranteed from the first inequality of (5.13). Thus, for each $\varepsilon>0$ sufficiently small

$$\liminf_{t\nearrow\infty}u(\cdot\,,t;u_0,v_0)\geqslant u_{[\mathfrak{L}_1+\gamma_1V_1(\Upsilon_\mu+\varepsilon),\lambda,\mathcal{X}_1,\Omega,\mathfrak{B}_1]}$$

and, therefore,

$$\liminf_{t\nearrow\infty}u(\cdot\,,t;u_0,v_0)\geqslant u_{[\mathfrak{L}_1+\gamma_1V_1\Upsilon_\mu,\lambda,\mathcal{X}_1,\Omega;\mathfrak{B}_1]}.$$

Similarly,

$$\liminf_{t\nearrow\infty}v(\cdot\,,t;u_0,v_0)\geqslant v_{[\mathfrak{L}_2+\gamma_2V_2\Xi_\lambda,\mu,\mathcal{X}_2,\Omega,\mathfrak{B}_2]},$$

where $v_{[\mathfrak{L}_2+\gamma_2V_2\Xi_\lambda,\mu,\mathcal{X}_2,\Omega,\mathfrak{B}_2]}$ stands for the unique positive solution of

$$\begin{cases}\big[\mathfrak{L}_2+\gamma_2V_2\Xi_\lambda\big]v=\mu v-\mathcal{X}_2v^2 & \text{in }\Omega,\\ \mathfrak{B}_2v=0 & \text{on }\partial\Omega.\end{cases}$$

This completes the proof of the theorem. □

Now, combining Theorem 5.3 together with Proposition 2.4 and Theorem 2.9, we obtain the following result.

COROLLARY 5.4. – *Suppose that*

$$\lambda > \sigma\big[\mathfrak{L}_1, \Omega^0_{V_1}, \mathfrak{B}_1\big(b_1, \Omega^0_{V_1}\big)\big] \quad \wedge \quad \mu > \sigma\big[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \Omega, \mathfrak{B}_2\big]. \tag{5.14}$$

Then, (5.1) *possesses a coexistence state and it is permanent for any* $\gamma_1 \geqslant 0$. *Similarly, if*

$$\lambda > \sigma[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \Omega, \mathfrak{B}_1] \quad \wedge \quad \mu > \sigma\big[\mathfrak{L}_2, \Omega^0_{V_2}, \mathfrak{B}_2\big(b_2, \Omega^0_{V_2}\big)\big], \tag{5.15}$$

then (5.1) *possesses a coexistence state and it is permanent for any* $\gamma_2 \geqslant 0$.

It should be noted that Corollary 5.4 is optimal. Moreover, thanks to Theorem 1.3 we obtain the following

THEOREM 5.5. – *Suppose* (2.8) *on* $\Gamma^1_1 \cap \partial\Omega^0_{V_1}$ *and* (5.14), *fix* γ_2 *and regard to* γ_1 *as a parameter. Then, the* u*-component of any coexistence state* $(u, v) = (u(\gamma_1), v(\gamma_1))$ *of* (5.2) *must satisfy*

$$\lim_{\gamma_1 \nearrow \infty} u(\gamma_1) = \begin{cases} u_{[\mathfrak{L}_1, \lambda, \mathcal{X}_1, \Omega^0_{V_1}, \mathfrak{B}_1(b_1, \Omega^0_{V_1})]} & \text{in } \Omega^0_{V_1}, \\ 0 & \text{in } \Omega \setminus \Omega^0_{V_1}. \end{cases} \tag{5.16}$$

Similarly, when (5.15) *is satisfied and* (2.8) *holds on* $\Gamma^2_1 \cap \partial\Omega^0_{V_2}$, *fixing* γ_1 *and regarding to* γ_2 *as a parameter gives*

$$\lim_{\gamma_2 \nearrow \infty} v(\gamma_2) = \begin{cases} v_{[\mathfrak{L}_2, \mu, \mathcal{X}_2, \Omega^0_{V_2}, \mathfrak{B}_2(b_2, \Omega^0_{V_2})]} & \text{in } \Omega^0_{V_2}, \\ 0 & \text{in } \Omega \setminus \Omega^0_{V_2}. \end{cases} \tag{5.17}$$

These convergences must be understood in the sense of Theorem 1.3.

Proof. – Suppose (5.14) and let (u, v) be a coexistence state of (5.2). Then, thanks to Theorem 2.16, it is easily shown that

$$u_{[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \lambda, \mathcal{X}_1, \Omega, \mathfrak{B}_1]} < u < u_{[\mathfrak{L}_1, \lambda, \mathcal{X}_1, \Omega, \mathfrak{B}_1]}$$

and

$$v_{[\mathfrak{L}_2 + \gamma_2 V_2 \Xi_\lambda, \mu, \mathcal{X}_2, \Omega, \mathfrak{B}_2]} < v < v_{[\mathfrak{L}_2, \mu, \mathcal{X}_2, \Omega, \mathfrak{B}_2]}.$$

Therefore, thanks again to Theorem 2.16, we have

$$u_{[\mathfrak{L}_1 + \gamma_1 V_1 \Upsilon_\mu, \lambda, \mathcal{X}_1, \Omega, \mathfrak{B}_1]} < u < u_{[\mathfrak{L}_1 + \gamma_1 V_1 v_{[\mathfrak{L}_2, \mu, \mathcal{X}_2, \Omega, \mathfrak{B}_2]}, \lambda, \mathcal{X}_1, \Omega, \mathfrak{B}_1]}.$$

Thanks to Theorem 1.3, passing to the limit as $\gamma_1 \nearrow \infty$ completes the proof of (5.16). The same argument can be easily adapted to prove the validity of (5.17) under condition (5.15). □

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