Équations aux dérivées partielles/*Partial Differential Equations*

Rough solutions of the Einstein vacuum equations

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Abstract   We announce a new result, contained in [10–12], concerning rough solutions to Einstein vacuum equations expressed relative to wave coordinates. Our methods blend paradifferential techniques with a geometric approach to Strichartz type inequlities, which takes advantage of the specific structure of the Einstein equations. This results in a gain of half a derivative relative to the classical result of [3] and [5]. To cite this article: S. Klainerman, I. Rodnianski, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 125–130. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Solutions peu régulières des équations d'Einstein dans le vide

Résumé   Nous annonçons un nouveau résultat concernant des solutions peu régulières des équations d'Einstein dans le vide exprimées en coordonnées d'ondes. Nos méthodes combinent les techniques paradifférentielles avec une approche géométrique pour les inégalités de type Strichartz, qui utilise la structure spécifique des équations d'Einstein. Cela permet de gagner une demi dérivée par rapport aux résultats classiques de [3] et [5]. Pour citer cet article : S. Klainerman, I. Rodnianski, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 125–130. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Les résultats classiques de Y. Choquet-Bruhat [3] et [5] montrent l'existence et l'unicité des solutions des équations d'Einstein dans le vide (1), (3), exprimées en coordonnées d'ondes (2). Le résultat le plus précis, obtenu par les méthodes classiques d'énergie et les inegalités de Sobolev, celui de [5], présuppose que les conditions initiales $\mathbf{g}(0)$, $\partial_t \mathbf{g}(0)$ vérifient (4), où $H^{s-1}(\Sigma)$ est l'espace classique de Sobolev sur la variété initiale Σ, avec $s > 2 + \frac{1}{2}$. En utilisant des techniques nouvelles, liées à des inégalités de type Strichartz pour des systèmes d'équations d'ondes à coefficients variables très peu réguliers, nous améliorons le résultat jusqu'à $s > 2$. Les premiers résultats de type Strichartz pour des équations d'ondes à coefficients très peu réguliers ont été obtenus dans les travaux récents fondamentaux de Smith [15], Bahouri–Chemin [1,2] et Tataru [19,18]. Les premières améliorations des résultats classiques pour des équations d'ondes quasilinéaires, obtenues par Bahouri–Chemin et Tataru, sont basées sur des inégalités de type Strichartz avec perte de régularité. Le résultat optimal dû à Tataru [18] conduit à $s > 2 + \frac{1}{6}$. Pour améliorer ce résultat, il est nécessaire de tenir compte de la structure non linéaire spécifique des équations. Dans [9] nous améliorons le résultat précédent pour obtenir $s > 2 + (2 - \sqrt{3})/2$, en tenant compte des propriétés des

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coefficients $g^{\alpha\beta}$ qui sont eux-mêmes solutions d'un système d'équations d'ondes du type étudié. Dans ce nouveau travail nous développons les idées de [9] pour prendre pleinement en compte la structure spécifique des équations d'Einstein en coordonnées d'ondes. Nous sommes convaincus que le résultat principal n'est pas optimal. L'exposant de Sobolev critique pour les équations d'Einstein est $s_c = \frac{3}{2}$. La résolution du problème de Cauchy compatible avec $s = s_c$ aurait pour conséquence un résultat de stabilité globale de la métrique de Minkowski beaucoup plus fort que celui de [4]. Ceci parait hors d'atteinte pour le moment. Un objectif plus raisonnable actuellement serait de prouver la « L^2 curvature conjecture » [7], qui correspond à $s = 2$.

1. Introduction

We consider the Einstein vacuum equations,

$$\mathbf{R}_{\alpha\beta}(\mathbf{g}) = 0, \tag{1}$$

where $\mathbf{g}$ is a four-dimensional Lorentz metric and $\mathbf{R}_{\alpha\beta}$ its Ricci curvature tensor. In wave coordinates x^α, $\alpha = 0, 1, 2, 3$,

$$\Box_{\mathbf{g}} x^\alpha = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\mu \left(\mathbf{g}^{\mu\nu} \sqrt{|\mathbf{g}|}\, \partial_\nu \right) x^\alpha = 0, \tag{2}$$

the Einstein vacuum equations take the reduced form, see [3,5],

$$\mathbf{g}^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}) \tag{3}$$

with N quadratic in the first derivatives $\partial\mathbf{g}$ of the metric. We consider the initial value problem along the spacelike hyperplane Σ given by $t = x^0 = 0$,

$$\nabla\mathbf{g}_{\alpha\beta}(0) \in \mathrm{H}^{s-1}(\Sigma), \qquad \partial_t \mathbf{g}_{\alpha\beta}(0) \in \mathrm{H}^{s-1}(\Sigma) \tag{4}$$

with ∇ denoting the gradient with respect to the space coordinates x^i, $i = 1, 2, 3$, and H^s the standard Sobolev spaces. We also assume that $\mathbf{g}_{\alpha\beta}(0)$ is a continuous Lorentz metric and $\sup_{|x|=r} |\mathbf{g}_{\alpha\beta}(0) - \mathbf{m}_{\alpha\beta}| \to 0$ as $r \to \infty$, where $|x| = (\sum_{i=1}^3 |x^i|^2)^{1/2}$ and $\mathbf{m}_{\alpha\beta}$ the Minkowski metric.

The following local existence and uniqueness result (well posedness) is well known (*see* [5] and the previous result of Choquet-Bruhat [3] for $s \geqslant 4$).

THEOREM 1.1. – *Considere the reduced equation* (3) *subject to the initial conditions* (4) *for some $s > 5/2$. Then there exists a time interval $[0, T]$ and unique (Lorentz metric) solution $\mathbf{g} \in \mathrm{C}^0([0, T] \times \mathbb{R}^3)$, $\partial\mathbf{g}_{\mu\nu} \in \mathrm{C}^0([0, T]; \mathrm{H}^{s-1})$ with T depending only on the size of the norm $\|\partial\mathbf{g}_{\mu\nu}(0)\|_{\mathrm{H}^{s-1}}$.*

We establish a significant improvement of this result bearing on the issue of minimal regularity of the initial conditions:

MAIN THEOREM. – *Consider a classical solution of the equations* (3) *for which* (1) *also holds.* [1] *We show* [2] *that the time T of existence depends in fact only on the size of the norm $\|\partial\mathbf{g}_{\mu\nu}(0)\|_{\mathrm{H}^{s-1}}$, for any fixed $s > 2$.*

Remark 1.2. – Theorem 1.1 implies the classical local existence result of [5] for asymptotically flat initial data sets Σ, g, k with ∇g, $k \in \mathrm{H}^{s-1}(\Sigma)$ and $s > \frac{5}{2}$, relative to a fixed system of coordinates. Uniqueness can be proved for additional regularity $s > 1 + \frac{5}{2}$. We recall that an initial data set (Σ, g, k) consists of a three dimensional complete Riemannian manifold (Σ, g), a 2-covariant symmetric tensor k on Σ verifying the constraint equations $\nabla^j k_{ij} - \nabla_i \operatorname{tr} k = 0$ and $R - |k|^2 + (\operatorname{tr} k)^2 = 0$ where ∇ is the covariant derivative, R the scalar curvature of (Σ, g).

Remark 1.3. – The Main Theorem ought to imply existence and uniqueness [3] for initial conditions with H^s, $s > 2$, regularity. To achieve this we only need to approximate a given H^s initial data set (i.e., $\nabla g \in H^{s-1}(\Sigma)$, $k \in H^{s-1}(\Sigma)$, $s > 2$) for the Einstein vacuum equations by classical initial data sets, i.e., $H^{s'}$ data sets with $s' > \frac{5}{2}$, for which Theorem 1.1 holds. The Main Theorem allows us to pass to the limit and derive existence of solutions for the given, rough, initial data set. We don't know however if such an approximation result for the constraint equations exists in the literature.

2. Main ideas of the proof

For convenience we shall also write the reduced equations (3) in the form of a system of quasilinear wave equations of the form,

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N(\phi, \partial\phi), \tag{5}$$

where $\phi = (\mathbf{g}_{\mu\nu})$, $N = N_{\mu\nu}$ and $\mathbf{g}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}(\phi)$. The classical local existence result for systems of wave equations of type (5) is based on energy estimates and the standard $H^s \subset L^\infty$ Sobolev inequality. Indeed using energy estimates and simple commutation inequalities one can show that,

$$\left\|\partial\phi(t)\right\|_{H^{s-1}} \leqslant E\left\|\partial\phi(0)\right\|_{H^{s-1}} \tag{6}$$

with $E = \exp\left(C \int_0^t \|\partial\phi(\tau)\|_{L_x^\infty}\,d\tau\right)$. By the classical Sobolev inequality,

$$E \leqslant \exp\left(Ct \sup_{0\leqslant\tau\leqslant t} \left\|\partial\phi(\tau)\right\|_{H^{s-1}}\,d\tau\right)$$

provided that $s > \frac{5}{2}$. The classical local existence result follows by combining this last estimate, for a small time interval, with the energy estimates (6). This scheme is very wasteful. To do better one would like to take advantage of the mixed $L_t^1 L_x^\infty$ norm appearing in the definition of E. If ϕ is a solution of the standard wave equation

$$\Box\phi = 0 \tag{7}$$

in Minkowski space, we have the following Strichartz type inequality:

$$\|\partial\phi\|_{L_t^2 L_x^\infty([0,T]\times\mathbb{R}^3)} \leqslant CT^\varepsilon \|\partial\phi(0)\|_{H^{1+\varepsilon}}, \tag{8}$$

with $\varepsilon > 0$ arbitrarily small. Based on this fact it was reasonable to hope that one can improve the Sobolev exponent in the classical local existence theorem from $s > \frac{5}{2}$ to $s > 2$. This can be easily done for solutions of semilinear wave equations, see [14]. In the case of general quasilinear wave equations of the form (5), however, the situation is far more difficult. One can no longer rely on the Strichartz inequality (8) for the flat D'Alembertian in (7); we need instead its extension to the operator $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ appearing in (5). Moreover, since the metric $\mathbf{g}^{\alpha\beta}$ depends on the solution ϕ, it can have only as much regularity as ϕ itself. This means that we have to confront the issue of proving Strichartz estimates for wave operators $\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$ with *very rough coefficients* $\mathbf{g}^{\alpha\beta}$. This issue was recently addressed in the pioneering works of Smith [15], Bahouri–Chemin [1,2] and Tataru [19,18], we refer to the introduction in [6] and [9] for a more thorough discussion of their important contributions. The results of Bahouri–Chemin and Tataru are based on establishing a Strichartz type inequality, *with a loss*, for wave operators with *very rough coefficients*. [4] The optimal result [5] in this regard, due to Tataru, see [18], requires a loss of $\sigma = \frac{1}{6}$. This leads to a proof of local well posedness for systems of type (5) with $s > 2 + \frac{1}{6}$.

To do better than that one needs to take into account the nonlinear structure of the equations. In [9] we were able to improve the result of Tataru by taking into account not only the expected regularity properties of the coefficients $\mathbf{g}^{\alpha\beta}$ in (5) but also the fact that they are themselves solutions to a similar system of equations. This allowed us to improve the exponent s, needed in the proof of well posedness of equations of type [6] (5), to $s > 2 + (2 - \sqrt{3})/2$. Our approach was based on a combination of the paradifferential

calculus ideas, initiated in [1] and [18], with a geometric treatment of the actual equations introduced in [6]. The main improvement was due to a gain of conormal differentiability for solutions to the Eikonal equations

$$H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \tag{9}$$

where the background metric H is a properly microlocalized and rescaled version of the metric $\mathbf{g}^{\alpha\beta}$ in (5). That gain could be traced down to the fact that a certain component of the Ricci curvature of H has a special form. More precisely denoting by L' the null geodesic vectorfield associated to u, $L' = -H^{\alpha\beta}\partial_\beta u\partial_\alpha$, and rescaling it in an appropriate fashion,[7] $L = bL'$, we found that the null Ricci component $\mathbf{R}_{LL} = \mathbf{Ric}(H)(L, L)$, verifies the remarkable identity:

$$\mathbf{R}_{LL} = L(z) - \frac{1}{2}L^\mu L^\nu \left(H^{\alpha\beta}\partial_\alpha\partial_\beta H_{\mu\nu}\right) + e, \tag{10}$$

where $z \leqslant \mathrm{O}(|\partial H|)$ and $e \leqslant \mathrm{O}(|\partial H|^2)$. Thus, apart from $L(z)$ which is to be integrated along the null geodesic flow generated by L, the only terms which depend of the second derivatives of H appear in $H^{\alpha\beta}\partial_\alpha\partial_\beta H$ and can therefore be eliminated with the help of the equations (5).

In this paper we develop the ideas of [9] further by taking full advantage of the the structure of the Einstein equations (1). An important aspect of our analysis here is that the term $L(z)$ appearing on the right-hand side of (10) vanishes identically. We make use of both the vanishing of the Ricci curvature of $\mathbf{g}$ and the wave coordinate condition (2). The other important new features are the use of energy estimates along the null hypersurfaces generated by the optical function u and a deeper use of the conormal properties of the null structure equations.

Our work is divided in three parts. In [10] we reduce the proof of the Main Theorem to the study of the asymptotic properties of the null Ricci coeficients χ and η, i.e., the main null components of the Hessian $\mathbf{D}^2 u$ of the optical function u defined by (9). The main steps are already quite well established, see [6] and [9]; they need to be adapted however to the present set-up.

Our second paper [11] is dedicated to the proof of the Asymptotics Theorem concerning the behavior of the null Ricci coefficients χ and η. We demonstrate that the optical function associated to Einstein metrics, or rather microlocalized versions of them, have better properties. This fact was already recognized in [4] where the construction of an optical function normalized at infinity played a crucial role in the proof of the global nonlinear stability of the Minkowski space. A similar construction, based on two optical functions, can be found in [8]. Here, we take the use of the special structure of the Einstein equations one step further by deriving unexpected regularity properties of optical functions which are essential in the proof of the Main Theorem. It was well known (*see* [4,6,9]) that the use of Codazzi equations combined with the Raychaudhuri equation for the trχ, the trace of null second fundamental form χ, leads to the improved estimate for the first angular derivatives of the traceless part of χ. A similar observation holds for another null component of the Hessian of the optical function, η. The role of the Raychaudhuri equation is taken by the transport equation the "mass aspect function" μ. In [11] we show, using the structure of the curvature terms in the main equations, how to derive new improved regularity estimates for the undifferentiated quantities $\hat\chi$ and η. In particular, in the case of the estimates for η we are lead to introduce a new nonlocal quantity tied to μ via a Hodge system.

Our last paper [12] is dedicated to the problem of Ricci defects $\mathbf{Ric}(H)$ arising after microlocalization of the Einstein metric $\mathbf{g}$. In particular, we derive a crucial estimate, needed in the proof of the Asymptotics Theorem, concerning the behavior of the spatial derivatives of $\mathbf{R}_{LL}(H)$. The proof of this estimate relies on the energy estimates on the null cones associated with the metric H. To make use of these estimates we establish remarkable algebraic cancellations in the terms comprising the component $\mathbf{R}_{LL}(H)$ written relative to our system of wave coordinates.

The key to the proof of our result is the following Strichartz inequality verified by a solution of (5),

$$\|\partial\phi\|_{\mathrm{L}^2_{[0,T]}\mathrm{L}^\infty_x} \leqslant C(B_0)\, T^\delta \tag{11}$$

for some $\delta > 0$. This is proved by a bootstrap argument by first assuming that

$$\|\partial\phi\|_{L_{[0,T]}^\infty H^{1+\gamma}} + \|\partial\phi\|_{L_{[0,T]}^2 L_x^\infty} \leqslant B_0. \tag{12}$$

The proof of (11) can be reduced, by a sequence of well established steps (*see* [10,9]), to a microlocal decay estimate for the wave operator $\Box_H$ where $H(t,x) = H_{(\lambda)}(t,x) = P_{<\lambda}\mathbf{g}(\lambda^{-1}t, \lambda^{-1}x)$ is our *effective background* Lorentz metric obtained by rescaling the microlocalized metric $P_\lambda g(t,x)$. The Littlewood–Paley projection $P_{<\lambda}$ cuts off the frequencies above the large dyadic parameter λ.

THEOREM 2.1. – *Let ψ be a solution of the equation,*

$$\Box_H \psi = 0 \tag{13}$$

on the time interval $[0, t_*]$ with $t_* \leqslant \lambda^{1-8\varepsilon_0}$. Assume that the initial data is given at $t = t_0 \in [0, t_*]$, supported in the ball $B_{1/2}(0)$ of radius $1/2$ centered at the origin. There exists a function $d(t)$, with $t_*^{1/q}\|d\|_{L^q([0,t_*])} \leqslant 1$ for some $q > 2$ sufficiently close to 2, an arbitrarily small $\delta > 0$ and a sufficiently large integer $m > 0$ such that for all $t \in [0, t_*]$ and the projection P on the frequencies of size 1,

$$\left\| P\,\partial\psi(t) \right\|_{L_x^\infty} \leqslant C(B_0)\left(\frac{1}{(1+|t-t_0|)^{1-\delta}} + d(t) \right) \sum_{k=0}^m \left\| \nabla^k \partial\psi(t_0) \right\|_{L_x^2}. \tag{14}$$

The proof of Theorem 2.1 relies on a purely geometric argument based on the construction of a modified Morawetz vectorfield K. We construct it with the help of the optical function u defined as the outgoing solution of the eikonal equation

$$H^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$$

with the boundary condition $u(\Gamma_t) = t$ on the time axis Γ_t, the geodesic of the unit future oriented timelike vectorfield T orthogonal to the hypersurfaces $t = c$, denoted Σ_t, passing through the origin. The null cones C_u defined as the level hypersurfaces of u and the 2-surfaces $S_{t,u} := \Sigma_t \cap C_u$.

We denote by N the vectorfield of unit normals to $S_{t,u}$ in Σ_t and define the null pair L, $\underline{L}$

$$L = T + N, \qquad \underline{L} = T - N.$$

The modified Morawetz vectorfield is constructed according to the formula

$$K = \underline{u}^2 L + u^2 \underline{L}, \qquad \underline{u} = 2t - u.$$

The standard generalized energy identity associated to K takes the form

$$\int_{\Sigma_t} \overline{Q}(K,T) = \int_{\Sigma_{t_0}} \overline{Q}(K,T) - \frac{1}{2}\int_{[t_0,t]\times\mathbb{R}^3} Q^{\alpha\beta}\,{}^{(K)}\bar{\pi}_{\alpha\beta} + \int_{[t_0,t]\times\mathbb{R}^3} \psi^2\Box_H(t), \tag{15}$$

where Q and $\overline{Q}$ are the standard and the modified energy momentum tensors, see [10] or [9], associated to the equation (13). Also,

$${}^{(K)}\bar{\pi}_{\mu\nu} = \mathcal{L}_K H_{\mu\nu} - 4t H_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu - 4t H_{\mu\nu}$$

is the modified deformation tensor of the vectorfield K.

We show that the decay estimate (14) follows from the the boundedness theorem:

$$\int_{\Sigma_t} \overline{Q}(K,T) \lesssim \int_{\Sigma_{t_0}} \overline{Q}(K,T).$$

This requires the control of the error term on the right-hand side of (15). For this we need good estimates for the null ${}^{(K)}\bar{\pi}$ which depend on the control of the null components χ, η of the Hessian $\mathbf{D}^2 u$. Most of the novelty of our work concerns these estimates. They are done in [11] and [12].

A related result for general systems of quasilinear wave equations was announced by H. Smith and D. Tataru.

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[1] In other words for any solution of the reduced equations (3) whose initial data satisfy the constraint equations, see [3] or [5]. The fact that our solutions verify (1) plays a fundamental role in our analysis.

[2] We assume however that T stays sufficiently small, e.g., $T \leqslant 1$. This a purely technical assumption which one should be able to remove.

[3] Properly speaking uniqueness holds, with $s > 2$, only for the reduced equations. Uniqueness for the actual Einstein equations requires one more derivative, see [5].

[4] The derivatives of the coefficients $\mathbf{g}$ are required to be bounded in $L_t^\infty H_x^{s-1}$ and $L_t^2 L_x^\infty$ norms, with s compatible with the regularity required on the right-hand side of the Strichartz inequality one wants to prove.

[5] Recently Smith–Tataru [17] have shown that the result of Tataru is indeed sharp.

[6] The result in [9] applies to general equations of type (5) not necessarily tied to (1). In [9] we have also made the simplifying assumptions $\mathbf{n} = 1$ and $\mathbf{v} = 0$.

[7] Such $\langle L, T \rangle_H = 1$ with T is the unit normal to the level hypersurfaces Σ_t associated to the time function t.

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