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Positivity of $L(\frac{1}{2}, \pi)$ for symplectic representations

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Abstract Let π a cuspidal generic representation of $\mathrm{SO}(2n+1)$. We prove that $L(\frac{1}{2}, \pi) \geqslant 0$. To cite this article: E. Lapid, S. Rallis, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 101–104. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Positivité de $L(\frac{1}{2}, \pi)$ pour représentations simplectiques

Résumé Soit π une représentation cuspidale géńerique de $\mathrm{SO}(2n+1)$. Nous prouvons que $L(\frac{1}{2}, \pi) \geqslant 0$. Pour citer cet article : E. Lapid, S. Rallis, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 101–104. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ where $\mathbb{A}$ is the adèles ring of a number field F. Suppose that π is self-dual. Then the standard L-function $L(s, \pi)$ is real for $s \in \mathbb{R}$ and positive for $s > 1$. By the generalized Riemann hypothesis we expect that $L(s, \pi) > 0$ for $s > \frac{1}{2}$ and in particular, $L(\frac{1}{2}, \pi) \geqslant 0$. However, the latter is unknown even in the case of quadratic Dirichlet characters. In general, if π is self dual then π is either *symplectic* or *orthogonal*, i.e., exactly one of the (partial) L-functions $L^S(s, \pi, \wedge^2)$ (exterior square) or $L^S(s, \pi, \mathrm{sym}^2)$ (symmetric square) has a pole at $s = 1$. In the first case, n is even and the central character of π is trivial [9].

Our main result in [13] is:

THEOREM 1. – *Let π be a symplectic cuspidal representation of* $\mathrm{GL}_n(\mathbb{A})$. *Then* $L(\frac{1}{2}, \pi) \geqslant 0$.

We remark that in the formulation of Theorem 1 we could take the partial L-function instead of the completed one.

In the case, $n = 2$, π is symplectic exactly when the central character of π is trivial. In this case more precise information is known about $L(\frac{1}{2}, \pi)$, at least in special cases (*cf.* [11]), and the theorem was proved before using a variant of Jacquet's relative trace formula [7]. Even for this case our proof is different. However, the relative trace formula may yield more information (*cf.* [8]).

The Tannakian formalism suggests that the symplectic (resp. orthogonal) representations are precisely the functorial images from groups whose L-group is a symplectic (resp. orthogonal) group. In fact, it has been proved [5,2] that generic cuspidal representations of $\mathrm{SO}(2n+1)$ are in one-to-one (functorial)

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correspondence with the set of families $\{\pi_1, \ldots, \pi_k\}$ of distinct cuspidal symplectic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ with $n_1 + \cdots + n_k = n$. As a consequence:

THEOREM 2. – *Let σ be a cuspidal generic representation of* $\mathrm{SO}(2n+1)(\mathbb{A})$. *Then* $L^S(\frac{1}{2}, \sigma) \geqslant 0$.

Here the (partial) L-function corresponds to the standard imbedding of the L-group Sp_n in GL_{2n}. We could have also taken the completed L-function as defined by Shahidi.

As a by-product of the proof we also obtain the following result.

THEOREM 3. – *Let π be a self-dual cuspidal representation of* $\mathrm{GL}_n(\mathbb{A})$. *Then the root numbers* $\varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2)$ *and* $\varepsilon(\frac{1}{2}, \pi, \wedge^2)$ *are equal to one.*

A priori one knows that these root numbers are ± 1. In [15] Prasad and Ramakrishnan, motivated by results of Fröhlich and Queyrut [4] and Deligne [3], conjectured that $\varepsilon(\frac{1}{2}, \pi) = 1$ for any orthogonal representation of GL_n. Theorem 3 is compatible with this conjecture and Langlands functoriality. We also remark that it is not difficult to prove that $\varepsilon(\frac{1}{2}, \pi \otimes \widetilde{\pi}) = 1$ for any cuspidal representation π of GL_n (*cf.* [1]).

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2. Reduction to a local statement

As mentioned before, Theorems 1 and 3 are proved using the theory of Eisenstein series on classical groups. Let G be a split connected classical group (symplectic or special orthogonal) of rank n. We identify GL_n with the Levi subgroup M of the Siegel parabolic subgroup $P = MU$ of G. Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and identify the induced space $I(\pi, s)$ with the space $\mathcal{A}_P(\pi, s)$ of automorphic forms φ on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$ such that the function $m \to |\det(m)|^{-s}\delta_P(m)^{-1/2}\varphi(mk)$ belongs to the space of π for any $k \in \mathbf{K}$, where δ_P is the modulus function of $P(\mathbb{A})$. We denote by $E(g, \varphi, s)$ (the meromorphic continuation of) the Eisenstein series for $\varphi \in I(\pi, s)$. We will be interested in the case where $E(\bullet, \varphi, s)$ has a pole at $s = \frac{1}{2}$. A necessary condition is that π is self-dual and that P is conjugate to its opposite (i.e., $G \neq \mathrm{SO}(4m+2)$). From now on we assume that these conditions are satisfied. Let $w \in G$ be such that the map $m \mapsto wmw^{-1}$ induces the involution $x^\sharp = w_n^{-1}\,{}^t x^{-1} w_n$ on GL_n where $(w_n)_{i,j} = (-1)^i \delta_{i+j,n+1}$. Let $E_{-1}(\bullet, \varphi)$ be the residue of $E(g, \varphi, s)$ at $s = \frac{1}{2}$. Up to a positive constant depending on normalization of measures, the inner product of residues of Eisenstein series is given by

$$\int_{G\backslash G(\mathbb{A})} E_{-1}(g, \varphi_1)\overline{E_{-1}(g, \varphi_2)}\,\mathrm{d}g = \int_{\mathbf{K}}\int_{M\backslash M(\mathbb{A})^1} \mathfrak{M}_{-1}\varphi_1(mk)\overline{\varphi_2(mk)}\,\mathrm{d}m\,\mathrm{d}k, \tag{1}$$

where $\mathfrak{M}_{-1}$ is the residue of the intertwining operator $\mathfrak{M}(s) : \mathcal{A}_P(\pi, s) \to \mathcal{A}_P(\pi, -s)$ at $s = \frac{1}{2}$.

Let $\pi^\sharp$ be the (abstract) representation of $M(\mathbb{A})$ on V_π defined by $\pi^\sharp(m)v = \pi(m^\sharp)v$. It is equivalent to the contragredient of π. Let $M(s) = M(\pi, s) : I(\pi, s) \to I(\pi^\sharp, -s)$ be the "abstract" intertwining operator. Since π is self-dual, and multiplicity one holds for GL_n, we have an intertwining operator $\iota = \iota_\pi : \pi^\sharp \to \pi$ which does not depend on the automorphic realization of π and which is given by $\iota(\varphi) = \varphi^\sharp$ where $\varphi^\sharp(m) = \varphi(m^\sharp)$. We write $\iota(s) = \iota(\pi, s)$ for the induced map $I(\pi^\sharp, s) \to I(\pi, s)$ given by $\left[\iota(s)(f)\right](g) = \iota(f(g))$. By our identifications we have $\mathfrak{M}(s) = \iota(-s) \circ M(s)$.

In the local case we can define ${\pi_v}^\sharp$ and the local intertwining operators $M_v(s) : I(\pi_v, s) \to I({\pi_v}^\sharp, -s)$ in the same way. If π_v is a local self-dual irreducible generic representation of $(\mathrm{GL}_n)_v$ then fixing an additive character ψ_v we may define an intertwining map $\iota_v = \iota_{\pi_v} : {\pi_v}^\sharp \to \pi_v$ by $\iota_v(W) = W^\sharp$ on the Whittaker model. This map does not depend on the choice of Whittaker model. Suppose that $\pi = \otimes_v \pi_v$ and $\psi =$ is a global additive character. Then we have $\iota_\pi = \prod_v \iota_{\pi_v}$.

Shahidi has defined normalization factors $m_v(\pi_v, s) = m_v(s)$ for the local intertwining operators [16]. (We suppress their dependence on ψ_v.) Thus we may write $M_v(\pi_v, s) = m_v(\pi_v, s)R_v(\pi_v, s)$ where $R_v(s) = R_v(\pi_v, s)$ are the normalized intertwining operators. Let $m(s) = m(\pi, s) = \prod_v m_v(\pi_v, s)$ and

$R(s) = \prod_v R_v(\pi_v, s)$ so that $M(s) = m(s)R(s)$. We have

$$m(s) = \begin{cases} \frac{L(s,\pi)}{\varepsilon(s,\pi)L(s+1,\pi)} \frac{L(2s,\pi,\wedge^2)}{\varepsilon(2s,\wedge^2,\pi)L(2s+1,\pi,\wedge^2)} = \frac{L(1-s,\pi)}{L(1+s,\pi)} \frac{L(1-2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)}, & G = \mathrm{Sp}_n, \\ \frac{L(2s,\pi,\mathrm{sym}^2)}{\varepsilon(2s,\pi,\mathrm{sym}^2)L(2s+1,\pi,\mathrm{sym}^2)} = \frac{L(1-2s,\pi,\mathrm{sym}^2)}{L(1+2s,\pi,\mathrm{sym}^2)}, & G = \mathrm{SO}(2n+1), \\ \frac{L(2s,\pi,\wedge^2)}{\varepsilon(2s,\pi,\wedge^2)L(2s+1,\pi,\wedge^2)} = \frac{L(1-2s,\pi,\wedge^2)}{L(1+2s,\pi,\wedge^2)}, & G = \mathrm{SO}(2n). \end{cases}$$

In particular, the residue m_{-1} at $s = \frac{1}{2}$ is given by

$$m_{-1} = \begin{cases} \frac{L(\frac{1}{2},\pi)}{\varepsilon(\frac{1}{2},\pi)L(\frac{3}{2},\pi)} \frac{\mathrm{res}_{s=1} L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)}, & G = \mathrm{Sp}_n, \\ \frac{\mathrm{res}_{s=1} L(s,\pi,\mathrm{sym}^2)}{\varepsilon(1,\pi,\mathrm{sym}^2)L(2,\pi,\mathrm{sym}^2)}, & G = \mathrm{SO}(2n+1), \\ \frac{\mathrm{res}_{s=1} L(s,\pi,\wedge^2)}{\varepsilon(1,\pi,\wedge^2)L(2,\pi,\wedge^2)}, & G = \mathrm{SO}(2n). \end{cases} \tag{2}$$

The Eisenstein series $E(\bullet, \varphi, s)$ has a pole at $s = \frac{1}{2}$ if and only if $m_{-1} \neq 0$. Note that $L(3/2, \pi) > 0$ and that if $\varepsilon(\frac{1}{2}, \pi) = -1$ then $L(\frac{1}{2}, \pi) = 0$ by the functional equation. Thus, Theorem 1 would follow, if we knew that $m_{-1} \geqslant 0$ in the first and last case. Moreover, the factor $\frac{\mathrm{res}_{s=1} L(s,\pi,\wedge^2)}{L(2,\pi,\wedge^2)}$ is positive since $L(s, \pi, \wedge^2)$ is holomorphic and non-zero for $\mathrm{Re}(s) > 1$ and real for $s \in \mathbb{R}$. Similarly for $\frac{\mathrm{res}_{s=1} L(s,\pi,\mathrm{sym}^2)}{L(2,\pi,\mathrm{sym}^2)}$. On the other hand $\varepsilon(s, \pi, \wedge^2)$, $\varepsilon(s, \pi, \mathrm{sym}^2)$ are exponential functions and $\varepsilon(\frac{1}{2}, \pi, \wedge^2) \cdot \varepsilon(\frac{1}{2}, \pi, \mathrm{sym}^2) = \varepsilon(\frac{1}{2}, \pi \otimes \pi) = 1$. Hence Theorem 3 would follow, if we knew in addition that $m_{-1} \geqslant 0$ in the second case. Therefore it remains to show that $m_{-1} \geqslant 0$ in all cases. Let $\mathfrak{B}(s) = \mathfrak{B}(\pi, s)$ be the operator $\iota(-s) \circ R(s) : I(\pi, s) \to I(\pi, \bar{s})^*$ where * denotes the Hermitian dual. It is Hermitian for $s \in \mathbb{R}$ and $\mathfrak{B}(\pi, 0)$ is an involution. Since $\mathfrak{M}_{-1} = m_{-1} \cdot \mathfrak{B}(\frac{1}{2})$ the relation, (1) yields that $\mathfrak{B}(\frac{1}{2})$ is semi-definite and has the same sign as m_{-1}. It remains to show that the sign is positive. In the case where π_v is everywhere unramified this follows from the fact that ι_v and R_v act trivially on the unramified vector. In the general case one knows by [12], Proposition 6.3 that $\mathfrak{B}(\pi, 0)$ has a nontrivial $+1$-eigenspace. It remains to show the following:

PROPOSITION 4. – *Suppose that $\mathfrak{B}(\pi, \frac{1}{2})$ is semi-definite. Then $\mathfrak{B}(\pi, 0)$ is definite (i.e., a scalar, necessarily ± 1), and has the same sign as $\mathfrak{B}(\pi, \frac{1}{2})$.*

This global statement follows from its local counterpart (with analogous notation).

3. Local analysis

Let π be a self-dual generic irreducible unitarizable representation of $\mathrm{GL}_n(F)$ where F is now a local field. We say that π is of G-type if $\mathfrak{B}(\pi, 0)$ is a scalar (necessarily ± 1). We first prove Proposition 4 in the square-integrable case. This requires an analysis of the reducibility points of $I(\pi, s)$ for π square-integrable, which involves among other things the theory of R-groups. Such an analysis was carried out by Shahidi, Tadic, Muić Jantzen, Goldberg and others. (See [17,16,18,14,10,6].) This analysis also shows that if π is tempered and $0 < s < \frac{1}{2}$ then $I(\pi, s)$ is irreducible.

To prove the local analogue of Proposition 4 for a general π, the following elementary lemma from linear algebra will be useful.

LEMMA 5. – *Let $\mathfrak{B}_\alpha$, $0 \leqslant \alpha \leqslant 1$, be a continuous family of Hermitian operators on a finite dimensional inner product space. Suppose that $\mathfrak{B}_0$ is positive semi-definite and that the rank of $\mathfrak{B}_\alpha$ is constant for $0 \leqslant \alpha < 1$. Then $\mathfrak{B}_1$ is positive semi-definite.*

Let π be as before. Since any generic irreducible representation of $\mathrm{GL}_n(F)$ is parabolically induced from essentially square-integrable representations, we may write $\pi = \Sigma \boxplus \Omega$ ($\boxplus$ stands for induction) where Σ is induced from mutually inequivalent, self-dual, square-integrable representations which are not of G-type, and Ω is induced from square integrable self-dual representations of G-type, and representations of the form $\rho_j \boxplus \rho_j^\vee$ where ρ_j is essentially square-integrable. Moreover, since π is unitarizable the central exponents of the ρ_j's are less than $\frac{1}{2}$ in absolute value.

The first step is to reduce to the case where π is tempered. By twisting the ρ_j's by unramified characters, we obtain a continuous "deformation" $\{\pi_\alpha\}$ of π into a tempered representation. The reduction is achieved by applying Lemma 5 to both families $\mathfrak{B}(\pi_\alpha, \frac{1}{2})$ and $\mathfrak{B}(\pi_\alpha, 0)$.

Suppose now that π is tempered. The main step is to show that $\Sigma = 0$. Indeed, if $\Sigma = 0$ then π is of G-type and the operators $\mathfrak{B}(\pi, s)$ are non-degenerate for $0 \leqslant s < \frac{1}{2}$ since $I(\pi, s)$ is irreducible for $0 < s < \frac{1}{2}$. We may then apply Lemma 5 once again. To prove that $\Sigma = 0$, we consider a family of intertwining operators $\mathfrak{B}'(\alpha)$ on $I(\Sigma|\bullet|^\alpha \boxplus \Omega|\bullet|^{1/2}, 0)$. As before, we may use Lemma 5 to deform α to 0. On the other hand, if $\mathfrak{B}'(0)$ were semi-definite, then the same would be true for $\mathfrak{B}(\Sigma, 0)$. However, by the theory of R-groups, $\mathfrak{B}(\Sigma, 0)$ is of order exactly two unless $\Sigma = 0$.

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