

# Sharp Hodge decompositions in two and three dimensional Lipschitz domains

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**Abstract** We identify the optimal range of coefficients  $s, p$  for which differential forms with coefficients in the Sobolev space  $L_s^p(\Omega)$  admit natural Hodge decompositions in arbitrary two and three dimensional Lipschitz domains  $\Omega$ . To cite this article: D. Mitrea, M. Mitrea, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 109–112. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## Décompositions de Hodge optimales pour les domaines lipschitziens en dimensions deux et trois

**Résumé** Nous identifions la gamme optimale des coefficients  $s, p$  pour lesquels les formes différentielles à coefficients dans l'espace de Sobolev  $L_s^p(\Omega)$  admettent des décompositions de Hodge naturelles, pour des domaines lipschitziens  $\Omega$  arbitraires de dimensions deux et trois. Pour citer cet article : D. Mitrea, M. Mitrea, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 109–112. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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### Version française abrégée

Nous utilisons la notation habituelle utilisée dans la géométrie différentielle. Plus spécifiquement, pour une variété réelle  $M$ , nous dénotons par  $d$  l'opérateur de différentiation extérieure, et  $\delta$  son adjoint formel. Puis, nous posons  $\wedge, \vee$  pour le produit extérieur et intérieur des formes différentielles, respectivement. Rappelons que  $\Omega \subset M$  s'appelle *lipschitzien* si on peut décrire  $\partial\Omega$  localement à l'aide des graphes des fonctions lipschitziennes.

À partir de l'espace de Sobolev habituel sur  $L_s^p(\Omega)$ . On note alors  $L_s^p(\Omega, \Lambda^\ell)$  l'espace des  $\ell$ -formes différentielles à coefficients dans  $L_s^p(\Omega)$ . Ensuite, si  $\nu$  désigne la normale unitaire à  $\partial\Omega$ , nous posons

$$H^{s,p}(\Omega; d_\ell) := \{u \in L_s^p(\Omega, \Lambda^\ell); du \in L_s^p(\Omega, \Lambda^{\ell+1})\}, \quad (1.1)$$

$$H_\wedge^{s,p}(\Omega; d_\ell) := \{u \in H^{s,p}(\Omega; d_\ell); \nu \wedge u = 0\}, \quad (1.2)$$

$$\mathcal{H}_\wedge^{s,p}(\Omega, \Lambda^\ell) := \{u \in H_\wedge^{s,p}(\Omega; d_\ell); \delta u = 0, du = 0\}. \quad (1.3)$$

De façon similaire, on peut introduire  $H^{s,p}(\Omega; \delta_\ell)$ ,  $H_\vee^{s,p}(\Omega; \delta_\ell)$ , et  $\mathcal{H}_\vee^{s,p}(\Omega, \Lambda^\ell)$  (on peut aussi employer (1.1)–(1.3) et l'isomorphisme de Hodge).

**THÉORÈME 1.1.** – Soit  $M$  une variété réelle, compacte, sans frontière et de dimension  $n \in \{2, 3\}$ . Supposons que  $M$  est munie d'une métrique riemannienne de classe  $C^{1,1}$ . Enfin, supposons que  $\Omega$  est

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un domaine lipschitzien arbitraire de  $M$ . Alors, il existe un  $p_\Omega \in [1, 2)$  tel que pour tout  $\ell \in \{0, 1, \dots, n\}$ , et tout couple  $s, p$  satisfaisant

$$\begin{aligned}
 &0 < \frac{1}{p} < 1, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad \text{et} \\
 &\frac{1}{2} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{p} - \frac{s}{2} < \frac{1}{2} \left( 1 + \frac{1}{p_\Omega} \right) \quad \text{si } n = 2, \\
 &\frac{2}{3} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{p} - \frac{s}{3} < \frac{1}{3} \left( \frac{2}{p_\Omega} + 1 \right) \quad \text{si } n = 3,
 \end{aligned} \tag{1.4}$$

on a

$$L_s^p(\Omega, \Lambda^\ell) = d_{\ell-1} H_\wedge^{s,p}(\Omega; d_{\ell-1}) \oplus \delta_{\ell+1} H^{s,p}(\Omega; \delta_{\ell+1}) \oplus \mathcal{H}_\wedge^{s,p}(\Omega, \Lambda^\ell), \tag{1.5}$$

$$L_s^p(\Omega, \Lambda^\ell) = d_{\ell-1} H^{s,p}(\Omega; d_{\ell-1}) \oplus \delta_{\ell+1} H_\vee^{s,p}(\Omega; \delta_{\ell+1}) \oplus \mathcal{H}_\vee^{s,p}(\Omega, \Lambda^\ell), \tag{1.6}$$

où les sommes directes sont topologiques.

L'indice  $p_\Omega$  ci-dessus est lié à l'exposant critique du problème de Dirichlet et de Neumann pour  $\Omega$  au sens de [1,2,10,8]. Grâce des contre-exemples simples dans des domaines avec des singularités coniques d'isolement, on voit que le théorème ci-dessus est optimal. La preuve utilise les intégrales singulières et a un caractère constructif.

Let  $M$  be a (smooth) Riemannian manifold, and assume that  $\mathcal{E}, \mathcal{F} \rightarrow M$  are two Hermitian vector bundles. Recall that a relatively compact domain  $\Omega \subset M$  is called *Lipschitz* provided  $\partial\Omega$  can be described in appropriate local coordinates by means of graphs of Lipschitz functions. For such a domain, the unit outward conormal  $\nu \in T^*M$  to  $\partial\Omega$  is well defined a.e. with respect to the surface measure  $dS$ .

Sobolev spaces of fractional index  $L_s^p(\Omega)$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$ , can then be introduced starting from the definitions in [7] (where the Euclidean case is discussed), and using a natural lifting procedure (based on smooth partitions of unity and pull-back; cf. also [8]). In a similar fashion, boundary Besov spaces  $B_s^{p,q}(\partial\Omega)$  can be defined for  $1 \leq p, q \leq \infty$ ,  $0 < |s| < 1$ . In particular, the trace operator  $\text{Tr} : L_s^p(\Omega) \rightarrow B_{s-1/p}^{p,p}(\partial\Omega)$  is well-defined and bounded whenever  $1 < p < \infty$  and  $1/p < s$ . We also set  $L_s^p(\Omega, \mathcal{E}) := L_s^p(\Omega) \otimes \mathcal{E}$ ,  $B_s^{p,q}(\partial\Omega, \mathcal{F}) := B_s^{p,q}(\partial\Omega) \otimes \mathcal{F}$ , etc.

For a first-order differential operator  $D : \mathcal{E} \rightarrow \mathcal{F}$  with sufficiently smooth coefficients we set

$$H^{s,p}(\Omega; D) := \{u \in L_s^p(\Omega, \mathcal{E}); Du \in L_s^p(\Omega, \mathcal{F})\}, \tag{1.7}$$

where all derivatives are considered in a distributional sense, and equip it with the natural graph norm.

If in local coordinates  $D = \sum_j a_j(x) \partial_{x_j} +$  lower order terms, where  $a_j$  are matrix valued, then we define  $\sigma(D) : T^*M \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F})$ , the *principal symbol* of  $D$  as  $\sigma(D)(\xi) = \sum_j a_j(x) \xi_j$ , if  $x \in M$ ,  $\xi \in T_x^*M$ . Occasionally, we also write  $\sigma(D, \xi)$  in place of  $\sigma(D)(\xi)$ . In particular, if  $D^t$  denotes the formal adjoint of  $D$ , Green's formula

$$\int_{\partial\Omega} \langle \sigma(D, \nu)u, v \rangle dS = \iint_\Omega \langle Du, v \rangle - \iint_\Omega \langle u, D^t v \rangle \tag{1.8}$$

holds for sufficiently smooth sections  $u, v$ . In turn, for each  $1 < p < \infty$ ,  $-1 + 1/p < s < 1/p$ , Green's formula allows us to define the bounded operator

$$\sigma(D, \nu) : H^{s,p}(\Omega; D) \longrightarrow B_{s-1/p}^{p,p}(\partial\Omega, \mathcal{F}) \tag{1.9}$$

by insisting that

$$\langle \sigma(D, \nu)u, \text{Tr} \varphi \rangle = \langle Du, \varphi \rangle - \langle u, D^t \varphi \rangle \tag{1.10}$$

holds whenever  $u \in H^{s,p}(\Omega; D)$ ,  $\varphi \in L^{p'}_{1-s}(\Omega, \mathcal{F})$ , and  $1/p + 1/p' = 1$ . Here  $\langle \cdot, \cdot \rangle$  stands for various natural duality pairings. Finally, we set

$$H^{s,p}_{\sigma(D)}(\Omega; D) := \{u \in H^{s,p}(\Omega; D); \sigma(D, \nu)u = 0\}. \tag{1.11}$$

We shall now specialize the discussion to the case when  $\mathcal{E}, \mathcal{F}$  are exterior powers of the tangent bundle of  $M$ , denoted in the sequel by  $\Lambda^\ell$ , and when  $D$  is either the exterior derivative operator  $d_\ell : \Lambda^\ell \rightarrow \Lambda^{\ell+1}$ , or its formal adjoint  $\delta_{\ell+1} : \Lambda^{\ell+1} \rightarrow \Lambda^\ell$ , for  $0 \leq \ell \leq \dim M$ . As is well known,  $\sigma(d_\ell) = \wedge$ ,  $-\sigma(\delta_\ell) = \vee$ , the exterior and the interior product of forms, respectively.

The spaces of  $L^p_s$ -regular harmonic forms with vanishing normal or tangential traces in  $\Omega$  are, respectively,

$$\mathcal{H}^{s,p}_\wedge(\Omega, \Lambda^\ell) := \{u \in L^p_s(\Omega, \Lambda^\ell); du = 0, \delta u = 0 \text{ in } \Omega, \nu \wedge u = 0\}, \tag{1.12}$$

$$\mathcal{H}^{s,p}_\vee(\Omega, \Lambda^\ell) := \{u \in L^p_s(\Omega, \Lambda^\ell); du = 0, \delta u = 0 \text{ in } \Omega, \nu \vee u = 0\}. \tag{1.13}$$

Next, recall that the Laplace–Beltrami operator  $\Delta$  on  $M$  is given in local coordinates, where the metric tensor reads  $g = \sum g_{jk} dx_j \otimes dx_k$ , by

$$\Delta u := (\det(g_{jk}))^{-1/2} \sum_j \partial_j \left( \sum_k g^{jk} (\det(g_{jk}))^{1/2} \partial_k u \right), \tag{1.14}$$

where we take  $(g^{jk})$  to be the matrix inverse of  $(g_{jk})$ . Let  $\nabla_{\text{tan}}, \partial_\nu$  denote, respectively, the tangential gradient and the normal derivative on  $\partial\Omega$ . For each  $1 < q < \infty$ , consider the estimate

$$\begin{aligned} \|\nabla u\|_{L^q(\partial\Omega)} &\leq C \min\{\|\nabla_{\text{tan}} u\|_{L^q(\partial\Omega)}, \|\partial_\nu u\|_{L^q(\partial\Omega)}\}, \\ &\text{uniformly for } u \text{ satisfying } \Delta u = 0 \text{ in } \Omega. \end{aligned} \tag{1.15}$$

Following the work in the Euclidean context from [1,10,2], it has been shown in [8] that in any Lipschitz domain  $\Omega$ , (1.15) always holds for some  $q > 2$ . For each Lipschitz domain  $\Omega \subset M$  it is then natural to set

$$\begin{aligned} p_\Omega &:= \text{the Hölder conjugate exponent of the supremum of all } q\text{'s} \\ &\text{for which (1.15) holds both in } \Omega \text{ and in } M \setminus \overline{\Omega}. \end{aligned} \tag{1.16}$$

Thus, in general,  $1 \leq p_\Omega < 2$ ; cf. [1,2,10], for the flat, Euclidean setting and [8] for Lipschitz subdomains of Riemannian manifolds. Also,  $p_\Omega = 1$  if  $\partial\Omega \in C^1$  [4], whereas for a Lipschitz polygon or polyhedron  $p_\Omega$  can be estimated in terms of the angles involved; cf. [6].

We next define  $\mathcal{H}_\Omega$  to be the region of all points in  $\mathbb{R}^2$  whose coordinates  $(s, 1/p)$  satisfy the following conditions:

$$\begin{aligned} 0 < \frac{1}{p} < 1, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad \text{and either} \\ \frac{1}{2} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{p} - \frac{s}{2} < \frac{1}{2} \left( 1 + \frac{1}{p_\Omega} \right) &\text{ if } n = 2, \quad \text{or} \\ \frac{2}{3} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{p} - \frac{s}{3} < \frac{1}{3} \left( \frac{2}{p_\Omega} + 1 \right) &\text{ if } n = 3. \end{aligned} \tag{1.17}$$

The above set of inequalities describes the points in the interior of the hexagon in figure 1 below.

To state our main result, we denote by  $b_\ell(\Omega)$  the  $\ell$ -th Betti number of  $\Omega$ .

**THEOREM 1.2 (Hodge decompositions).** – *Let  $M$  be a compact, boundaryless, smooth, manifold of real dimension  $n$ . Assume that  $M$  is equipped with a Riemannian metric tensor whose coefficients are of class  $C^{1,1}$ . Also, fix  $\Omega$ , an arbitrary Lipschitz subdomain of  $M$ . Then, if  $n = 2$  or  $n = 3$ , for any  $\ell \in \{0, 1, \dots, n\}$ , and any  $s, p$  as in (1.17),*

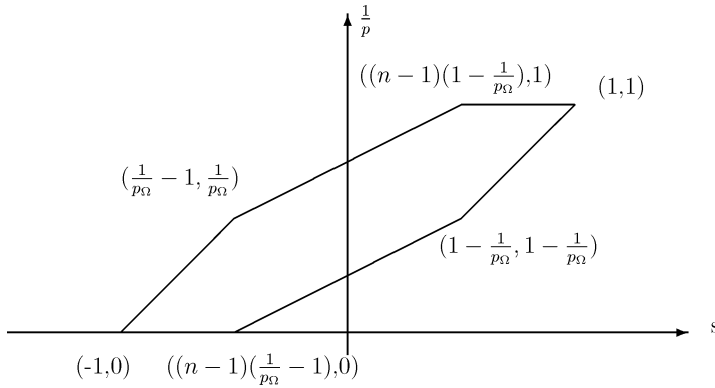


Figure 1. – The region  $\mathcal{H}_\Omega$  described in (1.17) for  $n = 2, 3$ .

$$L_S^p(\Omega, \Lambda^\ell) = d_{\ell-1}H_\Lambda^{s,p}(\Omega; d_{\ell-1}) \oplus \delta_{\ell+1}H^{s,p}(\Omega; \delta_{\ell+1}) \oplus \mathcal{H}_\Lambda^{s,p}(\Omega, \Lambda^\ell), \tag{1.18}$$

$$L_S^p(\Omega, \Lambda^\ell) = d_{\ell-1}H^{s,p}(\Omega; d_{\ell-1}) \oplus \delta_{\ell+1}H_V^{s,p}(\Omega; \delta_{\ell+1}) \oplus \mathcal{H}_V^{s,p}(\Omega, \Lambda^\ell), \tag{1.19}$$

where the direct sums (of closed linear subspaces of  $L_S^p(\Omega, \Lambda^\ell)$ ) are topological. Furthermore, the last summands in (1.18), (1.19) are finite dimensional. Their dimensions are  $b_\ell(\Omega)$  and  $b_{n-\ell}(\Omega)$ , respectively.

As a corollary, we have that Hodge decompositions for vector-fields with components in  $L^p(\Omega)$ ,  $1 < p < \infty$ , always hold for  $4/3 \leq p \leq 4$  in arbitrary two dimensional Lipschitz domains, and for  $3/2 \leq p \leq 3$  in arbitrary three dimensional Lipschitz domains. The case when  $s = 0$ ,  $p = 2$ , for which standard variational techniques work, is well known; see, e.g., [3,5].

By means of counterexamples it can be shown that the range of validity for (1.18)–(1.19) described in Theorem 1.2 is asymptotically sharp. Note that when  $\partial\Omega \in C^1$  (or even when  $v$  has vanishing mean oscillations),  $p_\Omega = 1$  so that the region  $\mathcal{H}_\Omega$  becomes the parallelogram described by  $1 < p < \infty$ ,  $-1 + 1/p < s < 1/p$ . In the case when  $\partial\Omega \in C^\infty$ , proving (1.18)–(1.19) can be done by reducing matters to solving certain regular elliptic boundary problems for the Hodge–Laplacian; this point of view has been adopted in [9].

We obtain the decompositions (1.18)–(1.19) in a constructive fashion, relying on singular integral operators; details as well as applications to PDE’s in nonsmooth domains will appear elsewhere.

What the corresponding situation for (1.18)–(1.19) is when  $n > 3$  remains an open problem at the moment.

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### References

[1] Dahlberg B., Arch. Rational Mech. Anal. 65 (1977) 275–288.  
 [2] Dahlberg B., Kenig C., Ann. Math. 125 (1987) 437–465.  
 [3] Dautray R., Lions J.-L., Mathematical Analysis and Numerical Methods for Science and Technology, Springer-Verlag, 1990.  
 [4] Fabes E., Jodeit M., Rivière N., Acta Math. 141 (1978) 165–186.  
 [5] Girault V., Raviart P.-A., Finite Element Methods for Navier–Stokes Equations, Springer-Verlag, 1986.  
 [6] Grachev N.V., Maz’ya V.G., Linköping Univ. Research Report LiTH-MAT-R-91-50.  
 [7] Jerison D., Kenig C.E., J. Functional Anal. 130 (1995) 161–219.  
 [8] Mitrea M., Taylor M., J. Functional Anal. 176 (2000) 1–79.  
 [9] Schwarz G., Hodge Decomposition – A Method for Solving Boundary Value Problems, LMN, Vol. 1607, Springer-Verlag, 1995.  
 [10] Verchota G., J. Functional Anal. 59 (1984) 572–611.