C. R. Acad. Sci. Paris, Ser. I 334 (2002) 649–654

Équations aux dérivées partielles/*Partial Differential Equations*

On completeness of root functions of elliptic boundary problems in a domain with conical points on the boundary

Youri V. Egorov [a], Vladimir A. Kondratiev [b], Bert-Wolfgang Schulze [c]

[a] **Laboratoire des mathématiques pour l'industrie et la physique, UMR 5640, Université Paul Sabatier, UFR MIG, 118, route de Narbonne, 31062, Toulouse cedex 4, France**

[b] **Mehmat Faculty, Lomonosov University, Vorob'evy Gory, 119899 Moscow, Russia**

[c] **Institute of Mathematics, Potsdam University, 601553 Potsdam, Germany**

Received 5 July 2001; accepted after revision 4 February 2002

Note presented by Pierre-Louis Lions.

Abstract We prove the completeness of the system of eigen and associated functions (i.e., root functions) of an elliptic boundary value problem in a domain, whose boundary is a smooth surface everywhere, except at a finite number of points, such that each point possesses a neighborhood, where the boundary is a conical surface. ***To cite this article: Y.V. Egorov et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 649–654.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur la complétude des fonctions propres et associées d'un problème au bord elliptique dans un domaine avec points coniques sur le bord

Résumé On montre que les fonctions propres et associées d'un problème au bord pour un opérateur elliptique d'ordre $2m$, défini dans un domaine dans $\mathbb{R}^n$ avec points coniques sur le bord, forment un système total. ***Pour citer cet article : Y.V. Egorov et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 649–654.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On considére un domaine borné Ω dans $\mathbb{R}^n$ de frontiére $\partial\Omega$, régulière en dehors d'un ensemble fini de points de structure conique. Soit L un opérateur, différentiel de type elliptique d'ordre $2m$ défini dans Ω. On suppose aussi qu'un ensemble d'opérateurs différentiels $B_j,\ j=1,\dots,m$, est défini sur la partie régulière de $\partial\Omega$, et que la condition de Lopatinski est vérifiée pour l'opérateur $\mathcal{L}=(L,B_1,\dots,B_m)$.

Notre résultat fondamental est le théorème suivant.

THÉORÈME. – *Supposons qu'il existe un ensemble de rayons* $\arg\lambda=\theta_i,\ i=1,\dots,N$*, du plan complèxe, de croissance minimal pour la résolvante du problème au bord correspondant à l'opérateur* $\mathcal{L}$*, et tels que*

E-mail addresses: egorov@mip.ups-tlse.fr (Y.V. Egorov); kondrat@vnmok.math.msu.su (V.A. Kondratiev); schulze@math.uni-potsdam.de (B.-W. Schulze).

S1631-073X(02)02320-8/FLA

les angles parmi de rayons voisins sont inférieurs à $\pi(2m-\gamma/2)/n$*, où* $0<\gamma<2m$*. Alors le spectre de l'opérateur* $\mathfrak{L}$ *est discret et le système des fonctions propres et associées du problème au bord forment un système complet dans* $\mathrm{L}_2(\Omega)$.

Les méthodes utilisent des téchniques élaborées dans [6,10–13,16,17] et [1].

1. Introduction

We prove here the completeness of the system of eigen and associated functions (i.e., root functions) of an elliptic boundary value problem in a domain whose boundary is a smooth surface everywhere except of a finite number of points such that each point possesses a neighborhood where the boundary is a conical surface.

The problem of completeness of the system of eigen and associated functions of the boundary value problems for elliptic operators in domains with smooth boundaries was studied in the articles by F.B. Browder [6–8], M.V. Keldysh [12], M.S. Agranovich [4], N.M. Krukovsky [14], S. Agmon [1], M. Schechter [15], M.S. Agranovich, R. Denk, and M. Faierman [5].

All the mentioned authors have used the M.V. Keldysh methods of [12]. We shall use them here also as well as the methods of T. Carleman as in [9], S. Agmon [1], and of [10,13,11,16,17].

2. Hypotheses

Let Ω be a bounded domain in $\mathbb{R}^n$, and $\partial\Omega$ be its boundary, $\overline{\Omega}$ be the closure of Ω. We shall use the standard notation:

$$x=(x_1,\dots,x_n),\quad D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},\quad \text{where } |\alpha|=\alpha_1+\cdots+\alpha_n,\ \alpha=(\alpha_1,\dots,\alpha_n).$$

Assume that $\partial\Omega$ is a surface of the class C^{2m} everywhere except of the point $x=0$ (we'll denote it O) and that it coincides in a neighborhood of the point O with a conical domain $K=\{x:\frac{x}{|x|}\in K'\}$, where K' is a domain on the unit sphere having a boundary of the class C^{2m}.

Consider a differential operator in Ω:

$$L(x,D)=\sum_{|\alpha|\leqslant 2m}a_\alpha(x)D^\alpha,$$

where $a_\alpha(x)$ are bounded measurable functions in $\overline{\Omega}$, and for $|\alpha|=2m$ they are continuous in $\overline{\Omega}\setminus O$.

The coefficients $a_\alpha(x)$ for $|\alpha|=2m$ have the form $a_\alpha(x)=a_{\alpha 0}(\frac{x}{|x|})+a_{\alpha 1}(x)$, in a neighborhood of the point O, where $\lim_{x\to 0}a_{\alpha 1}(x)=0$.

Set

$$B_j(x,D)=\sum_{|\alpha|\leqslant m_j}b_{\alpha j}(x)D^\alpha,\quad j=1,\dots,m,\ m_j<2m,$$

where $b_{\alpha j}(x)$ are functions of the class C^{2m-m_j} in $\overline{\partial\Omega}\setminus O$, and for $|\alpha|=m_j$ $b_{\alpha j}(x)=b_{\alpha j0}(\frac{x}{|x|})+b_{\alpha j1}(x)$, where $\lim_{x\to 0}b_{\alpha j1}(x)=0$.

Put $L'=\sum_{|\alpha|=2m}a_\alpha(x)D^\alpha$. We will suppose everywhere below that the operator L is *elliptic*, i.e., $L'(x,\xi)\neq 0,\ \xi\in\mathbb{R}^n\setminus 0,\ x\in\overline{\Omega}\setminus O$. If $n=2$, we assume also that the condition of regular ellipticity is fulfilled. Suppose that the operators:

$$L_0=\sum_{|\alpha|=2m}a_{\alpha 0}(\omega)D^\alpha,\qquad B_{j0}=\sum_{|\alpha|=m_j}b_{\alpha j0}(\omega)D^\alpha,$$

where (ρ, ω) is the spherical coordinate system with its center in O, satisfy the Lopatinsky condition on $\partial K' \setminus O$.

We shall consider the complex-valued functions defined in Ω. Together with the usual Sobolev spaces $\mathrm{H}^k(\Omega)$ we use special spaces $\mathrm{W}^k_\gamma(\Omega)$ consisting of the functions u such that:

$$\|u\|^2_{\mathrm{W}^k_\gamma(\Omega)} \equiv \sum_{|\alpha|\leqslant k} \int_\Omega |x|^{\gamma-2|\alpha|+2k} \left|D^\alpha u\right|^2 \mathrm{d}x < \infty.$$

Let us consider the boundary problem:

$$L(x, D)u = f(x) \quad \text{in } \Omega; \qquad B_j(x, D)u = 0 \quad \text{on } \partial\Omega,\ j = 1, \ldots, m. \tag{1}$$

The following operator pencil is very important for the study of the boundary problems in domains with a conical point on the boundary:

$$r^{-\lambda} L_0\big(r^\lambda \Phi(\omega)\big), \quad \omega \in K'; \qquad r^{-\lambda} B_{j0}\big(r^\lambda \Phi(\omega)\big), \quad \omega \in \partial K',$$

where

$$L_0 = \sum_{|\alpha|=m} a_{\alpha 0}(\omega) D^\alpha, \qquad B_{j0} = \sum_{|\alpha|=m_j} b_{\alpha j 0}(\omega) D^\alpha.$$

It is well known that the spectrum of the problem,

$$r^{-\lambda} L_0\big(r^\lambda \Phi(\omega)\big) = 0 \quad \text{in } K', \qquad r^{-\lambda} B_{j0}\big(r^\lambda \Phi(\omega)\big) = 0 \quad \text{on } \partial K' \tag{2}$$

is discrete.

The following theorem is proved in [13] (*see* also [11]):

THEOREM 1. – *If there are no points of the spectrum of the problem* (2) *on the line* $\operatorname{Re}\lambda = \frac{-\gamma+4m-n}{2}$ *with* $0 \leqslant \gamma < 2m$, *then*

$$\|u\|_{\mathrm{W}^{2m}_\gamma(\Omega)} \leqslant C\big(\|Lu\|_{\mathrm{W}^0_\gamma(\Omega)} + \|u\|_{\mathrm{L}_2(\Omega)}\big)$$

for all functions $u(x) \in \mathrm{W}^{2m}_\gamma(\Omega)$ *such that* $B_j u = 0,\ j = 1, \ldots, m,\ x \in \partial\Omega \setminus O$.

From now on we assume the hypothesis of Theorem 1.

3. Rays of minimal growth

Let us denote $\mathcal{L}$ the linear unbounded operator $\mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$, defined in $D_{\mathcal{L}} = \{u : u \in \mathrm{W}^{2m}_\gamma,\ B_j u = 0, j = 1, \ldots, m,\ x \in \partial\Omega \setminus O\}$, which transfers each element $u \in D_{\mathcal{L}}$ in Lu.

The hypotheses of Theorem 1 imply that $\mathcal{L}$ is a closed linear operator $\mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ and the dimensions of his kernel and cokernel are finite. If the spectrum of $\mathcal{L}$ is not the whole complex plane then it is discrete.

DEFINITION 1. – A ray $\arg\lambda = \theta$ of the complex plane λ is a ray of *minimal growth* for the resolvent $R(\lambda) = (\mathcal{L} - \lambda E)^{-1} : \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)$ of the operator $\mathcal{L}$, if the resolvent exists, for all λ on this ray with sufficiently big module and for all such λ we have $\|R(\lambda)\|_{\mathrm{L}_2\to\mathrm{L}_2} \leqslant C|\lambda|^{-\delta},\ \delta > 0$, where $C = \text{const} > 0$, $\delta = \text{const} > 0$.

THEOREM 2. – *Suppose that the following conditions are fulfilled*:

(1) $(-1)^m L_0(x, \xi)/|L_0(x, \xi)| \neq \mathrm{e}^{\mathrm{i}\theta},\ \xi \neq 0,\ \xi \in \mathbb{R}^n,\ x \in \Omega$.

(2) *At each point* $x \in \partial\Omega \setminus O$ *denote* ν *the normal vector to* $\partial\Omega$ *and let* $\xi \neq 0$ *be a real vector, orthogonal to* ν. *Let* $t_1^+(\xi, \lambda), \ldots, t_m^+(\xi, \lambda)$ *be the roots with positive imaginary parts of the polynomial in* t,

$$(-1)^m L_0(x, \xi + t\nu) - \lambda,$$

where λ *is a complex number such that* $\arg\lambda=\theta$. *Then the polynomials* $B_{j\nu}(x,\xi+t\nu)$, $j=1,\dots,m$, *are linearly independent modulo the polynomial* $\prod_{k=1}^{m}(t-t_k^+(\xi,\lambda))$.

(3) *The boundary value problem in the infinite cone* K,

$$L_0(\omega,x)u-\lambda u\equiv\sum_{|\alpha|=2m}a_\alpha(\omega)D^\alpha u-\mathrm{e}^{\mathrm{i}\theta}u=f\in \mathrm{W}_\gamma^0(K),$$

$$B_{j0}(\omega,D_x)u\equiv\sum_{|\alpha|=m_j}a_{\alpha j0}(\omega)D^\alpha u=0\quad\text{on } \partial K,$$

has a unique solution from the class $\mathrm{W}_\gamma^{2m}(K)\cap\mathrm{W}_\gamma^0(K)$ *for some* γ, $0\leqslant\gamma<2m$, *and*

$$\|u\|_{\mathrm{W}_\gamma^{2m}(K)}+\|u\|_{\mathrm{W}_\gamma^0(K)}\leqslant C\|f\|_{\mathrm{W}_\gamma^0(K)}.$$

Then the spectrum of the operator $\mathcal{L}$ *is discrete and the ray* $\arg\lambda=\theta$ *is a ray of minimal growth for* $R(\lambda,\mathcal{L})$.

Using Theorem A1.3 from [1] we prove the following:

THEOREM 3. – *Let* T *be a compact operator in* $\mathrm{L}_2(\Omega)$ *such that* $T\mathrm{L}_2(\Omega)\subset\mathrm{W}_\gamma^{2m}(\Omega)$, $2m>\gamma>0$. *Let* λ_j *be a sequence of nonzero eigenvalues of* T, *with regard to their multiplicity and* $R(\lambda,T)$ *be the resolvent of* T. *Then*

(1) $\sum_j|\lambda_j|^{n/(2m-\gamma/2)+\varepsilon}<\infty$ *for any* $\varepsilon>0$.

(2) *For any* $\varepsilon>0$ *there exists a sequence* $\rho_i\to 0$, $i=1,2\dots$, *such that* $R(\lambda,T)$ *is defined for* $|\lambda|=\rho_i$ *and*

$$\left\|R(\lambda,T)\right\|\leqslant\exp\left(|\lambda|^{-n/(2m-\gamma/2)^+\varepsilon}\right)\quad\text{for } |\lambda|=\rho_i,\ i=1,2\dots.$$

4. Principal result

Now we can state our principal result: the theorem on the completeness of the system of the root functions of an elliptic boundary problem in a domain with a conical point on its boundary. For domains with smooth boundaries this result was obtained in [1].

THEOREM 4. – *Suppose that there exist the rays* $\arg\lambda=\theta_i$, $i=1,\dots,N$, *in the complex plane which satisfy the conditions of Theorem* 2 *and the angles between the pairs of neighbor rays are less than* $\pi(2m-\gamma/2)/n$, *where* $0<\gamma<2m$. *Then the spectrum of the operator* $\mathcal{L}$ *is discrete and the root functions form a complete system in* $\mathrm{L}_2(\Omega)$.

Proof. – Theorem 2 implies that the spectrum of the operator $\mathcal{L}$ is discrete and each the ray $\arg\lambda=\theta_i$ is a ray of minimal growth for the resolvent $R(\lambda,\mathcal{L}):\mathrm{L}_2(\Omega)\to\mathrm{L}_2(\Omega)$. It means, in particular, that

$$\left\|R(\lambda,\mathcal{L})\right\|_{\mathrm{L}_2(\Omega)\to\mathrm{L}_2(\Omega)}=\mathrm{O}\left(|\lambda|^{-\delta}\right)\tag{3}$$

as $|\lambda|\to\infty$ and $\delta>0$.

Suppose that there exists a function $f^*\in\mathrm{L}_2(\Omega)$, orthogonal to all eigen and associated functions of the operator $\mathcal{L}$. We'll show that $f^*=0$. Then it will mean that the system of root functions is complete.

Suppose that the point $\lambda=0$ is regular for the operator $\mathcal{L}$. Consider the function

$$F(\lambda)=\left(f^*,R\left(\frac{1}{\lambda},T\right)f\right),\tag{4}$$

where $T=\mathcal{L}^{-1}$, $f\in\mathrm{L}_2(\Omega)$, $(\cdot,\cdot)$ is the scalar product in $\mathrm{L}_2(\Omega)$.

Since the resolvent of $\mathcal{L}$ is a meromorphic function with poles at the points of the spectrum of $\mathcal{L}$, the function F is analytic for λ which are not eigenvalues of $\mathcal{L}$. We shall use the following relation between the resolvents of the operators $\mathcal{L}$ and $\mathcal{L}^{-1}$:

$$R\left(\frac{1}{\lambda}, \mathcal{L}^{-1}\right) = \lambda I - \lambda^2 R(\lambda, \mathcal{L}). \tag{5}$$

Let λ_k be a pole of $R(\lambda, T)$. Consider the expansion

$$R(\lambda, T) f = \frac{\Phi_1}{(\lambda - \lambda_k)^j} + \frac{\Phi_2}{(\lambda - \lambda_k)^{j-1}} + \cdots + \frac{\Phi_j}{\lambda - \lambda_k} + \sum_{i=0}^{\infty} g_i (\lambda - \lambda_k)^i$$

in a neighborhood of λ_k. Here $j \geqslant 1$, $\Phi_1 \neq 0$, $\Phi_i \in \mathrm{L}_2(\Omega)$, $g_i \in \mathrm{L}_2(\Omega)$, $\Phi_1, \Phi_2, \ldots, \Phi_j$ is a chain of the associated functions. This expansion implies that λ_k is a regular point of $F(\lambda)$, since f^* is orthogonal to all Φ_i. Therefore, $F(\lambda)$ is an entire function.

The relations (3)–(5) imply that

$$\left|F(\lambda)\right| \leqslant C \exp\left(|\lambda|^{2-\delta}\right), \tag{6}$$

for $|\lambda| \to \infty$, $\arg \lambda = \theta_i$, $i = 1, \ldots, N$. Besides, Theorem 3 implies that for any $\varepsilon > 0$ there exists a sequence $r_j \to \infty$ such that

$$\left|F(\lambda)\right| \leqslant \exp\left(|\lambda|^{n/(2m-\gamma/2)-\varepsilon}\right), \tag{7}$$

for $|\lambda| = r_j$.

Consider $F(\lambda)$ in the closure of an angle between the rays $\arg \lambda = \theta_j$ and $\arg \lambda = \theta_{j+1}$. Its size is less than $\pi(2m - \gamma/2)/n$. Since $R(\lambda, T) = \lambda I - \lambda^2 R(\lambda, \mathcal{L})$ and the ray $\arg \lambda = \theta_i$ is a ray of minimal growth, we have inequality (6) on the sides of the angle and (7) on the sequence of the arcs going to infinity.

Choosing $\varepsilon > 0$ in (7) sufficiently small and applying the Fragmen–Lindelöff theorem we obtain that $|F(\lambda)| = \mathrm{O}(|\lambda|^{2-\delta})$ as $|\lambda| \to \infty$ in the whole complex plane. Therefore, $F(\lambda)$ is a linear function, i.e., $F(\lambda) = c_0 + c_1 \lambda$. On the other hand, we have $R(1/\lambda, T) = \lambda I + \lambda^2 T + \cdots$, and therefore,

$$F(\lambda) = \lambda(f^*, f) + \lambda^2 (f^*, Tf) + \cdots.$$

Since F is linear, we have $(f^*, Tf) = 0$ for all $f \in \mathrm{L}_2(\Omega)$. Since the range of the operator $\mathcal{L}$ is dense in $\mathrm{L}_2(\Omega)$, we have $f^* = 0$. Thus, the system of the root functions of the operator $\mathcal{L}$ is complete in $\mathrm{L}_2(\Omega)$. □

5. Some generalizations

The Theorem 4 implies the completeness in $\mathrm{W}^0_\gamma(Q)$ with $\gamma > 0$. Now we shall state some corollaries of Theorem 4.

COROLLARY 1. – *Let the conditions of Theorem* 4 *be fulfilled. Then the system of the root elements is, for* $\gamma \geqslant 0$, *dense in the space*

$$\widetilde{\mathrm{W}}^{2m}_\gamma = \left\{u \in \mathrm{W}^{2m}_\gamma(\Omega),\ Bju = 0 \text{ on } \partial\Omega,\ j = 1, \ldots, m\right\}.$$

Remark. – Since

$$\mathrm{W}^0_{\gamma_1 - 2m}(\Omega) \subset \mathrm{W}^{2m}_{\gamma_1}(\Omega)$$

and the space $\mathrm{C}^\infty_0(\Omega)$ is dense in any $\mathrm{W}^0_{\gamma_1 - 2m}(\Omega)$ with $\gamma_1 \geqslant 0$, we can conclude that the system of the root functions is dense in any space $\mathrm{W}^0_\gamma(\Omega)$ with $\gamma \geqslant -2m$.

The obtained results can be expanded to the spaces $\mathrm{L}_p(\Omega)$, $p \geqslant 1$.

Example 1. – Consider an elliptic operator of second order:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} a_i(x)u_{x_i} + a_0(x)u,$$

where $a_{ij}(x),\ a_i(x),\ a_0(x)$ are continuous real functions, which is defined on the set of C^2-functions satisfying the homogeneous Dirichlet conditions in a domain with a finite number of conical points on its boundary. In this case Theorem 4 is true but we cannot apply the methods using the quadratic form $(Lu, u)_{\mathrm{L}_2}$ as in [4,14], since the coefficients a_{ij} can be not differentiable.

Example 2. – Another example is given by the oblique derivative problem for the operator L from Example 1 (*see* [1] in the case of the smooth boundary).

References

[1] S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, Comm. Pure Appl. Math. 15 (1962) 119–147.
[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959) 623–727.
[3] M.S. Agranovich, Elliptic boundary problems, in: Partial Differential Equations IX, Encyclopedia of Mathematical Sciences, Vol. 79, Springer, 1991, pp. 1–144.
[4] M.S. Agranovich, On series with respect to root vectors of operators associated with forms having symmetric principal part, Funct. Anal. Appl. 28 (3) (1994) 151–167.
[5] M.S. Agranovich, R. Denk, M. Faerman, Weakly smooth nonselfadjoint spectral problems for elliptic boundary value problems, in: P. Demuth, E. Schrohe, B.-W. Schulze (Eds.), Spectral Theory, Microlocal Analysis, Singular Manifolds, Birkhäuser, 2000, pp. 138–199.
[6] F.E. Browder, On the eigenfunctions and eigenvalues of the general elliptic differential operator, Proc. Nat. Acad. Sci. USA 39 (1953) 433–439.
[7] F.E. Browder, Estimates and existence theorems for elliptic boundary value problems, Proc. Nat. Acad. Sci. USA 45 (1959) 365–372.
[8] F.E. Browder, On the spectral theory of strongly elliptic differential operators, Proc. Nat. Acad. Sci. USA 45 (1959) 1423–1431.
[9] T. Carleman, Über die Verteilung der Eigenwerte partieller Differentialgleichungen, Ber. der Sächs. Akad. Wiss. Leipzig, Mat. Nat. Kl. 88 (1936) 119–132.
[10] N. Dunford, J.T. Schwartz, Linear Operators, Vol. II, Interscience, New York, 1963.
[11] Yu.V. Egorov, B.-W. Schulze, Pseudo-Differential Operators, Singularities, Applications, Oper. Theory Adv. Appl., Vol. 93, Birkhäuser, 1997.
[12] M.V. Keldysh, On the eigenvalues and eigenfunctions of certain classes of non-selfadjoint equations, Dokl. AN SSSR 77 (1951) 11–14.
[13] V.A. Kondratiev, Boundary value problems for elliptic equations in domains with conical or singular points, Trudy Moskov. Mat. Obshch. 16 (1967) 209–292.
[14] N.M. Krukovsky, Theorems on the m-fold completeness of the generalized eigen- and associated functions from W_2^1 of certain boundary value problems for elliptic equations and systems, Differentsial'nye Uravneniya 12 (10) (1976) 1842–1851.
[15] M. Schechter, Remarks on elliptic boundary value problems, Comm. Pure Appl. Math. 12 (1959) 457–482.
[16] B.-W. Schulze, Pseudo-Differential Operators on Manifolds with Singularities, North-Holland, Amsterdam, 1991.
[17] B.-W. Schulze, Boundary Value Problems and Singular Pseudo-Differential Operators, Wiley, Chichester, 1998.