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Probabilités/*Probability Theory*

# Intersecting random half spaces with a cube

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**Abstract** We compute the typical number of points of the discrete cube $\{-1,1\}^N$ that belong to the intersection of $M$ random half-spaces, when $M$ is a small proportion of $N$. ***To cite this article: M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 807–809.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

### Volume de l'intersection d'un cube et de sous-espaces aleatoires

**Résumé** Nous calculons la cardinalité typique de l'intersection du cube discret $\{-1,1\}^N$ et de $M$ demi-espaces aleatoires, quand $M$ est une petite proportion de $N$. ***Pour citer cet article : M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 807–809.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Consider independent standard normal r.v. $(g_{i,k})_{i\leqslant N,k\leqslant M}$. Consider a number $\tau > 0$. The following subset of $\mathbb{R}^N$

$$U_k = \left\{ x = (x_i)_{i\leqslant N} \in \mathbb{R}^N;\ \frac{1}{\sqrt{N}} \sum_{i\leqslant N} g_{i,k} x_i \geqslant \tau \right\} \tag{1}$$

is a random half-space of $\mathbb{R}^N$. Its boundary is nearly at distance $\tau$ from the origin. Consider the discrete cube $\Sigma_N = \{-1,1\}^N$. We are interested in computing the "typical" cardinality of the set $\Sigma_N \cap \bigcap_{k\leqslant M} U_k$. (This typical cardinality is less than the average cardinality, that, of course, is easier to compute.)

To state the result, we consider the functions

$$H(x) = P(g \geqslant x), \qquad A(x) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{-x^2/2}}{H(x)},$$

where $g$ is standard Gaussian. We set $\rho = \max(\tau, 0)$.

LEMMA 1. – *There exists a constant $L$ with the following property. If $L\alpha\,\mathrm{e}^{L\rho^2} \leqslant 1$, the equations*

$$q = E\,\mathrm{th}^2(g\sqrt{r}), \quad r = \frac{\alpha}{1-q} E\left(A^2\left(\frac{\tau - g\sqrt{q}}{\sqrt{1-q}}\right)\right)$$

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*have a unique solution.*

Taking for $q, r$ the values determined above, we can define the function

$$F(\alpha,\tau) = \log 2 - \frac{r}{2}(1-q) + E\,\mathrm{ch}(g\sqrt{r}) + \alpha E \log H\left(\frac{\tau - g\sqrt{q}}{\sqrt{1-q}}\right).$$

THEOREM 1. – *There exists a constant $L$ with the following property. If $LM\,\mathrm{e}^{L\rho^2} \leqslant N$, then for each $t$ with $0 \leqslant Lt(1+\rho^2) \leqslant 1$, we have*

$$P\left(\left|\frac{1}{N}\log\left(\operatorname{card}\Sigma_N \cap \bigcap_{k\leqslant M} U_k\right) - F\left(\frac{M}{N},\tau\right)\right| \geqslant t\right) \leqslant L\exp\left(-\frac{Nt^2}{L}\right) \tag{2}$$

(*where we define* $\log 0 = -\infty$).

Another case of interest is when the r.v. $g_{i,k}$ are replaced by independent random signs $\eta_{i,k}$ ($P(\eta_{i,k} = \pm 1) = 1/2$), that is, when we replace (1) by

$$V_k = \left\{x \in \mathbb{R}^N;\ \frac{1}{\sqrt{N}}\sum_{i\leqslant N}\eta_{i,k}x_i \geqslant \tau\right\}. \tag{3}$$

In that case, we have the following weaker result.

THEOREM 2. – *There exists a constant $L$ with the following property. If $LM\,\mathrm{e}^{L\rho^2} \leqslant N$, then given $t$ with $0 < Lt(1+\rho^2) \leqslant 1$, there exists an integer $N(t,\rho)$, such that* (2) *holds for $N \geqslant N(t,\tau)$, where $U_k$ is replaced by $V_k$.*

To reformulate this result, given $t > 0$, for $N \geqslant N(t,\tau)$, we have

$$\exp\left(N\left(F\left(\frac{M}{N},\tau\right) - t\right)\right) \leqslant \operatorname{card}\left(\Sigma_N \cap \bigcap_{k\leqslant M} V_k\right) \leqslant \exp\left(N\left(F\left(\frac{M}{N},\tau\right) + t\right)\right)$$

unless we are very unlucky (the probability of failure is exponentially small).

The proof of these results is a complicated affair. Given a function $u$, one introduces the random Hamiltonian $H_{N,M}$ given for $\sigma = (\sigma_i)_{i\leqslant N}$ in $\Sigma_N$ by

$$-H_{N,M}(\sigma) = \sum_{k\leqslant M} u\left(\frac{1}{\sqrt{N}}\sum_{i\leqslant N} g_{i,k}\sigma_i\right) \tag{4}$$

and the corresponding Gibbs' measure $\langle\cdot\rangle$ on $\Sigma_N$. The situation of Theorem 1 "corresponds to the case where $u(x) = 0$ if $x \geqslant \tau$ and $u(x) = -\infty$ if $x < \tau$".

Physicists (and now mathematicians) have studied these systems with random Hamiltonian under the name of "spin glasses". The formula for $F(\alpha,\tau)$ was discovered (but by no means proved) by Gardner [2]. The present work is directly motivated by the recent result of Schcherbina and Tirozzi [3] who prove a result similar to Theorem 2 when the cube $\Sigma_N$ is replaced by the sphere $S_N = \{\sigma;\ \sum_{i\leqslant N}\sigma_i^2 = N\}$. A most remarkable feature of the result of [3] is that it is proved not only for small $\alpha$, but when $\tau > 0$, up to the largest possible value of $\alpha_c(\tau)$ of $\alpha$ (value that is also determined in their work).

Despite many similarities, there are deep differences between our result and the result of [3].

It can be argued that the crucial (and most difficult) step of the study of a system such as governed by an Hamiltonian like (4) is to prove that "the correlations of the spin asymptotically vanish", i.e., that in

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average over the randomness $|\langle\sigma_1\sigma_2\rangle - \langle\sigma_1\rangle\langle\sigma_2\rangle|$ is small for large $N$. The beauty of the approach of [3] is to start with an Hamiltonian as in (4), where the function $u$ is concave. One can then use tools such as the Brascamp–Lieb inequalities (whose relevance to this purpose we learned in [1]) to obtain effortlessly the vanishing of the spin correlations. No such route seems possible here, and this vanishing of correlations has to be obtained through delicate estimates, such as those of the forthcoming paper [6].

On the other hand, having to consider rather large values of $\alpha$ creates serious difficulties (that are solved in [3] through hard work) and these difficulties are absent here.

It is believed that there is a critical value $\alpha_c$ such that for $N$ large, the set $\Sigma_N \cap \bigcap_{k\leqslant M} U_k$ is typically non-empty (resp. typically empty) if $M/N \leqslant \alpha < \alpha_c$ (resp. $M/N \geqslant \alpha \geqslant \alpha_c$). On the other hand it is believed that the formula of Theorem 1 *fails* if $M/N$ is close to $\alpha_c$. Understanding what happens in that case is a serious long-term challenge.

Complete proofs of the present results (as well as alternative proofs of the results of [3]) will be contained in the forthcoming book [7].

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