

The Föppl–von Kármán plate theory as a low energy Γ -limit of nonlinear elasticity

Gero Friesecke ^a, Richard D. James ^b, Stefan Müller ^c

^a Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

^b Department of Aerospace Engineering and Mechanics, 107 Akerman Hall, University of Minnesota, Minneapolis, MN 55455, USA

^c Max-Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany

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Abstract

We show that the Föppl–von Kármán theory arises as a low energy Γ -limit of three-dimensional nonlinear elasticity. A key ingredient in the proof is a generalization to higher derivatives of our rigidity result [5] that for maps $v : (0, 1)^3 \rightarrow \mathbb{R}^3$, the L^2 distance of ∇v from a single rotation is bounded by a multiple of the L^2 distance from the set $\text{SO}(3)$ of all rotations. *To cite this article: G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 201–206.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

La théorie Föppl–von Kármán des plaques comme Γ -limite de l'élasticité non linéaire

Résumé

Nous montrons que la théorie Föppl–von Kármán des plaques émerge comme Γ -limite de la théorie de l'élasticité tridimensionnelle. La démonstration repose sur une généralisation aux dérivées d'ordre supérieur de notre résultat de rigidité [5] que pour des fonctions $v : (0, 1)^3 \rightarrow \mathbb{R}^3$, la distance L^2 de ∇v à une rotation est bornée par un multiple de la distance L^2 à l'ensemble $\text{SO}(3)$ des rotations. *Pour citer cet article: G. Friesecke et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 201–206.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans cette Note nous continuons notre récent travail [5,6] dédié à clarifier rigoureusement la relation entre l'élasticité non linéaire tridimensionnelle et les théories bidimensionnelles pour des domaines minces [9,1,3]. Suivant [5,6] le point de départ de notre approche est l'énergie élastique

$$E^h(w) = \int_{\Omega_h} W(\nabla w(z)) \, dz$$

d'une déformation $w : \Omega_h = S \times (-\frac{h}{2}, \frac{h}{2}) \rightarrow \mathbb{R}^3$. Pour des déformations avec $E^h \sim h$ (élongation finie du plan médian) respectivement $E^h \sim h^3$ (flexion finie sans élongation du plan médian) le passage à une théorie bidimensionnelle à été justifié rigoureusement, au sens de Γ -convergence : $\frac{1}{h}E^h$ tend vers une fonctionnelle de membrane [7,8] et $\frac{1}{h^3}E^h$ tend vers la fonctionnelle des plaques non linéaire de Kirchhoff [5,6] (voir aussi [2,12]).

E-mail addresses: gf@maths.warwick.ac.uk (G. Friesecke); james@umn.edu (R.D. James); sm@mis.mpg.de (S. Müller).

Ici nous analysons le cas des déformations avec $E^h \sim h^5$, et dérivons la théorie de Föppl–von Kármán. Le point essentiel de la preuve est une généralisation aux dérivées supérieures d’un résultat de rigidité pour des déformations w avec ∇w proches de $\text{SO}(3)$ donné dans [5, Theorem 2] (voir Theorem 3 ci-dessous). Ce résultat permet de linéariser le problème autour d’un mouvement rigide (voir Lemmas 4 et 5 ci-dessous).

Pour énoncer notre résultat, on se ramène à une région fixe par le changement de coordonnées $x = (z_1, z_2, \frac{z_3}{h})$, puisque $y : \Omega = S \times (-\frac{1}{2}, \frac{1}{2}) \rightarrow \mathbb{R}^3$. Introduisant la notation $x' = (x_1, x_2)$ et $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ pour les coordonnées et le gradient dans le plan, on a alors $\nabla w = (\nabla' y, \frac{1}{h} y_{,3}) =: \nabla_h y$ et $\frac{1}{h} E(w) = I^h(y) := \int_{\Omega} W(\nabla_h y) dx$. On suppose que la fonction W vérifie

$$W = 0 \quad \text{sur } \text{SO}(3), \quad W(F) \geq c \text{dist}^2(F, \text{SO}(3)), \quad c > 0, \tag{1}$$

$$W(QF) = W(F) \quad \forall Q \in \text{SO}(3), \quad W \text{ est } C^2 \text{ dans un voisinage de } \text{SO}(3). \tag{2}$$

Nous introduisons $Q_2(G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes e_3 + e_3 \otimes a)$, où $Q_3(F) = (\partial^2 W / \partial F^2)(\text{Id})(F, F)$ est le double de l’énergie d’élasticité linéaire. Dans le cas de l’élasticité isotropique on a $Q_3(F) = 2\mu |\text{sym } F|^2 + \lambda(\text{tr } F)^2$, $Q_2(G) = 2\mu |\text{sym } G|^2 + \frac{2\mu\lambda}{2\mu+\lambda}(\text{tr } G)^2$, où $\text{sym } F = (F + F^T)/2$. Pour $u \in W^{1,2}(S; \mathbb{R}^2)$ and $v \in W^{2,2}(S)$ nous considérons la fonctionnelle de Föppl–von Kármán définie par

$$I^0(u, v) = \int_S \frac{1}{2} Q_2 \left(\frac{1}{2} [\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v] \right) + \frac{1}{24} Q_2((\nabla')^2 v) dx'. \tag{3}$$

THÉORÈME 1. – Soit $S \subset \mathbb{R}^2$ un ouvert borné connexe lipschitzien. On suppose que la fonction W vérifie (1), (2). Alors, lorsque $h \rightarrow 0$, les fonctionnelles $h^{-4} I^h$ Γ -convergent vers I^0 . Plus précisément on a

- (i) Si $\limsup_{h \rightarrow 0} h^{-4} I^h(y^{(h)}) < \infty$ alors il existe des constantes $c^{(h)} \in \mathbb{R}$, $\bar{R}^{(h)} \in \text{SO}(3)$ telles que $\bar{R}^{(h)} \rightarrow \bar{R}$ and

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)} \rightarrow \bar{y} \quad \text{dans } W^{1,2}(\Omega; \mathbb{R}^3), \quad \bar{y}(x) = \begin{pmatrix} x' \\ 0 \end{pmatrix}, \tag{4}$$

$$\nabla_h \tilde{y}^{(h)} \rightarrow \text{Id} \quad \text{dans } L^2(\Omega; \mathbb{R}^{3 \times 3}), \tag{5}$$

$$\frac{1}{h} \int_{-1/2}^{1/2} \tilde{y}_3^{(h)}(\cdot, x_3) dx_3 \rightarrow v \quad \text{dans } W^{1,2}(S), \quad v \in W^{2,2}(S), \tag{6}$$

$$\frac{1}{h^2} \int_{-1/2}^{1/2} (\tilde{y}^{(h)' }(\cdot, x_3) - x') dx_3 \rightarrow u \quad \text{dans } W^{1,2}(S; \mathbb{R}^2), \tag{7}$$

$$\liminf_{h \rightarrow 0} \frac{1}{h^4} I^h(y^{(h)}) \geq I^0(u, v). \tag{8}$$

- (ii) Si $u \in W^{1,2}(S; \mathbb{R}^2)$ et $v \in W^{2,2}(S)$ alors il existe $\hat{y}^{(h)} \in W^{1,2}(\Omega; \mathbb{R}^3)$ telle que (4)–(7) soient vérifiées (où $\tilde{y}^{(h)}$ est remplacé par $\hat{y}^{(h)}$ et où $\bar{R}^{(h)} = \text{Id}$, $c^{(h)} = 0$) et

$$\lim_{h \rightarrow 0} \frac{1}{h^4} I^h(\hat{y}^{(h)}) = I^0(u, v). \tag{9}$$

Par les arguments classiques de la Γ -convergence, on obtient la convergence des applications (presque) minimisantes, voir Corollary 2 ci-dessous.

1. Introduction

In this article we continue our recent work [5,6] devoted to clarifying, in a mathematically rigorous way, the relationship between 3D nonlinear elasticity and 2D theories in thin domains [9,1,3].

As in [5,6] our starting point is the elastic energy

$$E^h(w) = \int_{\Omega_h} W(\nabla w(z)) dz \tag{10}$$

of a deformation $w : \Omega_h = S \times (-\frac{h}{2}, \frac{h}{2}) \rightarrow \mathbb{R}^3$. Deformations with $E^h \sim h$ (e.g., finite stretching of the midplane S) respectively $E^h \sim h^3$ (e.g., finite bending displacement leaving S unstretched) are rigorously known to be governed by membrane theory respectively nonlinear plate theory, in the sense that $\frac{1}{h}E^h$ Γ -converges to the membrane functional [7,8] and $\frac{1}{h^3}E^h$ to Kirchhoff's nonlinear plate functional [5,6] (for related work see [2,12]).

Here we analyze the case when $E^h \sim h^5$, and show that $\frac{1}{h^5}E^h$ Γ -converges to the Föppl–von Kármán theory of plates. Heuristically it is clear that this theory cannot be valid unless the overall displacement is small up to a single rigid motion. The key point in the proof is to justify and quantify this smallness from smallness of the elastic energy. This is achieved by a higher-derivative generalization of our rigidity estimate [5, Theorem 2] (Theorem 3 below). That suitable bounds on the scaled displacements would imply rigorous Γ -convergence results has been noticed independently by A. Raoult [13]. For a very different approach see [11]. Using similar arguments one can derive the full spectrum of limiting theories corresponding to the scaling $E^h \sim h^\alpha$, $\alpha \geq 3$. Details will appear elsewhere. Such a hierarchy of theories has been previously suggested in the literature based on formal asymptotic expansions, for recent contributions see [4,10].

To state our result it is convenient to work in a fixed domain $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$, change variables $x = (z_1, z_2, \frac{z_3}{h})$ and rescale deformations according to $y(x) = w(z(x))$ so that $y : \Omega \rightarrow \mathbb{R}^3$. We abbreviate $x' = (x_1, x_2)$ and use the notation $\nabla' y = y_{,1} \otimes e_1 + y_{,2} \otimes e_2$ for the in-plane gradient so that $\nabla w = (\nabla' y, h^{-1}y_{,3}) =: \nabla_h y$ and

$$\frac{1}{h}E(w) = I^h(y) := \int_{\Omega} W(\nabla_h y) \, dx. \tag{11}$$

We assume that the stored energy W is Borel measurable with values in $[0, \infty]$ and satisfies

$$W = 0 \quad \text{on } \text{SO}(3), \quad W(F) \geq c \, \text{dist}^2(F, \text{SO}(3)), \quad c > 0, \tag{12}$$

$$W(QF) = W(F) \quad \forall Q \in \text{SO}(3), \quad W \text{ is } C^2 \text{ in a neighbourhood of } \text{SO}(3). \tag{13}$$

Since the relevant deformation gradients will be close to $\text{SO}(3)$ we define

$$Q_2(G) = \min_{a \in \mathbb{R}^3} Q_3(G + a \otimes e_3 + e_3 \otimes a), \tag{14}$$

where $Q_3(F) = (\partial^2 W / \partial F^2)(\text{Id})(F, F)$ is twice the linearized energy and $G \in M^{2 \times 2}$. For isotropic elasticity we have $Q_3(F) = 2\mu |\text{sym } F|^2 + \lambda (\text{tr } F)^2$, $Q_2(G) = 2\mu |\text{sym } G|^2 + \frac{2\mu\lambda}{2\mu+\lambda} (\text{tr } G)^2$, where $\text{sym } F = (F + F^T)/2$ denotes the symmetric part of a square matrix F . For $u \in W^{1,2}(S; \mathbb{R}^2)$ and $v \in W^{2,2}(S)$ we introduce the Föppl–von Kármán functional

$$I^0(u, v) = \int_S \frac{1}{2} Q_2 \left(\frac{1}{2} [\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v] \right) + \frac{1}{24} Q_2((\nabla')^2 v) \, dx'. \tag{15}$$

THEOREM 1. – *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and that W satisfies (12), (13). Then the functionals $\frac{1}{h^4}I^h$ are Γ -convergent to I^0 . More precisely we have*

- (i) (compactness and lower bound) *If $\limsup_{h \rightarrow 0} h^{-4} I^h(y^{(h)}) < \infty$ then there exist a subsequence and constants $c^{(h)} \in \mathbb{R}^3$, $\bar{R}^{(h)} \in \text{SO}(3)$ such that $\bar{R}^{(h)} \rightarrow \bar{R}$ and*

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)} \rightarrow \bar{y} \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^3), \quad \bar{y}(x) = \begin{pmatrix} x' \\ 0 \end{pmatrix}, \tag{16}$$

$$\nabla_h \tilde{y}^{(h)} \rightarrow \text{Id} \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}), \tag{17}$$

$$\frac{1}{h} \int_{-1/2}^{1/2} \tilde{y}_3^{(h)}(\cdot, x_3) \, dx_3 \rightarrow v \quad \text{in } W^{1,2}(S), \quad v \in W^{2,2}(S), \tag{18}$$

$$\frac{1}{h^2} \int_{-1/2}^{1/2} \left(\begin{pmatrix} \tilde{y}_1^{(h)} \\ \tilde{y}_2^{(h)} \end{pmatrix}(\cdot, x_3) - x' \right) \, dx_3 \rightarrow u \quad \text{in } W^{1,2}(S; \mathbb{R}^2), \tag{19}$$

$$\liminf_{h \rightarrow 0} \frac{1}{h^4} I^h(y^{(h)}) \geq I^0(u, v). \tag{20}$$

(ii) (optimality of lower bound) *Given $u \in W^{1,2}(S; \mathbb{R}^2)$ and $v \in W^{2,2}(S)$ there exist $\hat{y}^{(h)} \in W^{1,2}(\Omega; \mathbb{R}^3)$ such that (16)–(19) hold (with $\tilde{y}^{(h)}$ replaced by $\hat{y}^{(h)}$, with $\bar{R}^{(h)} = \text{Id}$ and $c^{(h)} = 0$) and*

$$\lim_{h \rightarrow 0} \frac{1}{h^4} I^h(\hat{y}^{(h)}) = I^0(u, v). \tag{21}$$

As usual the above Γ -convergence result implies the convergence of (almost) minimizers. For simplicity we focus on the case without boundary conditions and we take into account dead-load forces $f^{(h)} : S \rightarrow \mathbb{R}^3$ by considering the functional

$$J^h(y) = \int_{\Omega} W(\nabla_h y) - f^{(h)} \cdot y \, dx = \frac{1}{h} \int_{\Omega_h} W(\nabla w) - f^{(h)} \cdot w \, dx. \tag{22}$$

We suppose that $f^{(h)}$ points in the x_3 direction and satisfies

$$f_1^{(h)} = f_2^{(h)} = 0, \quad \int_S f_3^{(h)}(x') \, dx' = 0, \quad \int_S x' f_3^{(h)}(x') \, dx' = 0 \tag{23}$$

and that $h^{-3} f_3^{(h)} \rightarrow f_3$ in $L^2(S)$.

COROLLARY 2. – *Let $y^{(h)}$ be a sequence of almost minimizers of J^h , i.e. $h^{-4}(J^h(y^{(h)}) - \inf J^h) \rightarrow 0$. Then there exist a subsequence and constants $c^{(h)} \in \mathbb{R}^3$, $\bar{R}^{(h)} \in \text{SO}(3)$ such that (16)–(19) hold and $\bar{R}^{(h)} \rightarrow \bar{R}$ as $h \rightarrow 0$. Moreover the triple (u, v, \bar{R}) , where u and v are the limits in (18) and (19), respectively, minimizes the functional*

$$J^0(\tilde{u}, \tilde{v}, \tilde{R}) = I^0(\tilde{u}, \tilde{v}) - \tilde{R}_{33} \int_S f_3 \tilde{v} \, dx' \tag{24}$$

among all triples in $W^{1,2}(S; \mathbb{R}^2) \times W^{2,2}(S) \times \text{SO}(3)$, and $\lim_{h \rightarrow 0} h^{-4} J^h(y^{(h)}) = J^0(u, v, \bar{R})$. In particular for $f_3 \neq 0$ we have $\bar{R}_{33} = \pm 1$.

Remark. – If $\bar{R}_{33} = 1$ then \bar{R} is an in-plane rotation and $y^{(h)}$ is close to $\bar{R}(x')$ (up to translation). If $\bar{R}_{33} = -1$ then \bar{R} is an in-plane rotation followed by a 180° degree out-of-plane rotation $R_0 = \text{diag}(-1, 1, -1)$. Since J^0 is invariant under the transformation $(u, v, R) \mapsto (u, -v, R_0 R)$ it suffices to consider the (conventional) situation $R_{33} = 1$.

2. Rigidity and smallness estimates

The key ingredient in the proofs of Theorem 1 and Corollary 2 is an optimal rigidity estimate which bounds the L^2 -distance of a gradient from a single rotation by the L^2 -distance from the set $\text{SO}(n)$ of all rotations ([5], Theorem 2). In thin domains $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$ it can be applied to cubes of size h and it also provides a difference quotient estimate for the rotation in neighbouring cubes. Using mollification on the scale h one can convert this into estimates for higher derivatives. The estimates (25) of the following theorem combine the interior and the boundary estimates in [6], while (26) and (27) follow from the Poincaré–Sobolev inequality and the trivial estimate $\|R - \tilde{R}\|_{L^p}^p \leq C \|R - \tilde{R}\|_{L^2}^2$.

THEOREM 3. – *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. Let $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $E = \int_{\Omega} \text{dist}^2(\nabla_h y, \text{SO}(3)) \, dx$. Then there exist maps $R : S \rightarrow \text{SO}(3)$ and $\tilde{R} : S \rightarrow \mathbb{R}^{3 \times 3}$, with $|\tilde{R}| \leq C$, and a constant $\bar{Q} \in \text{SO}(3)$ such that*

$$\|\nabla_h y - R\|_{L^2(\Omega)}^2 \leq CE, \quad \|R - \tilde{R}\|_{L^2(S)}^2 \leq CE, \quad \|\nabla' \tilde{R}\|_{L^2(S)}^2 \leq \frac{C}{h^2} E, \tag{25}$$

$$\|\nabla_h y - \bar{Q}\|_{L^2(\Omega)}^2 \leq \frac{C}{h^2} E. \tag{26}$$

If, in addition, $E \leq Ch^\beta$, with $\beta > 2$ then also

$$\|R - \bar{Q}\|_{L^p(S)}^2 \leq C_p h^{\beta-2}, \quad \text{where } p = \frac{2\beta}{\beta-2}. \tag{27}$$

To prove part (i) of Theorem 1 we first use the above rigidity estimates to establish the convergence results (16)–(19) (see Lemma 4). Then we identify the weak limit of the distortion $h^{-2}[(R^{(h)})^T \nabla_h y - \text{Id}]$ to obtain (20) (see Lemma 5).

LEMMA 4. – Suppose that $I^h(y^{(h)}) \leq Ch^4$. Then there exist maps $R^{(h)} : S \rightarrow \text{SO}(3)$, $\tilde{R}^{(h)} : S \rightarrow \mathbb{R}^{3 \times 3}$, with $|\tilde{R}^{(h)}| \leq C$ and constants $\bar{R}^{(h)} \in \text{SO}(3)$, $c^{(h)} \in \mathbb{R}^3$ such that $\tilde{y}^{(h)} = (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$ satisfies (18), (19) and

$$\|\nabla_h \tilde{y}^{(h)} - R^{(h)}\|_{L^2(\Omega)} \leq Ch^2, \quad \|R^{(h)} - \tilde{R}^{(h)}\|_{L^2(S)} \leq Ch^2, \tag{28}$$

$$\|R^{(h)} - \text{Id}\|_{L^4(S)} \leq Ch, \quad \|\nabla' \tilde{R}^{(h)}\|_{L^2(S)} \leq Ch, \tag{29}$$

$$\frac{\nabla_h \tilde{y}^{(h)} - \text{Id}}{h} \rightarrow A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3 \quad \text{in } L^2(\Omega), \tag{30}$$

$$2 \text{sym} \left(\frac{R^{(h)} - \text{Id}}{h^2} \right) \rightarrow A^2 \quad \text{in } L^p(S), \forall p < 2, \text{ and weakly in } L^2. \tag{31}$$

Proof. – Step 1 (normalization). Estimates (28) and (29) follow immediately from Theorem 3 with $\beta = 4$, since one can choose $\bar{R}^{(h)}$ so that (26) and (27) hold with $\bar{Q} = \text{Id}$. Applying an additional in-plane rotation of order h to $\tilde{y}^{(h)}$ and $R^{(h)}$ we may assume that $\int_\Omega y_{1,2}^{(h)} - y_{2,1}^{(h)} dx = 0$.

Step 2 (convergence of $A^{(h)} := h^{-1}(R^{(h)} - \text{Id})$). By (29) $A^{(h)} \rightharpoonup A$ in $L^4(S)$ for a subsequence. Trivially $A_{,3} = 0$. It follows from (28) that $h^{-1}(\tilde{R}^{(h)} - \text{Id}) \rightharpoonup A$ in $W^{1,2}(S)$. In particular $A \in W^{1,2}(S)$ and the convergence is strong in L^2 . Using again (28) we deduce that $A^{(h)} \rightarrow A$ in $L^2(S)$ and in all L^q with $q < 4$. In view of (28) this implies the convergence in (30).

Step 3 (convergence of $h^{-2} \text{sym}(R^{(h)} - \text{Id})$). We have $A^{(h)} + (A^{(h)})^T = -h(A^{(h)})^T A^{(h)}$. Hence $A + A^T = 0$ and after division by h we obtain (31) from the strong convergence of $A^{(h)}$.

Step 4 (convergence of the scaled normal and tangential deviations). The convergence (18) of the scaled normal component immediately follows from (30). Moreover $v_i = A_{3i}$ for $i = 1, 2$. Hence $v \in W^{2,2}$ as $A \in W^{1,2}$. Regarding (19) we define

$$u^{(h)}(x') = \int_{-1/2}^{1/2} \frac{(y^{(h)})'(x', x_3) - x'}{h^2} dx_3, \tag{32}$$

where $(y^{(h)})'$ denotes the components of $y^{(h)}$ in e_1 and e_2 direction. We see from (31) and (28) that $\text{sym} \nabla' u^{(h)}$ is bounded in L^2 . Korn's inequality and the normalization in step 1 yield an L^2 bound for $\nabla' u^{(h)}$ and hence (19).

Step 5 (identification of A). By steps 3 and 4 A is skew-symmetric, $A_{31} = v_{,1}$ and $A_{32} = v_{,2}$. Now (30) and (32) imply $hu_{1,2}^{(h)} \rightarrow A_{12}$. Hence $A_{12} = 0$. \square

LEMMA 5. – Let $R^{(h)}$ and $\tilde{y}^{(h)}$ be as in Lemma 4. Then

$$G^{(h)} := \frac{(R^{(h)})^T \nabla_h \tilde{y}^{(h)} - \text{Id}}{h^2} \rightharpoonup G \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}) \tag{33}$$

and the 2×2 submatrix G'' given by $G''_{ij} = G_{ij}$ for $1 \leq i, j \leq 2$ satisfies

$$G''(x', x_3) = G_0(x') + x_3 G_1(x'), \tag{34}$$

where

$$\text{sym}(G_0) = \frac{1}{2}(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v), \quad G_1 = -(\nabla')^2 v. \tag{35}$$

Moreover

$$\liminf_{h \rightarrow 0} I^h(\tilde{y}^{(h)}) \geq \int_{\Omega} \frac{1}{2} Q_3(G(x)) \, dx \geq I^0(u, v). \tag{36}$$

Proof. – To show that the limit matrix G'' is affine in x_3 we consider the difference quotients $H^{(h)}(x', x_3) = s^{-1}[G^{(h)}(x', x_3 + s) - G^{(h)}(x', x_3)]$. Multiply the definition of $G^{(h)}$ by $R^{(h)}$, take difference quotients and express the difference quotient acting on y by an integral over y_3 . This yields for $i, j \in \{1, 2\}$

$$\left(\frac{1}{h} \frac{1}{s} \int_0^s \frac{1}{h} \tilde{y}_{i,3}^{(h)}(x', x_3 + \sigma) \, d\sigma \right)_j = (R^{(h)} H^{(h)})_{ij}(x', x_3).$$

In view of (30) the left-hand side converges in $W^{-1,2}(S \times (-1, 1 - s))$ to $A_{i3,j}(x') = -v_{ij}(x')$. Since $R^{(h)} \rightarrow \text{Id}$ boundedly a.e. and $H^{(h)} \rightharpoonup H$ in L^2 this implies that $H_{ij} = -v_{ij}$. Thus G'' is affine in x_3 and G_1 has the form given in the lemma. In order to prove the formula for G_0 it suffices to study $G_0^{(h)}(x') = \int_{-1/2}^{1/2} G^{(h)}(x', x_3) \, dx_3$. Symmetrizing and taking the limit $h \rightarrow 0$ we obtain (35) with the help of (19), (31) and (30).

The first inequality in (36) follows by careful Taylor expansion [6]. For the second inequality one uses $Q_3 \geq Q_2$, expands the argument of Q_2 and integrates in x_3 . \square

Proof of Theorem 1. – Part (i) follows from Lemmas 4 and 5. Regarding part (ii) assume first that u and v are smooth. Then it suffices to take

$$\hat{y}^{(h)}(x', x_3) = \begin{pmatrix} x' \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^2 u \\ hv \end{pmatrix} - h^2 x_3 \begin{pmatrix} v,1 \\ v,2 \\ 0 \end{pmatrix} + h^3 x_3 d^{(0)} + \frac{h^3}{2} x_3^2 d^{(1)}, \tag{37}$$

where $d^{(0)}$ and $d^{(1)}$ are determined by the relation between Q_2 and Q_3 . If $u \in W^{1,2}$, $v \in W^{2,2}$ we consider suitable smooth approximations with $u^{(h)} \rightarrow u$ in $W^{1,2}$ and $v^{(h)} \rightarrow v$ in $W^{2,2}$. \square

Proof of Corollary 2 (sketch). – By (26) we can choose $\bar{R}^{(h)}$ and $c^{(h)}$ such that $\check{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)} - (x', hx_3)$ satisfies $\|\check{y}^{(h)}\|_{W^{1,2}}^2 \leq Ch^{-2} I^h(y^{(h)})$. Together with the trivial bound $\inf J^h \leq 0$ (use the test function $x \mapsto (x', hx_3)$) and (23) we easily deduce $I^h(y^{(h)}) \leq Ch^4$. Thus we can apply Theorem 1 and we conclude easily since $h^{-1} \check{y}^{(h)} \rightarrow (0, v)$. \square

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