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Géométrie analytique/*Analytic Geometry*

Holonomic systems with solutions ramified along a cusp

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Abstract We classify the holonomic systems of (micro) differential equations of multiplicity one along a singular Lagrangian irreducible variety contained in an involutive submanifold of maximal codimension. We show that their solutions are related to ${}_kF_{k-1}$ hypergeometric functions on the Riemann sphere. ***To cite this article: O. Neto, P.C. Silva, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 171–176.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Systèmes holonomes avec solutions ramifiées le long d'un cusp

Résumé On classifie les systèmes holonomes d'équations (micro) differentielles de multiplicité un dont le support est un espace analytique complexe Lagrangien, singulier, irréductible et contenu dans une sous-varieté lisse de codimension maximal. On montre que leur solutions sont en rapport avec des fonctions ${}_kF_{k-1}$ hypergeométriques sur la sphère de Riemann. ***Pour citer cet article : O. Neto, P.C. Silva, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 171–176.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans la suite k et n sont des entiers positifs tels que $2 \leqslant k \leqslant n-1$ et $(k,n)=1$. On définit $\vartheta = x\partial_x + (n/k)y\partial_y$. Soient I_t la matrice identité d'ordre t et N_t le bloc de Jordan nilpotent d'ordre t.

THÉORÈME 1. – *Soit L un $\mathbb{C}\{x\}$-module libre de dimension k. Soit ∇ un endomorphisme $\mathbb{C}$-linéaire de L tel que $\nabla(fu) = x(\mathrm{d}f/\mathrm{d}x)u + f\nabla u$ pour $f \in \mathbb{C}\{x\}$, $u \in L$. Soit p un endomorphisme $\mathbb{C}\{x\}$-linéaire de L tel que $[\nabla, p] = ((n-k)/k)p$ et $p^k = (n/k)^k x^{n-k}$.*

Alors il y a des nombres complexes λ_i, $i \in \mathbb{Z}$, et générateurs de L, u_i, $i \in \mathbb{Z}$, tels que (1) *soit vérifiée, $u_{i+k} = u_i$, $(u_0, \ldots, u_{k-1})$ soit une base de L, $\nabla u_i = \lambda_i u_i$ et $pu_i = (n/k)x^{\alpha_i}u_{i+1}$, $i \in \mathbb{Z}$.*

On va noter R la $\mathbb{C}$-algèbre $\mathbb{C}\{x, p, t_1, \ldots, t_{m-2}\}/(p^k - (n/k)^k x^{n-k})$. La dérivation $x\partial_x + ((n-k)/k)p\partial_p$ de $\mathbb{C}\{x, p, t_1, \ldots, t_{m-2}\}$ induit une dérivation Δ de R.

THÉORÈME 2. – *Soit L un R-module sans torsion de type fini et rang un. Soit ∇ un endomorphisme $\mathbb{C}$-linéaire de L tel que $\nabla(fu) = \Delta(f)u + f\nabla u$ pour $f \in R$, $u \in L$.*

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Si les ∂_{t_j}, $1 \leqslant j \leqslant m-2$, *agissent sur* L *comme des endomorphismes* $\mathbb{C}$*-linéaires tels que* $\partial_{t_j}(fv) = (\partial f/\partial t_j)v + f\partial_{t_j}v$, $1 \leqslant j \leqslant m-2$, $f \in R$, $v \in L$, *alors* L *est* $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$*-libre de rang* k. *De plus, il y a des nombres complexes* λ_i, $i \in \mathbb{Z}$, *et des générateurs de* L, v_i, $i \in \mathbb{Z}$, *tels que* $v_{i+k} = v_i$, $(v_0, \ldots, v_{k-1})$ *soit une base du* $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$*-module* L *et* (1), (5) *soient verifiées.*

Preuve. – Supposons $m = 2$. Puisque L est un R-module de type fini et R est un $\mathbb{C}\{x\}$-module de type fini, L est un $\mathbb{C}\{x\}$-module de type fini. Puisque $\mathbb{C}\{x\}$ est un domaine d'ideaux principaux et L est un $\mathbb{C}\{x\}$-module sans torsion, L est $\mathbb{C}\{x\}$-libre de dimension finie, disons l. Soit K le corps des fractions de R. Puisque les anneaux $\mathbb{C}\{x\}[x^{-1}][p]/(p^k - (n/k)^k x^{n-k})$ et $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R$ sont isomorphes et puisque $p^k - (n/k)^k x^{n-k}$ définit un polynôme irréductible sur le corps $\mathbb{C}\{x\}[x^{-1}]$, $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R \simeq K$. Puisque R est $\mathbb{C}\{x\}$-libre de rang k, K est un espace vectorial sur $\mathbb{C}\{x\}[x^{-1}]$ de dimension k. Puisque $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} L \simeq K \otimes_R L$ et L est un R-module de rang un, K est un espace vectoriel sur $\mathbb{C}\{x\}[x^{-1}]$ de dimension l. Ceci entrâine l'egalité $l = k$. Par le Théorème 1 on a le résultat pour $m = 2$. Le cas général est prouvé par induction en m. □

THÉORÈME 3. – *Les systèmes holonomes de multiplicité un sont réguliers holonomes.*

PROPOSITION 4. – *Soit* Λ *le germe en* $q \in T^*X \setminus X$ *d'un ensemble analytique Lagrangien, conique et irréductible. Soit* M *la fibre au point* q *d'un* $\mathcal{E}_X$*-module holonome de multiplicité un le long de* Λ. *Soit* N *le réseau canonique de* M (voir *Théorème* 5.16 *de* [3]). *Alors* $N/N(-1)$ *est un* $\mathcal{O}_{\Lambda,q}(0)$*-module sans torsion de type fini et rang un.*

THÉORÈME 5. – *Soient* c, λ'_i, $i = 0, \ldots, d-1$, *des nombres complexes tels que* $c \in (\mathbb{C} \setminus \mathbb{R}) \cup]0, 1[$, $\lambda'_i \neq 0$, *for all* i, *et* $\{\lambda'_i - \lambda'_j : 0 \leqslant i, j \leqslant d-1\} \cap (c\mathbb{N} + (1-c)\mathbb{N}) \subset \{0\}$. *Alors il y a une solution, unique, du problème microdifferentiel de Cauchy* (6) *où* $A_0 = \mathrm{diag}(\lambda'_0, \ldots, \lambda'_{d-1})$, $U \in M_d(\mathcal{E}_{X,(0,\mathrm{d}y)}(0))$, *et* $\mathcal{A}_{-1} \in M_d(\mathcal{E}_{X,(0,\mathrm{d}y)}(-1))$.

Étant donnés des nombres complexes λ_i, $i \in \mathbb{Z}$, tels que (1) soit vérifié, on note $\mathcal{M}_{(\lambda_i)}$ (resp. $\mathcal{L}_{(\lambda_i)}$) le $\mathcal{E}_{\mathbb{C}^m}$-module (resp. $\mathcal{D}_{\mathbb{C}^m}$-module) engendré par u_i, $i \in \mathbb{Z}$, vérifiant les rélations (7).

THÉORÈME 6. – *Étant donné un système d'équations microdifferentielles* $\mathcal{M}$ *de multiplicité un le long du conormal de l'hypersurface de* $\mathbb{C}^m$, $y^k = x^n$, *il y a des nombres complexes* λ_i, $i \in \mathbb{Z}$, *tels que* (1) *soit vérifiée et* $\mathcal{M}_{(0,\mathrm{d}y)}$ *soit isomorphe à* $(\mathcal{M}_{(\lambda_i)})_{(0,\mathrm{d}y)}$.

Preuve. – Soit $\mathcal{N}$ le réseau canonique de $\mathcal{M}$. La fibre au point $(0, \mathrm{d}y)$ de $\mathcal{O}_\Lambda(0)$ est égale à R. On va noter M, N et L les fibres au point $(0, \mathrm{d}y)$ des faisceaux $\mathcal{M}$, $\mathcal{N}$ et $\mathcal{N}/\mathcal{N}(-1)$ respectivement. Par la Proposition 4, L est un R-module sans torsion de type fini et rang un. L'operateur ϑ agisse sur R comme la dérivation Δ. De plus, $[\vartheta, p] = H_{\sigma(\vartheta)}(p) = ((n-k)/k)p$. En définissant $\nabla = \vartheta$, on conclût que L vérifie les conditions du Théorème 2. Soient v_i, $i \in \mathbb{Z}$, des générateurs pour L vérifiant (5). Il existent $u_i \in N$, $i \in \mathbb{Z}$, tels que $v_i = u_i + N(-1)$ et $u_{i+k} = u_i$, $i \in \mathbb{Z}$. Les u_i's engendrent le $\mathcal{E}_{X,(0,\mathrm{d}y)}(0)$-module N. De plus, $(\partial_x \partial_y^{-1})u_i + (n/k)x^{\alpha_i}u_{i+1}$ et $(\vartheta - \lambda_i)u_i \in N(-1)$. Par le Théorème 5 on peut supposer que $\vartheta u_i = \lambda_i u_i$. Par une variation de l'argument de la preuve du Théorème 1 $(\partial_x \partial_y^{-1})u_i + (n/k)x^{\alpha_i}u_{i+1} \in N(-l)$ pour tous $l \geqslant 1$. Donc $(\partial_x \partial_y^{-1})u_i = -(n/k)x^{\alpha_i}u_{i+1}$, $i \in \mathbb{Z}$. □

THÉORÈME 7. – *Soit* $\mathcal{L}$ *le germe à l'origine d'un* $\mathcal{D}_{\mathbb{C}^m}$*-module coherent de varieté caractéristique l'union du conormal de l'hypersurface* $y^k = x^n$ *avec la section nulle. Alors* $\mathcal{L}$ *a multiplicité un le long du conormal de* $y^k = x^n$ *et on a un isomorphisme de espaces vectoriels complexes* $\partial_y : \mathcal{L}_0 \to \mathcal{L}_0$ *si et seulement si il y a des nombres complexes* λ_i, $i \in \mathbb{Z}$, *tels que* (1), (11) *soient vérifiées et* $\mathcal{L}$ *soit isomorphe à* $\mathcal{L}_{(\lambda_i)}$.

Le Théorème 7 c'est une conséquence du Théorème 6 et du Théorème 8.6.19 de [2].

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On définit $v(x,y,t) = y^{-\lambda_0 k/n}u_0(x,y,t)$. Puisque $\vartheta v = 0$, v est constant le long des fibres de l'application $\gamma : (\mathbb{C}^2 \setminus \{(0,0)\}) \times \mathbb{C}^{m-2} \to \mathbb{P}^1$, définit par $\gamma(x,y,t) = (x^n : y^k)$. Alors il existe une fonction φ multivaluée holomorphe sur $\mathbb{P}^1 \setminus \{0,1,\infty\}$, telle que $u_0(x,y,t) = y^{\lambda_0 k/n}\varphi(y^k/x^n)$. De plus, φ est une solution d'une equation differentielle hypergeométrique ${}_kF_{k-1}$.

Let k, n be integers s.t. $2 \leqslant k \leqslant n-1$ and $(k,n)=1$. Let X be a complex manifold of dimension m. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Let Λ be a germ of a conic Lagrangian subvariety of $T^*X \setminus X$. By Theorem 8.3 of [7], if Λ is singular and irreducible and Λ is contained in an involutive submanifold of $T^*X \setminus X$ of codimension $m-1$, there exist a system of local coordinates $(x, y, t_1, \ldots, t_{m-2})$ s.t. Λ can be identified with the conormal of the hypersurface $y^k = x^n$. These are the singular Lagrangian varieties with milder singularities. We can find in [7], Theorem 8.6, the classification of the systems of microdifferential equations with simple characteristics along Λ. The purpose of this paper is to classify the systems of microdifferential equations of multiplicity one along Λ. As a consequence we obtain a classification theorem for $\mathcal{D}$-modules. For a topological approach to this problem cf. [5].

1. Main result

We denote the identity matrix of order t by I_t and the nilpotent Jordan block of order t by N_t.

LEMMA 1. – *Let $\alpha \in \mathbb{C}$ and $Y \in M_{\mu\times\nu}(\mathbb{C}\{x\}[x^{-1}])$ s.t. $(x(\mathrm{d}/\mathrm{d}x) - \alpha)Y = YN_\nu - N_\mu Y$. If $\alpha \notin \mathbb{Z}$, $Y = 0$. If $\alpha \in \mathbb{Z}$, $Y = Cx^\alpha$, where $C \in M_{\mu\times\nu}(\mathbb{C})$ s.t. $CN_\nu - N_\mu C = 0$.*

THEOREM 2. – *Let L be a free $\mathbb{C}\{x\}$-module of dimension k. Let ∇ be a $\mathbb{C}$-linear endomorphism of L such that $\nabla(fu) = x(\mathrm{d}f/\mathrm{d}x)u + f\nabla u$, $f \in \mathbb{C}\{x\}$, $u \in L$. Let p be a $\mathbb{C}\{x\}$-linear endomorphism of L s.t. $[\nabla, p] = ((n-k)/k)p$ and $p^k = (n/k)^k x^{n-k}$.*

There exist complex numbers λ_i, $i \in \mathbb{Z}$, and a system of generators of L, u_i, $i \in \mathbb{Z}$, s.t.

$$\lambda_{i+k} = \lambda_i, \qquad \alpha_i := \lambda_i - \lambda_{i+1} + (n-k)/k \quad \textit{is a nonnegative integer}, \tag{1}$$

$u_{i+k} = u_i$, $(u_0, \ldots, u_{k-1})$ is a basis of L, $\nabla u_i = \lambda_i u_i$ and $pu_i = (n/k)x^{\alpha_i}u_{i+1}$, $i \in \mathbb{Z}$.

Proof. – If $A = (a_{i,j})$, $B = (b_{i,j})$ are respectively the matrices of ∇, p w.r.t. a basis $(u_0, \ldots, u_{k-1})$ of L,

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n-k}{k}\right)B = [B, A]. \tag{2}$$

Assume the additional hypothesis A *is constant*. We can assume that A is the direct sum of l Jordan blocks A_r of size m_r and eigenvalue λ_r, $0 \leqslant r \leqslant l-1$, where $1 \leqslant l \leqslant k$. Consider the block decomposition $B = (B_{r,s})$, $0 \leqslant r, s \leqslant l-1$, $B_{r,s} \in \mathrm{M}_{m_r\times m_s}(\mathbb{C}\{x\})$. Let $A_r = \lambda_r I_{m_r} + N_{m_r}$ be the decomposition of the Jordan block A_r into semisimple and nilpotent parts. We get a block decomposition $[B, A] = (B_{r,s}A_s - A_r B_{r,s})$, $B_{r,s}A_s - A_r B_{r,s} = (\lambda_s - \lambda_r)B_{r,s} + B_{r,s}N_{m_s} - N_{m_r}B_{r,s}$. Hence $(x(\mathrm{d}/\mathrm{d}x) + \lambda_r - \lambda_s - (n-k)/k)B_{r,s} = B_{r,s}N_{m_s} - N_{m_r}B_{r,s}$. By Lemma 1 there are matrices $C_{r,s} \in \mathrm{M}_{m_r\times m_s}(\mathbb{C})$, $0 \leqslant r, s \leqslant l-1$, s.t. $B_{r,s} = C_{r,s}x^{\lambda_s - \lambda_r + (n-k)/k}$. Hence $B_{r,s} = 0$ or $\lambda_s - \lambda_r + (n-k)/k \in \mathbb{Z}$. By the assumptions on k and n,

$$B_{r,r} = 0 \quad \text{and} \quad B_{r_0,r_1}, B_{r_1,r_2}, \ldots, B_{r_{t-1},r_t} \neq 0 \Rightarrow B_{r_0,r_t} = 0 \quad \text{for } t = 2, \ldots, l-1. \tag{3}$$

In particular $l \geqslant 2$. Since $B^k = (n/k)^k x^{n-k} I_k$, there are integers $i_0, \ldots, i_{k-1}$, s.t. $0 \leqslant i_j \leqslant l-1$ and $B_{i_0,i_1}B_{i_1,i_2}\cdots B_{i_{k-1},i_0} \neq 0$. If $0 \leqslant r < s \leqslant k-1$, $i_r \neq i_s$. Otherwise there would exist a constant matrix $C \neq 0$ s.t. $B_{i_r,i_{r+1}}\cdots B_{i_{s-1},i_s} = x^{(s-r)(n-k)/k}C$ and $(s-r)(n-k)/k \in \mathbb{Z}$. Hence $l = k$ and $j \mapsto i_j$ defines a circular permutation of $\{0, \ldots, k-1\}$. We can assume $B_{j,i} \neq 0$ if $j \equiv i+1 \pmod k$, $i = 0, \ldots, k-1$. By (3) $B_{j,i} = 0$ if $j \not\equiv i+1 \pmod k$, $i = 0, \ldots, k-1$. Thus A is a diagonal matrix with eigenvalues λ_i, $0 \leqslant i \leqslant k-1$. Moreover, $pu_i = C_{i+1,i}x^{\alpha_i}u_{i+1}$, where $\alpha_i = \lambda_i - \lambda_{i+1} + (n-k)/k$, $i \in \mathbb{Z}$, and $\lambda_j = \lambda_i$ if $j \equiv i \pmod k$. By Lemma 1, $\alpha_i \in \mathbb{Z}$. Since $(u_l, \ldots, u_{l+k-1})$ is a basis of the $\mathbb{C}\{x\}$-module L, $\alpha_i \geqslant 0$. Up to a $\mathbb{C}$-linear change of basis, $C_{i+1,i} = n/k$ for $0 \leqslant i \leqslant k-1$. The reduction of the general case to the case A *is constant* is a variation on a standard manipulation of shearing transformations. □

It follows from (1) that $\alpha_{i+k} = \alpha_i$, for all i, and $\sum_{i=0}^{k-1} \alpha_i = n-k$.

Let R denote the $\mathbb{C}$-algebra $\mathbb{C}\{x, p, t_1, \ldots, t_{m-2}\}/(p^k - (n/k)^k x^{n-k})$. The derivation $x\partial_x + ((n-k)/k)p\partial_p$ of $\mathbb{C}\{x, p, t_1, \ldots, t_{m-2}\}$ induces a derivation Δ of R.

THEOREM 3. – *Let L be a finitely generated torsion-free R-module of rank one. Let ∇ be a $\mathbb{C}$-linear endomorphism of L s.t.*

$$\nabla(fu) = \Delta(f)u + f\nabla u, \quad f \in R,\ u \in L. \tag{4}$$

If ∂_{t_j}, $1 \leqslant j \leqslant m-2$, act on L as $\mathbb{C}$-linear endomorphisms and $\partial_{t_j}(fv) = (\partial f/\partial t_j)v + f\partial_{t_j}v$, $1 \leqslant j \leqslant m-2$, for $f \in R$ and $v \in L$, L is a free $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$-module of rank k. Moreover, there are complex numbers λ_i, $i \in \mathbb{Z}$, and a system of generators of L, v_i, $i \in \mathbb{Z}$, s.t. (1) *holds, $v_{i+k} = v_i$, $(v_0, \ldots, v_{k-1})$ is a basis of L as a $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$-module and*

$$\nabla v_i = \lambda_i v_i, \quad pv_i = (n/k)x^{\alpha_i} v_{i+1}, \quad \partial_{t_j} v_i = 0, \quad 1 \leqslant j \leqslant m-2. \tag{5}$$

Proof. – Assume $m = 2$. Since L is a finitely generated R-module and R is a finitely generated $\mathbb{C}\{x\}$-module, L is a finitely generated $\mathbb{C}\{x\}$-module. Since $\mathbb{C}\{x\}$ is a principal ideal domain and L is a torsion-free $\mathbb{C}\{x\}$-module, L is a finitely free $\mathbb{C}\{x\}$-module. Let l be the dimension of the $\mathbb{C}\{x\}$-module L. Let K be the quotient field of R. Since the rings $\mathbb{C}\{x\}[x^{-1}][p]/(p^k - (n/k)^k x^{n-k})$ and $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R$ are isomorphic and $p^k - (n/k)^k x^{n-k}$ is irreducible over $\mathbb{C}\{x\}[x^{-1}]$, $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R$ is a field. Hence $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R \tilde{\to} K$. Since R is $\mathbb{C}\{x\}$-free of rank k, K is a $\mathbb{C}\{x\}[x^{-1}]$-vector space of dimension k. Since $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} L$ is isomorphic to $K \otimes_R L$ and L is an R-module of rank one, K is a $\mathbb{C}\{x\}[x^{-1}]$-vector space of dimension l. Hence $l = k$. The result follows from Theorem 2.

The general case is proved by induction in m. Assume $m = 3$. Set $t = t_1$. The $\mathbb{C}\{x, p\}$-module $\widetilde{L} = L/(t)L$ verifies the assumptions of the theorem for $m = 2$. Let $\widetilde{v_i}$, $0 \leqslant i \leqslant k-1$, be a basis of $\widetilde{L}$. Choose v_i, $0 \leqslant i \leqslant k-1$, s.t. $\widetilde{v_i} = v_i + (t)L$. Let M be the $\mathbb{C}\{x, t\}$-module generated by the v_i's. Since $L = M + (t)^l L$ for all l, $M = L$ (*see* [6], Proposition II. 1.1.3.). If $\sum_{i=1}^{k} a_i v_i = 0$, $a_i \in (t)$ for $0 \leqslant i \leqslant k-1$. Hence $a_i \in (t)^l$ for $0 \leqslant i \leqslant k-1$, $l \geqslant 1$. Therefore v_i, $0 \leqslant i \leqslant k-1$, is a basis of L. After performing a base change coming from the solution of Cauchy problem we can assume that $(\partial v_i/\partial t) = 0$, $0 \leqslant i \leqslant k-1$. Since $\nabla v_i - \lambda v_i$, $pv_i - (n/k)x^{\alpha_i} v_{i+1} \in (t)L$, relations (5) hold. □

THEOREM 4. – *Holonomic systems of multiplicity one are regular holonomic.*

Proof. – A nonvanishing section u of a holonomic $\mathcal{E}$-module $\mathcal{M}$ of multiplicity one along Λ is a local generator of $\mathcal{M}$. Set $\overline{\mathcal{M}} = \mathcal{E}_X(0)u/\mathcal{E}_X(-1)u$. By the definition of multiplicity of [6] (Appendix D), there is a dense open subset U of the support of u s.t. $(I_\Lambda \otimes \overline{\mathcal{M}})|_U = 0$. Hence $u|_U$ is a generator with simple characteristics of $\mathcal{M}|_U$. Therefore $\mathcal{M}$ is regular holonomic at a generic point of Λ. □

PROPOSITION 5. – *Let $\Lambda \subset T^*X \setminus X$ be the germ at a point q of an irreducible conic Lagrangian variety. Let M be the fiber at q of a holonomic $\mathcal{E}_X$-module of multiplicity one along Λ. Let N be its canonical lattice* (see [3], *Theorem* 5.16). *Then $N/N(-1)$ is a finitely generated torsion free $\mathcal{O}_{\Lambda,q}(0)$-module of rank one.*

THEOREM 6. – *Consider the microdifferential Cauchy problem*

$$[cx\partial_x + y\partial_y, U] - [A_0, U] - \mathcal{A}_{-1}U = 0, \qquad \sigma_0(U)\big((0, \mathrm{d}y)\big) = I_d, \tag{6}$$

where $U \in M_d(\mathcal{E}_{X,(0,\mathrm{d}y)}(0))$, $c \in (\mathbb{C} \setminus \mathbb{R}) \cup]0, 1[$, $A_0 \in M_d(\mathbb{C})$ and $\mathcal{A}_{-1} \in M_d(\mathcal{E}_{X,(0,\mathrm{d}y)}(-1))$. If A_0 is semisimple with eigenvalues $\lambda'_0, \ldots, \lambda'_{d-1}$ verifying $\{\lambda'_i - \lambda'_j : 0 \leqslant i, j \leqslant d-1\} \cap (c\mathbb{N} + (1-c)\mathbb{N}) \subset \{0\}$ there is one and only one solution of (6).

Set $\vartheta = x\partial_x + (n/k)y\partial_y$. Let λ_i, $i \in \mathbb{Z}$, be complex numbers s.t. (1) holds. We denote by $\mathcal{M}_{(\lambda_i)}$ the $\mathcal{E}_{\mathbb{C}^m}$-module given by the generators u_i, $i \in \mathbb{Z}$, and relations

$$u_{i+k} = u_i, \quad (\vartheta - \lambda_i)u_i = 0, \quad \partial_x u_i = -(n/k)x^{\alpha_i}\partial_y u_{i+1}, \quad \partial_{t_j} u_i = 0, \quad 1 \leqslant j \leqslant m-2. \tag{7}$$

We denote by $\mathcal{L}_{(\lambda_i)}$ the $\mathcal{D}_{\mathbb{C}^m}$-module given by the same sets of generators and relations.

THEOREM 7. – *Let X be a complex manifold of dimension m. Let Λ be the germ at $q \in T^*X$ of an irreducible conic Lagrangian variety contained in an involutive submanifold of $T^*X \setminus X$ of codimension $m-1$. Given a system of microdifferential equations $\mathcal{M}$ of multiplicity one along Λ, there are complex numbers λ_i, $i \in \mathbb{Z}$, s.t.* (1) *holds and, after a convenient quantized contact transformation, the germ at q of $\mathcal{M}$ is isomorphic to $\mathcal{M}_{(\lambda_i)}$.*

Proof. – Let $\mathcal{N}$ be the canonical lattice of $\mathcal{M}$. By the remarks in the first paragraph of this Note the germ at $(0, \mathrm{d}y)$ of $\mathcal{O}_\Lambda(0)$ equals R. Let M, N and L denote, respectively, the germs at $(0, \mathrm{d}y)$ of the sheaves $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{N}/\mathcal{N}(-1)$. By Proposition 5 L is a finitely generated torsion free R-module of rank one.

Since $[\vartheta, \mathcal{E}_X(0)] \subset \mathcal{E}_X(0)$ and $[\vartheta, \mathcal{I}_\Lambda(-1)] \subset \mathcal{I}_\Lambda(-1)$, the operator ϑ acts on R as a derivation by $\vartheta(f) = \sigma_0([\vartheta, P]) = \{\sigma(\vartheta), \sigma_0(P)\} = H_{\sigma(\vartheta)}(f)$, where $P \in \mathcal{E}_X(0)$ s.t. $\sigma_0(P) = f$. The Hamiltonian vector field $H_{\sigma(\vartheta)}$ equals $x\partial_x + (n/k)y\partial_y + ((n-k)/k)p\partial_p - \eta\partial_\eta$. Hence ϑ acts on R as the derivation Δ and $[\vartheta, p] = H_{\sigma(\vartheta)}(p) = ((n-k)/k)p$. By the regularity conditions $\vartheta\mathcal{N}(k) \subset \mathcal{N}(k)$ and $\partial_{t_j}\mathcal{N}(k) \subset \mathcal{N}(k)$ for $k \in \mathbb{Z}$ and $1 \leqslant j \leqslant m-2$. If $u \in \mathcal{N}$, $v = u + \mathcal{N}(-1)$, $P \in \mathcal{E}_X(0)$ and $f = \sigma_0(P)$, $\vartheta Pu = [\vartheta, P]u + P\vartheta u$. Setting $\nabla = \vartheta$ we deduce that (4) holds. Hence L verifies the conditions of Theorem 3.

Let v_i, $i \in \mathbb{Z}$, be a system of generators of L s.t. $v_{i+k} = v_i$ and (5) holds. There are $u_i \in N$, $i \in \mathbb{Z}$, s.t.

$$v_i = u_i + N(-1) \quad \text{and} \quad u_{i+k} = u_i, \quad i \in \mathbb{Z}. \tag{8}$$

The u_i's generate the $\mathcal{E}_{X,(0,\mathrm{d}y)}(0)$-module N. Moreover,

$$\left(\partial_x\partial_y^{-1}\right)u_i + (n/k)x^{\alpha_i}u_{i+1}, \quad (\vartheta - \lambda_i)u_i \in N(-1). \tag{9}$$

By Theorem 6 we can choose the u_i's verifying (8), (9) and s.t. $\vartheta u_i = \lambda_i u_i$. A variation on the argument of the proof of Theorem 2 shows that $(\partial_x\partial_y^{-1})u_i + (n/k)x^{\alpha_i}u_{i+1} \in N(-l)$ for all $l \geqslant 1$. Hence $(\partial_x\partial_y^{-1})u_i = -(n/k)x^{\alpha_i}u_{i+1}$, $i \in \mathbb{Z}$. □

2. $\mathcal{D}$-modules

Given a ring R and an element $\varepsilon \in R$, we use Pochammer's notation $(\varepsilon)_j = \varepsilon(\varepsilon+1)\cdots(\varepsilon+j-1)$.

THEOREM 8. – *Let $\mathcal{L}$ be the germ at the origin of a coherent $\mathcal{D}_{\mathbb{C}^m}$-module with characteristic variety equal to the union of the conormal of the hypersurface $y^k = x^n$ with the zero section. Then $\mathcal{L}$ has multiplicity one along the conormal of $y^k = x^n$ and*

$$\partial_y : \mathcal{L}_0 \to \mathcal{L}_0 \tag{10}$$

is an isomorphism of complex vector spaces if and only if there are complex numbers λ_i, $i \in \mathbb{Z}$, verifying (1) *s.t. $\mathcal{L}$ is isomorphic to $\mathcal{L}_{(\lambda_i)}$ and*

$$\lambda_i \notin (n/k)\{-1, -2, \ldots\}, \quad 0 \leqslant i \leqslant k-1. \tag{11}$$

Proof. – Let us show that the condition (11) is necessary. Set $q = (0, \mathrm{d}y)$. By Theorem 8.6.19 of [2] and Theorem 7, $\mathcal{L}$ is isomorphic to some $\mathcal{D}$-module $\mathcal{L}_{(\lambda_i)}$. Hence

$$y\partial_y u_l = (k/n)\lambda_l u_l + x^{\alpha_l+1}\partial_y u_{l+1}, \quad \partial_{t_j}u_l = 0, \quad l = 0, \ldots, k-1, \quad j = 1, \ldots, m-2. \tag{12}$$

It follows from (12) and (7) that we have an isomorphism of complex vector spaces

$$(\mathcal{L}_{(\lambda_i)})_0 \cong \bigoplus_{i=0}^{k-1}(\mathbb{C}\{x\}[\partial_y] \oplus y\mathbb{C}\{x, y\})u_i. \tag{13}$$

Set $V=\bigoplus_{l=0}^{k-1}(\mathbb{C}\{x\}[\partial_y]\oplus y\mathbb{C}\{x\}[y])u_l$. Set $\delta_{l,i}=i+\alpha_l+\cdots+\alpha_{l+i-1}$, $0\leqslant l\leqslant k-1$, $i\geqslant 0$. Assume that (10) is injective. We will show by induction in r that $\lambda_l\notin(n/k)\{-1,-2,\dots,-r\}$. Set

$$Q_{j,l}=x^{\delta_{l,j+1}}u_{l+j+1}+\sum_{i=1}^{j}k\frac{\lambda_{l+i}}{n}x^{\delta_{l,i}}R_{j-i,l+i},\qquad R_{j,l}=\left(k\frac{\lambda_l}{n}+j+1\right)^{-1}\left(y^{j+1}u_l-Q_{j,l}\right),\tag{14}$$

for $0\leqslant l\leqslant k-1$, $0\leqslant j\leqslant r$. Since

$$\partial_y\left(y^{r+1}u_l-Q_{r,l}\right)=\left(\lambda_l(k/n)+r+1\right)y^r u_l\tag{15}$$

and since $Q_{r,l}$ is a $\mathbb{C}\{x\}$-linear combination of y^ju_l, $0\leqslant j\leqslant r$, $0\leqslant l\leqslant k-1$, $\lambda_l(k/n)+r+1\neq 0$. Assume that (11) holds. There are complex numbers $b_{l,r,s}$, $s\in\mathbb{N}$, s.t.

$$(\partial_y y)_s u_l=\left((k\lambda_l/n)+1\right)_s u_l+\sum_{r=1}^{s}b_{l,r,s}x^{\delta_{l,r}}\partial_y^r u_{l+r}.\tag{16}$$

Since $(\partial_y y)_s=(\partial_y)^s y^s$, $((k/n)\lambda_l+1)_s\partial_y^{-s}u_l=y^su_l-\sum_{r=1}^{s}b_{l,r,s}x^{\delta_{l,r}}\partial_y^{r-s}u_{l+r}\in\bigoplus_{l=0}^{k-1}\mathbb{C}\{x,y\}u_l$. Let W_{-s} $[V_{-s}]$, $s\in\mathbb{N}$, be the $\mathbb{C}\{x,y\}$-submodule of $\bigoplus_{l=0}^{k-1}\mathbb{C}\{x,y\}u_l$ [$\mathbb{C}\{x\}[y]$-submodule of V] generated by $\partial_y^{-s}u_l$, $l=0,\dots,k-1$. By the definition of W_s, $\bigoplus_{l=0}^{k-1}\mathcal{E}_{\mathbb{C}^m,q}(s)u_l$ is contained in $W_s+\bigoplus_{l=0}^{k-1}\mathcal{E}_{\mathbb{C}^m,q}(s-1)u_l$, for all $s\leqslant 0$. Hence $\bigoplus_{l=0}^{k-1}\mathcal{E}_{\mathbb{C}^m,q}(0)u_l\subset W_0+\bigoplus_{l=0}^{k-1}\mathcal{E}_{\mathbb{C}^m,q}(s)u_l$, for all $s\geqslant 0$.

By [6], Proposition II.1.1.3, $\bigoplus_{l=0}^{k-1}\mathbb{C}\{x,y\}u_l=\bigoplus_{l=0}^{k-1}\mathcal{E}_{\mathbb{C}^m,q}(0)u_l$. Hence the inclusion $(\mathcal{L}_{(\lambda_i)})_0\hookrightarrow(\mathcal{M}_{(\lambda_i)})_q$ is surjective. Let Φ denote the $\mathbb{C}\{x\}$-linear endomorphism of V defined by $\Phi(\partial_y^{j+1}u_l)=\partial_y^ju_l$, $\Phi(y^ju_l)=R_{j,l}$, $j\geqslant 0$. Notice that (10) induces a $\mathbb{C}\{x\}$-linear endomorphism of V. Moreover,

$$\partial_yV_{-s}\subset V_{-s+1},\quad \partial_yW_{-s}\subset W_{-s+1}\quad\text{and}\quad\Phi(V_{-s})\subset V_{-s-1},\quad s\geqslant 0.\tag{17}$$

By (15), $\Phi(\partial_y y^ju_l)=y^ju_l$ for $j\geqslant 1$. Hence the kernel of (10) is contained in W_{-s} for $s\geqslant 0$. Thus (10) is injective. By (17) and (14), $\partial_y\Phi(y^ju_l)=y^ju_l$ for $0\leqslant l\leqslant k-1$, $j\geqslant 0$. Hence $\partial_y\mathcal{L}_0+W_{-s}=\mathcal{L}_0$ for all s. By Proposition II.1.1.3 of [6], (10) is surjective. The result follows from Theorem 8.6.19 of [2]. □

Set $v(x,y,t)=y^{-\lambda_0k/n}u_0(x,y,t)$. Since $\vartheta v=0$, v is constant along the fibers of the map $\gamma:(\mathbb{C}^2\setminus\{(0,0)\})\times\mathbb{C}^{m-2}\to\mathbb{P}^1$ defined by $\gamma(x,y,t)=(x^n:y^k)$. Hence there is a multivalued holomorphic function φ on $\mathbb{P}^1\setminus\{0,1,\infty\}$, s.t. $u_0(x,y,t)=y^{\lambda_0k/n}\varphi(y^k/x^n)$. Set $\delta_x=x\partial_x$ and $\delta_y=y\partial_y$. Notice that $(\delta_x\varphi)\circ\gamma=-n\delta_z(\varphi\circ\gamma)$, $(\delta_y\varphi)\circ\gamma=k\delta_z(\varphi\circ\gamma)$ and $\delta_xu_i=-(n/k)x^{\alpha_i+1}\partial_yu_{i+1}$.

Since $\left[(y^k/x^n)\prod_{j=0}^{k-2}\left(\delta_x-\sum_{i=0}^{j}\alpha_i-j-1\right)\delta_x-(-(n/k))^ky^k\partial_y^k\right]u_0=0$,

$$\left[\prod_{j=0}^{k-1}\left(\delta_z-\tfrac{j}{k}+\tfrac{\lambda_0}{n}\right)-z\delta_z\prod_{j=0}^{k-2}\left(\delta_z+\tfrac{1}{n}\sum_{i=0}^{j}(\alpha_i+j+1)\right)\right]\varphi=0.$$

Therefore φ is a solution of a ${}_kF_{k-1}$ hypergeometric differential equation (*see* [4,1]).

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