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Équations aux dérivées partielles/*Partial Differential Equations*

Positive solutions for slightly super-critical elliptic equations in contractible domains

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Abstract We give examples of bounded domains Ω, even contractible, having the following property: there exists $\bar{k}(\Omega)$ such that, for every integer $k \geqslant \bar{k}(\Omega)$, problem $P(\varepsilon,\Omega)$ below, for $\varepsilon > 0$ small enough, has at least one solution blowing up as $\varepsilon \to 0$ at exactly k points. We also prove that the blow-up points tend to some points of $\partial\Omega$ as $k \to \infty$. ***To cite this article: R. Molle, D. Passaseo, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 459–462.***

Solutions positives pour l'équation $\Delta u + u^{(n+2)/(n-2)+\varepsilon} = 0$ en ouverts contractibles

Résumé On donne des exemples d'ouverts bornés Ω, même contractibles, satisfaisant la propriété suivante : il existe $\bar{k}(\Omega)$ tel que, pour tout $k \geqslant \bar{k}(\Omega)$, le problème $P(\varepsilon,\Omega)$ ci-dessous, pour $\varepsilon > 0$ suffisamment petit, a des solutions qui pour $\varepsilon \to 0$ explosent exactement en k points. On prouve aussi que ces points convergent vers des points de $\partial\Omega$ quand $k \to \infty$. ***Pour citer cet article : R. Molle, D. Passaseo, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 459–462.***

Let us consider the problem

$$P(\varepsilon,\Omega) \qquad \begin{cases} \Delta u + u^{(n+2)/(n-2)+\varepsilon} = 0 \quad \text{in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 3$, and ε is a small positive parameter.

It is well known that $P(\varepsilon,\Omega)$ has no solution if $\varepsilon \geqslant 0$ and Ω is starshaped (see [11]), while (see [9]) it has solution for all $\varepsilon \geqslant 0$ if Ω is, for example, an annulus (when $\varepsilon < 0$, $P(\varepsilon,\Omega)$ is solvable in any bounded domain Ω).

For $\varepsilon = 0$, in [4] the existence of a solution is proved for domains with small holes; in [1] this result is extended to all domains having "nontrivial" topology (in a suitable sense). This nontriviality condition (which covers a large class of domains) is only sufficient for the solvability but not necessary as shown by some examples of contractible domains Ω such that $P(0,\Omega)$ has solutions (see [5,7,12]).

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For $\varepsilon > 0$ large enough, nonexistence results have been proved also in some domains having nontrivial topology in the sense of [1] (see [13]); on the other hand, existence and multiplicity results hold for all $\varepsilon > 0$ in the same contractible domains considered in [12] (see [14]).

When $\varepsilon \to 0$, some concentration phenomena occur, which have been first investigated in the subcritical case, i.e. when $\varepsilon \to 0^-$ (see [3,15,8,2], etc.). In particular, in [2], multi-peak solutions are found, blowing-up as $\varepsilon \to 0^-$ at some points, which are critical points of suitable functions defined in terms of the Green and Robin functions in Ω.

In [6] similar phenomena are described in the super-critical case; in domains with small holes, for $\varepsilon > 0$ small enough, it is proved the existence of a finite number of solutions blowing-up as $\varepsilon \to 0^+$ at some pairs of points localized near the holes.

In this paper our aim is to give some examples showing, in particular, that these concentration phenomena for super-critical problems occur even in some contractible domains (notice that the solutions obtained in [14] for all $\varepsilon > 0$ do not blow-up as $\varepsilon \to 0$). In order to construct such examples, we consider some domains having radial symmetry with respect to a pair of co-ordinates and, for every integer k large enough, we prove the existence of solutions blowing-up as $\varepsilon \to 0^+$ at exactly k points, regularly placed around circles, whose distance from $\partial\Omega$ tends to 0 as $k \to \infty$. Thus we obtain, in particular, that the number of geometrically distinct solutions tends to infinity as $\varepsilon \to 0^+$. Notice that, as proved in [2], for $\varepsilon < 0$ the blow-up points remain uniformly away from $\partial\Omega$ and, for k large enough, there are not solutions blowing-up at k points as $\varepsilon \to 0^-$.

It is worth pointing out that the domains we consider in this paper (unlike [5,7,12] etc.) are not required to be close to domains having different topology (for example, the parameters r and σ in Theorem 1 are not required to be small).

Let us first consider a simple example. For all $\sigma > 0$ and $r \in]0,1[$, set

$$\Omega_r^\sigma = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; r < |x| < 1,\ \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} > \sigma x_n \right\}.$$

We look for solutions to $P(\varepsilon, \Omega_r^\sigma)$ of the form

$$u_{k,\varepsilon}(x) = \left[n(n-2)\right]^{(n-2)/4} \sum_{i=1}^{k} \left(\frac{\lambda_{k,\varepsilon}\varepsilon^{1/(n-2)}}{\lambda_{k,\varepsilon}^2 \varepsilon^{2/(n-2)} + |x - \xi_{i,k,\varepsilon}|^2} \right)^{(n-2)/2} + \theta_{k,\varepsilon}(x), \tag{1}$$

where $\theta_{k,\varepsilon} \to 0$ uniformly as $\varepsilon \to 0$, $\lambda_{k,\varepsilon}$ is a concentration parameter and the concentration points $\xi_{i,k,\varepsilon} \in \Omega_r^\sigma$ have the form $\xi_{i,k,\varepsilon} = (\rho_{k,\varepsilon}\cos(2\pi/k)i, \rho_{k,\varepsilon}\sin(2\pi/k)i, x_{3,k,\varepsilon}, \dots, x_{n,k,\varepsilon})$ for $i = 1, \dots, k$.

THEOREM 1. – *For all $\sigma > 0$ and $r \in]0,1[$, there exist $\bar{k} = \bar{k}(r,\sigma)$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0$, $\forall k \geqslant \bar{k}$, such that, for all $k \geqslant \bar{k}$ and $\varepsilon \in]0, \varepsilon_k[$, $P(\varepsilon, \Omega_r^\sigma)$ has at least two solutions $u_{k,\varepsilon}^{(1)}$ and $u_{k,\varepsilon}^{(2)}$ of the form* (1). *The corresponding concentration points $\xi_{i,k,\varepsilon}^{(1)}$ and $\xi_{i,k,\varepsilon}^{(2)}$ satisfy $x_{3,k,\varepsilon}^{(j)} = x_{4,k,\varepsilon}^{(j)} = \cdots = x_{n-1,k,\varepsilon}^{(j)} = 0$ for $j = 1, 2$ and*

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} \rho_{k,\varepsilon}^{(1)} = \frac{\sigma r}{\sqrt{1+\sigma^2}}, \qquad \lim_{k\to\infty}\lim_{\varepsilon\to 0} x_{n,k,\varepsilon}^{(1)} = \frac{r}{\sqrt{1+\sigma^2}},$$
$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} \rho_{k,\varepsilon}^{(2)} = r, \qquad \lim_{k\to\infty}\lim_{\varepsilon\to 0} x_{n,k,\varepsilon}^{(2)} = 0.$$

The concentration parameters $\lambda_{k,\varepsilon}^{(1)}$ and $\lambda_{k,\varepsilon}^{(2)}$ behave as follows: $\lim_{\varepsilon\to 0}\lambda_{k,\varepsilon}^{(j)} > 0$, $\forall k \geqslant \bar{k}$, $j = 1, 2$; $\lim_{k\to\infty}\lim_{\varepsilon\to 0}\lambda_{k,\varepsilon}^{(j)} = 0$ *for* $j = 1, 2$.

Sketch of the proof. – Let us set $S_r^\sigma=\{(\rho,x_n)\in\mathbb{R}^2\mid r^2<\rho^2+x_n^2<1,\ \rho>\sigma x_n,\ \rho>0\}$ and consider the function $\psi_k:S_r^\sigma\times\mathbb{R}^+\to\mathbb{R}$ defined by

$$\psi_k(\rho,x_n,\Lambda)=\frac{\Lambda^2}{2}\left\{\sum_{i=1}^{k}H(\xi_{i,k},\xi_{i,k})-2\sum_{1\leqslant i<j\leqslant k}G(\xi_{i,k},\xi_{j,k})\right\}+k\lg\Lambda,$$

where $\xi_{i,k}=(\rho\cos\frac{2\pi}{k}i,\rho\sin\frac{2\pi}{k}i,0,\dots,0,x_n)$, G denotes the Green function of $-\Delta$ in Ω_r^σ and H its regular part.

Using the method introduced in [2] and [15], suitably adapted to the super-critical case (see [6] for example), and taking also into account the symmetry of Ω_r^σ with respect to the co-ordinates $x_3,x_4,\dots,x_{n-1}$, the problem reduces to finding critical points of the function ψ_k, which persist with respect to small C^1 perturbations. Clearly $\psi_k(\rho,x_n,\Lambda)=k[\frac{\Lambda^2}{2}\gamma_k(\rho,x_n)+\lg\Lambda]$, with $\gamma_k(\rho,x_n)=H(\xi_{1,k},\xi_{1,k})-\sum_{i=2}^k G(\xi_{1,k},\xi_{i,k})$. Taking into account the properties of H and G, it is easy to verify that $\gamma_k(\rho,x_n)\to+\infty$ as $\operatorname{dist}(\xi_{1,k},\partial\Omega)\to 0$ and that $\lim_{k\to\infty}\gamma_k(\rho,x_n)=-\infty$, $\forall(\rho,x_n)\in S_r^\sigma$.

Notice that any critical point for ψ_k must satisfy the condition $\Lambda^2=-1/\gamma_k(\rho,x_n)$, which is possible only if $\gamma_k(\rho,x_n)<0$; a direct computation shows that finding critical points for ψ_k is equivalent to finding critical points (ρ,x_n) for γ_k, such that $\gamma_k(\rho,x_n)<0$.

Now the crucial step is to observe that, if we set, for example, $c_k=|\gamma_k((r+1)/2,0)|$, then we obtain $\lim_{k\to\infty}\frac{1}{c_k}\gamma_k(\rho,x_n)=-((r+1)/2\rho)^{n-2}$ $\forall(\rho,x_n)\in S_r^\sigma$. Moreover, we have $\lim_{k\to\infty}\inf\{\frac{1}{c_k}\gamma_k(\rho,x_n)\mid(\rho,x_n)\in S_r^\sigma,\ \rho=\tilde\rho\}=-((r+1)/2\tilde\rho)^{n-2}$, for all $\tilde\rho\in]0,1[$. Therefore, for all $\tilde\rho\in]\frac{\sigma r}{\sqrt{1+\sigma^2}},r[$, the minimum $\min\{\gamma_k(\rho,x_n)\mid(\rho,x_n)\in S_r^\sigma,\ \rho<\tilde\rho,\ x_n>0\}$ is achieved for k large enough and the minimum points must converge to $(\sigma r/\sqrt{1+\sigma^2},r/\sqrt{1+\sigma^2})$ as $k\to\infty$.

Now observe that $\lim_{k\to\infty}\inf\{\frac{1}{c_k}\gamma_k(\rho,0)\mid r<\rho<1\}=-((r+1)/2r)^{n-2}$ and that $\lim_{k\to\infty}\sup\{\frac{1}{c_k}\gamma_k(\rho,x_n)\mid(\rho,x_n)\in S_r^\sigma,\ \rho^2+x_n^2=\tilde r^2,\ x_n\leqslant r/\sqrt{1+\sigma^2}\}=-((r+1)/2\tilde r)^{n-2}$, $\forall\tilde r\in]r,1[$. It follows that for k large enough there exists at least another critical point for γ_k (a saddle point) which converges to $(r,0)$ as $k\to\infty$.

Finally, notice that these two critical points, we get for γ_k, both correspond to negative critical values (which tend to $-\infty$ as $k\to\infty$); the corresponding critical points for ψ_k, of the form $(\rho,x_n,\sqrt{-1/\gamma_k(\rho,x_n)})$, persist with respect to small C^1 perturbations, so they give rise to solutions which behave as described in Theorem 1 when $\varepsilon\to 0$ and $k\to\infty$. □

Remark 1. – When $\varepsilon=0$, for all $r\in]0,1[$ there exists $\sigma(r)>0$ such that $P(0,\Omega_r^\sigma)$ has solution for all $\sigma\in]0,\sigma(r)]$ (see [12]), while it is natural to expect that it has no solution if σ is large enough. On the contrary, Theorem 1 holds for all $\sigma>0$ and gives solutions which do not converge to solutions of $P(0,\Omega_r^\sigma)$ since they vanish as $\varepsilon\to 0$.

We describe now some results (whose proof is reported in [10]) which extend Theorem 1.

THEOREM 2. – *Let Ω be a bounded domain of $\mathbb{R}^n$ and assume that there exist $a,b\in\mathbb{R}$ and two functions $\rho_1,\rho_2:[a,b]\to[0,+\infty[$ such that $\overline{\Omega}=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n\mid a\leqslant x_n\leqslant b,\ \rho_1^2(x_n)\leqslant\sum_{i=1}^{n-1}x_i^2\leqslant\rho_2^2(x_n)\}$. Let $\bar x_n\in[a,b]$ satisfy $\rho_1(\bar x_n)>0$ and assume that there exists a neighbourhood $I(\bar x_n)$ of $\bar x_n$ such that $\rho_1(\bar x_n)<\rho_1(x_n)$, $\forall x_n\in I(\bar x_n)\setminus\{\bar x_n\}$ or $\bar x_n\in]a,b[$ and $\rho_1(\bar x_n)>\rho_1(x_n)$, $\forall x_n\in I(\bar x_n)\setminus\{\bar x_n\}$. Then there exist $\bar k=\bar k(\Omega)$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k>0$, $\forall k\geqslant\bar k$, such that, for all $k\geqslant\bar k$ and $\varepsilon\in]0,\varepsilon_k[$, $P(\varepsilon,\Omega)$ has at least one solution $u_{k,\varepsilon}$ of the form* (1) *satisfying $x_{3,k,\varepsilon}=x_{4,k,\varepsilon}=\cdots=x_{n-1,k,\varepsilon}=0$ and* $\lim_{k\to\infty}\lim_{\varepsilon\to 0}x_{n,k,\varepsilon}=\bar x_n$, $\lim_{k\to\infty}\lim_{\varepsilon\to 0}\rho_{k,\varepsilon}=\rho_1(\bar x_n)$. *Moreover,* $\lim_{\varepsilon\to 0}\lambda_{k,\varepsilon}>0$, $\forall k\geqslant\bar k$ *and* $\lim_{k\to\infty}\lim_{\varepsilon\to 0}\lambda_{k,\varepsilon}=0$.

Remark 2. – Consider two balls of $\mathbb{R}^n$, $B(c_1,r_1)$ and $B(c_2,r_2)$, such that $\overline{B(c_1,r_1)}\subset B(c_2,r_2)$. Then Theorem 2 clearly applies when $\Omega=B(c_2,r_2)\setminus\overline{B(c_1,r_1)}$ (and r_1 is not required to be small enough).

Notice that the symmetry of Ω with respect to the co-ordinates $x_3, x_4, \ldots, x_{n-1}$ (we use in Theorems 1 and 2) is not really essential to get solutions. What we really need, in order to find solutions of the form (1), is the radial symmetry of Ω with respect to x_1 and x_2 (i.e., $(x_1, \ldots, x_n) \in \Omega$ if and only if $(0, \sqrt{x_1^2 + x_2^2}, x_3, \ldots, x_n) \in \Omega$). In fact, the following theorem holds.

THEOREM 3. – *Let Ω be a bounded domain of $\mathbb{R}^n$, radially symmetric with respect to x_1 and x_2 and set $S_\Omega = \{(\rho, x_3, \ldots, x_n) \in \mathbb{R}^{n-1} \mid \rho > 0,\ (0, \rho, x_3, \ldots, x_n) \in \Omega\}$. Moreover consider the function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $\varphi(\rho, x_3, \ldots, x_n) = \rho$ and let $(\bar\rho, \bar x_3, \ldots, \bar x_n)$, with $\bar\rho > 0$, be an "essential" critical point for the function φ constrained on $\overline{S_\Omega}$, according to a suitable definition* (*see* [10] *and also Remark* 3)*. Then, for all $k \geqslant \bar k = \bar k(\Omega)$, there exists $\varepsilon_k > 0$ such that, for all $\varepsilon \in]0, \varepsilon_k[$, $P(\varepsilon, \Omega)$ has at least one solution $u_{k,\varepsilon}$ of the form* (1)*, satisfying $\lim_{k\to\infty} \lim_{\varepsilon\to 0} \rho_{k,\varepsilon} = \bar\rho$, $\lim_{k\to\infty} \lim_{\varepsilon\to 0} x_{i,k,\varepsilon} = \bar x_i$ for $i = 3, 4, \ldots, n$. Moreover, $\lim_{\varepsilon\to 0} \lambda_{k,\varepsilon} > 0$, $\forall k \geqslant \bar k$ and $\lim_{k\to\infty} \lim_{\varepsilon\to 0} \lambda_{k,\varepsilon} = 0$.*

Remark 3. – The "essential" critical points for φ constrained on $\overline{S_\Omega}$ are special points of the boundary of S_Ω. For example, if $(\bar\rho, \bar x_3, \ldots, \bar x_n)$ is, in a suitable neighbourhood, the only minimum point for φ constrained on $\overline{S_\Omega}$, then it is an "essential" critical point in the sense we need in Theorem 3. This theorem applies to a large class of domains. For example, let S be a bounded domain of R^{n-h} with $1 \leqslant h \leqslant n-1$, such that $\overline{S} \subset]0, +\infty[\times \mathbb{R}^{n-h-1}$, and set $\Omega_S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid ([\sum_{i=1}^{h+1} x_i^2]^{1/2}, x_{h+2}, \ldots, x_n) \in S\}$ (nontrivial domains of this type, like solid tori, have been considered in [13]). Then Theorem 3 gives solutions of $P(\varepsilon, \Omega_S)$, blowing-up at k points as $\varepsilon \to 0$ (other solutions, one can easily find exploiting the radial symmetry of Ω_S with respect to all the variables $x_1, \ldots, x_{h+1}$, present a different behaviour).

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