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Contractive liftings and the commutator

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Abstract This paper presents the solution to a problem proposed by B.Sz.-Nagy about extending the commutant lifting theorem to the case when the underlying operators do not intertwine. The main theorem establishes minimal norm liftings of certain commutators. The proof is constructive. ***To cite this article: C. Foias et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 431–436.***

Dilatations contractive et le commutateur

Résumé Cet article présente la solution d'un problème posé par B.Sz.-Nagy sur l'extension du théorème de dilatation des commutants au cas des opérateurs sous-jacents qui ne commutent pas. Le théorème principal établit des dilatations aux normes minimales de certains commutateurs. La démonstration est constructive. ***Pour citer cet article : C. Foias et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 431–436.***

Version française abrégée

En 1968 Sz.-Nagy [6] a posé le problème de l'extension du théorème de dilatation des commutants au cas des opérateurs sous-jacents qui ne commutent pas. Cet article présente une solution à ce problème. Pour être précis, rappelons d'abord le théorème de la dilatation des commutants. Soit $\mathcal{H}$, $\mathcal{H}'$ et $\mathcal{K}'$ des espaces d'Hilbert. Ici $\mathcal{H}'$ est un sous-espace de $\mathcal{K}'$, et P' la projection orthogonale de $\mathcal{K}'$ sur $\mathcal{H}'$. Rappelons aussi que U' sur $\mathcal{K}'$ est une *dilatation isométrique* de T' sur $\mathcal{H}'$ si U' est une isométrie sur $\mathcal{K}'$ et $P'U' = T'P'$. Le résultat suivant est une des nombreuses formulations équivalentes du théorème de dilatation des commutants ; voir Section VII.1 dans [1].

THÉORÈME 0.1. – *Soit $A : \mathcal{H} \to \mathcal{H}'$ une contraction entrelaçant une isométrie T sur $\mathcal{H}$ et une contraction T' sur $\mathcal{H}'$, c'est-à-dire, $T'A = AT$. Soit U' sur $\mathcal{K}'$ une dilatation isométrique de T'. Alors il existe une contraction B de $\mathcal{H}$ dans $\mathcal{K}'$ telle que $P'B = A$ et $U'B = BT$.*

Le résultat suivant est notre solution au problème de B. Sz.-Nagy.

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THÉORÈME 0.2. – *Soit $A:\mathcal{H}\to\mathcal{H}'$ une contraction, T sur $\mathcal{H}$ une isométrie, et T' sur $\mathcal{H}'$ une contraction. Soit U' sur $\mathcal{K}'$ une dilatation isométrique de T'. Alors il existe une contraction B de $\mathcal{H}$ sur $\mathcal{K}'$ telle que*

$$P'B = A \quad et \quad \|U'B - BT\| \leqslant \sqrt{2}\|T'A - AT\|^{1/2}.$$

La constante $\sqrt{2}$ ci-dessus est optimale. Le théorème de dilatation des commutants est un cas particulier du théorème précédent. Notre démonstration est constructive. Le Théorème 0.2 est un cas particulier du résultat suivant établit aussi dans cette note.

THÉORÈME 0.3. – *Soit $A:\mathcal{H}\to\mathcal{H}'$ une contraction, $\mathcal{H}_0$ un sous-espace de $\mathcal{H}$, et $R:\mathcal{H}_0\to\mathcal{H}$ une contraction. Soit U' sur $\mathcal{K}'$ une dilatation isométrique de T' sur $\mathcal{H}'$. Alors il existe une contraction B de $\mathcal{H}$ sur $\mathcal{K}'$ telle que*

$$P'B = A \quad et \quad \|U'BR - B|\mathcal{H}_0\| \leqslant \sqrt{2}\|T'AR - A|\mathcal{H}_0\|^{1/2}.$$

Finalement, on remarque que le Théorème 0.3 peut être considéré comme une version relaxée du Théorème 0.2 donnant le théorème de dilatation relaxée de [3].

1. Introduction

In [4] Sarason encompassed many classical norm constrained interpolation problems in a representation theorem of operators commuting with special contractions. Motivated by this paper Sz.-Nagy and Foias [5] obtained a purely geometrical extension now known as the commutant lifting theorem. The commutant lifting theorem turned out to be a useful tool in solving completion and extension problems in mathematics and engineering [1,2]. To recall the commutant lifting theorem, let $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{K}'$ be Hilbert spaces. Here $\mathcal{H}'$ is a subspace of $\mathcal{K}'$, and we let P' be the orthogonal projection of $\mathcal{K}'$ onto $\mathcal{H}'$. Recall that U' on $\mathcal{K}'$ is an *isometric lifting* of T' on $\mathcal{H}'$ if U' is an isometry on $\mathcal{K}'$ and $P'U' = T'P'$. The following result is one of several equivalent formulations of the commutant lifting theorem; see Section VII.1 in [1].

THEOREM 1.1. – *Let A mapping $\mathcal{H}$ into $\mathcal{H}'$ be a contraction intertwining an isometry T on $\mathcal{H}$ with a contraction T' on $\mathcal{H}'$, that is, $T'A = AT$. Let U' on $\mathcal{K}'$ be an isometric lifting of T'. Then there exists a contraction B from $\mathcal{H}$ into $\mathcal{K}'$ such that $P'B = A$ and $U'B = BT$.*

In 1968 Sz.-Nagy [6] posed the problem of extending the commutant lifting theorem to the case when the operator A does not intertwine T with T'. The problem is to construct a contraction B to minimize the distance $\|U'B - BT\|$. The following result is our solution to this problem.

THEOREM 1.2. – *Let A mapping $\mathcal{H}$ into $\mathcal{H}'$ be a contraction, T on $\mathcal{H}$ an isometry, and T' on $\mathcal{H}'$ a contraction. Furthermore, let U' on $\mathcal{K}'$ be an isometric lifting of T'. Then there exists a contraction B from $\mathcal{H}$ into $\mathcal{K}'$ such that*

$$P'B = A \quad and \quad \|U'B - BT\| \leqslant \sqrt{2}\|T'A - AT\|^{1/2}. \tag{1}$$

The above theorem contains the commutant lifting theorem as an obvious special case. The constant $\sqrt{2}$ in (1) is optimal, since for some simple examples one has always equality in (1).

Theorem 1.2 has a homogeneous version which appears when one does not require the operator A to be a contraction. For this case Theorem 1.2 implies that we can find an operator B from $\mathcal{H}$ into $\mathcal{K}'$ with $\|B\| = \|A\|$ such that

$$P'B = A \quad \text{and} \quad \|U'B - BT\| \leqslant \sqrt{2}\big(\|A\|\|T'A - AT\|\big)^{1/2}. \tag{2}$$

We will derive Theorem 1.2 as a special case of the following more general result.

THEOREM 1.3. – *Let A mapping $\mathcal{H}$ into $\mathcal{H}'$ be a contraction, $\mathcal{H}_0$ a subspace of $\mathcal{H}$, and R a contraction mapping $\mathcal{H}_0$ into $\mathcal{H}$. Furthermore, let U' on $\mathcal{K}'$ be an isometric lifting of T' on $\mathcal{H}'$. Then there exists a contraction B from $\mathcal{H}$ into $\mathcal{K}'$ such that*

$$P'B = A \quad \textit{and} \quad \|U'BR - B|\mathcal{H}_0\| \leqslant \sqrt{2}\|T'AR - A|\mathcal{H}_0\|^{1/2}. \tag{3}$$

Theorem 1.3 can be viewed as a relaxed version of Theorem 1.2. As a corollary it yields the relaxed lifting theorem in [3]. Our proof of Theorem 1.3 is constructive, that is, we construct a contraction B satisfying (3); see Section 3 below.

2. Lifting operators modulo a perturbation

In this section we will show that Theorem 1.2 is an easy consequence of Theorem 1.3. Actually, we shall prove a bit more. We shall use Theorem 1.3 to derive the following analog of Theorem 1.2 which corresponds to the Treil–Volberg generalization [8] of the commutant lifting theorem.

THEOREM 2.1. – *Let A be an operator from $\mathcal{H}$ into $\mathcal{H}'$, let T on $\mathcal{H}$ be an operator satisfying $I \leqslant T^*T$, and let T' on $\mathcal{H}'$ be a contraction. Furthermore, let U' on $\mathcal{K}'$ be an isometric lifting of T'. Then there exists an operator B from $\mathcal{H}$ into $\mathcal{K}'$ with $\|B\| = \|A\|$ such that*

$$P'B = A \quad \textit{and} \quad \|U'B - BT\| \leqslant \sqrt{2}\|T\|\big(\|A\|\|T'A - AT\|\big)^{1/2}. \tag{4}$$

Proof. – Without loss of generality we may assume that $\|A\| = 1$. Set $\mathcal{H}_0 = T\mathcal{H}$, and define R from $\mathcal{H}_0$ into $\mathcal{H}$ by $R(Th) = h$ for each $h \in \mathcal{H}$. The fact that $\|Th\| \geqslant \|h\|$ implies that the space $\mathcal{H}_0$ is closed and that R is a well defined contraction. By Theorem 1.3 there exists an operator B from $\mathcal{H}$ into $\mathcal{K}'$ with $\|B\| = \|A\| = 1$ such that

$$\begin{aligned}\|U'B - BT\| &= \|U'BRT - BT\| \leqslant \|U'BR - B|\mathcal{H}_0\|\|T\| \\ &\leqslant \sqrt{2}\|T\|\|T'AR - A|\mathcal{H}_0\|^{1/2}.\end{aligned}$$

The inequality $\|Th\| \geqslant \|h\|$ also yields $\{h \in \mathcal{H} \mid \|Th\| \leqslant 1\} \subset \{h \in \mathcal{H} \mid \|h\| \leqslant 1\}$. Hence

$$\begin{aligned}\|T'AR - A|\mathcal{H}_0\| &= \sup_{\|Th\|\leqslant 1} \|T'ARTh - ATh\| \\ &\leqslant \sup_{\|h\|\leqslant 1} \|T'Ah - ATh\| = \|T'A - AT\|.\end{aligned}$$

By combining this with the previous inequality we obtain the desired norm estimate. This completes the proof. □

It is clear that Theorem 1.2 (when T is an isometry and $\|A\| \leqslant 1$) is a particular case of Theorem 2.1.

3. Proof of Theorem 1.3

We split the proof of Theorem 1.3 into three parts.

PART 1. Our starting point is the intertwining relation

$$T'AR - A|\mathcal{H}_0 = \Delta. \tag{5}$$

Recall that A mapping $\mathcal{H}$ into $\mathcal{H}'$ is a contraction, and $\mathcal{H}_0$ is a subspace of $\mathcal{H}$, while R is a contraction from $\mathcal{H}_0$ into $\mathcal{H}$ and T' is a contraction on $\mathcal{H}'$. Finally, Δ is an operator from $\mathcal{H}_0$ into $\mathcal{H}'$.

Let U' on $\mathcal{K}'$ be an isometric lifting of T'. Our aim is to construct a contraction B from $\mathcal{H}$ into $\mathcal{K}'$ such that

$$P'B = A \quad \text{and} \quad \|U'BR - B|\mathcal{H}_0\| \leqslant \sqrt{2}\|\Delta\|^{1/2}. \tag{6}$$

Recall that all minimal isometric lifting are isomorphic; see Theorem I.4.1 in [7] or Proposition IV.1.3 in [2]. So without loss of generality (*cf.* [2], Section IV.2) assume that U' is the Sz.-Nagy–Schäffer minimal isometric lifting of T', that is,

$$U' = \begin{bmatrix} T' & 0 & 0 & 0 & \cdots \\ D_{T'} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \mathcal{D}_{T'} \\ \mathcal{D}_{T'} \\ \mathcal{D}_{T'} \\ \vdots \end{bmatrix}. \tag{7}$$

Here $D_{T'}$ is the defect operator $(I - T'^*T')^{1/2}$ and $\mathcal{D}_{T'}$ is the closure of $D_{T'}\mathcal{H}'$.

PART 2. Let $D_A = (I - A^*A)^{1/2}$ and $\mathcal{D}_A$ be the closure of $D_A\mathcal{H}$. Set $X = T'AR$. Since the operators T', A and R are contractive, the operator X is also a contraction. Thus for each $h_0 \in \mathcal{H}_0$ we have

$$\|D_X h_0\|^2 = \|D_{T'}ARh_0\|^2 + \|D_A R h_0\|^2 + \|D_R h_0\|^2 \geqslant \|D_{T'}ARh_0\|^2 + \|D_A R h_0\|^2. \tag{8}$$

Let A_0 be the operator from $\mathcal{H}_0$ into $\mathcal{H}'$ defined by $A_0 = A|\mathcal{H}_0$. Eq. (5) implies that $X = \Delta + A_0$. Since X and A_0 are contractions, we have $\|\Delta\| \leqslant 2$. Set $\gamma = 2\|\Delta\|$. Then both $\Delta^*\Delta$ and $A_0^*\Delta + \Delta^*A_0$ have norm at most equal to γ. In particular, $D_{\Delta,\gamma} = (\gamma I - \Delta^*\Delta)^{1/2}$ on $\mathcal{H}_0$ is well defined. Next, notice that

$$X^*X = A_0^*A_0 + A_0^*\Delta + \Delta^*A_0 + \Delta^*\Delta.$$

Thus for each $h_0 \in \mathcal{H}_0$ we have

$$\begin{aligned} \|D_X h_0\|^2 &= \big((I - X^*X)h_0, h_0\big) \leqslant \big((I - A_0^*A_0)h_0, h_0\big) + \big((\gamma I - \Delta^*\Delta)h_0, h_0\big) \\ &= \|D_A h_0\|^2 + \|D_{\Delta,\gamma} h_0\|^2. \end{aligned}$$

Combining this with (8) yields

$$\left\| \begin{bmatrix} D_A \\ D_{\Delta,\gamma} \end{bmatrix} h_0 \right\| \geqslant \left\| \begin{bmatrix} D_{T'}AR \\ D_A R \end{bmatrix} h_0 \right\| \quad (h_0 \in \mathcal{H}_0). \tag{9}$$

Let $\widetilde{D}_A$ be the operator from $\mathcal{H}_0$ into $\mathcal{H}$ defined by $\widetilde{D}_A = D_A|\mathcal{H}_0$. Since $D_A\mathcal{H}_0$ is contained in $\mathcal{D}_A$, there exists a 2×2 contractive operator matrix

$$\begin{bmatrix} C & P \\ Z & Q \end{bmatrix} : \begin{bmatrix} \mathcal{D}_A \\ \mathcal{H}_0 \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix} \tag{10}$$

satisfying

$$\begin{bmatrix} C & P \\ Z & Q \end{bmatrix} \begin{bmatrix} \widetilde{D}_A \\ D_{\Delta,\gamma} \end{bmatrix} = \begin{bmatrix} D_{T'}AR \\ D_A R \end{bmatrix}. \tag{11}$$

Clearly, the operator $[C^* \ Z^*]^*$ mapping $\mathcal{D}_A$ into $\mathcal{D}_{T'} \oplus \mathcal{D}_A$ is a contraction. According to Lemma 3.1 below the operator

$$\Gamma = \begin{bmatrix} C \\ CZ \\ CZ^2 \\ \vdots \end{bmatrix} : \mathcal{D}_A \to \ell_+^2(\mathcal{D}_{T'}) \tag{12}$$

is also a contraction.

PART 3. In this part we will construct a contraction B satisfying (3). Consider the operator B given by

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ \ell_+^2(\mathcal{D}_{T'}) \end{bmatrix}. \tag{13}$$

Notice that $\mathcal{K}' = \mathcal{H}' \oplus \ell^2_+(\mathcal{D}_{T'})$. Clearly, $P'B = A$. Since Γ is a contraction, B is a contraction from $\mathcal{H}$ into $\mathcal{K}'$; see Lemma IV.1.1 in [1]. Using $\widetilde{D}_A = D_A|\mathcal{H}_0$ along with the form of U' in (7) we obtain

$$U'BR - B|\mathcal{H}_0 = \begin{bmatrix} \Delta \\ D_{T'}AR - \Gamma_0\widetilde{D}_A \\ \Gamma_0 D_A R - \Gamma_1\widetilde{D}_A \\ \Gamma_1 D_A R - \Gamma_2\widetilde{D}_A \\ \vdots \end{bmatrix}, \tag{14}$$

where $\Gamma_j = CZ^j$ for all integers $j \geqslant 0$. Eq. (11) yields $D_{T'}AR = C\widetilde{D}_A + PD_{\Delta,\gamma}$. In other words,

$$D_{T'}AR - \Gamma_0\widetilde{D}_A = PD_{\Delta,\gamma}. \tag{15}$$

Using $D_A R = Z\widetilde{D}_A + QD_{\Delta,\gamma}$ from (11), we arrive at

$$\Gamma_j D_A R = CZ^j D_A R = CZ^{j+1}\widetilde{D}_A + CZ^j QD_{\Delta,\gamma}.$$

This readily implies that

$$\Gamma_j D_A R - \Gamma_{j+1}\widetilde{D}_A = CZ^j QD_{\Delta,\gamma} \quad (j \geqslant 0). \tag{16}$$

Now fix $h_0 \in \mathcal{H}_0$. Using (14), (15) and (16) we see that

$$\left\|(U'BR - B|\mathcal{H}_0)h_0\right\|^2 = \|\Delta h_0\|^2 + \|PD_{\Delta,\gamma}h_0\|^2 + \sum_{k=0}^{\infty}\left\|CZ^k QD_{\Delta,\gamma}h_0\right\|^2.$$

Recall that the operator Γ in (12) is a contradiction. Because $[P^*\ \ Q^*]^*$ is also a contraction, we obtain

$$\begin{aligned}\left\|(U'BR - B|\mathcal{H}_0)h_0\right\|^2 &\leqslant \|\Delta h_0\|^2 + \|PD_{\Delta,\gamma}h_0\|^2 + \|QD_{\Delta,\gamma}h_0\|^2 \\ &\leqslant \|\Delta h_0\|^2 + \|D_{\Delta,\gamma}h_0\|^2 = \gamma\|h_0\|^2.\end{aligned}$$

Hence $\|U'BR - B|\mathcal{H}_0\| \leqslant \sqrt{\gamma}$. Since $\gamma = 2\|\Delta\|$, this completes the proof. □

Notice that by choosing $R = T^*|\mathcal{H}_0$ where $\mathcal{H}_0 = T\mathcal{H}$, then the contraction B in (13) satisfies (1) in Theorem 1.2. The following result was used in the proof of Theorem 1.3.

LEMMA 3.1. – *Let C mapping $\mathcal{X}$ into $\mathcal{Y}$ and Z on $\mathcal{X}$ be operators such that*

$$\begin{bmatrix} C \\ Z \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \tag{17}$$

is a contraction. Then the operator

$$\Gamma = \begin{bmatrix} C \\ CZ \\ CZ^2 \\ \vdots \end{bmatrix} : \mathcal{X} \to \ell^2_+(\mathcal{Y}) \tag{18}$$

is a contraction.

Proof. – Let x be a vector in $\mathcal{X}$. Because $[C^*\ \ Z^*]^*$ is a contraction, $\|Cx\|^2 + \|Zx\|^2 \leqslant \|x\|^2$. By applying the previous inequality with $Z^k x$ in place of x we see that $\|CZ^k x\|^2 \leqslant \|Z^k x\|^2 - \|Z^{k+1}x\|^2$. For any positive integer n this implies that

$$\sum_{k=0}^{n}\|CZ^k x\|^2 \leqslant \sum_{k=0}^{n}\left(\|Z^k x\|^2 - \|Z^{k+1}x\|^2\right) = \|x\|^2 - \|Z^{n+1}x\|^2 \leqslant \|x\|^2.$$

Since this holds for each n, the vector Γx is in $\ell^2_+(\mathcal{Y})$ and $\|\Gamma x\| \leqslant \|x\|$. This completes the proof. □

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