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Second order Hamilton–Jacobi–Bellman inequalities

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Abstract This work is devoted to the study of a class of Hamilton–Jacobi–Bellman inequalities which come from an optimal control problem where the state equation is a stochastic variational inequality. We show that the value function, which minimizes the cost, is a viscosity solution of the studied equation. This approach is made by perturbing the initial problem. Then we prove the uniqueness of the inequality. ***To cite this article: A. Zălinescu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 591–596.***

Inéquations de Hamilton–Jacobi–Bellman de deuxième ordre

Résumé Cette note est consacrée à l'étude des inéquations aux dérivées partielles du type Hamilton–Jacobi–Bellman issues d'un problème de contrôle optimal où l'équation d'état est une inéquation variationnelle stochastique. En fait, on démontre que la fonction minimisant la fonctionnelle de coût est une solution de viscosité pour l'équation étudiée. Cette approche est menée par une méthode de perturbation du problème initial. L'unicité de la solution de viscosité est également prouvée. ***Pour citer cet article : A. Zălinescu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 591–596.***

Version française abrégée

Existence et unicité pour des inéquations du type Hamilton–Jacobi du premier ordre ont été montrées dans plusieurs articles, par exemple [5] ou [6]. Ici, en partant d'un problème de contrôle stochastique, on prolonge ces résultats aux inéquations de deuxième ordre.

Soit $\nu = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geqslant 0}, W)$ un *système de référence* composé d'un espace probabilisé complet $(\Omega, \mathcal{F}, P)$, d'une filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfaisant les conditions habituelles de complétude et de continuité à droite, et d'un $\{\mathcal{F}_t\}$-mouvement brownien d'-dimensionnel W. Etant donné un espace métrique compact U, on note $\mathcal{A}_\nu$ l'ensemble des processus $u(\cdot)$ $\{\mathcal{F}_t\}$-progressivement mesurable à valeurs dans U. Les éléments $u(\cdot) \in \mathcal{A}_\nu$ sont appelés *contrôles admissibles.*

Etant donné un horizon de temps $T > 0$, on considère le système contrôlé défini par l'inéquation variationnelle stochastique suivante :

$$\begin{cases} \mathrm{d}X_s + \partial\varphi(X_s)\,\mathrm{d}s \ni b(s, X_s, u_s)\,\mathrm{d}s + \sigma(s, X_s, u_s)\,\mathrm{d}W_s, & s \in [t, T]\,; \\ X_t = x, \end{cases} \tag{1}$$

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dont les coefficients $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$, $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ et $\sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d'}$ satisfont :

(H1) φ est convexe et s.c.i. ;
(H2) $\mathrm{Int}(\mathrm{Dom}\,\varphi) \neq \emptyset$;
(H3) b et σ sont Lipschitz en $x \in \mathbb{R}^d$, uniformément en $(t,u) \in [0,T] \times U$;
(H4) b et σ sont uniformément continues.

Pour tout $(t,x) \in [0,T] \times \mathrm{Dom}\,\varphi$ et $u(\cdot) \in \mathcal{A}_\nu$, il existe une unique solution forte

$$\big(X^{t,x,u}, \eta^{t,x,u}\big) \in \mathrm{L}^2_{ad}\big(\Omega; \mathrm{C}\big([t,T];\mathbb{R}^d\big) \times \big(\mathrm{L}^2_{ad}\big(\Omega; \mathrm{C}\big([t,T];\mathbb{R}^d\big)\big) \cap \mathrm{L}^1\big(\Omega; BV\big([t,T];\mathbb{R}^d\big)\big)\big)$$

de l'équation (1) (cf. [1]), le processus à variation borné $\eta^{t,x,u}$ satisfaisant formellement (puisque il n'est pas nécessairement absolument continu) la relation « $\mathrm{d}\eta_s^{t,x,u} \in \partial\varphi(X_s^{t,x,u})\,\mathrm{d}s$ », décrite rigoureusement par

$$\int_s^{s'} \big\langle z - X_r^{t,x,u}, \mathrm{d}\eta_r^{t,x,u} \big\rangle + \int_s^{s'} \varphi\big(X_r^{t,x,u}\big)\,\mathrm{d}r \leqslant (s'-s)\varphi(z), \quad \forall t \leqslant s \leqslant s' \leqslant T,\ \forall z \in \mathbb{R}^d.$$

Initialement défini pour $x \in \mathrm{Dom}\,\varphi$, on peut prolonger $X^{t,x,u}$ ($t \in [0,T]$, $u(\cdot) \in \mathcal{A}_\nu$) à $x \in \overline{\mathrm{Dom}\,\varphi}$, en utilisant la propriété de continuité de la solution de l'équation (1) par rapport aux données initiales.

On considère le problème de Mayer associé à l'équation d'état. Etant donnée une fonction $h : \mathbb{R}^d \to \mathbb{R}$ continue et à croissance au plus polynômiale, on définit

$$V(t,x) := \inf_{\nu, u \in \mathcal{A}_\nu} \mathbb{E}h\big(X_T^{t,x,u}\big), \quad (t,x) \in [0,T] \times \overline{\mathrm{Dom}\,\varphi}.$$

Cette fonction est aussi continue et à croissance au plus polynômiale. Dans ce travail on caractérise la fonction V comme l'unique solution de viscosité de l'inéquation de Hamilton–Jacobi–Bellman

$$\begin{cases} \dfrac{\partial v}{\partial t} + \inf\limits_{u\in U} \mathcal{L}\big(t,x,u,Dv,D^2v\big) \in \big\langle \partial\varphi(x), Dv \big\rangle & \text{dans }]0,T[\, \times \overline{\mathrm{Dom}\,\varphi}, \\ v(T,\cdot) = h \quad \text{sur } \overline{\mathrm{Dom}\,\varphi}, \end{cases} \tag{2}$$

avec

$$\mathcal{L}(t,x,u,q,Q) := \frac{1}{2}\,\mathrm{tr}\,\sigma\sigma^{\mathbf{T}}(t,x,u)Q + \big\langle b(t,x,u), q \big\rangle, \quad (t,x,u,q,Q) \in [0,T] \times \mathbb{R}^d \times U \times \mathbb{R}^d \times \mathcal{S}_d.$$

Pour démontrer que V est solution de l'équation (2), on utilise la méthode de « pénalisation » du problème initial, en considérant l'équation

$$\begin{cases} \mathrm{d}X_s + \nabla\varphi_\varepsilon(X_s)\,\mathrm{d}s = b(s,X_s,u_s)\,\mathrm{d}s + \sigma(s,X_s,u_s)\,\mathrm{d}W_s, \quad s \in [t,T]; \\ X_t = x, \end{cases} \tag{3}$$

où $\varphi_\varepsilon(x)$ désigne la regularisation de Yosida de φ,

$$\varphi_\varepsilon(x) := \inf\big\{\tfrac{1}{2\varepsilon}|z-x|^2 + \varphi(z);\ z \in \mathbb{R}^d\big\},\ x \in \mathbb{R}^d,$$

qui est convexe, différentiable, et son gradient $\nabla\varphi_\varepsilon$ est Lipschitzien de norme $1/\varepsilon$ [2]. En vertue de cettes propriétés, l'équation (3) admet une unique solution $X^{\varepsilon,t,x,u} \in \mathrm{L}^2_{ad}(\Omega; \mathrm{C}([t,T];\mathbb{R}^d))$ qui converge vers $X^{t,x,u}$ dans cet espace. Le calcul montre que la fonction valeur pénalisée

$$V_\varepsilon(t,x) := \inf_{\nu, u \in \mathcal{A}_{t\nu}} \mathbb{E}h\big(X_T^{\varepsilon,t,x,u}\big)$$

converge uniformément vers V sur les compacts dans $[0,T] \times \overline{\mathrm{Dom}\,\varphi}$. Comme V_ε est une solution de viscosité (cf. [4]) de l'équation

$$\begin{cases} \dfrac{\partial V_\varepsilon}{\partial t} + \inf\limits_{u\in U} L\big(t,x,u,DV_\varepsilon,D^2V_\varepsilon\big) - \langle \nabla\varphi_\varepsilon, DV_\varepsilon \rangle = 0 & \text{dans }]0,T[\, \times \mathbb{R}^d, \\ V_\varepsilon(T,\cdot) = h \quad \text{sur } \mathbb{R}^d, \end{cases} \tag{4}$$

un passage à la limite permet d'obtenir l'existence d'une solution de l'équation (2).

Afin de démontrer l'unicité, on suit l'approche développée dans [3]. Les difficultés posées par la présence de la sous-différentielle dans l'équation sont surmontées en utilisant sa propriété de monotonie.

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THÉORÈME 1. – *Sous les hypothèses introduites ci-dessus, V est une solution de viscosité de l'équation* (2). *Si, en plus, b et σ sont Lipschitz en $t \in [0, T]$, uniformément en $(x, u) \in \mathbb{R}^d \times U$, alors V est l'unique solution de viscosité dans la classe des fonctions continues à croissance au plus polynômiale.*

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1. Introduction

The aim of this paper is to prove existence and uniqueness for a class of Hamilton–Jacobi–Bellman inequality of second order. Such equations have already been studied, for example in [5] or [6], but for the first order case.

Let $\nu = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geqslant 0}, W)$ be a *reference system* composed of a complete probability space $(\Omega, \mathcal{F}, P)$, a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions of completness and right-continuity, and a d'-dimensional $\{\mathcal{F}_t\}$-Brownian motion W. Let U be a compact metric space. By $\mathcal{A}_\nu$ we denote the set of all U-valued $\{\mathcal{F}_t\}$-progressively measurable processes $u(\cdot)$. The elements of $\mathcal{A}_\nu$ are called *admissible controls.*

Given a finite time horizon $T > 0$, we consider a stochastic control system described by the following stochastic variational inequality:

$$\begin{cases} \mathrm{d}X_s + \partial\varphi(X_s)\,\mathrm{d}s \ni b(s, X_s, u_s)\,\mathrm{d}s + \sigma(s, X_s, u_s)\,\mathrm{d}W_s, & s \in [t, T]; \\ X_t = x, \end{cases} \tag{5}$$

where the functions $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$, $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d'}$ are supposed to satisfy the following standard assumptions:

(H1) φ is convex and l.s.c.;
(H2) $\mathrm{Int}(\mathrm{Dom}\,\varphi) \neq \emptyset$;
(H3) b and σ are Lipschitz in $x \in \mathbb{R}^d$, uniformly in $(t, u) \in [0, T] \times U$;
(H4) b and σ are uniformly continuous.

DEFINITION 1.1. – We say that the pair of stochastic processes $(X, \eta) \in \mathrm{L}^2_{ad}(\Omega; \mathrm{C}([t, T]; \mathbb{R}^d)) \times (\mathrm{L}^2_{ad}(\Omega; \mathrm{C}([t, T]; \mathbb{R}^d)) \cap \mathrm{L}^1(\Omega; BV([t, T]; \mathbb{R}^d)))$ is a (strong) solution of Eq. (5) if almost surely

$$\eta_t = 0, \quad X_s + \eta_s = x + \int_t^s b(r, X_r, u_r)\,\mathrm{d}r + \int_t^s \sigma(r, X_r, u_r)\,\mathrm{d}W_r, \quad \forall s \in [t, T],$$

and

$$\int_s^{s'} \langle z - X_r, \mathrm{d}\eta_r\rangle + \int_s^{s'} \varphi(X_r)\,\mathrm{d}r \leqslant (s' - s)\varphi(z), \quad \forall t \leqslant s \leqslant s' \leqslant T,\ \forall z \in \mathbb{R}^d.$$

Under (H1)–(H4) (see [1]), for each $(t, x) \in [0, T] \times \mathrm{Dom}\,\varphi$ and $u(\cdot) \in \mathcal{A}_\nu$, there exists a unique strong solution $(X^{t,x,u}, \eta^{t,x,u})$ of Eq. (5). It depends continuously on the initial data. More precisely, for some constant C depending only on T, d, b and σ, we have

$$\mathbb{E} \sup_{s\in[t\vee t', T]} \left|X_s^{t,x,u} - X_s^{t',x',u}\right|^2 \leqslant C\left[|x - x'|^2 + \left(1 + |x|^2\right)|t - t'|\right]. \tag{6}$$

This estimate allows us to extend $X^{t,x,u}$ continuously to $x \in \overline{\mathrm{Dom}\,\varphi}$. Moreover, for every $p \geqslant 1$ there exists a positive constant C_p, depending only on T, d, L, p and $d(0, \mathrm{Dom}\,\varphi)$, such that

$$\mathbb{E} \sup_{s\in[t,T]} \left|X_s^{t,x,u}\right|^p \leqslant C_p\left(1 + |x|^p\right), \quad (t, x) \in [0, T] \times \overline{\mathrm{Dom}\,\varphi},\ u(\cdot) \in \mathcal{A}_\nu. \tag{7}$$

Let $h : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with polynomial growth and

$$V(t, x) := \inf_{\nu, u\in\mathcal{A}_\nu} \mathbb{E}h\left(X_T^{t,x,u}\right), \quad (t, x) \in [0, T] \times \overline{\mathrm{Dom}\,\varphi}.$$

The objective is to characterize this continuous value function with polynomial growth via the following Hamilton–Jacobi–Bellman inequalities:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \inf\limits_{u\in U} \mathcal{L}\big(t,x,u,Dv,D^2v\big) \in \big\langle \partial\varphi(x), Dv\big\rangle & \text{in }]0,T[\times \overline{\mathrm{Dom}\,\varphi}, \\ v(T,\cdot) = h \quad \text{on } \overline{\mathrm{Dom}\,\varphi}, \end{cases} \tag{8}$$

where, for $(t,x,u,q,Q) \in [0,T]\times\mathbb{R}^d\times U\times\mathbb{R}^d\times\mathcal{S}_d$,

$$\mathcal{L}(t,x,u,q,Q) := \frac{1}{2}\,\mathrm{tr}\,\sigma\sigma^{\mathbf{T}}(t,x,u)Q + \langle b(t,x,u), q\rangle.$$

Let us define

$$\partial\varphi_*(x;q) := \liminf_{(x',q')\to(x,q),\, x^*\in\partial\varphi(x')} \big\langle x^*, q'\big\rangle, \quad (x,q)\in\overline{\mathrm{Dom}\,\varphi}\times\mathbb{R}^d$$

and $\partial\varphi^*(x;q) := -\partial\varphi_*(x;-q)$.

DEFINITION 1.2. – We say that $v \in USC(]0,T]\times\overline{\mathrm{Dom}\,\varphi})$ ($\in LSC(]0,T]\times\overline{\mathrm{Dom}\,\varphi})$, resp.) is a sub-(super)solution of (8) if $v(T,\cdot)\leqslant h$ ($\geqslant$, resp.) on $\overline{\mathrm{Dom}\,\varphi}$ and if we have

$$\frac{\partial\Psi}{\partial t}(t,x) + \inf_{u\in U} L\big(t,x,u,D\Psi(t,x),D^2\Psi(t,x)\big) \geqslant \partial\varphi_*\big(x;D\Psi(t,x)\big) \quad (\leqslant \partial\varphi^*(x;D\Psi(t,x)),\ \text{resp.}) \tag{9}$$

whenever $\Psi\in C^{1,2}(]0,T[\times\overline{\mathrm{Dom}\,\varphi})$ such that $v-\Psi$ has a local maximum (minimum, resp.) point in $(t,x)\in\,]0,T[\times\overline{\mathrm{Dom}\,\varphi}$.

A viscosity solution will be a function which is both a subsolution and a supersolution.

2. Existence and uniqueness

The method used to prove that V is a viscosity solution for (8) is that of 'penalization'. We perturb the state equation by replacing φ by its Yosida regularization φ_ε, and we formulate the Mayer problem for this penalized equation by keeping the same final cost. Since φ_ε has a good behavior (differentiable, Lipschitz continuous), the penalized value function will be a solution of a Hamilton–Jacobi–Bellman equation. Passing to the limit, we obtain the following result:

THEOREM 2.1. – *Under the assumptions* (H1)–(H4) *and the assumptions on h, V is a viscosity solution of Eq.* (8).

In order to prove uniqueness, we impose a supplementary condition:
(H5) b and σ are Lipschitz in $t\in[0,T]$, uniformly in $(x,u)\in\mathbb{R}^d\times U$.

THEOREM 2.2. – *Suppose that the assumptions* (H1)–(H5) *hold and let $v\in USC(]0,T]\times\overline{\mathrm{Dom}\,\varphi})$ be a subsolution of* (8) *and $w\in LSC(]0,T]\times\overline{\mathrm{Dom}\,\varphi})$ a supersolution of Eq.* (8). *If v and w have polynomial growth, then $v\leqslant w$ on $]0,T]\times\overline{\mathrm{Dom}\,\varphi}$.*

Proof. – Let L be a constant which is greater than the upper bound of $|b(\cdot,0,\cdot)|+|\sigma(\cdot,0,\cdot)|$ on $[0,T]\times U$ and also greater then the Lipschitz constants of b and σ in (H3) and (H5).

Without restricting generality, we may assume that $\varphi(x)\geqslant\varphi(0)=0$, $\forall x\in\mathbb{R}^d$ and $0\in\mathrm{Int}(\mathrm{Dom}\,\varphi)$. This will imply that

$$\inf_{n\in N_{\overline{\mathrm{Dom}\,\varphi}}(x)} \langle n,x\rangle > 0, \quad \forall x\in\partial(\mathrm{Dom}\,\varphi), \tag{10}$$

$$\langle x^*,x\rangle \geqslant 0, \quad \forall x\in\mathrm{Dom}\,\varphi,\ \forall x^*\in\partial\varphi(x). \tag{11}$$

As v and w have polynomial growth, there exist $p\geqslant 1$ and $K>0$ such that

$$\big|v(t,x)\big| + \big|w(t,x)\big| \leqslant K\big(1+|x|^p\big), \quad \forall (t,x)\in\,]0,T]\times\overline{\mathrm{Dom}\,\varphi}. \tag{12}$$

Set $\mu := 4pL[1+4pL]$ and consider the equation

$$\begin{cases} -\mu v + \dfrac{\partial v}{\partial t} + \inf_{u\in U} L\big(t,x,u,Dv,D^2v\big) \in \big\langle \partial\varphi(x), Dv\big\rangle & \text{in }]0,T[\times \overline{\mathrm{Dom}\,\varphi}, \\ v(T,\cdot) = \mathrm{e}^{\mu T} h & \text{on } \overline{\mathrm{Dom}\,\varphi}. \end{cases} \tag{13}$$

In order to prove that $v \leqslant w$, it is sufficient to suppose that v, respectively w, is a subsolution, respectively, a supersolution of (13) and they satisfy (12). Indeed, the mapping $v \to \mathrm{e}^{\mu t} v$ transforms the subsolutions (supersolutions) of (8) in subsolutions (supersolutions) of (13).

We suppose, by absurd, that there exists $(t_0,x_0) \in]0,T]\times \overline{\mathrm{Dom}\,\varphi}$ such that $\theta := v(t_0,x_0) - w(t_0,x_0) > 0$, and put, for $\alpha > 0$, $\varepsilon > 0$ and $(t,x,s,y) \in (]0,T]\times \overline{\mathrm{Dom}\,\varphi})^2$:

$$\Psi_\alpha(t,x,s,y) := \frac{\alpha}{2}\big(|x-y|^2 + |t-s|^2\big) + \frac{\theta t_0}{8}\left(\frac{1}{t}+\frac{1}{s}\right) + \varepsilon|x|^{2p} + \varepsilon|y|^{2p},$$

$$\Phi_{\alpha,\varepsilon}(t,x,s,y) := v(t,x) - w(s,y) - \Psi_\alpha(t,x,s,y).$$

Let $M_{\alpha,\varepsilon} := \sup_{(]0,T]\times\overline{\mathrm{Dom}\,\varphi})^2} \Phi_{\alpha,\varepsilon}$, which is finite. Because of (12), there exists $\hat\xi \in (]0,T]\times \overline{\mathrm{Dom}\,\varphi})^2$, $\hat\xi \equiv (\hat t,\hat x,\hat s,\hat y)$, such that $\Phi_{\alpha,\varepsilon}(\hat\xi) = M_{\alpha,\varepsilon}$ (we did not mention the dependence of ε and α, for the sake of simplicity). By a standard argument one can prove that if $\varepsilon \leqslant \varepsilon_0 := \theta/(4|x_0|^{2p})$, then

$$v(\hat t,\hat x) - w(\hat s,\hat y) - \varepsilon\big(|\hat x|^{2p} + |\hat y|^{2p}\big) \geqslant M_{\alpha,\varepsilon} \geqslant \frac{1}{2}\theta, \quad \forall \alpha > 0, \tag{14}$$

$$\lim_{\alpha\to\infty} \alpha\big(|x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 + |t_{\alpha,\varepsilon} - s_{\alpha,\varepsilon}|^2\big) = 0, \tag{15}$$

and $\hat\xi \in (]0,T[\times \overline{\mathrm{Dom}\,\varphi})^2$ if α greater than a certain $\alpha_\varepsilon > 0$. From now on we will take $\varepsilon \leqslant \varepsilon_0$, $\alpha \geqslant \alpha_\varepsilon$.

Put $\beta(x) := 2p|x|^{2p-2}x$ and $B(x) := 2p|x|^{2p-2}\big[\frac{2p-2}{|x|^2} x\otimes x + I\big]$, for $x \in \mathbb{R}^d$. By Theorem 3.2 in [3], there exist $\widehat Q, \widehat R \in \mathcal{S}_d$ such that

$$\left(\frac{\partial\Psi_\alpha}{\partial t}(\hat\xi), D_x\Psi_\alpha(\hat\xi) + \varepsilon\beta(\hat x), \widehat Q + \varepsilon B(\hat x)\right) \in \overline{\mathcal{P}}^{1,2,+}_{\overline{\mathrm{Dom}\,\varphi}} v(\hat t,\hat x), \tag{16}$$

$$\left(-\frac{\partial\Psi_\alpha}{\partial s}(\hat\xi), -D_y\Psi_\alpha(\hat\xi) - \varepsilon\beta(\hat y), \hat R - \varepsilon B(\hat y)\right) \in \overline{\mathcal{P}}^{1,2,-}_{\overline{\mathrm{Dom}\,\varphi}} w(\hat s,\hat y), \tag{17}$$

and satisfying

$$\begin{pmatrix} \widehat Q & 0 \\ 0 & -\widehat R \end{pmatrix} \leqslant 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{18}$$

Thanks to the lower semicontinuity of $\partial\varphi_*$ (which clearly implies the upper semicontinuity of $\partial\varphi^*$) and the fact that v and w are subsolution and supersolution respectively, of Eq. (13), the following relations hold (we have suppressed the arguments of functions, in order to shorten the formulas):

$$-\mu v + \frac{\partial\Psi_\alpha}{\partial t} + \inf_{u\in U} L(\hat t,\hat x,u,D_x\Psi_\alpha + \varepsilon\beta, \hat Q + \varepsilon B) \geqslant \partial\varphi_*(\hat x; D_x\Psi_\alpha + \varepsilon\beta), \tag{19}$$

$$\mu w - \frac{\partial\Psi_\alpha}{\partial s} + \inf_{u\in U} L(\hat s,\hat y,u,-D_y\Psi_\alpha - \varepsilon\beta, \hat R - \varepsilon B) \leqslant \partial\varphi^*(\hat y; -D_y\Psi_\alpha - \varepsilon\beta). \tag{20}$$

An important aspect of the proof consists in the following lemma, whose proof we skip.

LEMMA 2.3. – *If* $x \in \mathrm{Int}(\mathrm{Dom}\,\varphi)$, $q \in \mathbb{R}^d$, *then* $\partial\varphi_*(x;q) = \inf_{x^*\in\partial\varphi(x)}\langle x^*, q\rangle$. *This equality also holds if* $x \in \partial(\mathrm{Dom}\,\varphi)$, $q \in \mathbb{R}^d$ *and* $\inf_{n\in N_{\overline{\mathrm{Dom}\,\varphi}}(x)}\langle n, q\rangle > 0$.

We are going to apply it. Relation (10) yields

$$\inf_{n\in N_{\overline{\mathrm{Dom}\,\varphi}}(\hat{x})}\left\langle n, D_x\Psi_\alpha(\hat{\xi})+\varepsilon\beta(\hat{x})\right\rangle = \inf_{n\in N_{\overline{\mathrm{Dom}\,\varphi}}(\hat{x})}\left\langle n, \alpha(\hat{x}-\hat{y})+2\varepsilon p|\hat{x}|^{2p-2}\hat{x}\right\rangle$$
$$\geqslant 2\varepsilon p|\hat{x}|^{2p-2}\inf_{n\in N_{\overline{\mathrm{Dom}\,\varphi}}(\hat{x})}\langle n,\hat{x}\rangle > 0,$$

if $\hat{x}\in\partial(\mathrm{Dom}\,\varphi)$. Hence

$$\partial\varphi_*\left(\hat{x}; D_x\Psi_\alpha(\hat{\xi})+\varepsilon\beta(\hat{x})\right) = \inf_{x^*\in\partial\varphi(\hat{x})}\left\langle x^*, \alpha(\hat{x}-\hat{y})+\varepsilon\beta(\hat{x})\right\rangle.$$

Analogously,

$$\partial\varphi^*\left(\hat{y}; -D_y\Psi_\alpha(\hat{\xi})-\varepsilon\beta(\hat{y})\right) = \sup_{y^*\in\partial\varphi(\hat{y})}\left\langle y^*, \alpha(\hat{x}-\hat{y})-\varepsilon\beta(\hat{y})\right\rangle.$$

If $x^*\in\partial\varphi(\hat{x})$ and $y^*\in\partial\varphi(\hat{y})$, then, by (11) and from the fact that $\partial\varphi$ is a monotone operator,

$$\left\langle x^*, \alpha(\hat{x}-\hat{y})+\varepsilon\beta(\hat{x})\right\rangle \geqslant \left\langle x^*, \alpha(\hat{x}-\hat{y})\right\rangle \geqslant \left\langle y^*, \alpha(\hat{x}-\hat{y})\right\rangle \geqslant \left\langle y^*, \alpha(\hat{x}-\hat{y})-\varepsilon\beta(\hat{y})\right\rangle,$$

hence $\partial\varphi_*(\hat{x}; D_x\Psi_\alpha(\hat{\xi})+\varepsilon\beta(\hat{x})) \geqslant \partial\varphi^*(\hat{y}; -D_y\Psi_\alpha(\hat{\xi})-\varepsilon\beta(\hat{y}))$. From (18), (19) and (20) we conclude

$$\mu\left(v(\hat{t},\hat{x})-w(\hat{s},\hat{y})\right)+\frac{\theta t_0}{8}\left(\frac{1}{\hat{t}^2}+\frac{1}{\hat{s}^2}\right)$$
$$\leqslant 3\left(L+\alpha L^2\right)\left(|\hat{x}-\hat{y}|^2+|\hat{t}-\hat{s}|^2\right)+4pL[1+4pL]\varepsilon\left(1+|\hat{x}|^{2p}+|\hat{y}|^{2p}\right).$$

Due to the choice of μ and relation (14), this gives

$$\tfrac{1}{2}\theta \leqslant 3\left(L+\alpha L^2\right)\left(|\hat{x}-\hat{y}|^2+|\hat{t}-\hat{s}|^2\right)+\varepsilon\mu.$$

Finally, letting $\alpha\to\infty$, and then $\varepsilon\to 0$, we obtain a contradiction. □

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